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A Note Concerning Topography and Inertial Currents[1]

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ABSTRACT

A simple steady dynamical model of inertial currents incorporating the effects of bottom topography is studied. The results of this analysis and those of preceding investigations indicate that topography may exert considerable influence on the structure of the Gulf Stream, on its separation point from the coastline, and on its subsequent meander pattern.

Recent studies indicate that topography may exert considerable influence on the meanders of the Gulf Stream and on its point of separation from the coastline (Greenspan, 1962; Warren, 1962). This note comments briefly on the extensions, limitations, and implications of a recently developed theory concerning the effects of topography on inertial currents. A comparison is made of some of its features with those of other theories on inertial currents.

First, previous results (Greenspan, 1962) that presented a criterion for the existence of steady inertial boundary layers are generalized. The present analysis is based on a simple model consisting of depth-averaged dynamic equations for either a single- or double-layer ocean (Charney, 1955). In this

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model it is assumed that the fluid is incompressible and inviscid and further, that the primary circulation results wholly or in part from an existing wind-stress system.

A careful boundary-layer analysis shows that the steady inertial currents are governed by two laws: the conservation of mass,

$$H\vec{q} = \hat{k} \times \nabla \psi(x, y); \tag{1}$$

and the conservation of potential vorticity, or its constancy, along streamlines,

$$\frac{\omega + f}{H} = P(\psi). \tag{2}$$

From eqs. (1) and (2) the single dimensionless equation for the lateral variation of the streamfunction, ψ, through the current is

$$\frac{1}{HJ}\frac{\partial}{\partial z}\frac{1}{HJ}\frac{\partial \psi}{\partial z} + \frac{f(\eta)}{H} = P(\psi). \tag{3}$$

In eqs. (1)–(3), $\vec{q}$ is the particle velocity, f is the planetary vorticity, ω is the relative vorticity of the column, $\hat{k}$ is a unit vector perpendicular to the plane of motion, z is the stretched boundary-layer coordinate normal to the physical boundary, η is an orthogonal coordinate measuring distance along the coast, H is the depth, P is an arbitrary function, and J is the Jacobian of the transformation from a fixed cartesian reference frame (Greenspan, 1962). It is specifically assumed that the boundary radius of curvature is large compared to the width of the current and that the depth can vary appreciably across the stream, i.e. $J = J(\eta)$, $H = H(z, \eta)$.

The basic criterion for the existence of a steady inertial current results from the asymptotic requirement that the streamfunction approach, through an exponential decay, a prescribed interior state at the edge of the current,

$$\lim_{z \to \infty} \psi(z, \eta) = \Psi_I(\eta).$$

The criterion states that:

$$\left(\frac{\partial}{\partial \eta}\Psi_I\right)\frac{\partial}{\partial \eta}(f/H) = -\vec{q}_I \cdot \hat{n}\,\frac{\partial}{\partial \eta}(f/H) > 0, \tag{4}$$

where $\vec{q}_I \cdot \hat{n}$ is the normal velocity of the primary (interior) flow at the boundary. This result was derived previously for only the case in which H does not vary rapidly through the layer. However, the same criterion remains valid under the more general condition, $H = H(z, \eta)$, and even for the two-layer fluid model, in which $H = H(\psi, \eta)$. The proofs, which are exercises in partial differentiation, are presented in the Appendix, since it is the interpretation of the result that concerns us here.

Most theories based on simple models of wind-stress-driven oceanic circulation (Carrier and Robinson, 1962; Charney, 1955; Morgan, 1956) have predicted separation at the point of zero-normal interior velocity, $\vec{q_I} \cdot \hat{n}$, though this has not usually been stated explicitly. Instead, because of the simplicity of the assumed stress distributions, the result appears as the position of maximum wind-stress curl (Carrier and Robinson, 1962) or the point of maximum stream transport (Charney, 1955; Morgan, 1956).

However, the average wind-stress curl is an interpreted statistical quantity subject to wide variation and, in particular, it probably has little direct relationship to the more consistent point of separation of the Gulf Stream from the coastline, such as that observed in the immediate neighborhood of Cape Hatteras. The consistency of this separation position, the obvious reversal of the coastline curvature in this region, and the bending of the boundary away from the stream rather than toward it, all suggest that this separation phenomenon is associated with some local effect such as topography. The second factor of the criterion, $\partial/\partial\eta\,(f/H)$, could then be most significant, for the conditions necessary for the sustenance of a steady inertial current are violated when the depth increases rapidly in the direction of flow.

The oceanic depth topography in the Cape Hatteras area is extremely interesting (Fig. 1). The 10-, 100-, and 1000-fathom lines almost coelesce just south of the Cape, thereby producing a very steep depth gradient—of the order of one mile in 30 in the direction of the current. The Gulf Stream, as it flows along the edge of the continental shelf (a topographic limitation?), is literally forced into deep water by the convergence of the depth-contour lines. (The reversal in boundary curvature must follow such a convergence, for the deep-water contours are relatively insensitive to the geological processes that alter the coastline.)

Consideration of a single-layer model of the Gulf Stream, in which H is the actual oceanic depth, leads to the conclusion [from eq. (4)] that the nature of an inertial current must change when the depth suddenly increases. In other words, some or all of the conditions under which the criterion of eq. (4) was established are violated in the process. The stream may then become unsteady, inviolably three-dimensional, or a free jet no longer attached to the boundary.

The implications in a two-layer model (density stratification in the simplest form) are not as clear-cut as in the single-layer model and must still be studied. Basically, the influence of topography, if any, on the depth of the topmost layer requires clarification, and the assumption of no motion in the lower layer needs review. In a two-layer model, the extent to which the Gulf Stream is topographically controlled remains an open question. (Perhaps this control is effected through some intermediary means.) A host of other questions can

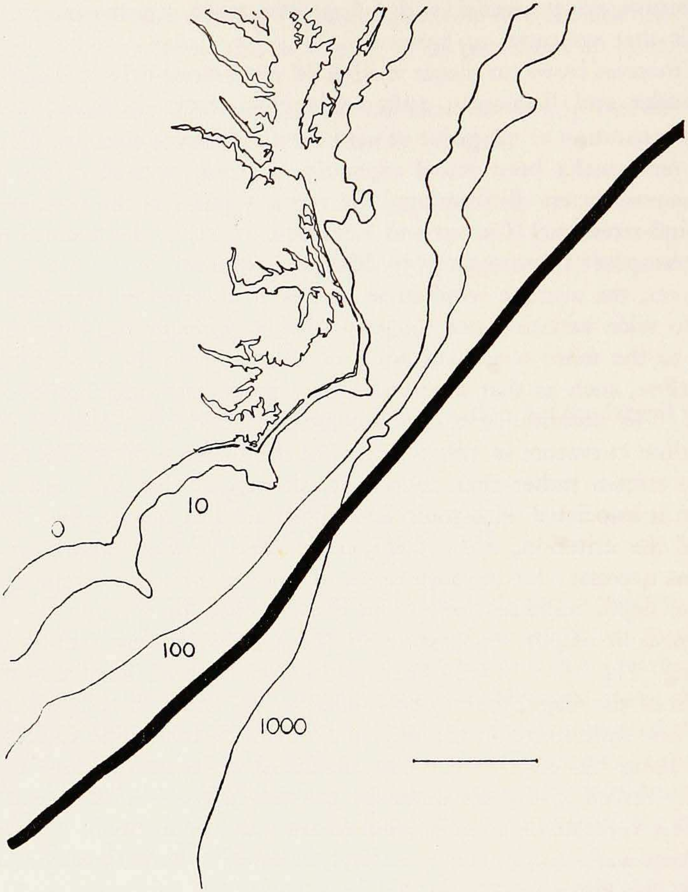

Figure 1. Bottom Topography in the vicinity of Cape Hatteras showing the coastline and the 10-, 100-, and 1000-fathom lines. The broad line indicates the approximate position of the axis of the Gulf Stream. The scale bar denotes 100 km.

be posed just as easily, but here the emphasis is on the possible role of depth variations in the separation process.

The Kuroshio Current and the general topography off the coast of Japan exhibit characteristics similar to those off Cape Hatteras. Off Japan the current is also forced into deep water by the convergence of contour lines, and the coastline again exhibits a reversal of curvature. That such a reversal of curvature occurs at approximately the same latitude in both Atlantic and Pacific seems an odd coincidence.

Warren (1962) has investigated the effects of topography on the Gulf Stream north of Cape Hatteras. His study, based on a somewhat systematic reduction of the general equations of motion, shows that the current, a free jet in this region, can indeed be controlled by the topography. The basic physical arguments proceed as follows. A free jet, flowing northward with deep water to the east, experiences either an increase in mean velocity or an increase in breadth as it moves into shallower water, in order to preserve a given mass transport. Either change results in an imbalance in which the Coriolis force exceeds the pressure gradient; the stream is then returned to deeper water. A stream moving into deeper water is forced back for similar reasons. The forces induced by topography thus tend to stabilize the motion of the current.

Warren's calculations show that the meanders of the Gulf Stream can result from depth variations and that they are an inherent feature of a steady topographic theory. Of importance here is the fact that the steady equation governing the mean motion of a free inertial jet is essentially a consequence of the two conservation laws [equations (1) and (2)] and that it can be derived from a single-layer model. Most of the details in the following reduction are omitted in the interest of brevity.

The vorticity in a meandering jet with large curvature is approximately given by

$$\omega \simeq Kv + \frac{\partial v}{\partial z},$$

where K is the streamline curvature, v the fluid velocity, and z a spanwise coordinate. As a first approximation, assume the conditions in midstream to be essentially uniform, with $vH = Q$ = constant for mass conservation. Eq. (2) becomes

$$\bar{K}Q + f\bar{H} = P(\bar{\psi})\,\bar{H}^2,$$

where

$$\bar{K} = \bar{y}''\,[1 + (\bar{y}')^2]^{-\frac{3}{2}};$$

all quantities are evaluated at some central streamline, $\bar{\psi}\,(\bar{x}, \bar{y})$. The origin and orientation of the $(\bar{x}, \bar{y})$ coordinate system may be chosen conveniently. A better approximation is achieved by improving the foregoing averaging technique. This is accomplished by integrating across the entire stream width, $-\infty < z < \infty$. Let $Q = \int_{-\infty}^{\infty} vH dz$, $M = \int_{-\infty}^{\infty} v^2 H dz$, etc., so that, for example, $\int_{-\infty}^{\infty} Kv^2 H dz = \bar{K}M$; eq. (2) then becomes

$$\bar{K}M + fQ = \bar{P}Q\bar{H}, \tag{5}$$

if the term arising from $\partial v/\partial z$ is neglected. Here $\bar{K}, f, \bar{H}$ are functions of position; the remaining parameters are appropriate constants. Since this procedure is essentially that used by Warren, the resultant equations are all structurally similar and are expressions of the same basic meander mechanism.

The steady theory, however, requires a time-varying initial condition (the position of the stream somewhere just north of the Cape) in order to provide for different meander patterns. This, in effect, assumes that there is sufficient time for the flow to achieve a near steady-state before the initial conditions change substantially. In other words, the time required for the fluid to traverse the entire meander path (a few days) should be small compared to the characteristic time or period, T, during which the initial conditions vary. The period, T, must then be at least of the order of several weeks.

What is the source of the time dependence of the initial position? One possibility arises from the fact that the same theory used to study the meanders also indicates a change in form of the Gulf Stream just south of the Cape, as evidenced by the failure of the criterion in eq. (4). Thus, this simple model requires a time-dependent initial condition just north of the Cape, but it also provides a separation point, or at least a change in the nature of the current just south of Cape Hatteras. Whether or not the transition can be described by purely dynamical considerations (time dependent?) is unknown; also unknown is the exact cause of the slow oscillation in the initial position of the meander pattern (instability?).

There are a few further comments on the single-layer model that are pertinent. Let ψ_- and ψ_+ denote the bounding streamlines of a free jet in an otherwise quiescent fluid and let the depths corresponding to these boundaries be $H_\pm$; then,

$$P(\psi_\pm) = f/H_\pm,$$

since the vorticity is assumed to be zero at the lateral extremes. The function $P(\psi)$ must now be prescribed, because the preceding relation serves to define the depths $H_\pm$. A variation in P from one edge of the current to the other can be accounted for by a change in either depth or the Coriolis parameter, f. Therefore a jet directed northward generally requires a lateral depth variation [if $P(\psi_-) \neq P(\psi_+)$]; a drift into deeper water results as the latitude increases. Once the current enters a region of uniform depth, the stream may turn eastward and broaden so that the change in f compensates for the constancy of H.

Finally, note that eq. (3) is sufficiently general to yield free jet solutions for a varying bottom depth, without the inclusion of viscosity. Take $\mathcal{J} = 1$,

$$H = \frac{1}{2}(H_+ + H_-) + \frac{1}{2}(H_+ - H_-)\tanh \lambda z,$$

and let

$$P(\psi) - f/H = A \sin 2\pi \left(\frac{\psi - \psi_-}{\psi_+ - \psi_-}\right),$$

so that

$$\int_{\psi_-}^{\psi_+} [P(\psi) - f/H]\, d\psi = 0.$$

It follows that

$$\frac{\psi - \psi_-}{\psi_+ - \psi_-} = \frac{2}{\pi} \tan^{-1} C \exp\left[\left(\frac{A\pi}{\psi_+ - \psi_-}\right)^{\frac{1}{2}} \left(\frac{1}{2}[H_+ + H_-] + \frac{1}{2\lambda}[H_+ - H_-]\, ln \cosh \lambda z\right)\right],$$

where λ, A, and C are constants, and y is taken to be a parameter. Although jet-like solutions are obtainable, the theory is quite incapable of providing an accurate structure, nor can the precise limits of the stream width be determined. To do this, a more sophisticated model may be necessary.

If the various results of the single-layer topographic theory are collected and if the sum total is applied to the Gulf Stream, then the following speculative description emerges. From Florida to Cape Hatteras, the Gulf Stream is essentially an inertial current that is fed by a primary interior flow and is constrained by topography to flow along the edge of the continental shelf. Just south of Cape Hatteras, the stream is forced into deep water by the convergence of depth-contour lines. The conditions for the sustenance of the steady current are then violated and the stream enters some sort of very complicated transition, from a steady inertial current to a free jet, which is quite possibly time-dependent and three-dimensional. To the north of Cape Hatteras, the stream, now a free jet, acquires a favorable topographic climate, but its initial position is indefinite (or time-dependent) as a result of the transition processes. The current proceeds northward, controlled principally by topography (the lateral depth gradient); the meander pattern is essentially steady in any time interval of the order of a week or less. The stream eventually drifts into deeper water, broadening and turning more to the east as the depth becomes more nearly uniform. The circulation is completed by the breakup of the jet and by the return of the fluid in essentially a single gyre pattern. In this last phase, the mid-Atlantic ridge may be another important topographic consideration.

This description differs from that proposed by Carrier and Robinson, for their theory requires a double-gyre oceanic circulation, an internal zonal jet at the latitude of maximum wind-stress curl, and a radical departure of the current from the coastline. Warren's success in explaining the meander pattern together with the other previously mentioned criticisms make the wind-stress theory of doubtful applicability to the separation phenomenon.

Of course there are many criticisms of a purely topographic theory, for too many variables, processes, and effects have been neglected or ignored in this incomplete model, and much remains to be done. [Charney's investigation of the two-layer model (1955) definitely shows the importance of density stratification.] However, the description is sufficiently suggestive to warrant greater emphasis and continued study on the role of topography on inertial currents.

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APPENDIX

In the first case—$H = H(z,\eta) = H_\infty(\eta) - h(z,\eta)$—let $ds = HJdz$ and $\psi = \varphi(z,\eta) + \Psi_I(\eta)$, so that eq. (3) becomes

$$\varphi_{ss} = F(\varphi + \Psi_I) - f/H. \tag{A}$$

Since $\lim\limits_{s\to\infty} \varphi = \lim\limits_{z\to\infty} \varphi = 0$,

$$F[\Psi_I(\eta)] = f/H_\infty(\eta),$$

and

$$\varphi_{ss} = F(\varphi + \Psi_I) - F(\Psi_I) + f/H_\infty - f/H_\infty - h),$$

or, asymptotically,

$$\varphi_{ss} \simeq F'(\Psi_I)\varphi - fh/H_\infty^2. \tag{B}$$

Exponentially decaying solutions exist for only $F'(\psi_I) > 0$, which is equivalent to the stated criterion.

In the case of a two-fluid layer, the depth in general depends on the streamfunction, $H = H(\psi, \eta)$. Let

$$T(\psi,\eta) = F(\psi) - f/H. \tag{C}$$

Then the asymptotic analysis requires $T_\psi(\Psi_I, \eta) > 0$ for the existence of exponentially decaying solutions. But

$$T_\psi(\Psi_I,\eta) = F'(\Psi_I) + (f/H^2) H_\psi. \tag{D}$$

However $F(\Psi_I) = f/H(\Psi_I, \eta)$, so that

$$F'(\Psi_I)\frac{\partial \Psi_I}{\partial \eta} = f'/H - (f/H^2)\left(H_{\psi_I} + H_\eta \frac{\partial \Psi_I}{\partial \eta}\right). \tag{E}$$

Eqs. (D) and (E) yield the criterion.