On Thermally Maintained Circulation in a Closed Ocean Basin[1]

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ABSTRACT

A three-dimensional model of a thermohaline-maintained circulation is presented for a closed basin of finite depth on the beta plane. The circulation, which conserves potential vorticity, consists of a westward drift at all levels in the open ocean and a return flow that is accomplished in a swift eastward-flowing current. A comparison is made to Worthington's recent water-mass analysis of deep circulation in the North Atlantic Basin.

Introduction. Worthington (1965) has recently performed a water-mass analysis of a substantial portion of the deep Gulf-Stream system. The circulation consistent with his analysis is an approximately semicircular gyre with a strong north-south asymmetry. This circulation pattern was recognized by the present authors as strikingly consistent with the conservation of potential vorticity (Robinson 1965) in the entire motion. We demonstrate this fact by presenting a class of theoretical circulations of similar form. These flows are of interest in their own right, since, to the best of our knowledge, they represent the first analytical solutions for nonlinear, three-dimensional circulation in an enclosed basin of variable density. It is further remarkable that such diverse attacks on the deep-circulation problem yield similar results. The approach via water-mass analysis has the advantage of dealing directly with real oceanic data; the advantage in the mathematical formulation lies in our ability to provide details of the circulation patterns (e.g. vertical-velocity distributions) that are consistent with the field equations for the conservation of mass, momentum, heat, and salt.

We model the deep water of an ocean basin as a Boussinesq fluid on an enclosed beta plane of finite depth. We neglect diffusive processes and hence allow no interchange of momentum or heat with the region above the thermocline. The quasilinear, and quasigeostrophic,[2] three-dimensional problem is

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2. Here the vertical component of vorticity in the potential-vorticity equation is evaluated with velocity components that are geostrophic.

solved for the case of a semicircular basin of constant depth, where the potential vorticity is conserved in a relatively simple manner. The principal results indicate a slow westward drift at all levels, with a swift returning current over the northern arc of the basin.

We are cognizant of the fact that such a simple model cannot represent the full complexity of (diffusive) processes that drive the circulation of a real ocean basin. However, it does represent the manner in which vorticity, once introduced by diffusion from the horizontal or vertical boundaries of the basin, tends to be conserved and hence is a significant isolatable process in the theory of oceanic circulation.

2. *Basic Equations of the Model.* We consider a semicircular ocean basin that has as the bounding diameter the latitudinal circle designated by Θ_0. Let the basin be of constant depth, H, and extend a maximum distance, R, north from Θ_0 latitude [cf. Fig. 1]. (The symbol R_0 denotes the radius of the earth.)

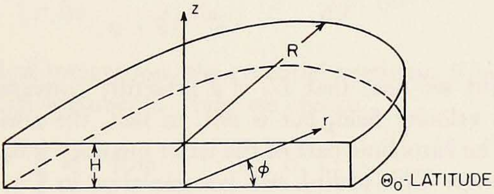

Figure 1. Model of semicircular ocean basin.

We use the beta plane, Boussinesq approximation to the equations of motion, and write the Coriolis parameter as $f' = f_0 + \beta y$, where y is the distance measured north from the latitudinal circle Θ_0. Let the circular-polar coordinates be r', φ, z, and, in dimensionless units, we measure distance in

$$r = r'/R\varphi, \qquad \zeta = z/H. \tag{1}$$

The corresponding dimensionless velocity components will be

$$u = u_r/U_0, \qquad v = u_\varphi/U_0, \qquad w = (R/HU_0)u_z. \tag{2}$$

In the conservation equations we also use the dimensionless pressure, p, and density anomaly, θ, defined as

$$p = \varrho'/\varrho_0 D_0 f_0 R, \qquad \theta = \sigma_t \varrho_0/\Delta\varrho, \tag{3}$$

where $\Delta\varrho$ is the total vertical density difference in the ocean basin.

We now define

$$U_0 \equiv gH\Delta\varrho/\varrho_0 f_0 R, \quad \varepsilon \equiv U_0/f_0 R, \quad \beta^* = (R/R_0)\operatorname{cotan}(\Theta_0), \quad f \equiv 1 + \beta^* r \sin\varphi, \tag{4}$$

and write the conservation equations as

$$\varepsilon\left[\vec{v}\cdot\nabla u - \frac{v^2}{r}\right] - fv = -\frac{\partial p}{\partial r},$$

$$\varepsilon\left[\vec{v}\cdot\nabla v + \frac{uv}{r}\right] + fu = -\frac{1}{r}\frac{\partial p}{\partial\varphi},$$

$$\theta = -\frac{\partial p}{\partial\zeta}, \tag{5}$$

$$\nabla\cdot\vec{v} = 0,$$

$$\vec{v}\cdot\nabla\theta = 0.$$

At this point we note that U_0 is a presently convenient quantity for the scaling of the velocity field, but is not, in fact, the anticipated amplitude of the motion. The baroclinic part of the latter quantity is based on the horizontal density difference. This scaling will become clear in § 3.

We shall seek solution of the equations under the condition that the normal component of velocity vanishes at a level surface, $\zeta = 1$, the bottom, $\zeta = 0$, and the vertical walls enclosing the semicircular basin. The virtue of this geometry is its simplicity; its value as a model of subthermocline flow is moot. The flow is maintained by density differences across the surfaces of the basin. A detailed examination of (5) reveals a third-order system of equations that has as a first integral the potential-vorticity function. In addition to the density-boundary condition, we need to specify the potential-vorticity integral. In fact, simple forms of this integral will prove to be relevant for dealing with a closed-circulation problem.

The potential-vorticity equation is most simply derived after a quasi-Lagrangian transformation of the equations that interchanges the dependent and independent rôles of θ and ζ. The resulting equations are [vide Robinson 1965 for details of the transformation and derivation of (7)]

$$\varepsilon\left[\vec{v}\cdot\nabla_h u - \frac{v^2}{r}\right] - fv = -\partial\Pi/\partial r,$$

$$\varepsilon\left[\vec{v}\cdot\nabla_h v + \frac{uv}{r}\right] + fu = -\partial\Pi/r\partial\varphi, \tag{6}$$

$$\zeta = \partial \Pi / \partial \theta , \tag{6}$$

$$\frac{\partial \zeta}{\partial \theta} u = - \partial \psi / \partial r ,$$

$$\frac{\partial \zeta}{\partial \theta} v = \partial \psi / r \partial \varphi ,$$

$$w = \vec{u} \cdot \nabla_h \zeta ,$$

where $\Pi = p + \theta\,\zeta$. In the above, ∇_h is the horizontal component of vector del operator taken at constant θ, and a stream-like function, ψ, has been defined to satisfy the transformed form of the mass-continuity equation. The vertical component of the vorticity equation can now be integrated to yield the potential-vorticity equation,

$$\varepsilon \left[\frac{1}{r} \frac{\partial}{\partial r} r v - \frac{1}{r} \frac{\partial}{\partial \varphi} u \right] + f = \frac{\partial \zeta}{\partial \theta} P(\psi, \theta) . \tag{7}$$

The free functional of integration, the potential vorticity, $P(\psi, \theta)$, in general depends on both of its arguments. Here we choose

$$\mathrm{P} = 1 + c\psi , \qquad c = 0, 1, -1 , \tag{8}$$

where all possible choices of linear functions of ψ are expressed in (8). We remark that a primary effect of the θ-dependence of P (ignored here) is to determine the vertical-density stratification that is uncoupled to the motion field. Since we are primarily interested in the horizontal streamline structure of the solution, (8) was chosen as the simplest mathematical structure. At the same time, the dependence of P on ψ couples the functional dependence to the motion field, and such a choice qualitatively illustrates the effect of this coupling. The θ-dependence of P has been discussed elsewhere (Spiegel, personal communication[3]).

3. *Circulation in a Closed Basin.* Since our problem is now homogeneous with respect to the boundary conditions, the motion is driven by the inhomogeneous term in the potential-vorticity equation. The solution that we exhibit is based on the assumption that $\beta^* < 1$, which is consistent with the validity of the β-plane approximation to the equations. For the case of interest in comparing our results with Worthington's, $\beta^* = 1/4$, (cf. § 4 below). Hence a power-series expansion of the equations and boundary conditions in β^* is allowed. We define the variables

3. From Ph. D. thesis in preparation.

$$\begin{aligned} \Pi &= -\theta^2/2+\beta^*\Pi_1+o(\beta^{*2}), & v &= \beta^* v_1+\ldots\ldots\ldots\ldots, \\ \zeta &= \theta+\beta^*\zeta_1+\ldots\ldots\ldots, & \psi &= \beta^*\psi_1+\ldots\ldots\ldots\ldots, \\ u &= \beta^* u_1+\ldots\ldots\ldots\ldots, & w &= \beta^{*2}w_2+\ldots\ldots\ldots\ldots \end{aligned} \quad (9)$$

Substituting the expansion (9) into (6) and (8), we obtain the quasigeostrophic potential-vorticity equation,

$$\varepsilon\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\Pi_1+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\Pi_1\right]-C\Pi_1+\frac{\partial^2\Pi_1}{\partial\theta^2}=-r\sin\varphi, \quad (10)$$

where Π_1 satisfies the boundary conditions

$$w_2=\frac{1}{r}\left[\frac{\partial\Pi_1}{\partial\varphi}\frac{\partial^2\Pi_1}{\partial r\partial\theta}-\frac{\partial\Pi_1}{\partial r}\frac{\partial^2\Pi_1}{\partial\varphi\partial\theta}\right]=0 \quad \text{at} \quad \theta=0,1\,. \quad (11)$$

In (10) and (11) we have made an error of $o(\varepsilon\beta^*)$, which is to be small. The boundary conditions for w_2 should be satisfied at two levels of ζ; however, to the order of expansion here considered, $\theta=\zeta+o(\beta^*)$ and the first-order solution formally must satisfy the boundary conditions at constant θ.

The substitution

$$\Pi_1=\pi_1(r,\theta)\sin\varphi \quad (12)$$

is consistent with the problem as posed. This allows the boundary conditions (11) to be integrated as (for details, see Robinson 1965: § 3.2):

$$\begin{aligned} \pi_1 &= k_1\frac{\partial\pi_1}{\partial\theta} \quad \text{at} \quad \theta=1\,, \\ \pi_1 &= k_0\frac{\partial\pi_1}{\partial\theta} \quad \text{at} \quad \theta=0\,. \end{aligned} \quad (13)$$

The constants, $k_{1,0}$, will be related to details of subsequent solutions for π_1. The quasigeostrophic state of the circulation is calculated from

$$\begin{aligned} u_1 &= \frac{-\cos\varphi}{r}\pi_1\,, \\ v_1 &= \sin\varphi\,\partial\pi_1/\partial r\,, \\ \zeta_1 &= -\sin\varphi\,\partial\pi_1/\partial\theta. \end{aligned} \quad (14)$$

The solution for π_1 can be represented as

$$\pi_1 = r\hat{\pi}_1(\theta) + \sum_1^\infty A_n \frac{Z_1[\alpha_n r]}{Z[\alpha_n]} \varphi_n(\theta), \tag{15}$$

where $\hat{\pi}_1$ is any solution of

$$\hat{\pi}_{1\theta\theta} - C\hat{\pi}_1 = -1, \tag{16}$$

and satisfies boundary conditions (13). We have used the definition $Z_1(\alpha_n r)$ as the first-order Bessel function of the complex argument, $\alpha_n r$, which is bounded at $r \to 0$. The complete set of functions, $\varphi_n(\theta)$, on the interval [0,1] is

$$\varphi_n = \left[\frac{\sin\lambda_n\theta}{k_0\lambda_n} + \cos\lambda_n\theta\right], \tag{17}$$

and λ_n are generated from

$$\tan\lambda_n = \frac{\lambda_n[-k_1 + k_0]}{1 + k_0 k_1 \lambda_n^2}, \quad \alpha_n^2 + (\lambda_n^2 - C)/\varepsilon = 0. \tag{18}$$

The coefficients A_n are determined from satisfying the mass-conservation condition at $r = 1$ (i.e. no normal velocity), whence

$$A_n = -\int_0^1 \varphi_n \hat{\pi}_1(\theta)\, d\theta \Big/ \int_0^1 \varphi_n^2(\theta)\, d\theta. \tag{19}$$

At this time we point out that, for a typical ocean basin, $\varepsilon \ll 1$. Hence, essentially two distinct types of circulation can occur in the basin that conserves potential vorticity. If all α_n determined from (18) are imaginary ($\lambda_n^2 > C$), (15) implies a boundary layer of $o(\varepsilon^{1/2})$ at $r \to 1$, the northern arc of the basin. A theorem has been proved (Spiegel, personal communication) that this class of solutions corresponds to a slow westward drift at *all levels* in a major portion of the basin, with a swift return flow over the northern arc [cf. Fig. 2]. If, however, k_0, k_1 are chosen such that α_n is real for some $n = q$ (it can be shown that there are at most two α_q that can be real for any choice of k_0, k_1), then for ε small, $Z_1(\alpha_q r)$ represents a series of small horseshoe-like gyres over the major portion of the ocean. Fig. 3 illustrates the lowest eigenfunction in this case. In general, the lowest eigenfunction will dominate all the flow in the basin, with the strongest gyre in the northern part of the basin. For certain values of k_0, k_1, this is not the case, and a complicated superposition of Figs. 2a and 3 will result. These complex-valued eigenvalues are particularly intriguing, because they give rise to motion of an intermediate scale in the interior basin; this motion has a wave length of the order of 100 km with a strong bias to the north-south components of circulation (cf. Crease 1962 for measurement

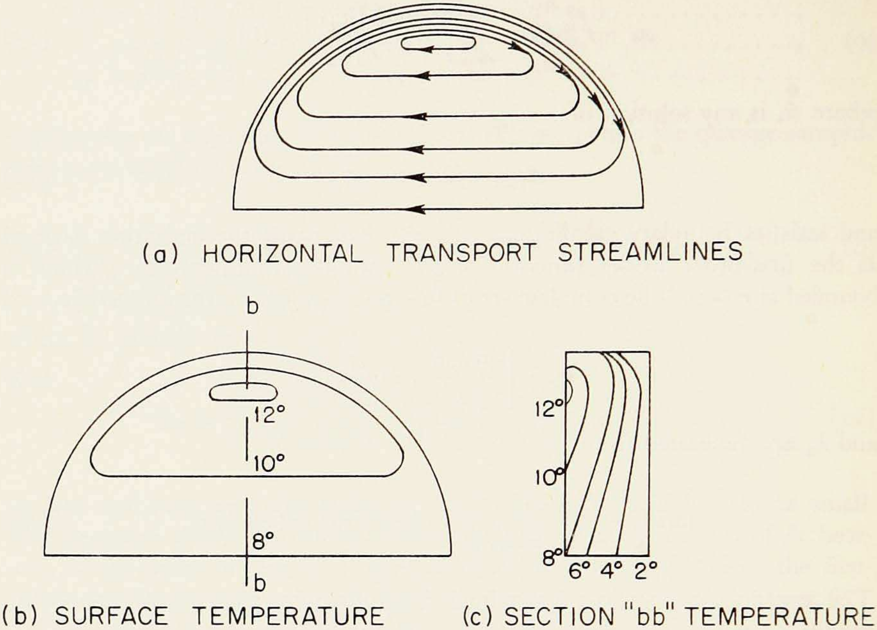

Figure 2. Schematic presentation of solution.

of motions of this scale). We fully realize that the latter remarks are speculative in nature, and there is very little reason to believe that such small-scale circulation is really part of a coherent structure that spans the whole basin. On the other hand, the ghost of this simplified phenomenon may haunt the much more complex real ocean flows.

4. *Deep Circulation in the North Atlantic Basin.* To relate our results to Worthington's water-mass analysis, we consider the part of the basin in the North Atlantic that is directly south of the Gulf-Stream system. The latitudinal

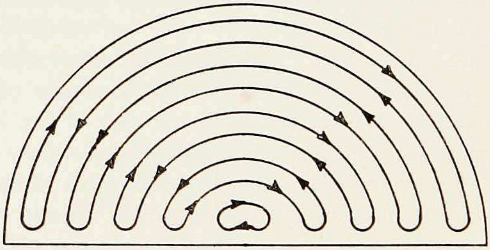

Figure 3. Horizontal transport of α_1 (real) eigenfunction.

circle of 30° is the diameter, and the northern arc is composed of the 4000-m sea-bottom topography on the eastern coast of North America, extending to 40°N. The eastern portion of the arc is completed by the Newfoundland Ridge at 53°W. We consider the circulation of the water mass that lies between 2°C and 12°C in this basin. The values of the parameters are

$$\Delta\varrho/\varrho_0 \cong 2\times10^{-3}, \quad H\simeq 3\times10^5\,\mathrm{cm}, \quad R\simeq 10^8\,\mathrm{cm},$$

$$f_0 \simeq 3/4\times10^{-4}\,\mathrm{sec}^{-1}, \quad U_0\simeq 80\,\mathrm{cm/sec}, \tag{20}$$

$$\beta^*\simeq 1/4, \quad \varepsilon\simeq 10^{-2}.$$

Under these numbers the upper bound for the magnitude of the deep flow is of the order of 20 cm/sec in the major portion of the gyre, and the returning flow is of the order of 200 cm/sec at its maximum. The width of the returning current is of the order of 100 km. The total transport is of the order of 30×10^6 m^3/sec. Actual calculations for the detailed structure of the flow reveal that these velocity and transport estimates are high by at least a factor of two or three. Moreover, the numerical values of the field quantities depend on the choice of k_0, k_1, and C. In particular, note that the characteristic horizontal density difference (and therefore the thermal wind shear) is a function or these quantities.

Fig. 4 represents Worthington's schematic circulation derived from water-mass analysis and geostrophic calculations. To infer this pattern, Worthington

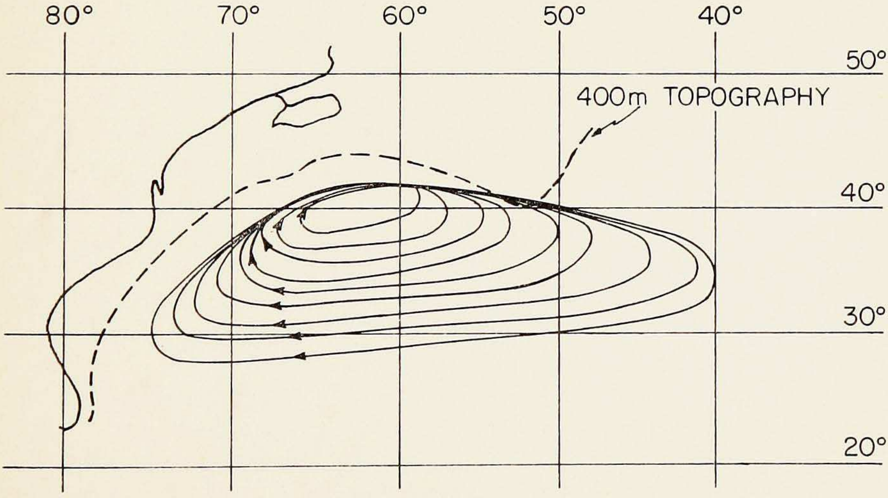

Figure 4. Transport of water that lies between 2°C and 12°C in the Gulf-Stream system. Each transport line represents 10^6 m^3/sec according to Worthington (1965).

has explicitly assumed that water transported to the east by the Gulf Stream returns to the west at the same temperature range. Thus, his empirical analysis is consistent with potential-vorticity conservation, and the resemblance to our results, though striking, is understandable.

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