Inertial Oscillations in an Ekman Layer Containing a Horizontal Discontinuity Surface[1]

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ABSTRACT

Solutions are obtained for systems of equations that determine the development of velocity profiles in an infinitely deep fluid system subjected to Coriolis accelerations and composed of two layers of viscous fluid differing in density, viscosity, and geostrophic velocity. The density, viscosity, and horizontal pressure gradient are assumed to remain constant in space and time within each layer and to differ discontinuously at the horizontal interface between the layers. In one case considered, the initial state is one of geostrophic imbalance in one of the layers. In another case, the initial state is one of geostrophic balance in both layers, but the geostrophic velocities are allowed to vary periodically in time in either or both layers.

The boundary conditions imposed are that the velocities remain bounded at large (vertical) distance from the interface. The interface conditions imposed are that the shearing stress and velocity be continuous at the interface for all time.

The solutions contain height-dependent transient terms that approach the two-layer steady-state solutions given by Bjerknes et al. (1933) for large time. However, they also contain terms representing a permanent inertial oscillation that has height-constant amplitude and phase at all finite values of height (with zero height defined as the interface).

If the viscosity of the lower layer is assumed to be infinite and its horizontal pressure gradient is assumed to be zero, the solutions yield zero velocity at the interface, and the terms representing the permanent inertial oscillation disappear at all finite heights. At infinite height, the solutions with both sets of assumptions give the "circle of inertia" characteristic of a nonviscous atmosphere.

If typical atmosphere-ocean values of density, eddy viscosity, and velocity are taken in the first case, the amplitude of the permanent oscillation is of the order of 1% of the geostrophic wind but is comparable in magnitude to the interface current speed.

It should be noted that these relatively large and persistent oscillations in the oceanic velocities can arise, even in the absence of changes in the oceanic horizontal pressure gradients.

Imposition of a zero-velocity lower boundary condition at finite depth in the lower layer yields a solution that goes to a steady interface current at large time. However, imposition of such a condition at depths as great as five times the Ekman "depth of frictional influence,"

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or greater, results in insignificant reduction of the amplitude of the oscillating transient term over periods long compared with the observed period of variation in synoptic-scale atmospheric horizontal pressure gradients. Thus, even in this case, the oscillation in the interface velocity may be regarded as quasipermanent.

These solutions are compared with that obtained by Ekman (1905) for a single (oceanic) layer of infinite depth, in which a time-constant wind stress is specified at the upper boundary. In Ekman's solution, the amplitude of the oscillating transient term is reduced significantly within a period of three pendulum days.

i. *Introduction.* Nonlinear models of the atmosphere-ocean planetary boundary layer are being developed at The Travelers Research Center, Inc., and these models will be numerically integrated on electronic computers. Particular emphasis is being placed on the vertical eddy-transfer processes in this layer, on time scales of the order of one hour, since this is the order of the anticipated time step in the numerical integration. An essential component of such models is the formulation of air-ocean interface conditions on the momentum, according to the time scales considered.

It was felt that a preliminary study of the behavior of simpler models having the same interface conditions would be useful. Comparison of one-layer model solutions with two-layer model solutions should be of special interest, since the subject of the over-all study is to be the interactions of the planetary boundary layers of the air and ocean rather than the reaction of either to arbitrarily specified momentum characteristics of the other.

The simplest relevant linear models that include the physical processes of interest—the eddy-transfer of momentum in the layers and on the time scales considered—appear to be those based on the classical time- and space-constant eddy viscosity formulation of Ekman (1905). Ekman obtained solutions for the steady-state drift current in a single-layer model in which the upper boundary condition required that the eddy stress in the ocean layer be equal to a specified wind stress at the air-ocean interface. Another solution for a single-layer steady-state problem was obtained by Åkerblom (see, e.g., Berry et al. 1945), but his solution was for a model having a lower boundary condition appropriate for viscous flow over a rigid lower boundary. Steady-state solutions for a two-layer system with conditions imposing continuity of the eddy stress and mean velocity at the interface have been demonstrated by Bjerknes et al. (1933). These conditions appear to be appropriate for our studies and are the first that will be investigated. A solution to a corresponding time-dependent problem, i.e., the development of the drift current from a specified initial state, was also given by Ekman (1905). Similarly, the development from specified initial conditions of Åkerblom's atmospheric wind spiral has been considered—as a special case in a much broader study—by Ooyama in Blackadar et al. (1957).

Their one-layer time-dependent solutions approached those of the corresponding steady-state problems at large times; i.e., oscillations with the inertial period, arising from the rotation of the coordinate system, were damped, albeit slowly. No previously reported solution for the time-dependent problem corresponding to the steady-state problem considered by Bjerknes et al. has been found. Yet it is the case that is of interest to us. In our preliminary study we have obtained such solutions, which are presented and discussed below.

ii a. *Equations for Constant Geostrophic Velocities.* We wish to find the solution $u_j(z,t)$, $v_j(z,t)$, $j = 1, 2$ to the following problem:

$$\frac{\varrho_j}{K_j}\frac{\partial u_j}{\partial t} - \frac{f\varrho_j}{K_j} v_j - \frac{\partial^2 u_j}{\partial z^2} = -\frac{f\varrho_j}{K_j} v_{gj}\,, \tag{1 a}$$

$$\frac{\varrho_j}{K_j}\frac{\partial v_j}{\partial t} + \frac{f\varrho_j}{K_j} u_j - \frac{\partial^2 v_j}{\partial z^2} = \frac{f\varrho_j}{K_j} u_{gj}\,, \tag{1 b}$$

where the subscript $_1$ refers to the upper layer, and the subscript $_2$ refers to the lower layer. The time- and space-constant parameters are: ϱ = density, K = (dynamic) eddy viscosity, f = Coriolis parameter, and u_g and v_g, the components of the geostrophic velocities $\vec{v}_g$, are step functions in z and t; $z = 0$ is the value of the height coordinate at the interface between the layers. The physical idealizations used in obtaining equations (1) from the equations of motion for a fluid in a rotating coordinate system are listed by many authors (e.g., see Defant 1961).

Associated with equations (1) are the initial conditions

$$u_1(z,0) = v_1(z,0) = 0, \qquad u_2(z,0) = v_2(z,0) = 0, \tag{2}$$

the interface conditions,

$$u_1(0,t) = u_2(0,t), \qquad v_1(0,t) = v_2(0,t), \tag{3}$$

$$\left. K_1 \frac{\partial u_1}{\partial z}\right|_{z=0} = \left. K_2 \frac{\partial u_2}{\partial z}\right|_{z=0},$$
$$\left. K_1 \frac{\partial v_1}{\partial z}\right|_{z=0} = \left. K_2 \frac{\partial v_2}{\partial z}\right|_{z=0}, \tag{4}$$

and the requirements that u_1, v_1 be bounded as $z \to +\infty$ and that u_2, v_2 be bounded as $z \to -\infty$. Details of the solution procedure are given in Appendix A. The solutions obtained are

$$\vec{v}_1(z,t) = -\frac{1}{2}\left[1 + \left(\varrho_1 K_1/\varrho_2 K_2\right)^{1/2}\right]^{-1} (\vec{v}_{g1} - \vec{v}_{g2}) \tag{5 a}$$

$$\Bigg\{ e^{(1+i)z(\varrho_1 f/2K_1)^{1/2}} \operatorname{erfc}\left[\frac{1}{2} z(\varrho_1/K_1 t)^{1/2} + (1+i)(ft/2)^{1/2}\right] +$$

$$e^{-(1+i)z(\varrho_1 f/2K_1)^{1/2}} \operatorname{erfc}\left[\frac{1}{2} z(\varrho_1/K_1 t)^{1/2} - (1+i)(ft/2)^{1/2}\right] -$$

$$2e^{-ift} \operatorname{erfc}\left[\frac{1}{2} z(\varrho_1/K_1 t)^{1/2}\right]\Bigg\} + \vec{v}_{g1}(1 - e^{-ift}),$$

and

$$\vec{v}_2(z,t) = \frac{1}{2}\left[1 + \left(\varrho_2 K_2/\varrho_1 K_1\right)^{1/2}\right]^{-1} (\vec{v}_{g1} - \vec{v}_{g2}) \tag{5 b}$$

$$\Bigg\{ e^{-(1+i)z(\varrho_2 f/2K_2)^{1/2}} \operatorname{erfc}\left[-\frac{1}{2} z(\varrho_2/K_2 t)^{1/2} + (1+i)(ft/2)^{1/2}\right] +$$

$$e^{(1+i)z(\varrho_2 f/2K_2)^{1/2}} \operatorname{erfc}\left[-\frac{1}{2} z(\varrho_2/K_2 t)^{1/2} - (1+i)(ft/2)^{1/2}\right] -$$

$$2e^{-ift} \operatorname{erfc}\left[-\frac{1}{2} z(\varrho_2/K_2 t)^{1/2}\right]\Bigg\} + \vec{v}_{g2}(1 - e^{-ift}),$$

where the complementary error function is defined as the complex line integral

$$\operatorname{erfc}\zeta = 2(\pi)^{-1/2}\int_{\zeta}^{\infty} e^{-\xi^2} d\xi, \tag{6}$$

with the path of integration subject to the restriction that arg $\xi \to \mu$ with $|\mu| < \pi/4$ as $\xi \to \infty$ along the path; $\mu = \pi/4$ is allowed if $\mathrm{Re}(\xi^2)$ is bounded to the left.

ii b. *Solutions for Time-dependent Geostrophic Velocities.* The procedure described in Appendix A may be used to solve the problem described by equations (1) through (4) for the case in which geostrophic velocity, $\vec{v}_g$, is permitted to vary with time; in particular, we now let

$$\begin{aligned} \vec{\tilde{v}}_{g1} &= \vec{v}_{g1}(1 - \cos\omega t), \\ \vec{\tilde{v}}_{g2} &= 0. \end{aligned} \tag{7}$$

In this case, $\vec{v}_1$ has the form

$$\vec{v}_1(z,t) = -\frac{\vec{v}_{g1}}{2}\left[1+(\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1}\left\{e^{(1+i)z(\varrho_1 f/2K_1)^{1/2}}\right.$$

$$\operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t)^{1/2}+(1+i)(ft/2)^{1/2}\right]+e^{-(1+i)z(\varrho_1 f/2K_1)^{1/2}}$$

$$\operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t)^{1/2}-(1+i)(ft/2)^{1/2}\right]-[2\omega^2/(\omega^2-f^2)]e^{-ift}$$

$$\operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t)^{1/2}\right]+\frac{f}{2(\omega-f)}e^{-i\omega t}\left(e^{-(1+i)z[\varrho_1(f-\omega)/2K_1]^{1/2}}\right.$$

$$\operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t)^{1/2}-(1+i)[(f-\omega)t/2]^{1/2}\right]+e^{(1+i)z[\varrho_1(f-\omega)/2K_1]^{1/2}}$$

$$\left.\operatorname{erfc}\frac{1}{2}z(\varrho_1/K_1 t)^{1/2}+(1+i)[(f-\omega)t/2]^{1/2}\Big]\right)-\frac{f}{2(\omega+f)}e^{i\omega t} \qquad (8)$$

$$\left(e^{-(1+i)z[\varrho_1(f+\omega)/2K_1]^{1/2}}\operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t)^{1/2}-(1+i)[(f+\omega)t/2]^{1/2}\right]+\right.$$

$$\left.\left.e^{(1+i)z[\varrho_1(f+\omega)/2K_1]^{1/2}}\operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t)^{1/2}+(1+i)[(f+\omega)t/2]^{1/2}\right]\right)\right\}+$$

$$\vec{v}_{g1}\left\{\left[1-[\omega^2/\omega^2-f^2)]e^{-ift}\right]+[f^2/(\omega^2-f^2)]\cos\omega t-i[f\omega/(\omega^2-f^2)]\sin\omega t\right\}.$$

In the corresponding expression for $\vec{v}_2(z,t)$, the last terms, which depend only on time, do not appear because of the assumption that $\vec{v}_{g2} = 0$.

iii a. *The Case $\vec{v}_{g1}$ Constant, $\vec{v}_{g2} = 0$.* It is of particular interest to examine the behavior of the solutions given in equations (5) as time becomes large and to see whether a limit exists as $t \to \infty$. For this purpose, we use the fact that

$$\lim_{\zeta\to+\infty}\operatorname{erfc}\left[\frac{\alpha}{\zeta}\pm\zeta(\pi/2)^{1/2}(1+i)\right]=\begin{cases}0\\2\end{cases}, \qquad (9)$$

according to whether the sign is plus or minus, where α is a real constant and ζ a real variable. Then

$$\lim_{t\to\infty} \vec{v}_1(z,t) = \vec{v}_{g1}\left\{1 - [1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}]^{-1} e^{-(1+i)z(\varrho_1 f/2K_1)^{1/2}}\right\}$$
$$+ \lim_{t\to\infty} \vec{v}_{g1} e^{-ift}\left\{[1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}]^{-1} \operatorname{erfc}\left[\frac{1}{2} z(\varrho_1/K_1 t)^{1/2}\right] - 1\right\}; \tag{10}$$

this expression is made up of a term representing a permanent inertial oscillation (for which no limit exists) superimposed upon a term corresponding to the two-layer steady-state solution of Bjerknes et al. (1933). A result similar to equation (10) is obtainable for the lower layer.

The behavior of the solutions at the outer boundaries is demonstrated by the limits

$$\lim_{z\to\infty} \vec{v}_1(z,t) = \vec{v}_{g1}(1 - e^{-ift}), \tag{11 a}$$

$$\lim_{z\to-\infty} \vec{v}_2(z,t) = 0, \tag{11 b}$$

which can be found by using the fact that

$$\lim_{\zeta\to\infty} \operatorname{erfc}\zeta = 0, \tag{12}$$

provided the path of integration is taken so that $|\arg\zeta| < \pi/4$ as $\zeta \to \infty$. The solutions (11) correspond to those obtained for a frictionless fluid.

From the identity

$$\operatorname{erfc}(\zeta) + \operatorname{erfc}(-\zeta) = 2, \tag{13}$$

it is seen from equation (5 a) that, at the common surface of the layers,

$$\vec{v}_1(0,t) = \vec{v}_{g1}(1 - e^{-ift})\left\{1 - \left[1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1}\right\}, \tag{14}$$

and since

$$\left[1 + (\varrho_2 K_2/\varrho_1 K_1)^{1/2}\right]^{-1} = 1 - \left[1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1},$$

the corresponding expression for $\vec{v}_2(0, t)$ reduces to the form (14) as required by the interface condition.

iii b. *One-layer Model (K_2 Infinite, $\vec{v}_{g2} = 0$).* The solution for the one-layer problem with $\vec{v}_1(0, t) = \vec{v}_1(z, 0) = 0$ and $\vec{v}_1$ bounded as $x \to +\infty$ is obtainable from the more general solution (5 a) by requiring the geostrophic velocity in the lower layer to vanish and by taking the viscosity K_2 to be infinitely large. The resulting form is

$$\vec{v}_1(z,t) = -\frac{1}{2}\vec{v}_{g1}\left\{e^{-(1+i)z(\varrho_1 f/2K_1)^{1/2}}\operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t) - (1+i)(ft/2)^{1/2}\right] + e^{(1+i)z(\varrho_1 f/2K_1)^{1/2}}\operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t) + (1+i)(ft/2)^{1/2}\right] - 2e^{-ift}\operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t)^{1/2}\right]\right\} + \vec{v}_{g1}(1-e^{-ift}). \quad (15)$$

Note that

$$\lim_{\xi\to 0} \operatorname{erfc}(\xi) = 1.$$

Thus, the terms containing the factor e^{-ift} go to zero in the limit as t approaches infinity, for all finite z.

Therefore, no permanent oscillation exists and the solution approaches the steady-state solution; i.e.,

$$\lim_{t\to+\infty} \vec{v}_1(z,t) = \vec{v}_{g1}\left[1 - e^{-(1+i)z(\varrho_1 f/2K_1)^{1/2}}\right]. \quad (16)$$

At infinite height, the solution (15) approaches the limit given in (11 a).

iii c. *The Case* $\vec{v}_{g1} = \vec{v}_{g1}(1-\cos\omega t)$, $v_{g2} = 0$. The solutions in this case again contain undamped oscillations since

$$\lim_{t\to\infty} \vec{v}_1(z,t) = \vec{v}_{g1}\left\{1 - \left[1+(\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1} e^{-(1+i)z(\varrho_1 f/2K_1)^{1/2}}\right\} + \lim_{t\to\infty} v_{g1}e^{-ift}\left[\omega^2(\omega^2-f^2)\right]\left\{\left[1+(\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1} \operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t)^{1/2}\right] - 1\right\} + \lim_{t\to\infty} \vec{v}_{g1}\frac{f}{4}\left[1+(\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1} \left\{\frac{e^{i\omega t}}{\omega+f}\left(e^{-(1+i)z[\varrho_1(f+\omega)/2K_1]^{1/2}}\operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t)^{1/2} - (1+i)[f+\omega)t/2]^{1/2}\right]\right) - \frac{e^{-i\omega t}}{\omega-f}\left(e^{-(1+i)z[\varrho_1(f-\omega)/2K_1]^{1/2}} \operatorname{erfc}\left[\frac{1}{2}z(\varrho_1/K_1 t)^{1/2} - (1+i)[(f-\omega)t/2]^{1/2}\right]\right)\right\} + \lim_{t\to\infty}\vec{v}_{g1} \left([f^2/(\omega^2-f^2)]\cos\omega t - [if\omega/(\omega^2-f^2)]\sin\omega t\right). \quad (17)$$

Similarly, the expression for $\vec{v}_2$ has no limit, since terms involving e^{-ift} and $e^{\pm i\omega t}$ are again present.

It may be shown that, as the frequency of the geostrophic velocity oscillation, ω, approaches natural frequency, f, of the system, the solution remains bounded and resonance does not occur.

iii d. *Model With Zero-velocity Lower Boundary Condition at Finite Depth.* A question arises about the effect on the solutions of the imposition of a no-slip boundary condition at the bottom of the lower sublayer. A solution has been obtained for the u component of the interface velocity for this case (Appendix B). The resulting form is

$$
\begin{aligned}
u_1(0,t) = u_2(0,t) = &-\left[1 + (\varrho_2 K_2/\varrho_1 K_1)^{1/2}\right]^{-1} v_{g1} \sin ft - \\
&\left[1 + (\varrho_2 K_2/\varrho_1 K_1)^{1/2}\right]^{-1}\left[1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1} v_{g1} \sum_{k=1}^{\infty}\left\{\left[1 + (\varrho_2 K_2/\varrho_1 K_1)^{1/2}\right]^{-1} - \right. \\
&\left.\left[1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1}\right\}^{k-1} \cdot \operatorname{Im}\left\{2e^{-ift}\operatorname{erfc}\left[kH(\varrho_2/K_2 t)^{1/2}\right] - \right. \\
&\left(e^{(1+i)2kH(f\varrho_2/2K_2)^{1/2}}\operatorname{erfc}\left[kH(\varrho_2/K_2 t)^{1/2} + (1+i)(ft/2)^{1/2}\right] + \right. \\
&\left.\left. e^{-(1+i)2kH(f\varrho_2/2K_2)^{1/2}}\operatorname{erfc}\left[kH(\varrho_2/K_2 t)^{1/2} - (1+i)(ft/2)^{1/2}\right]\right)\right\},
\end{aligned}
\tag{18}
$$

where H is the depth of the (rigid) lower boundary.

It can be shown that the series is uniformly convergent for sufficiently large t so that the limit of the solution is obtainable by taking the sum of the limits of the individual terms. Thus,

$$
\begin{aligned}
\lim_{t\to\infty} u_{1,2}(0,t) = &\ 2\left[1 + (\varrho_2 K_2/\varrho_1 K_1)^{1/2}\right]^{-1}\left[1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1} \\
&v_{g1}\sum_{k=1}^{\infty}\left\{\left[1 + (\varrho_2 K_2/\varrho_1 K_1)^{1/2}\right]^{-1} - \left[1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1}\right\}^{k-1} \\
&\cdot \operatorname{Im} e^{-(1+i)2kH(f\varrho_2/2K_2)^{1/2}}.
\end{aligned}
\tag{19}
$$

As in the one-layer case discussed in iii b, the terms containing the factor e^{-ift} go to zero in the limit as t approaches infinity.

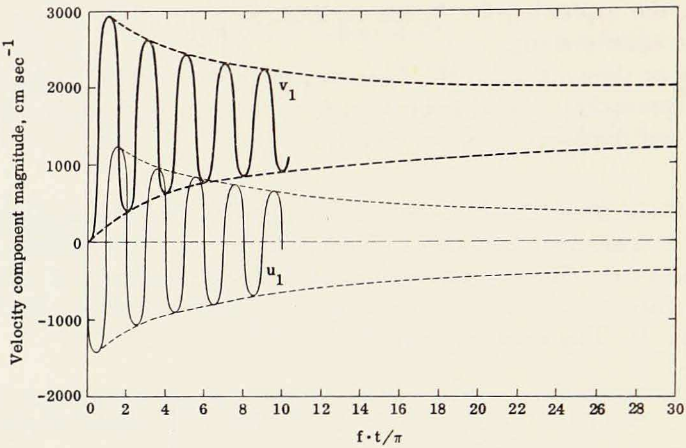

Figure 1. Two-layer model velocity components at the height $z = \pi[2K_i/\varrho_i f]^{1/2}$.

iv. *Typical Solution Values.* Numerical calculations have been performed on the IBM 7094 to evaluate the velocity components for the two-layer models, using the values of the parameters given below:

$$\text{Layer 1:} \quad u_{g1} = 0, \quad v_{g1} = 1.5 \times 10^3$$
$$\varrho_1 = 1.15 \times 10^{-3}, \quad K_1 = 4.3 \times 10,$$

$$\text{Layer 2:} \quad u_{g2} = v_{g2} = 0$$
$$\varrho_2 = 1.0, \quad K_2 = 4.3 \times 10^2,$$

with $f = 10^{-4}$; here all quantities are given in the appropriate c.g.s. units.

In Fig. 1 the velocities for the upper layer are plotted as functions of time at the level $z = (2K_1/\varrho_1 f)^{1/2}\pi$. Both u_1 and v_1 are seen to damp slowly while oscillating about the values of their corresponding geostrophic velocities. The u_1, v_1 curves for the one-layer model were found to be nearly the same as those in Fig. 1, though damping at a slightly faster rate.

From (14) it is seen that the velocities at the interface are simple non-damping cosine and sine functions of period $2\pi/f$. These are shown in Fig. 2, with u oscillating about zero and v oscillating about

$$\left\{1 - [1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}]^{-1}\right\} v_{g1}.$$

Fig. 3 illustrates the spirals obtained by plotting those components of the solutions that do not contain the factor e^{-ift}; these nonoscillating terms, $\bar{u}_j, \bar{v}_j$, approach the steady solutions rapidly. The values are shown for $t = 10\pi/f$.

Fig. 4 again presents the u component, $u_I(0, t)$, of the interface velocity

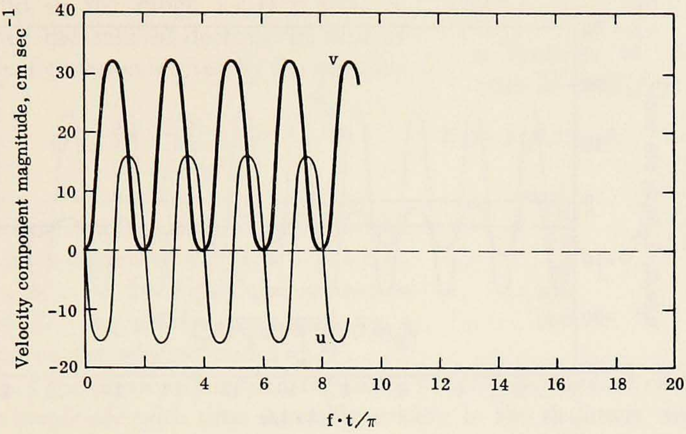

Figure 2. Solutions at the interface $z = 0$, plotted against time.

for the infinite-layer case as given in (14) together with the corresponding velocity component, $u_F(0, t)$, for the finite-depth model given by (18) for the depth-ratio value

$$H' \equiv H(f\varrho_2/2K_2)^{1/2} = 1,$$

and for all other parameters listed above. The limiting value, U_F, for infinite time given by (19) is indicated by the dashed line, about which the function $u_F(0, t)$ oscillates.

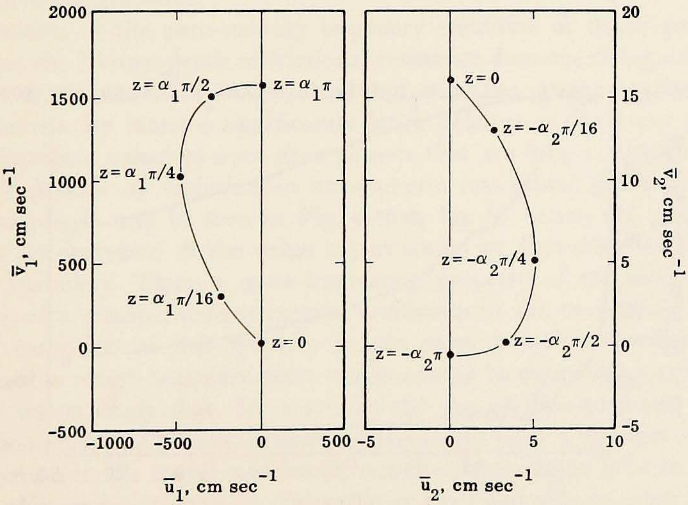

Figure 3. Nonoscillating terms of the solution for the two-layer model at time $t = 10\pi/f$. The quantity α_i is defined by $\alpha_i = (2K_i/\varrho_i f)^{1/2}$, $i = 1, 2$.

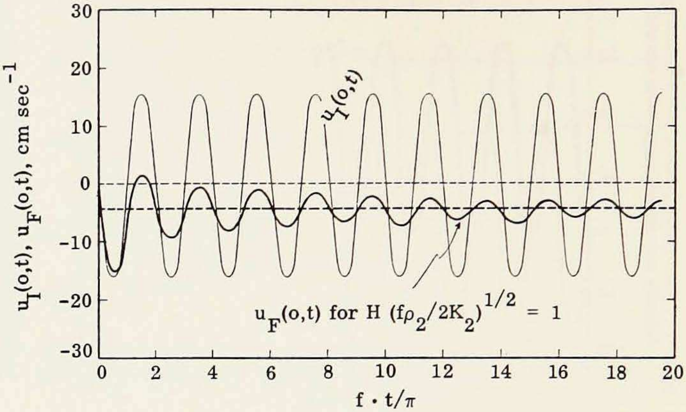

Figure 4. Interface velocity u component for infinite layer model and finite depth model, with $H(f\varrho_2/2K_2)^{1/2} = 1.0$.

In Fig. 4, two major differences between u_I and u_F are immediately evident. First is the tendency of u_F to approach a nonzero limit, U_F, at large time. Second is a general decrease in apparent amplitude of the oscillating u_F with time, although an irregularity in the rate of this decrease may be seen. This is more evident in Fig. 5. Investigation shows that this irregularity cannot be ascribed to approximations in the numerical evaluation procedure, as was first suspected.

Values of $U_F = \lim_{t\to\infty} u_F(0,t)$ are listed in Table I for several values of the

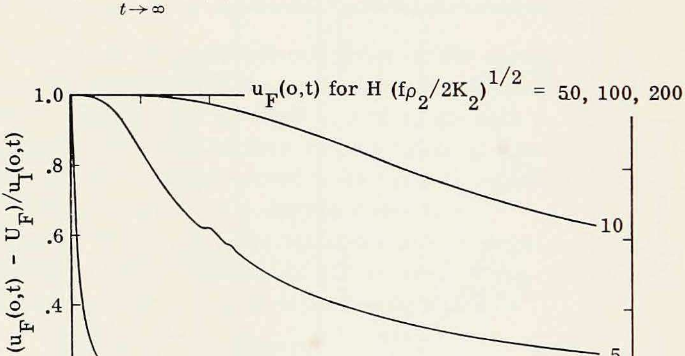

Figure 5. The ratio $U' = [u_F(0,t) - U_F]/u_I(0,t)$ for selected values of $H' = H(f\varrho_2/2K_2)^{1/2}$.

depth ratio in the range $1 \leq H' \leq 200$. A measure of the rate of decrease in time of the range of variation is given by the quantity

$$U' = \frac{u_F(0,t) - U_F}{u_I(0,t)}$$

for successive maxima and minima in the two solutions occurring for $ft/\pi = j + 0.5$, $j = 0, 1, 2, \ldots$. In Fig. 5 the successive values of U' are shown connected by a smooth curve, for selected values of H'.

Table I. $U_F = \lim_{t\to\infty} u_F(0,t)$ as a function of the depth ratio $H' = H(f\varrho_2/2K_2)^{1/2}$.

$H(f\varrho_2/2K_2)^{1/2} = H'$	$\lim_{t\to\infty} u_F(0,t)$
1	-4.27
5	7.78×10^{-4}
10	-5.93×10^{-8}
50	0
100	0
200	0

In Fig. 5 the previously noted irregularity in the general trend of decreasing apparent amplitude with time is readily evident in the shallower depth cases. Because of the greater evidence of the property in the shallow depth cases and because of the apparent tendency of irregularities to appear later in greater depth cases, it is believed that this feature of the solution represents the appearance at the interface of numerically significant disturbances due to bottom effects.

v. *Discussion.* One of the interesting aspects of the solutions obtained above is the fact that inertial oscillations may arise in either layer because of changes in the gradient flow in the other layer. This effect has greater practical significance with respect to the oceanic layer because of the large amplitude of such an induced oscillation relative to other components of the flow and because of the presumably more rapid and frequent time changes in atmospheric pressure gradients.

Imposition of the zero-velocity boundary condition at depth greater than five times the Ekman depth of frictional resistance does not change an essential property of the two-layer model solutions, in that the range of variation in the interface velocity remains significantly large (relative to the mean magnitude of the interface velocity) over time periods that are long compared with the observed periods of variation in atmospheric horizontal pressure gradients. For example, it may be seen in Fig. 5 that, for $H' = 10$, the parameter U' has not yet decreased to the value 0.5 at about 25 days after initial time (in middle latitudes). Thus, a most interesting property of the solutions is the presence of a quasipermanent inertial oscillation in the two-layer models.

The extension of this property to the more complex numerical models envisioned in future boundary-layer studies cannot be rigorously justified. However, it seems likely that, for example, the use of time-constant upper and lower boundary conditions in such models could alter a significant transient phenomenon in the model solutions. Since the phenomenon is of an oscillatory nature, this alteration may be acceptable or even desirable in some studies. In any case, the investigator should be aware of this possibility.

The models with either a single layer or two coupled layers that are discussed here contain an extreme simplification in the requirement for a height- and time-constant eddy viscosity within each layer. A better simulation of the actual atmosphere-ocean boundary layer could be obtained with a three-layer model having an atmospheric layer, an upper oceanic mixed layer with relatively large eddy viscosity, and an underlying deep oceanic layer with smaller eddy viscosity. One would expect, however, that this model would support more persistent inertial oscillations than those predicted by the rigid-bottom model described in § iii d. Still, the following discussion of the expected properties of observed wind-current profiles in the light of the contrasting single layer and two-layer model solutions remains somewhat speculative.

We may summarize the numerical estimates given in § iv as predicting the following gross properties of drift-current boundary-layer wind systems. The inertial oscillation in the solution for the planetary boundary layer of the atmosphere has relatively small amplitude (about 1 % when compared with gradient-level wind speeds). If such an oscillation were present in observed profiles, it would be extremely difficult to detect, given the present accuracy of observation. Furthermore, the solutions for the one-layer and two-layer models do not differ greatly in the numerical values of the amplitude of the oscillating term over considerable time periods. For example, for the parameter values given in § iv, the amplitude of the damped inertial oscillation in the one-layer model has only been reduced to about 7/8 of the value of the permanent oscillation in the two-layer model after more than 75 pendulum days.

However, the amplitude of the oscillation in the solution for the wind-drift current layer is of the same magnitude as the surface-current vector itself. Pronounced differences between the two-layer solution and Ekman's (1905) solution for the time-dependent wind-drift current are evident. There is present a large, depth-constant oscillating component in the infinite depth two-layer solution, at large time, while the amplitude of the damped inertial oscillation in Ekman's solution would be reduced to less than 0.5 of the surface-current magnitude by the end of the first pendulum day. The contrast between the solutions is, of course, due to the difference in the boundary conditions imposed at the air-sea interface. Therefore, if the two-layer solutions are at all realistic, we would expect that, even over the open oceans in relatively steady gradient-wind conditions, the characteristic angle of deflection between the wind stress and surface drift predicted by the steady-state theory would appear only in the mean, with individual surface-drift directions appearing to range from a direction roughly parallel to that of the wind stress (anemometer-level wind) to a direction perpendicular (*cum sole*) to that of the wind stress.

The characteristic decrease in the steady-state drift current with depth would not be observable in layers of depth-constant eddy viscosity because of the depth-constant, relatively large amplitude of the inertial oscillation.

These particular deviations from the steady-state theory, if observed, could therefore be due to the presence of the inertial oscillations, not to the essential unreality of the constant viscosity assumption. It could be predicted that such deviations would be frequently observed with both the single-layer and two-layer theories in regions of frequent change in the geostrophic wind. The models would differ in that the two-layer model would predict that the inertial oscillation would be more persistent under time-constant geostrophic conditions than would be expected from the single-layer model solutions.

None of these predictions is contradicted by the few observational studies summarized, for example, by Defant (1961), but neither are they fully confirmed.

In some of these observations, e.g., those of Gustafson and Kullenberg, the damping rate predicted by one-layer theories is difficult to reconcile with observed oscillations of essentially undiminished amplitude over three or four pendulum days. Moreover, the fact that the amplitude variations that are observed occur over periods characteristic of synoptic-scale variations in the geostrophic winds is not inconsistent with the two-layer theory.

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APPENDIX A

Solution of the Equations with Constant Geostrophic Velocities. Letting L represent the Laplace transform operator

$$\int_0^\infty e^{-st}[\;]\,dt\,,$$

we set

$$U_j(z,s) = L[u_j(z,t)],$$
$$V_j(z,s) = L[v_j(z,t)],$$

and assume that the functions of z and t behave so that limiting operations on the variables u_j, v_j carry over to their transforms; i.e., so that

$$L\,[\lim_{z\to\alpha} f(z,t)] = \lim_{z\to\alpha} F(z,s)\,.$$

Application of L to equations (1) yields the subsidiary equations

$$\frac{\varrho_j}{K_j}\,s\,U_j(z,s) - \frac{\varrho_j}{K_j}\,u_j(z,0) - \frac{f\varrho_j}{K_j}\,V_j(z,s) - \frac{\partial^2}{\partial z^2}\,U_j(z,s) = -\frac{1}{s}\frac{f\varrho_j}{K_j}\,v_{gj}\,, \quad (20)$$

$$\frac{\varrho_j}{K_j} s V_j(z_j s) - \frac{\varrho_j}{K_j} v_j(z, 0) + \frac{f\varrho_j}{K_j} U_j(z, s) - \frac{\partial^2}{\partial z^2} V_j(z, s) = \frac{1}{s} \frac{f\varrho_j}{K_j} u_{gj} \qquad (21)$$

in the transforms U_j, V_j, which are functions of z and the complex transform variable s; here the initial values of u_j and v_j that appear all vanish by the conditions given in equation (2).

If the vector $[U_j, V_j]$ is denoted by W_j, equations (20) and (21) can be written in matrix form as

$$\frac{\partial^2 W_j}{\partial z^2} - A_j W_j = B_j , \qquad (22)$$

where

$$A_j = \frac{\varrho_j}{K_j} \begin{bmatrix} s & -f \\ f & s \end{bmatrix} \quad \text{and} \quad B_j = \frac{1}{s} \frac{f\varrho_j}{K_j} \begin{bmatrix} v_{gj} \\ -u_{gj} \end{bmatrix} ;$$

thus the problem is reduced to solving a second-order ordinary differential equation in W_j with coefficients independent of z. The general solution to equation (22) is

$$W_j(z, s) = e^{z(A_j)^{1/2}} \begin{bmatrix} \gamma_j^1(s) \\ \gamma_j^2(s) \end{bmatrix} + e^{-z(A_j)^{1/2}} \begin{bmatrix} \delta_j^1(s) \\ \delta_j^2(s) \end{bmatrix} - A_j^{-1} B_j , \qquad (23)$$

where the γ's and δ's are functions of the transform variable and are to be determined by the conditions imposed at the interface and at $\pm\infty$.

Since A_j is a normal matrix, the spectral theorem may be applied to obtain a matrix $R_j = (A_j)^{1/2}$; viz.,

$$(A_j)^{1/2} = (\lambda_j)^{1/2} E_1 + (\mu_j)^{1/2} E_2 , \qquad (24)$$

where E_1, E_2 are the projection operators in the spectral decomposition of A_j and λ_j, μ_j are the eigenvalues of A_j. Since $\lambda_j = (\varrho_j/K_j)(s - if)$ and $\mu_j = (\varrho_j/K_j)(s + if)$, it follows that the corresponding eigenvectors are scalar multiples of [1, i] and [1, $-i$]. The matrix representation for the E's can be determined, since the columns of these matrices are the projections of the canonical basis vectors [1, 0] and [0, 1].

Then, from equation (24)

$$(A_j)^{1/2} = \frac{1}{2} \begin{bmatrix} (\lambda_j)^{1/2} + (\mu_j)^{1/2} & -i[(\lambda_j)^{1/2} - (\mu_j)^{1/2}] \\ i[(\lambda_j)^{1/2} - (\mu_j)^{1/2}] & (\lambda_j)^{1/2} + (\mu_j)^{1/2} \end{bmatrix} . \qquad (25)$$

Here $(A_j)^{1/2}$ is not uniquely determined, since there are two possible choices for the square root of each of the eigenvalues.

The solution given in equation (23) is in terms of $e^{\pm z(A_j)^{1/2}}$ so that a matrix representation must be found for these exponential functions, and it can be shown that

$$e^{z(A_j)^{1/2}} = \begin{bmatrix} \dfrac{e^{z(\lambda_j)^{1/2}} + e^{z(\mu_j)^{1/2}}}{2} & \dfrac{e^{z(\lambda_j)^{1/2}} - e^{z(\mu_j)^{1/2}}}{2i} \\ -\dfrac{e^{z(\lambda_j)^{1/2}} - e^{z(\mu_j)^{1/2}}}{2i} & \dfrac{e^{z(\lambda_j)^{1/2}} + e^{z(\mu_j)^{1/2}}}{2} \end{bmatrix}. \tag{26}$$

Then, since

$$A_j^{-1} B_j = \frac{f}{s(s^2+f^2)} \begin{bmatrix} s & f \\ -f & s \end{bmatrix} \begin{bmatrix} v_{gj} \\ -u_{gj} \end{bmatrix}, \tag{27}$$

the general solution W_j to equation (22) is

$$W_j(z,s) = \begin{bmatrix} \gamma_j^1 \dfrac{e^{z(\lambda_j)^{1/2}} + e^{z(\mu_j)^{1/2}}}{2} + \gamma_j^2 \dfrac{e^{z(\lambda_j)^{1/2}} - e^{z(\mu_j)^{1/2}}}{2i} \\ -\gamma_j^1 \dfrac{e^{z(\lambda_j)^{1/2}} - e^{z(\mu_j)^{1/2}}}{2i} + \gamma_j^2 \dfrac{e^{z(\lambda_j)^{1/2}} + e^{z(\mu_j)^{1/2}}}{2} \end{bmatrix} +$$

$$\begin{bmatrix} \delta_j^1 \dfrac{e^{-z(\lambda_j)^{1/2}} + e^{-z(\mu_j)^{1/2}}}{2} + \delta_j^2 \dfrac{e^{-z(\lambda_j)^{1/2}} - e^{-z(\mu_j)^{1/2}}}{2i} \\ -\delta_j^1 \dfrac{e^{-z(\lambda_j)^{1/2}} - e^{-z(\mu_j)^{1/2}}}{2i} + \delta_j^2 \dfrac{e^{-z(\lambda_j)^{1/2}} + e^{-z(\mu_j)^{1/2}}}{2} \end{bmatrix} - \tag{28}$$

$$\frac{f}{s(s^2+f^2)} \begin{bmatrix} s & f \\ -f & s \end{bmatrix} \begin{bmatrix} v_{gj} \\ -u_{gj} \end{bmatrix}.$$

We now assume that the square roots of the eigenvalues are chosen so that the real parts are positive. Then, so that the solution remains bounded as $z \to \pm\infty$, it is necessary to take

$$\gamma_1^1 = \gamma_1^2 = 0\,, \tag{29}$$

and

$$\delta_2^1 = \delta_2^2 = 0\,. \tag{30}$$

We note that, if the square roots had been chosen to possess negative real parts, the other four coefficients would have had to be zero instead (i.e., $\gamma_2^1 = \gamma_2^2 = \delta_1^1 = \delta_1^2 = 0$); due to the symmetry of elements in the first two terms, the choice is immaterial; if the square roots were taken so that the real parts were of opposite sign, all γ's and δ's would vanish, and the solution would degenerate to $A_j^{-1} B_j$. From equations (28), (29), and (30), it is seen that four coefficients remain to be determined by the interface conditions given in equations (3) and (4). The U_j's and V_j's can then be reduced to the following forms:

$$U_j(z,s) = C_j \Bigg\{ \Big[(u_{g1} - i v_{g1}) - (u_{g2} - i v_{g2})\Big] \Bigg[\frac{2 e^{-\alpha_j z (s-if)^{1/2}}}{s} - \frac{2 e^{-\alpha_j z (s-if)^{1/2}}}{s-if}\Bigg] + \Big[(u_{g1} + i v_{g1}) - (u_{g2} + i v_{e2})\Big] \qquad (31)$$
$$\Bigg[\frac{2 e^{-\alpha_j z (s+if)^{1/2}}}{s} - \frac{2 e^{-\alpha_j z (s+if)^{1/2}}}{s+if}\Bigg]\Bigg\} - \frac{f v_{gj}}{s^2+f^2} + \frac{f^2 u_{gj}}{s(s^2+f^2)},$$

$$V_j(z,s) = i C_j \Bigg\{ \Big[(u_{g1} - i v_{g1}) - (u_{g2} - i v_{g2})\Big] \Bigg[\frac{2 e^{-\alpha_j z (s-if)^{1/2}}}{s} - \frac{2 e^{-\alpha_j z (s-if)^{1/2}}}{s-if}\Bigg] - \Big[(u_{g1} + i v_{g1}) - (u_{g2} + i v_{g2})\Big] \qquad (32)$$
$$\Bigg[\frac{2 e^{-\alpha_j z (s+if)^{1/2}}}{s} - \frac{2 e^{-\alpha_j z (s+if)^{1/2}}}{s+if}\Bigg]\Bigg\} + \frac{f^2 v_{gj}}{s(s^2+f^2)} + \frac{f u_{gj}}{s^2+f^2}, \qquad j = (1,2)$$

where

$$C_1 = -\frac{1}{4}\Big[1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}\Big]^{-1}, \qquad C_2 = \frac{1}{4}\Big[1 + (\varrho_2 K_2/\varrho_1 K_1)^{1/2}\Big]^{-1}$$
$$\alpha_1 = (\varrho_1/K_1)^{1/2}, \qquad \alpha_2 = -(\varrho_2/K_2)^{1/2}.$$

Application of the inverse transform,

$$L^{-1} = \left(\frac{1}{2\pi i}\right) \int_{c-i\infty}^{c+i\infty} e^{st}[\;]\,ds, \qquad c > 0,$$

to the U_j's and V_j's in (31) and (32) yields the solutions u_j and v_j, $j = 1, 2$ (see Erdélyi 1954), with the further result that

$$\vec{v}_j(z,t) = u_j(z,t) + i v_j(z,t)$$

defines a complex function of the real variables z and t. Letting

$$\vec{v}_{gj} = u_{gj} + i v_{gj},$$

we obtain the solutions (5a and (5b) [see § ii a].

APPENDIX B

The Interface Velocity for a Lower Layer of Finite Depth with $u_{g1} = u_{g2} = v_{g2} = 0$. Since we wish to determine the velocity component $u_1^1(0, t)$ for the case in which $u_{g1} = u_{g2} = v_{g2} = 0$, it is necessary [see equation (28)] to invert the transform

$$U_1(0, s) = \gamma_1^1 + \delta_1^1 - \frac{f}{s^2 + f^2} v_{g1}, \tag{33}$$

where γ_1^1 and δ_1^1 are determined by the boundary conditions

(i) W_1 bounded as $z \to \infty$,
(ii) $W_1(0, s) = W_2(0, s)$,
(iii) $K_1 \, \partial W_1(0, s)/\partial z = K_2 \, \partial W_2(0, s)/\partial z$, and
(iv) $W_2(-H, s) = 0$,

so that the velocities will vanish at the bottom ($z = -H$) of the lower layer. As before, condition (i) implies that $\gamma_1^1 = \gamma_1^2 = 0$; the remaining conditions yield six equations in the six unknowns δ_1^1, δ_1^2, δ_2^1, δ_2^2, γ_2^1, γ_2^2. Solution of these equations for δ_1^1 yields the desired transform in terms of infinite series as

$$\begin{aligned} U_1(0, s) = {} & \frac{1}{2} \frac{f v_{g1}}{s^2 + f^2} \left[1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1} \Bigg\{ \Bigg[1 + (1 - \beta)\, e^{-2H[(\varrho_2/K_2)(s+if)]^{1/2}} \cdot \\ & \cdot \sum_{j=0}^{\infty} (-\beta)^j e^{-2jH[(\varrho_2/K_2)(s+if)]^{1/2}} \Bigg] + \Bigg[1 + (1 - \beta)\, e^{-2H[(\varrho_2/K_2)(s-if)]^{1/2}} \cdot \\ & \cdot \sum_{j=0}^{\infty} (-\beta)^j e^{-2jH[(\varrho_2/K_2)(s-if)]^{1/2}} \Bigg] \Bigg\} - \frac{i}{2} \frac{f^2 v_{g1}}{s(s^2 + f^2)} \left[1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1} \\ & \left[1 + (\varrho_2 K_2/\varrho_1 K_1)^{1/2}\right]^{-1} \Bigg\{ 2 e^{-2H[(\varrho_2/K_2)(s+if)]^{1/2}} \sum_{j=0}^{\infty} (-\beta)^j e^{-2jH[(\varrho_2/K_2)(s+if)]^{1/2}} - \\ & 2 e^{-2H[(\varrho_2/K_2)(s-if)]^{1/2}} \sum_{f=0}^{\infty} (-\beta)^j e^{-2jH[(\varrho_2/K_2)(s-if)]^{1/2}} \Bigg\} - \frac{f}{s^2 + f^2} v_{g1}, \end{aligned} \tag{34}$$

where

$$\beta = \left[1 + (\varrho_1 K_1/\varrho_2 K_2)^{1/2}\right]^{-1} - \left[1 + (\varrho_2 K_2/\varrho_1 K_1)^{1/2}\right]^{-1}.$$

Since the series is uniformly convergent in s, it can be inverted term by term to yield $u_1(0, t)$ as given in (18). From the continuity condition (ii), $u_1(0, t)$ is equal to $u_2(0, t)$ so that (18) gives the required velocity at the interface.

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