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On the Dynamical Formulation of the Large-scale Momentum Exchange between Atmosphere and Ocean[1]

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ABSTRACT

The vorticity equations for the vertically integrated momentum are derived for both the ocean and atmosphere, subject to the hydrostatic approximation, with the ocean's surface treated as a material interface. The familiar wind-stress curl is the dominant momentum exchange mechanism for large- and synoptic-scale circulations, with the air-pressure torque on the sea surface becoming of comparable importance for motion scales of the order of 10^2 km. The effect of the divergence of the integrated mass flux is among the larger terms of the vorticity equation for all scales of transient motion considered.

1. *Introduction.* In most studies on the dynamics of oceanic and atmospheric currents, a number of assumptions are made about the vertical density distribution, the bottom configuration, and the behavior of the free surface, in addition to more specific modeling approximations. Due to the relative scarcity of adequate observations on oceanic flow in depth, the technique of vertical integration is usually employed to render the problem two-dimensional and is often accompanied by the assumptions of homogeneity and incompressibility.

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The corresponding atmospheric analysis is represented by the familiar barotropic models designed for large-scale dynamical prediction. This technique is particularly useful for the present purpose of examining the large-scale circulation—producing momentum exchanges across the ocean-atmosphere interface, subject to the hydrostatic approximation. In particular, the analysis will be independent of thermodynamic processes in either the ocean or atmosphere, and independent of the equations of state. On the other hand, the analysis does not consider the heat exchange at the ocean's surface which may have an important indirect effect on the interface momentum flux. A complete theory of ocean-atmosphere interaction must ultimately consider both the heat and momentum exchanges.

2. *Vertically Integrated Dynamic Equations for the Ocean.* The equations of large-scale oceanic motion may be written

$$\frac{\partial \varrho u}{\partial t} = -\nabla\cdot(\varrho u \boldsymbol{v}) - \frac{\partial}{\partial z}(\varrho u w) + f\varrho v - \frac{\partial p}{\partial x} + A\nabla^2 u + \frac{\partial \tau_x}{\partial z}, \tag{1}$$

$$\frac{\partial \varrho v}{\partial t} = -\nabla\cdot(\varrho v \boldsymbol{v}) - \frac{\partial}{\partial z}(\varrho v w) - f\varrho u - \frac{\partial p}{\partial y} + A\nabla^2 v + \frac{\partial \tau_y}{\partial z}, \tag{2}$$

$$\frac{\partial p}{\partial z} = -\varrho g, \tag{3}$$

$$\frac{\partial \varrho}{\partial t} = -\nabla\cdot\varrho\boldsymbol{v} - \frac{\partial \varrho w}{\partial z}, \tag{4}$$

where u, v, and w are the speeds along the eastward-, northward-, and upward-directed axes x, y, and z, respectively; ϱ is the density, p the pressure, f the Coriolis parameter, g gravity, A a horizontal eddy-diffusion coefficient, and τ_x, τ_y the components of the stress $\boldsymbol{\tau} = \boldsymbol{i}\tau_x + \boldsymbol{j}\tau_y$, with $\boldsymbol{i}$ and $\boldsymbol{j}$ unit vectors along x and y. Here also $\nabla = \boldsymbol{i}(\partial/\partial x) + \boldsymbol{j}(\partial/\partial y)$ and $\boldsymbol{v} = \boldsymbol{i}u + \boldsymbol{j}v$. We shall not consider here the oceanic equation of state or the thermodynamic energy equation, and we shall also not consider the astronomical tidal forces. The boundary conditions accompanying (1)–(4) are

$$w_\zeta = \boldsymbol{v}_\zeta\cdot\nabla\zeta + \frac{\partial \zeta}{\partial t}, \tag{5}$$

$$w_{-h} = \boldsymbol{v}_{-h}\cdot\nabla(-h) + \frac{\partial(-h)}{\partial t}, \tag{6}$$

representing the continuity of surface-water particles and the bottom kinematic boundary condition. Here ζ is the sea-surface elevation (relative to an undisturbed reference level $z = 0$), and $-h$ is the ocean's depth (relative to $z = 0$).

Denoting integration over the total depth of the ocean by the operator $\langle\ \rangle$, defined as

$$\langle(\,)\rangle = \int_{-h}^{\zeta} (\,)\, dz, \tag{7}$$

and with use of the conditions (5) and (6), we find from (1)–(4) the integrated forms

$$\frac{\partial\langle\varrho u\rangle}{\partial t} = -\nabla\cdot\langle\varrho u \boldsymbol{v}\rangle + f\langle\varrho v\rangle - \frac{\partial\langle p\rangle}{\partial x} + p_a\frac{\partial\zeta}{\partial x} - p_b\frac{\partial(-h)}{\partial x} + \langle A\nabla^2 u\rangle + \tau_x^w - \tau_x^b, \tag{8}$$

$$\frac{\partial\langle\varrho v\rangle}{\partial t} = -\nabla\cdot\langle\varrho v \boldsymbol{v}\rangle - f\langle\varrho u\rangle - \frac{\partial\langle p\rangle}{\partial y} + p_a\frac{\partial\zeta}{\partial y} - p_b\frac{\partial(-h)}{\partial y} + \langle A\nabla^2 v\rangle + \tau_y^w - \tau_y^b, \tag{9}$$

$$p_a - p_b = -\langle\varrho\rangle g, \tag{10}$$

$$\frac{\partial\langle\varrho\rangle}{\partial t} = -\nabla\cdot\langle\varrho\boldsymbol{v}\rangle. \tag{11}$$

Here we have written $p_a = p_\zeta$ for the atmospheric pressure at the ocean's surface, $p_b = p_{-h}$ for the bottom pressure, and τ^w and τ^b for the surface (wind) and bottom stress components, respectively. The system (8)–(11) is a general form of the vertically integrated equations that is well suited to the present analysis. Their form is similar to that given by Fofonoff (1962) for the steady state with the neglect of variations in the sea-surface height, ζ.

If we were considering surges or wind-driven tides, the above equations would probably be the most convenient formulation and would facilitate the introduction of further approximations (Hansen 1956, Fischer 1959, Groen and Groves 1962). The wind-driven currents on an oceanic scale, on the other hand, are characterized by approximate geostrophic equilibrium. In such a case it is convenient to consider the vorticity equation, as has been widely done in the modeling of the large-scale atmospheric motions. Taking the curl of (8) and (9), we thus find

$$\frac{\partial}{\partial t}\operatorname{curl}\langle\varrho\boldsymbol{v}\rangle = -\beta\langle\varrho v\rangle + \operatorname{curl}\boldsymbol{\tau}^w + \operatorname{curl}\langle A\nabla^2\boldsymbol{v}\rangle + \operatorname{curl}\boldsymbol{N} + f\frac{\partial\langle\varrho\rangle}{\partial t} + \mathcal{J}(p_a,\zeta) + \mathcal{J}(p_b,h) - \operatorname{curl}\boldsymbol{\tau}^b, \tag{12}$$

where curl ( ) = $\boldsymbol{k}\cdot\nabla\times(\;)$, $\beta = \partial f/\partial y$, $\boldsymbol{\tau}^{w\,(b)} = \boldsymbol{i}\tau_x^{w\,(b)} + \boldsymbol{j}\tau_y^{w\,(b)}$, and $\boldsymbol{N} = \boldsymbol{i}\nabla\cdot\langle\varrho u\boldsymbol{v}\rangle + \boldsymbol{j}\nabla\cdot\langle\varrho v\boldsymbol{v}\rangle$.

The first two terms on the right-hand side of (12) are the β and wind-stress curl effects first considered by Sverdrup (1947) as an approximation to the vorticity balance in the open ocean. The third term represents the effect of the lateral-eddy diffusion, and its retention permits a description of the intensified currents characteristic of the western ocean as first shown by Munk (1950) for the steady state. The fourth term in (12) represents the nonlinear inertial terms in the equation of motion; these terms have been incorporated into inertial theories of the western boundary current by Morgan (1956) and Charney (1955). The nonlinear terms have generally not been considered in the dynamics of the open ocean, although they have been treated by Bryan (1963) in a numerical study of nondivergent flow. The term $f\partial\langle\varrho\rangle/\partial t$ in (12) represents the effect of net divergence (of the vertically integrated mass transport) through (11). The term $J(p_a, \zeta)$ is the torque exerted on the ocean's surface by the atmospheric pressure. Similarly, $J(p_b, h)$ is a pressure torque exerted by the ocean on the ocean's bottom, and together with the last term of (12), it represents the local momentum vorticity exchange with the underlying earth.

If we neglect variations in the atmospheric pressure p_a, and in the ocean's surface ζ, omit the inertial term, and linearize the remaining terms with respect to a mass transport stream function and a mean density, we obtain a formulation given by Welander (1959). Derivations of the integrated vorticity equation in which the variations in depth and density are considered explicitly, however, lead to relatively complicated formulations, as emphasized by Fortak (1962). Equation (12), on the other hand, is a consise and general statement of the oceanic vorticity balance that is suited to the examination of the large-scale atmosphere-ocean dynamical interaction.

3. *Vertically Integrated Dynamic Equations for the Atmosphere.* The system (1)–(4) may be applied without formal change to the atmosphere, again without reference to the thermodynamics. In analogy with (7), we may then denote vertical integration over the total atmospheric depth by the operator $\langle\langle\;\rangle\rangle$, defined as

$$\langle\langle(\;)\rangle\rangle = \int_\zeta^\infty (\;)\,dz. \tag{13}$$

If we apply this operator to (1)–(4) with the boundary condition (5) at $z = \zeta$, and apply the conditions $\boldsymbol{\tau}\to 0, \varrho w \to 0, \partial p/\partial t \to 0$ as $z\to\infty$, then we obtain the integrated system

$$\frac{\partial\langle\langle\varrho u\rangle\rangle}{\partial t} = -\nabla\cdot\langle\langle\varrho u\boldsymbol{v}\rangle\rangle + f\langle\langle\varrho v\rangle\rangle - \frac{\partial\langle\langle p\rangle\rangle}{\partial x} - \\ - p_a\frac{\partial\zeta}{\partial x} + \langle\langle A\nabla^2 u\rangle\rangle - \tau_x^w, \tag{14}$$

$$\frac{\partial\langle\langle\varrho v\rangle\rangle}{\partial t} = -\nabla\cdot\langle\langle\varrho v \boldsymbol{v}\rangle\rangle - f\langle\langle\varrho u\rangle\rangle - \frac{\partial\langle\langle p\rangle\rangle}{\partial y} - \\ - p_a\frac{\partial\zeta}{\partial y} + \langle\langle A\nabla^2 v\rangle\rangle - \tau_y^w, \tag{15}$$

$$p_a = g\langle\langle\varrho\rangle\rangle, \tag{16}$$

$$\frac{\partial\langle\langle\varrho\rangle\rangle}{\partial t} = -\nabla\cdot\langle\langle\varrho\boldsymbol{v}\rangle\rangle. \tag{17}$$

Taking now the curl of (14) and (15), we find

$$\frac{\partial}{\partial t}\operatorname{curl}\langle\langle\varrho\boldsymbol{v}\rangle\rangle = -\beta\langle\langle\varrho v\rangle\rangle - \operatorname{curl}\boldsymbol{\tau}^w + \operatorname{curl}\langle\langle A\nabla^2\boldsymbol{v}\rangle\rangle + \\ + \operatorname{curl}\boldsymbol{M} + f\frac{\partial\langle\langle\varrho\rangle\rangle}{\partial t} - \mathcal{J}(p_a,\zeta), \tag{18}$$

where

$$\boldsymbol{M} = \boldsymbol{i}\nabla\cdot\langle\langle\varrho u\boldsymbol{v}\rangle\rangle + \boldsymbol{j}\nabla\cdot\langle\langle\varrho v\boldsymbol{v}\rangle\rangle$$

in analogy with $\boldsymbol{N}$ in (12). A similar derivation for a rigid lower boundary, without frictional or diffusion effects, has been made by Starr and Gates (1954).

4. *The Momentum Exchange Mechanisms.* The large-scale momentum exchange processes between the ocean and atmosphere may now be identified as those terms in both (12) and (18) having opposite signs. We may thus denote the interaction terms

$$I_1 = \operatorname{curl}\boldsymbol{\tau}^w, \tag{19}$$

$$I_2 = \mathcal{J}(p_a,\zeta), \tag{20}$$

taken to be positive when momentum vorticity is transferred from the atmosphere to the ocean. I_1 represents the frictional stress exerted on the ocean's surface by the wind, and is negative (i.e., anticyclonic) over much of the ocean. I_2 represents a local torque exerted by the air on the disturbed oceanic surface, analogous to that exerted by the atmosphere on mountains, for example, although in the present case the ocean's surface is free to move.

In lieu of a scale theory for the complete dynamical equations, we may at least make a scale analysis of the oceanic vorticity equation (12) in order to estimate the relative magnitudes of the interaction effects. We may thus introduce: a characteristic speed V; a characteristic horizontal scale L (L_a for the atmosphere), a period $T = L/c$, where c is a characteristic phase speed; a density R; and a characteristic depth H. A similar scaling for the oceanic equations of motion has been made by Fofonoff (1962). Neglecting the last

Table I. Relative magnitude of terms in the integrated momentum vorticity equation.

Term of (12)	Dimensionless order	Relative magnitude*: Large-scale (quasi-steady; $L \sim r_e$)	Synoptic scale (transient; $L \sim 0.1\ r_e$)	Small-scale (transient; $L \sim 0.01\ r_e$)
$(\partial/\partial t)$ curl $\langle \varrho \boldsymbol{v} \rangle$	$(c/V)\ (H/L)\ Ro$	6×10^{-11}	6×10^{-8}	6×10^{-4}
$\beta\langle \varrho v \rangle$	$(\beta L^2/V)\ (H/L)\ Ro$	2×10^{-5}	2×10^{-5}	2×10^{-5}
curl $\boldsymbol{\tau}^w$	$\lvert\boldsymbol{\tau}^w\rvert\ (fL_a RV)^{-1}$	10^{-6}	10^{-5}	10^{-5}
curl $\langle A\nabla^2 \boldsymbol{v} \rangle$	$(H/L)\ Ro\ Re^{-1}$	6×10^{-10}	6×10^{-7}	6×10^{-4}
curl $\boldsymbol{N}$	$(H/L)\ Ro$	6×10^{-8}	6×10^{-6}	6×10^{-3}
$f\partial\langle \varrho \rangle/\partial t$	$(H/L)\ (c/V)$	6×10^{-7}	6×10^{-5}	6×10^{-3}
$\mathcal{J}\,(p_a, \zeta)$	$\lvert\Delta p_a\rvert\,\lvert\Delta\zeta\rvert\,(fLL_a RV)^{-1}$	3×10^{-8}	3×10^{-6}	3×10^{-5}

* For the values $H = 4$ km, $\beta = 10^{-13}$ cm^{-1} sec^{-1}, $f = 10^{-4}$ sec^{-1}, $R = 1$ g cm^{-3}, $\lvert\boldsymbol{\tau}^w\rvert =$ 1 dyne cm^{-2}, $\lvert\Delta p_a\rvert = 10$ mb, and $\lvert\Delta\zeta\rvert = 1$ m.

two terms of (12), the terms of the integrated momentum vorticity equation (12) assume the relative magnitudes shown in Table I. Here $Ro = V(fL)^{-1}$ is the Rossby number, $Re = LVRA^{-1}$ is a Reynolds number associated with the lateral-eddy viscosity, Δp_a is a typical variation in the surface pressure p_a on the scale L_a, and $\Delta\zeta$ is a typical variation in the ocean-surface height on the scale L.

For the large-scale oceanic circulation (i.e., for the nearly steady flow on an oceanic scale), we may estimate $L \sim r_e$ (the earth's radius), $L \sim L_a$, $H/L \sim 6\times10^{-4}$, $V \sim 10$ cm sec^{-1}, $c/v \sim 10^{-3}$, $Ro \sim 10^{-4}$, and $Re \sim 10^2$ (with $A\varrho^{-1} = 6\times10^7$ cm^2 sec^{-1}). The dominant terms in (12) then are $\beta\langle\varrho v\rangle$, curl $\boldsymbol{\tau}^w$, and $f\partial\langle\varrho\rangle/\partial t$, as shown in Table I. The importance of the β and stress terms for this scale of motion has been known since the work of Sverdrup (1947). The relative magnitude of the term $f\partial\langle\varrho\rangle/\partial t$ suggests that divergence effects are important in the transient behavior of the larger scales of oceanic motion, as, for example, in the adjustment process to an imposed wind stress. Provision for this effect has not generally been made in oceanic models, although Burger (1958) has shown that the corresponding effect in the atmosphere is important for the larger-scale motions. The interaction I_2 is relatively unimportant on this scale, as are the nonlinear and diffusion terms.

For synoptic-scale oceanic motions, say with $L \sim 0.1\ r_e$, we may estimate $c/V \sim 10^{-2}$, $V \sim 10$ cm sec^{-1}, $L \sim L_a$, $H/L \sim 6\times10^{-3}$, $Ro \sim 10^{-3}$, and $Re \sim 10$ (with $A\varrho^{-1} = 6\times10^7$ cm^2 sec^{-1}); the resulting relative magnitudes of the terms of (12) are shown in Table I. Again the dominant terms are the β, wind-stress and divergence effects, but now the nonlinear (inertial) term curl $\boldsymbol{N}$ and the lateral viscous term are of somewhat greater relative importance than on the larger scale. Bryan (1963) has examined the role of these effects for a non-divergent transient model and found that they exert a critical influence on the formation and behavior of synoptic-scale current eddies. However, the generally neglected interaction term I_2 appears to be of relative importance on this scale, and may play a role comparable to that of the inertial and viscous effects.

For even smaller-scale motions, say $L \sim 0.01\ r_e$, with $c/V \sim 10^{-1}$, $V \sim 1$ m sec^{-1}, $L \sim L_a$, $H/L \sim 6 \times 10^{-2}$, $Ro \sim 10^{-1}$, and $Re \sim 10$, the interaction I_2 may become comparable to the stress interaction I_1, and to the β term; but all of these terms are now the smaller ones in the vorticity equation, as indicated in Table I. This apparent importance of the air-pressure torque $\mathcal{J}(p_a, \zeta)$ on the smaller-scale motions is consistent with the evidently major role played by pressure effects in lake-surge development reported by Platzman (1958); however, in his case the characteristic water depth was about 10^{-2} times that of the present analysis. We note from Table I, however, that all terms *except* I_1 and I_2 are proportional to H; hence a reduction of H would increase the importance of I_1 and I_2 relative to the other terms in the vorticity equation, while leaving their size relative to each other unchanged.

From this analysis we may at least conclude that the familiar curl of the wind stress is the dominant momentum interaction on the large and synoptic scales, but it should be supplemented on scales of the order of 10^2 km by the pressure torque on the ocean's surface. Under the conditions assumed, the action of mean divergence is relatively important on all scales considered. In order to treat this effect, as well as to determine I_2, consideration should be given to oceanic models with a variable free surface. In comparing the several terms of (12), it should be recalled that thermodynamic effects have not been considered, a caution that has previously been sounded by Welander (1959). The variations in the structure and thickness of the upper warm ocean layer effectively reduce the action of surface-height variations on the deeper water, for example, and the vertical integration of the present analysis masks this effect. In order to treat this effect, as well as to consider the important ocean-atmosphere thermal interactions, representation of the vertical structure of the ocean would seem to be required.

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