# *On the Wave-induced Difference in Mean Sea Level Between the Two Sides of a Submerged Breakwater*[1]

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ABSTRACT

Very simple formulae are derived for the difference in mean level between the two sides of a submerged breakwater when waves are incident on it at an arbitrary angle. The formulae apply also to waves undergoing refraction due to changes in depth and to waves in open channel transitions.

When sea waves approach a submerged breakwater or an offshore sand bar, the mean level of the water on the far side of the bar or breakwater is commonly observed to be higher than on the side from which the waves are incident. The purpose of this note is to show that the difference in mean water level can be calculated very simply in certain circumstances, once the height of the incident waves and the coefficient of reflection are both known.

The situation is as shown in Fig. 1. A submerged "breakwater" separates two uniform regions in which the undisturbed depths are $h_1$ and $h_2$, say. Waves of amplitude $a_1$ are propagated from the left and are incident (not necessarily normally) on the "breakwater." There is a transmitted wave of amplitude $a_2$ and a reflected wave of amplitude $a_1'$.

If the steepness of the waves is sufficiently small everywhere, then the coefficients of transmission and reflection, namely

$$T = a_2/a_1 \quad \text{and} \quad R = a_1'/a_1 \ ,$$

are nearly independent of $a_1$. The coefficients $R$ and $T$ may be determined by experiment or, in some ideal cases, by the linear theory of water waves. (For some examples, see the REFERENCES.) In the neighborhood of the breakwater

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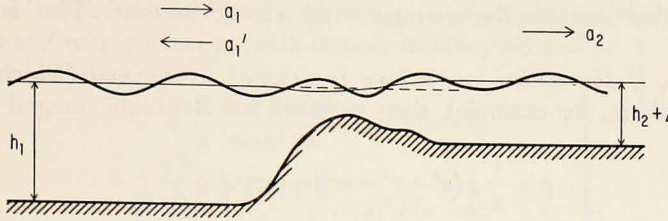


Figure 1.

itself the waves are not generally sinusoidal; nevertheless, the motion everywhere fluctuates harmonically with time, say with period $2\pi/\sigma$. The wavelength, $2\pi/k_1$, of the waves on the near side of the breakwater is related to the frequency $\sigma$ and to the local depth by the usual relationship

$$\sigma^2 = gk_1 \tanh k_1 h_1 ,$$

and similarly for the waves on the far side.

These results can be derived from the well-known small-amplitude theory of water waves. However, on the two sides of the barrier there will be a difference, $\Delta \bar{\zeta}$, in mean surface level that is of second order in the wave amplitude. We shall see that $\Delta \bar{\zeta}$, though of second order, can be determined directly from the first approximation in the following way.

Let $x$ and $y$ be horizontal coordinates and $z$ be measured vertically upward from the still-water level. Let $u$, $v$, and $w$ denote the corresponding components of velocity and $p$ the pressure. Let $\varrho$ and $g$ denote the density and the acceleration of gravity, both assumed constant. The free surface is denoted by $z = \zeta(x, y, t)$. Neglecting viscous forces, we then have two simple relationships (cf Longuet-Higgins and Stewart 1964).

First, consider the flux of *vertical momentum* into a vertical column of water of unit cross section contained between $z = 0$ and $z = \zeta$. The flux upward through the base of the column equals $(p + \varrho w^2)$ evaluated at $z = 0$. The flux through the upper surface of the column is zero. The flux of vertical momentum through the *sides* of the column, which is $\varrho uw$ per unit area, is of third order when integrated over the height $\zeta$ of the column. Hence this can be neglected. The total flux of vertical momentum into the column is therefore

$$(p + \varrho w^2)_{z=0} \, .$$

This is opposed by gravity, which produces a downward force, $\varrho g \zeta$. But since the motion is periodic, the vertical momentum within the column remains, on average, unchanged. Thus, on taking mean values we have

$$\overline{(p + \varrho w^2)}_{z=0} - \varrho g \bar{\zeta} = 0 \, , \tag{A}$$

where a bar denotes the average with respect to time. This is our first equation.

Second, *if the motion everywhere is assumed irrotational* (which excludes wave breaking, for example), then we have the Bernoulli integral

$$p + \frac{1}{2}\varrho(u^2 + v^2 + w^2) + \varrho g z + \varrho \frac{\partial \varphi}{\partial t} = 0 .$$

Here $\varphi$ denotes the velocity potential, which includes an arbitrary function of the time, $t$. If we take time-averages in this equation and set $z = 0$, we have

$$\bar{p}_{z=0} + \frac{1}{2}\varrho\,(\overline{u^2 + v^2 + w^2})_{z=0} + C = 0 , \tag{B}$$

$C$ being at most a constant.

From the two equations (A) and (B) we may eliminate the pressure to obtain the basic relationship

$$g\bar{\zeta} = -\frac{1}{2}\varrho\,(\overline{u^2 + v^2 - w^2})_{z=0} + C . \tag{C}$$

From this relationship it is very easy to determine the difference in mean surface level, $\bar{\zeta}$, at two different points $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$, say. Clearly the constant $C$ is immaterial, so we have

$$\varrho g(\bar{\zeta}_1 - \bar{\zeta}_2) = -\frac{1}{2}\varrho\,[(\overline{u^2 + v^2 - w^2})_{z=0}]_2^1 .$$

Thus, in the present problem, $\Delta\bar{\zeta}$ is given by

$$\Delta\bar{\zeta} = \frac{1}{2g}[(\overline{u^2 + v^2 - w^2})_{z=0}]_2^1 . \tag{D}$$

This is the simple relationship promised earlier.

Now, in a wave of amplitude $a$ traveling in some direction that makes an angle $\theta$ with the $x$-axis, the components of orbital velocity are given by

$$\left.\begin{aligned}
u &= \frac{a\sigma\cos\theta}{\sinh kh}\cosh k(z-h)\cos(kx' - \sigma t + \varepsilon)\\
v &= \frac{a\sigma\sin\theta}{\sinh kh}\cosh k(z-h)\cos(kx' - \sigma t + \varepsilon)\\
w &= \frac{a\sigma}{\sinh kh}\sinh k(z-h)\sin(kx' - \sigma t + \varepsilon) ,
\end{aligned}\right\}$$

where $x' = x\cos\theta + y\sin\theta$ and $\varepsilon$ denotes a constant phase. On squaring the velocities and taking averages with respect to time, we find

$$\left.\begin{aligned} \overline{u^2} &= \frac{1}{2}\frac{a^2\sigma^2\cos^2\theta}{\sinh^2 kh}\cosh^2 k(z-k) \\ \overline{v^2} &= \frac{1}{2}\frac{a^2\sigma^2\sin^2\theta}{\sinh^2 kh}\cosh^2 k(z-h) \\ \overline{w^2} &= \frac{1}{2}\frac{a^2\sigma^2}{\sinh^2 kh}\sinh^2 k(z-h), \end{aligned}\right\}$$

so

$$\overline{u^2+v^2-w^2} = \frac{1}{2}\frac{a^2\sigma^2}{\sinh^2 kh}. \tag{E 1}$$

Using the relationship that $\sigma^2 = gk\tanh kh$ locally, we then have

$$\frac{1}{2g}\left(\overline{u^2+v^2-w^2}\right) = \frac{a^2 k}{2\sinh 2kh}. \tag{E 2}$$

If two systems of waves are present (as on the seaward side of the breakwater), then in place of $a^2$ we shall have $(a_1^2 + a_1'^2)$. There will also be a contribution from the *product* terms, proportional to $a_1 a_1'$. However, on averaging with respect to the horizontal coordinates, $x, y$, as well as with respect to $t$, we find that these product terms vanish.

From equations (D) and (E 1), (E 2) we deduce that in the present situation

$$\Delta\bar{\zeta} = \frac{\sigma^2}{4g}\left(\frac{a_1^2 + a_1'^2}{\sinh^2 k_1 h_1} - \frac{a_2^2}{\sinh^2 k_2 h_2}\right), \tag{F 1}$$

or alternatively,

$$\Delta\bar{\zeta} = \frac{(a_1^2 + a_1'^2)k_1}{2\sinh 2k_1 h_1} - \frac{a_1^2 k_2}{2\sinh 2k_2 h}. \tag{F 2}$$

When the depths $h_1$ and $h_2$ on the two sides of the breakwater are equal, then we have simply

$$\Delta\bar{\zeta} = \frac{(a_1^2 + a_1'^2 - a_2^2)k}{2\sinh 2kh}, \tag{G}$$

where $k = k_1 = k_2$; $h = h_1 = h_2$. Since $a_2^2 \le a_1^2$, the right-hand side is non-negative, showing that the difference in level is then positive in general.

The outstanding feature of this result is that in deep water, if both $k_1 h_1$ and $k_2 h_2$ are large,

$$\Delta\bar{\zeta} = 0\,.$$

In other words, the difference in level is essentially a finite-depth effect.

In shallow water, where both $k_1 h_1$ and $k_2 h_2$ are small, equation (F 2) becomes

$$\Delta \bar{\zeta} = \frac{a_1^2 + a_1'^2}{4h_1} - \frac{a_2^2}{4h_2}, \tag{H}$$

and, if $h_1 = h_2 = h$, then

$$\Delta \bar{\zeta} = \frac{a_1^2 + a_1'^2 - a_2^2}{4h}. \tag{I}$$

Of course these results are subject to the usual limitations of the small-amplitude theory of surface waves, in particular that

$$ak \ll 1 \quad \text{and} \quad ak \ll (kh)^3$$

in each particular region. In addition, the loss of energy by friction or other means (such as breaking) must not be so great as to affect the results.

Nevertheless, the formulae are so simple and their application so straightforward that it would seem worthwhile to check their range of validity by experiments in the laboratory.

## REFERENCES AND BIBLIOGRAPHY

BARTHOLOMEUSZ, E. F.
1958. The reflexion of long waves at a step. Proc. Cambridge philos. Soc., *54*: 106–118.

BIESEL, F., and B. LE MÉHAUTÉ
1955. Étude theorique de la reflexion de la houle sur certains obstacles. La Houille Blanche, *10*: 130–138.

BOURODIMOS, E. L., and A. T. IPPEN
1966. Wave reflection and transmission in open channel transitions. Hydrodynamics Lab. Rep., Mass. Inst. Techn., No. 98 (August); 201 pp.

COOPER, R. I. B., and M. S. LONGUET-HIGGINS
1950. An experimental study of the pressure variations in standing water waves. Proc. Roy. Soc., (A) *206*: 424–435.

DEAN, R. G.
1964. Long wave modification by linear transitions. Proc. Amer. Soc. civ. Engrs., *90* (WW1): 1–29.

DEAN, R. G., and F. URSELL
1959. Interaction of a fixed, semi-immersed circular cylinder with a train of surface waves. Hydrodynamics Lab. Rep., Mass. Inst. Techn., No. 37; unpublished.

DEAN, W. R.
1945. On the reflexion of surface waves by a submerged plane barrier. Proc. Cambridge philos. Soc., *41*: 231–238.

1947. On the reflexion of surface waves by a submerged circular cylinder. Proc. Cambridge philos. Soc., *44*: 483–491.

HASKIND, M. D.
1948. The pressure of waves on a barrier. Inzhen. Sb., *4*: 147–160.

JEFFREYS, H.
1944. Motion of waves in shallow water. Wave Rep., U. K. Ministry of Supply; unpublished.

JOHN, F.
1948. Waves in the presence of an inclined barrier. Comm. appl. Math., *1*: 149–200.

JOLAS, P.
1960. Passage de la houle sur un seuil. La Houille Blanche, *15*: 148–152.

KREISEL, G.
1949. Surface waves. Quart. appl. Math., *7*: 21–44.

LAMB, SIR HORACE
1932. Hydrodynamics. 6th ed. Cambridge Univ. Press. 738 pp.

LEVINE, H.
1957. Scattering of surface waves on an infinitely deep fluid, pp. 712–716. Proc. Symposium on Behaviour of Ships in a Waterway. Wageningen, 1957. 1045 pp.

LEVINE, H., and E. RODEMICH
1958. Scattering of surface waves on an ideal fluid. Tech. Rep. Appl. Math. Stat., Stanford Univ., No. 78; 64 pp.

LONGUET-HIGGINS, M. S., and R. W. STEWART
1964. Radiation stresses in water waves: a physical discussion, with applications. Deep-sea Res., *11*: 529–562.

MACAGNO, E. O.
1954. Houle dans un canal présentant un passage en charge. La Houille Blanche, *9*: 10–37.

NEWMAN, J. N.
1965. Propagation of water waves over an infinite step. J. fluid Mech., *23*: 399–415.

TAKANO, K.
1959a. Effet de passage d'une houle linéaire plane sur un seuil. C. R. Acad. Sci. Paris, *248*: 1768–1771.
1959b. Effects de second order d'un seuil semi-indefini sur une houle irrotationelle. C. R. Acad. Sci. Paris, *249*: 622–624.
1960. Effets d'un obstacle parallélépipédique sur la propagation de la houle. La Houille Blanche, *15*: 247–267.
1963. Effet du second order d'un obstacle parallélépipédique sur la propagation de la houle dans un canal. Rec. oceanogr. Wks. Jap., *7*: 9–17.

URSELL, F.
1947a. The effect of a fixed vertical barrier on surface waves in deep water. Proc. Cambridge philos. Soc., *43*: 374–382.
1947b. The reflection of waves from a submerged low reef. Admiralty Res. Lab., Teddington, Rep. R5/103.41/W (unpublished).
1950a. Surface waves on deep water in the presence of a submerged circular cylinder, I. Proc. Cambridge philos. Soc., *46*: 141–152.
1950b. Surface waves on deep water in the presence of a submerged circularcylinder, II. Proc. Cambridge philos. Soc., *46*: 153–158.