Some Properties of Thermocline Equations in a Subtropical Gyre

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ABSTRACT

The functional dependence of solutions of a theoretical model of the subtropical oceanic thermocline upon the physical parameters and boundary conditions is exhibited: (1) in a limited range by a boundary layer approximation; and (2) in a wider range by numerical solutions of the nonlinear equations.

1. *Introduction.* In attempting to apply the theory of the oceanic thermocline (Robinson and Stommel, 1959) to the real ocean (*e. g.*, in order to obtain an estimate of the amplitude of transport of abyssal circulation; Arons and Stommel, 1960), the approximate solutions given by Robinson and Stommel do not seem to be adequate. Meant only to exhibit qualitative features of the thermocline, they were not really intended to be used for quantitative calculation. Moreover, in those cases where the wind-stress was included, the calculations, in addition to being very approximate, were nevertheless sufficiently involved to obscure the qualitative dependence of the structure of the thermocline on the parameters. For example, one would like to feel familiar enough with the solutions to know how an increase in downward velocity specified at the bottom of the Ekman layer affects the amplitude of upward velocity beneath the thermocline.

The calculations in this paper are performed on equations (14) and (15) of the Robinson and Stommel (1959) paper, with the choice $a = b = \frac{1}{3}$ and setting $\vartheta_y = 0$:

$$K\vartheta'' - W\vartheta' = 0\,, \tag{1.1}$$

$$W'' + \zeta\vartheta' = 0\,. \tag{1.2}$$

The first equation is nonlinear; the second is linear but has the nonconstant coefficient ζ. The independent variable is ζ; both of the dependent variables,

ϑ and W, are functions of ζ; and K is a physical parameter. The primes denote ordinary derivatives with respect to ζ. The range of variation of ζ is from $\zeta = 0$ (at the sea-surface) to $\zeta = \zeta_b$ (at the sea-bottom). The restriction to positive ζ is equivalent to restricting attention to anticyclonic (subtropical) gyres. The form of transformation with ζ in the negative range would be more appropriate to a discussion of cyclonic (subpolar) gyres, but these numerical calculations have not yet been made. The boundary conditions are:

$$\text{at } \zeta = 0: W = W(0), \text{ a constant} \qquad \vartheta = \vartheta(0), \text{ a constant;} \tag{1.3}$$

$$\text{at } \zeta = \zeta_b: W = 0 \qquad \vartheta = 0\,. \tag{1.4}$$

The reader should refer to the earlier paper by Robinson and Stommel (1959) for all definitions, the introduction of the similarity transformation, and a discussion of the physical meaning of K.

2. *Numerical Solution.* Returning to equations (1.1) and (1.2), without approximation, we define the new variable

$$Z = -W''/\zeta\,, \tag{2.1}$$

and they may be written

$$W\vartheta' = K\vartheta'' \tag{2.2}$$

$$Z = \vartheta'\,. \tag{2.3}$$

Eliminating ϑ, we obtain

$$Z' = \frac{W}{K}Z\,. \tag{2.4}$$

We assume (following a method described in Hartree, 1958) that there is a sequence of functions W_0, $W_1 \ldots W_n$, $W_{n+1}, \ldots$, and, of course, following from these, a sequence Z_0, Z_1, $\ldots$ Z_n, Z_{n+1}. Suppose we have a trial form of the solution W_0, which satisfies the boundary conditions at both top ($\zeta = 0$) and bottom ($\zeta = \zeta_b$) but which is not a true solution of the differential eq. (2.4). Then we may write the approximate equation

$$Z_1' = \frac{W_0}{K} Z_1\,, \tag{2.5}$$

which is linear in Z_1; then we solve for Z_1 and obtain W_1 from (2.1) by quadratures subject to the exact boundary conditions. We iterate this process until we finally reach a state where $Z_{n+1} = Z_n$ and $W_{n+1} = W_n$ to any degree of precision. The iterations are performed on an electronic computing machine. We may then consider that W_n is the solution of the problem. The boundary conditions are incorporated as follows.

Consider
$$Z'_{n+1} = \frac{W_n}{K} Z_{n+1}, \tag{2.6}$$

where W_n is considered to be a *known* function of ζ between $\zeta = 0$ and $\zeta = \zeta_b$. Then
$$Z_{n+1} = A\Psi_n, \tag{2.7}$$

where we write $\Psi_n = \exp\left[\int_0^\zeta (W_n(\zeta)/K)\,d\zeta\right]$ and where A is an integration constant.

The temperature boundary conditions are imposed by integrating (2.3):
$$\int_0^{\zeta_b} Z_{n+1}\,d\zeta = \int_0^{\zeta_b} \vartheta'\,d\zeta = -\vartheta_0. \tag{2.8}$$

Thus A is determined:
$$A = -\vartheta_0 / \int_0^{\zeta_b} \Psi_n\,d\zeta. \tag{2.9}$$

Making use of the definition of Z in (2.1), we write
$$W''_{n+1} = -A\zeta\Psi_n. \tag{2.10}$$

By quadrature we write the solution
$$W_{n+1} = -A\int_0^\zeta\int_0^\zeta \zeta\Psi_n\,d\zeta^2 + B\zeta + C. \tag{2.11}$$

The two constants of integration, B and C, are determined by the known boundary condition $W_{n+1} = W(0)$ at $\zeta = 0$ and $W_{n+1} = 0$ at $\zeta = \zeta_b$; the final form of the solution is
$$W_{n+1} = -A\left[\int_0^\zeta\int_0^\zeta \zeta\Psi_n\,d\zeta^2 - \frac{\zeta}{\zeta_b}\int_0^{\zeta_b}\int_0^{\zeta_b} \zeta\Psi_n\,d\zeta^2\right] + W(0)(1-\zeta/\zeta_b). \tag{2.12}$$

The calculation is repeated until finally $W_{n+1} = W_n$. The temperature field, ϑ, and the geostrophic meridional component of velocity, v, have also been computed from W for different choices of the parameter, and the results are displayed in Figs. 1–4, which are arranged so as to show the dependence of the solution on the parameters.

In practice we have found that there is a tendency for successive values of W_n to oscillate about the asymptotic limit; thus, for example, $W_n \geq W_\infty$ and $W_{n+1} \leq W_\infty$. Instead of using W_n to compute W_{n+1}, we have therefore used $\frac{1}{2}(W_{n-1} + W_n)$ as the function to compute W_{n+1}, which, for reasons not entirely clear, greatly improves the convergence of the iterative process.

3. *Limiting Cases.* There are two simple limiting cases which are useful to consider: for large K ($K \to \infty$) and for small K ($K \to 0$).

(a) Large K. For very large K it is clear that the temperature must be given by the conductive state: *i. e.*, $\vartheta'' = 0$, and hence

$$\vartheta = \vartheta_0(1 - \tau), \tag{3.1}$$

where $\tau = \zeta/\zeta_b$, which satisfies both boundary conditions on ϑ, and eq. (1.1). The eq. (1.2) may be directly integrated and the two constants of integration determined by the boundary conditions (1.3) and (1.4) on W. The solution is

$$W(\tau) = W(0) - \left[W(0) + \frac{\vartheta_0 \zeta_b^2}{b}\right]\tau + \left(\frac{\vartheta_0 \zeta_b^2}{b}\right)\tau^3. \tag{3.2}$$

(b) Small K. As K approaches zero, it is clear from (1.1) that ϑ' must vanish everywhere except at some depth $\zeta = \zeta_t$ where ϑ'' is very large locally.

Thus the solution of (1.1) is:

$$\begin{aligned} \vartheta &= \vartheta_0 \text{ in } 0 \le \zeta < \zeta_t, \\ \vartheta &= 0 \text{ in } \zeta_t < \zeta \le \zeta_b. \end{aligned} \tag{3.3}$$

Integrating (1.2) across the discontinuity in ϑ at $\zeta = \zeta_t$, we obtain

$$W' \Big|_{\zeta_t - \Delta\zeta}^{\zeta_t + \Delta\zeta} = -\int_{\zeta_t - \Delta\zeta}^{\zeta_t + \Delta\zeta} \zeta\vartheta' d\zeta = \zeta_t \vartheta_0. \tag{3.4}$$

The upper limit on the left-hand side vanishes and the lower is simply $-W(0)/\zeta_t$, hence we can compute the depth of the discontinuity in ϑ and W':

$$\zeta_t = \sqrt{\frac{W(0)}{\vartheta_0}}. \tag{3.5}$$

This is the depth of the thermocline for $K = 0$. The solution for W is

$$\begin{aligned} W &= W(0)(1 - \zeta/\zeta_t) \quad && 0 \le \zeta < \zeta_t, \\ W &= 0 && \zeta_t < \zeta \le \zeta_b, \end{aligned} \tag{3.6}$$

which is good for only $W(0) > 0$. When $W(0) < 0$, there is no layer of discontinuity, and over the whole range of depth

$$\vartheta = 0, \tag{3.7}$$

$$W = W(0)(1 - \zeta/\zeta_b). \tag{3.8}$$

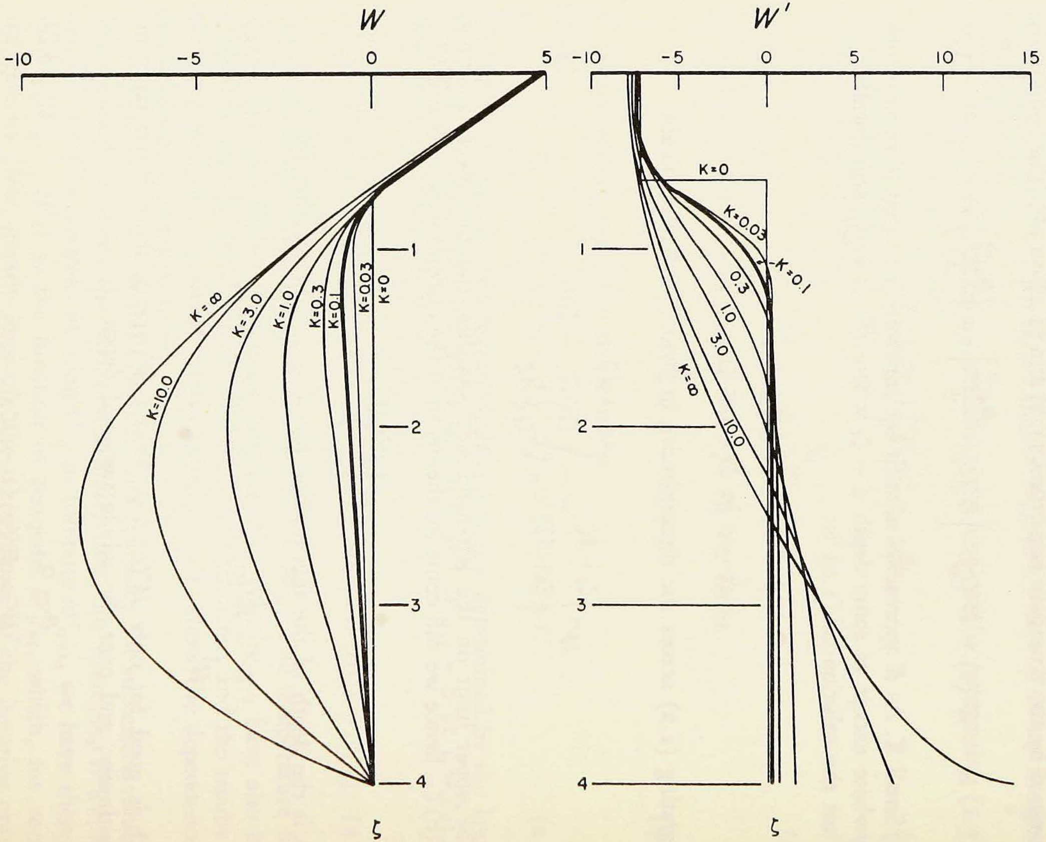

Figure 1. The distribution of W, W', and T, with transformed depth ζ; plotted for various choices of the conductivity K.

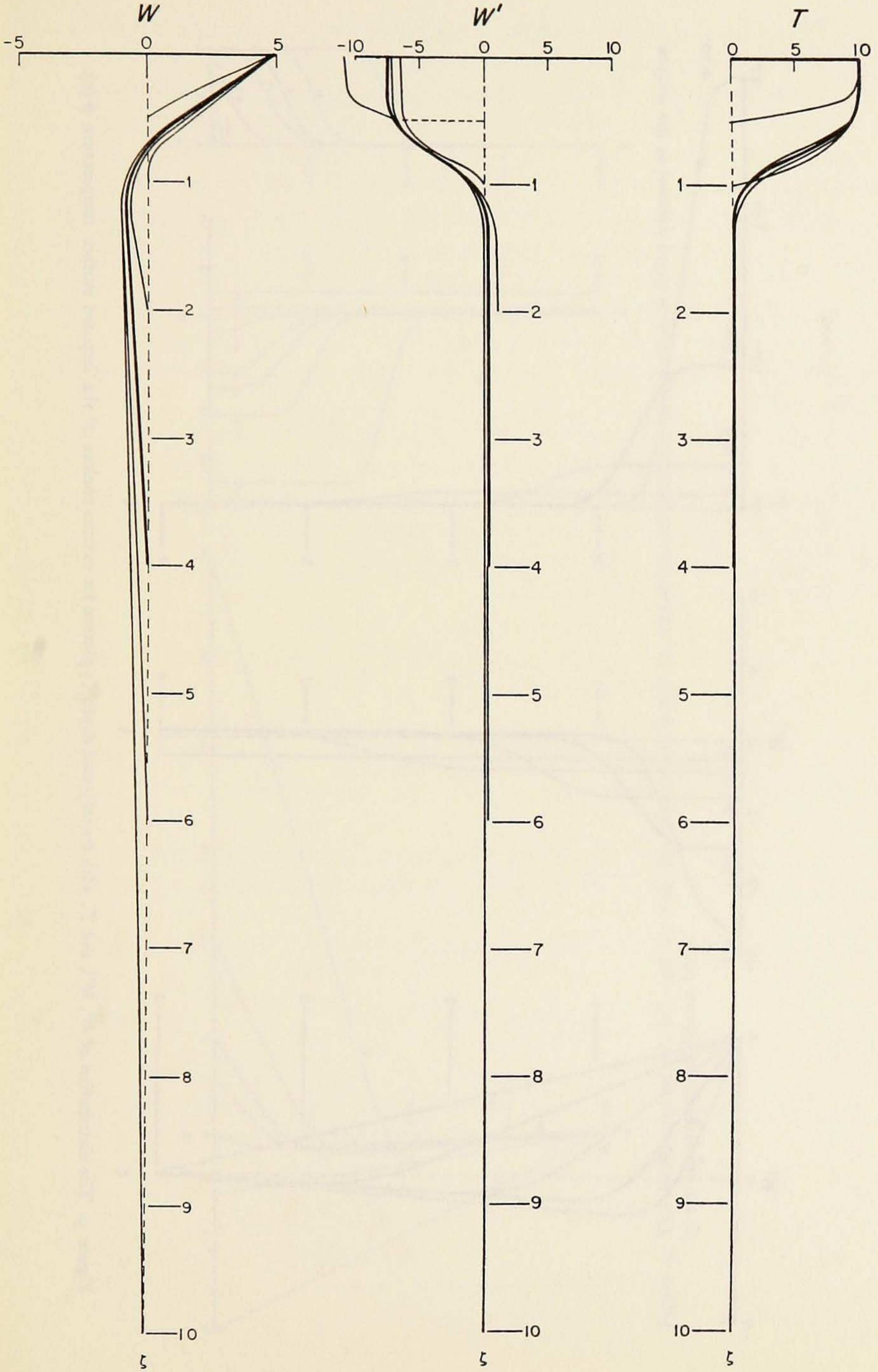

Figure 2. The distribution of W, W', and T, with the transformed depth ζ; plotted for various choices of the transformed depth of the bottom ζ_b.

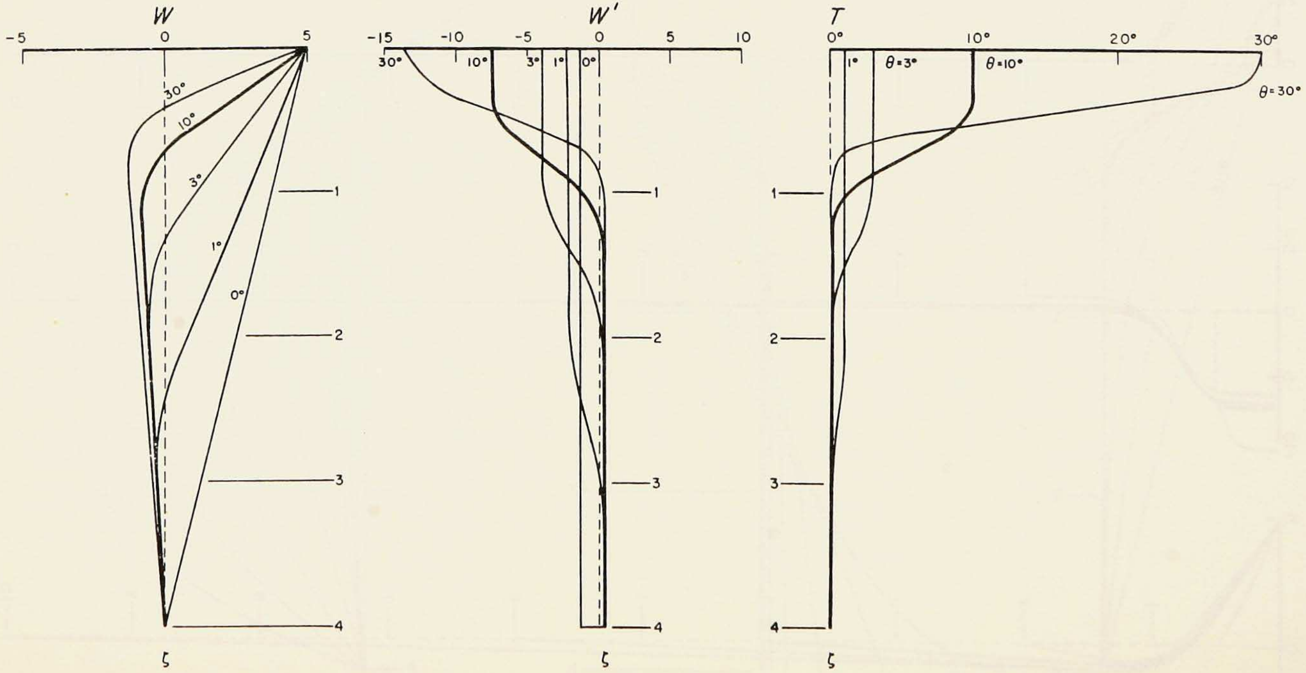

Figure 3. The distribution of W, W', and T, with transformed depth ζ; plotted for various choices of the imposed surface temperature $\vartheta(0)$.

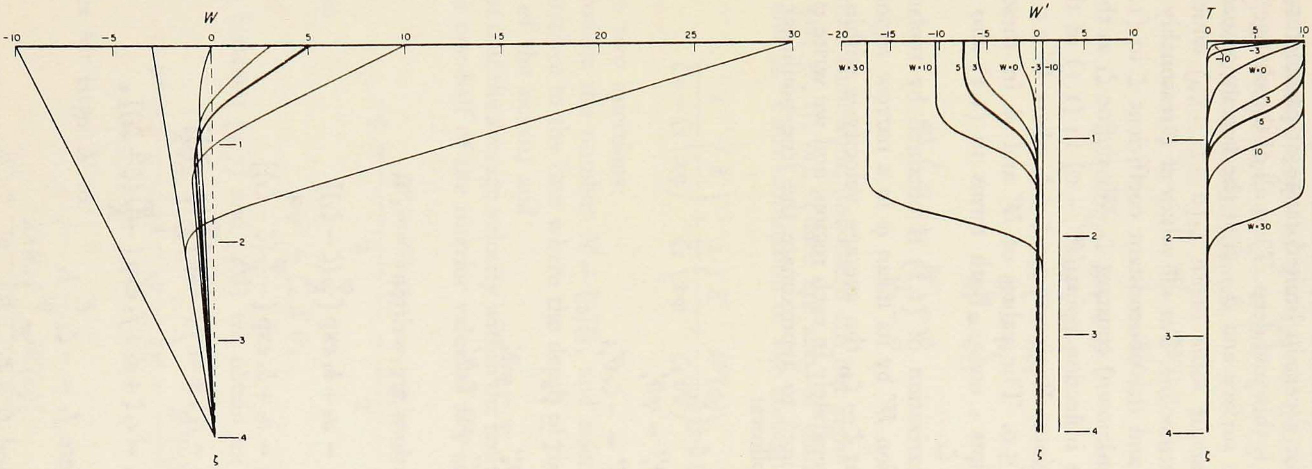

Figure 4. The distribution of W, W', and T, with the transformed depth ζ; plotted for various choices of the value of $W(0)$ imposed at the surface by the convergence of Ekman layer.

4. *The Thin Deep Thermocline*. Asymptotic Case $K \to 0$, $W > 0$. We anticipate that there is—for a certain limited range of parameters—a regime in which the thermocline is thin and deep. Thus all of the change in temperature between $\vartheta = \vartheta_0$ at the surface and $\vartheta = 0$ at the bottom actually occurs in a very narrow interval about some depth $\zeta_t [0 \leq \zeta_t \leq \zeta_b]$ which we can call the "depth" of the thermocline. For all values of ζ reasonably different from ζ_t, therefore, $\vartheta' = 0$, and the nonconstant coefficient ζ in (1.2) can be replaced by the (as yet unknown) constant ζ_t. We define ζ_t as the depth where $W = 0$, that is, at the inflection point $(\vartheta'' = 0)$ in (1.1) of the variable ϑ; and we define two regions: Region 1, $0 \leq \zeta \leq \zeta_t$ where $W \geq 0$ and Region 2, $\zeta_t \leq \zeta \leq \zeta_b$ where $W \leq 0$. The values of W and ϑ in these two regions are denoted by subscripts $_1$ and $_2$. Both terms in (1.1) also vanish except in the vicinity of $\zeta = \zeta_t$.

An appropriate linearization of (1.1) is obtained by replacement of the actually variable function W by its mean φ in a narrow region. Because W actually changes sign at $\zeta = \zeta_t$, the average velocity φ within the boundary layer must be formed separately in each region, and we write $\varphi = \varphi_1 = -\varphi_2$. The linear equations used to approximate the true nonlinear equations are therefore written as follows:

Region 1: $0 \leq \zeta \leq \zeta_t$

$$K\vartheta_1'' = \varphi\vartheta_1' \tag{4.1a}$$

$$W_1'' = -\zeta_t \vartheta_1'; \tag{4.2a}$$

Region 2: $\zeta_t \leq \zeta \leq \zeta_b$

$$K\vartheta_2'' = -\varphi\vartheta_2' \tag{4.1b}$$

$$W_2'' = -\zeta_t \vartheta_2'. \tag{4.2b}$$

Solutions of these equations are written

$$\vartheta_1 = a_1 + b_1 \exp\left[\frac{\varphi}{K}(\zeta - \zeta_t)\right] \tag{4.3}$$

$$\vartheta_2 = a_2 + b_2 \exp\left[-\frac{\varphi}{K}(\zeta - \zeta_t)\right] \tag{4.4}$$

$$W_1 = c_1\zeta + d_1 + f_1 \exp\left[\frac{\varphi}{K}(\zeta - \zeta_t)\right] \tag{4.5}$$

$$W_2 = c_2\zeta + d_2 + f_2 \exp\left[-\frac{\varphi}{K}(\zeta - \zeta_t)\right], \tag{4.6}$$

where $f_1 = -\zeta_t \dfrac{K}{\varphi} b_1$

and $f_2 = \zeta_t \dfrac{K}{\varphi} b_2$.

We anticipate that the exponential terms will only contribute in the neighborhood of the boundary layer at $\zeta = \zeta_t$, so that evaluation of constants $(a_1 b_1 c_1 d_1 a_2 b_2 c_2 d_2)$ of integration at $\zeta = 0$ and $\zeta = \zeta_b$ is particularly simple. The boundary conditions are as follows:

$$\text{at } \zeta = 0 \quad W_1 = W(0), \quad \vartheta_1 = \vartheta(0);$$

$$\text{at } \zeta = \zeta_t \quad W_1 = 0, \quad W_2 = 0, \quad W_1' = W_2', \quad \vartheta_1 = \vartheta_2, \quad \vartheta_1' = \vartheta_2';$$

$$\text{at } \zeta = \zeta_b \quad W_2 = 0, \quad \vartheta_2 = 0 .$$

There are nine conditions, and there are but eight independent constants of integration (a_i, b_i, c_i, d_i). There are also, of course, two undetermined parameters, φ and ζ_t. Thus elimination of the constants $a_i \ldots f_i$ leads to the relation

$$\frac{1}{\zeta_b - \zeta_t}\left(\frac{K}{2\varphi}\right) + \frac{1}{\zeta_t}\left(\frac{K}{2\varphi} + \frac{W(0)}{\zeta_t \vartheta_0}\right) = 1 . \qquad (4.7)$$

We now make two assertions:

(1) We introduce the number $N = \zeta_b/\zeta_t$, and assert that it is large; that is, we limit ourselves to the case where the depth of the thermocline is much less than that of the ocean; and

(2) We assert that the average velocity within the boundary layer in region 2, $-\varphi$, is equal to one-half of the interior value of W_2 as $\zeta \to \zeta_t$ (see Fig. 1) or from (4.6)

$$-\varphi = \frac{W_2^{\text{interior}}(\zeta_t)}{2} = \frac{c_2\,\zeta_t + d_2}{2} .$$

This reduces to

$$4\,\varphi^2 = \zeta_t\,K\,\vartheta_0 . \qquad (4.8)$$

Eliminating ζ_t between (4.7) and (4.8) we obtain an equation in φ alone:

$$\varphi^4 = \frac{K^2\vartheta_0}{8}\left[\left(1 + \frac{1}{N-1}\right)\varphi + \frac{W(0)}{2}\right], \qquad (4.9)$$

which simplifies for large N to

$$\varphi^4 = \frac{K^2\vartheta_0}{8}\left[\varphi + \frac{W(0)}{2}\right]. \qquad (4.10)$$

When solutions of this equation are obtained, then one can work back and express all other constants in terms of φ, and the problem is solved. It is physically a little more meaningful to introduce again the quantity

$$\lambda \equiv \frac{2\varphi}{W(0)} = \frac{W_2 \text{ interior } (\zeta_t)}{W(0)}, \tag{4.11}$$

which is the ratio of the upwelling velocity just beneath the thermocline to the downward velocity forced by the wind-stress just beneath the Ekman layer. The equation (4.10) is then expressed as

$$\lambda^4 = \mathcal{N}(\lambda + 1), \tag{4.10a}$$

where $\mathcal{N} = \dfrac{K^2 \vartheta_0}{[W(0)]^3}$, a dimensionless number.

It is evident that

$$\text{for } \mathcal{N} \ll 1, \quad \lambda = \mathcal{N}^{\frac{1}{4}}$$
$$\text{and } \mathcal{N} \gg 1, \quad \lambda = \mathcal{N}^{\frac{1}{3}}.$$

The parametric dependence of various quantities can now be easily expressed in terms of the original physical parameter in the extreme range of $\mathcal{N} \ll 1$:

	$\mathcal{N} \ll 1$
λ	$K^{\frac{1}{2}} \vartheta_0^{\frac{1}{4}} [W(0)]^{-\frac{3}{4}}$
2φ	$K^{\frac{1}{2}} \vartheta_0^{\frac{1}{4}} [W(0)]^{\frac{1}{4}}$
ζ_t	$\vartheta_0^{-\frac{1}{2}} [W(0)]^{\frac{1}{2}}$
K/φ	$2K^{\frac{1}{2}} \vartheta_0^{-\frac{1}{4}} [W(0)]^{-\frac{1}{4}}$
$K/\zeta_t\varphi$	$2K^{\frac{1}{2}} \vartheta^{\frac{1}{4}} [W(0)]^{-\frac{3}{4}}$

To recapitulate the meaning of the quantities in the left-hand column: λ is the ratio of amplitude of the upwelling velocity just beneath the thermocline to the amplitude of the downward component of velocity forced by the wind just beneath the Ekman layer; ζ_t is the depth of the thermocline (inflection point in the vertical temperature curve, and depth at which vertical component of velocity vanishes: *n. b.*, the depth ζ_t must be scaled by the similarity transformation to restore it to original physical dimensions); K/φ is a measure of the thickness of the thermocline layer measured in either direction from ζ_t (also in transformed coordinates); and $K/\varphi\zeta_t$ is the ratio of the thickness of the thermocline to the depth of the thermocline. For $\mathcal{N} \gg 1$ there is no true thermal boundary layering. The simple boundary layer theory, then, does not apply in this range, and we must restrict our attention to the range $N \ll 1$. The ratio of upward flow beneath the thermocline to downward Ekman flow, λ, is small in this case, as is seen above. The upward flow for a fixed Ekman flow increases with the mixing parameter K and the increased surface temperature, ϑ_0. The depth of the thermocline, ζ_t, in this range is independent of mixing parameter K and increases with increased downward Ekman flow and decreased surface temperature ϑ_0. The thickness of the thermocline, K/φ, does depend upon the mixing, and as we would expect, it is greater for greater mixing parameter K.

5. *Discussion of Results.* In order to discuss the nature of the results, it is helpful to begin by considering the case $W(0) = 5$, $T(0) = 10°C$, $K = 0.1$ cm²/sec, and $\zeta_b = 4$, shown in Fig. 1 by the heavy curves. This corresponds to the choice of a midlatitude station distant from the eastern wall more or less in the region of the western center of a subtropical high pressure gyre.

The quasivelocity $W(0) = 5$ corresponds to a downward velocity of about 5 cm day^{-1} just beneath a convergent Ekman layer. The surface temperature is taken as 10°C instead of 18°–20°C because, in the real ocean, salinity reduces the actual density stratification to about half of that which would obtain in the case of temperature alone. The value of the vertical eddy conductivity, $K = 0.1$ cm^2 sec^{-1}, is taken for convenience of discussion—for, as we shall see, it is rather difficult to decide on a value that could be honestly called characteristic. The value of $\zeta_b = 4$ represents a bottom depth of about 4000 m, but on account of the similarity transformation it is a pseudodepth, not having dimensions of length. Thus it is only at this particular point in the ocean (western central subtropical gyre) that the scales of W and ζ can be interpreted as cm day^{-1} and kilometers respectively.

These values of the parameters provide a convenient point of departure for the discussion. As can be seen from the heavy curves in Fig. 1, there is a clearly marked thermocline, with inflection point in temperature T at about $\zeta_t = 0.75$. The vertical component of pseudovelocity W also vanishes at this point. At shallower depths W is nearly linear, and T is almost a constant 10°. For $\zeta > \zeta_t$, $W \leq 0$, which means that the deep water moves upward against the water descending from the Ekman layer at the top. The maximum upward flow occurs at about $\zeta = 1.50$; and since the extremum in W corresponds to the zero of the meridional horizontal component of velocity, we refer to this depth as the level of no meridional motion and denote it by ζ_n. For $\zeta < \zeta_n$ the meridional horizontal component of the velocity is directed equatorward; at deeper levels, $\zeta > \zeta_n$, it is directed poleward. The other, lighter, curves in Fig. 1 represent cases with different values of mixing parameter K, all other parameters and boundary conditions remaining unchanged.

As is seen, the limiting case of no mixing, $K = 0$, is a simple two-layer case with a sharp discontinuity in temperature at $\zeta \approx 0.75$. W is linear in the upper layer and vanishes in the lower. There is no horizontal flow in the lower layer. On the other hand, by increasing K above $K = 0.1$, a thicker thermocline is produced and larger amplitudes of W are forced in the deep water. The value of ζ_t becomes slightly smaller, the value of ζ_n greater. The limiting case of $K = \infty$ corresponds to a linear temperature distribution with depth; the W distribution for $K = \infty$ is also drawn in Fig. 1. This case is only of interest in checking the consistency of the numerical solutions, because in such an extreme case, the simple idea of fixing the bottom temperature is a physically unrealistic boundary condition. We see therefore, as we might have suspected, that increased mixing thickens the thermocline and increases the amplitude of the deep circulation.

Formally, amplitude of the transport of deep circulation beneath the level of no meridional motion ζ_n is proportional to $-W(\zeta_n)$, whereas the transport driven by the wind is proportional to $W(0)$. Thus in a certain sense we can speak of a ratio λ of thermohaline to wind-driven circulation defined by $\lambda = \frac{-W(\zeta_n)}{W(0)}$. Comparison of the temperature curves in Fig. 1 with actual subtropical soundings suggests that the appropriate choice of K is in the range $0.1 \leq K \leq 1.0$, but it is difficult to be much more precise. The corresponding range of $W(\zeta_n)$ is about $-0.8 \geq W(\zeta_n) \geq -2.5$, which means that it is not easy to fix the amplitude of transport of the deep circulation, by inspection of the T curves, to much within a factor of 3. The ratio λ seems to lie somewhere in the range $0.15 \leq \lambda \leq 0.5$.

In Fig. 2, we revert to the case with $K = 0.1$ and hold all other parameters constant, except the depth to the bottom, ζ_b, which we permit to vary greatly in order to see its influence on the solutions. When ζ_b is made small, $\zeta_b < 4$, the amplitude of the deep circulation, $-W(\zeta_n)$, is greatly reduced; but when the depth $\zeta_b > 4$, there is very little influence on the ζ_n, $W(\zeta_n)$, λ or the form of the temperature distribution. In these respects it seems as though the depth, $\zeta_b = 4$, is already effectively infinite so far as the thermocline is concerned. However, in the deep water the amplitude of the meridional horizontal component of velocity, V, which is proportional to W', is very much reduced.

In Fig. 3 we again revert to the original case, but now we vary the surface temperature, $\vartheta(0)$. For $\vartheta(0) > 10°$, the thermocline depth, ζ_t, and reference level, ζ_n, are much reduced, but $-W(\zeta_n)$, λ, and V when $\zeta > \zeta_n$ are much increased. It seems reasonable that increasing the temperature should increase the transport of the thermohaline portion of the circulation; but it is only upon reflection that the shallower thermocline seems other than paradoxical.

In Fig. 4 we again revert to the original case, shown by heavy curves, and we now vary $W(0)$. Up until this case we have always taken $W(0) = 5$, so that the wind-driven part of the transport was always the same. Now it can vary. Consider increasing $W(0)$ to 10 and then to 30, *i. e.*, the Ekman convergence is made stronger and pushes more water downward. This deepens the thermocline (increases ζ_t) and the level of no meridional motion, ζ_n. It has a tendency to increase the deep water upwelling, $W(\zeta_n)$. On the whole, however, in the range $0.0 \leq W(0) \leq 10.0$, the amplitude of the thermohaline circulation is not strongly influenced by variation of $W(0)$. For negative values of $W(0)$—corresponding to the divergent Ekman layer drawing water upward from below, there is virtually no thermocline, the entire water column is cold, the circulation is entirely wind-driven, and the meridional component of velocity V extends all the way to the bottom.

In this discussion we have not explored all the interesting features of the numerical solutions. Thus, for example, if we were to construct a figure analogous to Fig. 1—with the parameter K variable but beginning with a choice of $W(0) = 5$ and $\vartheta(0) = 3°$, for example, then the family of curves for each K is rather different; ζ_t increases as K increases (opposite to the behavior of Fig. 1), and the value of $-W(\zeta_n)$ is not a monotonically increasing function of K.

It can be shown that $\partial\zeta_t(K)/\partial K > 0$ if $\zeta_t(0)/\zeta_b > \frac{1}{5}$, and $\partial\zeta_t(K)/\partial K < 0$ if $\zeta_t(0)/\zeta_b < \frac{1}{5}$. The latter case is more typical of the true subtropical oceanic case, *i. e.*, of a thin thermocline in an effectively infinite deep ocean.

The results of the numerical theory for $\mathcal{N} \ll 1$ (section 4) compare reasonably well with the asymptotic theory of section 4. To show this we compute the curve for W by the asymptotic theory for the same case as that about which the preceding discussion was pivoted. Then $\mathcal{N} = 8 \times 10^{-4}$, $\lambda = \mathcal{N}^{\frac{1}{4}} = 0.167$. The values of the various essential quantities computed are given thus:

	By numerical theory	By asymptotic theory
λ	0.172	0.167
$W(\zeta_n) \equiv 2\varphi$ . .	− 0.86	− 0.83
ζ_t	0.72	0.71
$\vartheta(\zeta_t) \equiv \vartheta_1$	5.4	5.0

6. *Acknowledgements*. The first thoughts about the subject of this paper arose following completion of the manuscript of the "The Gulf-Stream" book (p. 170 eq. 27) in the summer of 1955. The idea of using this particular form of coupling the vorticity equation to the thermal wind equation was applied to actual hydrographic data a year later (Stommel, 1956), and at the time, Dr. George Morgan and one of us (H.S.) discussed informally the idea of doing a boundary-layer analysis of the type given in Section 5. Also Dr. James Crease made some numerical computations of the kind presented here in Section 2. Neither of these preliminary investigations was pursued very far because there was not an adequate similarity transformation.

We were working with the idea that the isotherms near the thermocline all had a constant but undetermined slope, thus enabling us to transform the

partial vorticity equation to ordinary form. In retrospect that does not seem to be a bad idea, but at the time it seemed rather vague and uncomfortable. After Robinson found a usable transformation, it was possible to follow up both of these earlier investigations, and that is essentially what we have done here.

Dr. P. Welander and Dr. A. Robinson have succeeded in obtaining a similarity transformation including diffusion and all advective effects; however, their results were not available in final form for purposes of detailed comparison at the time this paper was submitted for publication.

The idea that subpolar gyres belong to the class of solutions with ζ negative is due to Dr. N. P. Fofonoff.

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