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A TEST OF MIXING LENGTH THEORIES IN A COASTAL PLAIN ESTUARY[1]

BY

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ABSTRACT

Observations of vertical and horizontal variations in salinity and velocity in the James River estuary have been previously employed in the indirect determination of the vertical eddy flux of salt, $\langle v_2's' \rangle$ (Pritchard, 1954). This term is employed here to compute a mixing length after the definition of Prandtl. It is shown that this *observed* mixing length is qualitatively similar to a *theoretical* one formulated from the geometry of the system and a stability parameter related to the density stratification of the system. Several alternate hypotheses regarding the *theoretical* mixing length are tested. It is found that better quantitative agreement is obtained when the influence of surface wind waves is included in the formulation of the *theoretical* mixing length.

Prandtl (1925) proposed the mixing length theory of turbulence in an attempt to relate the rate of momentum transport to the mean flow pattern of an incompressible fluid. Despite its questionable reality concerning the mixing process, this concept has proven extraordinarily fruitful in the development of fluid mechanics and consequently has been utilized in related fields to describe the turbulent transport of properties other than momentum.

The purpose of this work is to demonstrate that the mixing length associated with vertical diffusion of salt in a stratified estuary may be formulated in terms of the stability of the system and of an adiabatic mixing length determined by the geometry of the system. This mixing length will be shown to be qualitatively compatible with the corresponding one provided by Prandtl's definition. An improved correspondence results when the effects of surface wind waves are included in the formulation of the adiabatic mixing length.

Development. Consider a stratified coastal plain estuary in which there exists a mean circulation pattern similar to that described by

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Pritchard (1952). Here the lateral variations in velocity and salinity are negligibly small so that the flow pattern is essentially two dimensional, consisting of a seaward flow in the upper portion of the water column and a landward flow in the lower portion. Pritchard (1954) has shown that, in this system, the salt balance is maintained primarily by a longitudinal advective salt flux and a vertical nonadvective salt flux. It is the mixing length associated with this latter flux with which we are concerned. A left-handed co-ordinate system is employed in which the x_1, x_2, and x_3 -axes are directed horizontally seaward, vertically downward and laterally across the estuary, respectively.

Consider first the mixing length for salt as given by Prandtl's proposal. Let the vertical flux of salt at a point in the estuary be expressed as

$$\langle v'_2 s' \rangle = \gamma \sqrt{\overline{v'_2{}^2}} \times \sqrt{\overline{s'^2}}, \tag{1}$$

where $\sqrt{\overline{v'_2{}^2}}$ and $\sqrt{\overline{s'^2}}$ are the local root mean squares of the turbulent fluctuations in velocity and salt, respectively, and where γ is the correlation between them. If, as Prandtl suggests, a turbulent fluctuation in property may be described in terms of a linear dimension and of a mean gradient of that property, we have for $\sqrt{\overline{s'^2}}$

$$\sqrt{\overline{s'^2}} = l_s \frac{\overline{\partial s}}{\partial x_2}. \tag{2}$$

Further, it is hypothesized that a negative correlation must exist between the velocity fluctuations perpendicular and parallel to the direction of mean flow; that is,

$$\sqrt{\overline{v'_2{}^2}} = -c\sqrt{\overline{v'_1{}^2}}. \tag{3}$$

By the same argument that led to (2), we write

$$\sqrt{\overline{v'_2{}^2}} = -cl_v \frac{\overline{\partial v_1}}{\partial x_2}. \tag{4}$$

Assuming for this system that $l_s = l_v = l$ and by substituting (2) and (4) into (1), we have for the vertical flux of salt

$$\langle v'_2 s' \rangle = -\gamma c l^2 \left| \frac{\overline{\partial v_1}}{\partial x_2} \right| \frac{\overline{\partial s}}{\partial x_2}. \tag{5}$$

Letting $\eta^2 = \gamma c$ be a constant in (5), we have

$$\eta^2 l^2 = - \frac{\langle v'_2 s' \rangle}{\left| \dfrac{\overline{\partial v_1}}{\partial x_2} \right| \cdot \dfrac{\overline{\partial s}}{\partial x_2}} \tag{6}$$

from which the mixing length, l, may be calculated provided that the remaining quantities are known. For future reference, let this value of l be called the *observed* mixing length.

Consider now the derivation of l in terms of an adiabatic mixing length and a stability function. For an unstratified system, Prandtl (1932) found that the mixing length in the immediate vicinity of a solid boundary is nearly a linear function of the shortest distance from the boundary; that is

$$l_\alpha = \kappa(x_2), \tag{7}$$

where l_α is the adiabatic (unstratified) mixing length, κ the universal turbulence constant, and x_2 the distance from the boundary. Montgomery (1943) generalized Prandtl's geometrical assumption so that the adiabatic mixing length for cylinders of infinite length became a linear function of the reciprocal of the average "nearness," in the plane normal to the cylinder, to the boundary. Since the ratio of width to water depth is large (200:1) for the shallow coastal plain estuary in which we are interested, the system may be regarded as being composed essentially of two parallel plane boundaries: namely, the water surface, and the river bottom. For this geometrical setup, neglecting the roughness length, Montgomery's mixing length, l_α, is found to be

$$l_\alpha = \frac{\kappa x_2}{h} \{h - x_2\}, \tag{8}$$

where h is the boundary separation (in our case the depth of water). Clearly, l_α is symmetrical about the point of mid-depth.

It is evident that the mixing length in a homogeneous system will be unlike that in a stratified system. A stable stratification will provide a damping effect on vertical turbulent motions since a portion of the energy in excess of that required to maintain the mean field is used to perform work against the density gradient. The reverse is true for an unstable stratification. Rossby and Montgomery (1935) have pointed out that for a stable stratification the mixing length is reduced; furthermore they have proposed a relationship between the adiabatic and stratified lengths.

They state that the turbulent kinetic energy per unit mass for the case of neutral stability must be equal to the sum of the turbulent kinetic energy plus the potential energy per unit mass for the stable case. The turbulent kinetic energies for the two cases are proportional to

$$l_\alpha{}^2 \left| \overline{\frac{\partial v_1}{\partial x_2}} \right|^2 \quad \text{and} \quad l \cdot \left| \overline{\frac{\partial v_1}{\partial x_2}} \right|^2,$$

where the mean shear is assumed to be the same in the two cases. Here l_α is the mixing length for adiabatic conditions (neutral stability) and l is the mixing length for stable conditions. The potential energy for the stable case is shown to be proportional to

$$\frac{g}{\sigma_t} \frac{\partial \sigma_t}{\partial x_2} l^2,$$

where σ_t has the conventional meaning of the density less unity times one thousand. (Note: Rossby and Montgomery treated the atmospheric case, where the potential temperature θ takes the place of σ_t.)

The expression obtained by Rossby and Montgomery relating l for the stable case with l_α is then in our notation

$$l = l_\alpha(1 + \beta Ri)^{-\frac{1}{2}}, \tag{9}$$

where Ri is the Richardson number, given by

$$Ri = \frac{g}{\sigma_t} \frac{\partial \sigma_t}{\partial x_2} \bigg/ \left| \frac{\overline{\partial v_1}}{\partial x_2} \right|^2$$

and where β is an undetermined proportionality factor.

Holzman (1943) cited other references to show evidence for a critical value of Ri above which turbulence is suppressed. He argued that l should go to zero at some finite value of Ri, a condition not satisfied by eq. (9). Holzman proposed an alternate relationship which does provide for a critical value of Ri, and further, it is more amenable to mathematic treatment. His expression, in our notation, is

$$l = l_\alpha(1 - \beta Ri)^{\frac{1}{2}}. \tag{10}$$

As stated above, Rossby and Montgomery assumed that the turbulent kinetic energy for the adiabatic case was equal to the turbulent kinetic energy plus the potential energy due to the displacement over the mixing length for the stable case. An alternate proposal is that the energies *per unit length of displacement* are equal. This proposition leads to the expression

$$l = l_\alpha(1 + \beta Ri)^{-1}. \tag{11}$$

In eq. (8), l_α is based on the assumption of rigid planar boundaries at both the bottom and surface of the estuary. Actually, turbulent energy is introduced at the free water surface by wind waves. In this case the adiabatic mixing length should be larger in the near surface layers than the values given by eq. (8). It would appear reasonable to conclude that this increase should be related to the

orbital motion induced by surface wind waves. The orbital motion induced by surface waves is proportional to the expression

$$e^{-2\pi x_2/L},$$

where L is the wave length of the surface waves and x_2 the depth, as before. A possible expression for the adiabatic mixing length is then

$$l_\alpha = \frac{\kappa x_2}{h} \{h - x_2\}\{1 + ae^{-2\pi x_2/L}\}. \tag{12}$$

Here a is an undetermined proportionality factor.

The term $\kappa x_2\{h - x_2\}/h$ in eq. (12) is the same as in eq. (8) and depends only on the geometry of the system. This term is symmetrical about mean depth. The term $\{1 + ae^{-2\pi x_2/L}\}$ depends on the characteristics of the surface wave motion, and the effect of this term on l_α decreases rapidly with depth. Fig. 1 gives an example of l_α computed from both eq. (8) and (12). In this computation κ has been taken as 0.40, h as 7.5 m, L as 5 m, and a as 3.17. Note that eq. (8) gives an l_α which is symmetrical about a well defined maximum at mid-depth while inclusion of the effect of surface wind waves extends this area of maximum l_α into the upper half of the water column. In this particular case the two computations for l_α are essentially the same below mid-depth.

For purposes of comparison with the *observed* mixing length as determined by eq. (6), we designate the l determined by any of the equations (9), (10), or (11) as the *theoretical* mixing length. This designation will hold whether eq. (8) or eq. (12) is employed in determining l_α.

We will now compare the *observed* mixing length given by eq. (6) with the *theoretical* mixing lengths given by each of the equations (9), (10) and (11), using data collected in the James River estuary.

Eq. (6) contains the unknown constant η, and eqs. (9), (10) and (11) contain the unknown constant β. The procedure employed in this comparison involves the simultaneous solution of the two subject equations at two depths (3.5 m and 5 m) in order to obtain the constants η and β. The correspondence of the two l's at other depths is then examined.

The data here employed were collected in the James River estuary during three periods of several days each during the summer of 1950. The station is located some 17 miles upstream from the mouth of the James. For further information on the physical and chemical properties of this estuary, refer to Pritchard (1952, 1954). The data employed in the evaluation are presented in detailed reduced form by

Pritchard and Kent (1953) under the station designation J–17. From this latter reference the required values of $\langle v'_2 s' \rangle$, $\overline{\partial s}/\partial x_2$, σ_t, $\partial \sigma_t/\partial x_2$, $|\overline{\partial v_1}/\partial x_2|$, and $|\overline{\partial v_1}/\partial x_2|^2$ can be obtained directly or computed from auxiliary data.

We have first taken all three periods together as representing one set of data. The average values of the pertinent parameters were employed in comparing the *observed* with the *theoretical* mixing length.
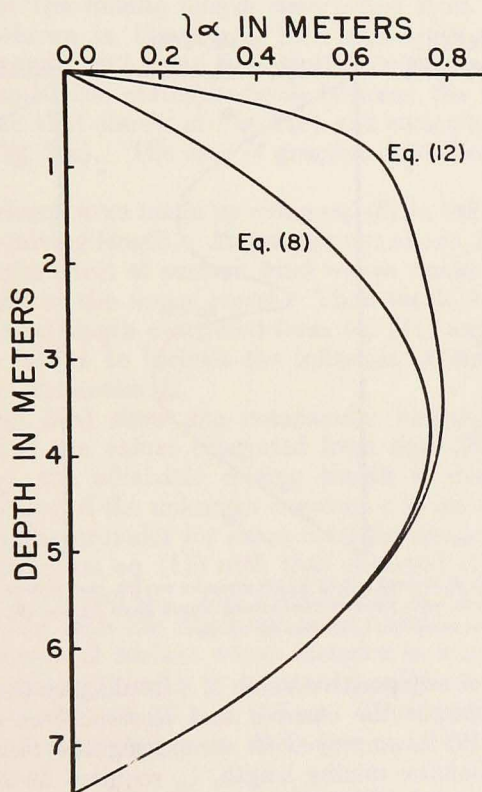

Figure 1. The adiabatic mixing length, l_α, as a function of depth as computed from eq. (8), which includes the effect of geometry only, and from eq. (12), which also includes the effect of surface wind waves.

In this initial comparison eq. (8) was employed in obtaining l_α. In comparing eq. (6), the *observed* mixing length, with eq. (9), the *theoretical* mixing length proposed by Rossby and Montgomery, we found that it was not possible to find a simultaneous solution at two depths which gave real values for η and β. In order to make some

comparison, a value 0.253 was chosen for η, and β was then found by assuming that the two pertinent equations were the same at the single depth of 3.5 m.   Results are shown in Fig. 2(a).   Here the *theoretical* is presented as a solid curve while the *observed* is plotted as discrete points for each one-half meter.   Note that the *observed* mixing length is in general larger than the *theoretical*, except at 3.5 m, where agreement was forced by the method of determining the proportionality constant β.
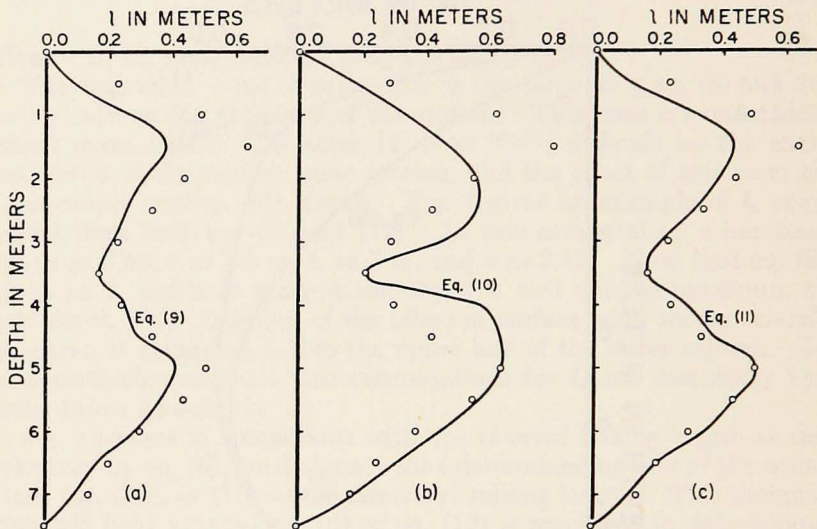

Figure 2. Comparison of *observed* mixing length given by eq. (6), plotted here as discrete points at each one-half meter depth, with the *theoretical* mixing length given by eqs. (9), (10) and (11). For these determinations, the parameter l_α is given by eq. (8).

The selection of any positive value of η would give the same relative disagreement between the *observed* and *theoretical* as shown in Fig. 2(a), when eq. (9) is employed in computing the *theoretical* mixing length.   The adiabatic mixing length, l_α, required as a parameter in using eq. (9), was computed from eq. (8) for this comparison.   Thus when l_α is determined on the basis of boundary geometry only, without considering the influence of wind waves, there is only fair correspondence between the mixing length based on Rossby and Montgomery's relationship and the one based on observation.

Comparison of the *observed* with the *theoretical* mixing length based on the relationship proposed by Holzman is shown in Fig. 2(b).   Eqs. (6) and (10) were solved simultaneously at 3.5 and 5 m depth to obtain values for the parameters η and β.   Except at these two points

of forced correspondence, the correspondence between the two lengths is not particularly good.

The value of β which results from this comparison is 0.412×10^{-3}. Eq. (10) provides for a critical value of Ri at which turbulence becomes completely damped: that is, at which the mixing length goes to zero. This condition is met when βRi equals unity: that is, when Ri equals 2.43×10^3.

Comparison of the mixing length determined from eq. (11) with the *observed* is shown in Fig. 2(c). Here again (6) and (11) were solved simultaneously at 3.5 and 5 m depth to obtain values of η and β. At other depths the correspondence between the two lengths is much better than that shown in Fig. 2(b) and somewhat better than that shown in Fig. 2(a). The area of greatest difference occurs in the upper layers.

These comparisons were made by using eq. (8) in the determination of the adiabatic mixing length. As pointed out above, it is reasonable to assume that the effect of surface wind waves would be to increase the mixing length in the upper layers. Thus the lack of agreement between the mixing length computed from eq. (11) and that *observed* may be due to failure to include the influence of surface waves in determining the parameter l_α.

Figs. 3(a) and 3(b) show the comparison between the *observed* mixing length and the values computed from eqs. (9) and (11), respectively, when the adiabatic mixing length is determined from eq. (12). The value of the unknown constant a in eq. (12) was taken at 3.17. This value provides for exact correspondence of the mixing length determined from eq. (11) with that observed at the maximum which occurs at 1.5 m depth. The wave length L was taken as 5 m, a value compatible with the observed mean wind.

With the influence of surface waves included in the determination of the adiabatic mixing length, it becomes possible to solve eqs. (6) and (9) simultaneously at 1.5 and 3.5 m. Correspondence between the mixing length computed from these two equations is reasonably good at all depths.

The excellent agreement shown in Fig. 3(b) between the mixing lengths determined from eqs. (11) and (6) is somewhat forced, since the constants η and β were determined to give exact correspondence at depths of 3.5 and 5.0 m; and the proportionality factor a in eq. (12) was evaluated so as to give exact agreement at 1.5 m. However, the correspondence at depths other than these three is much better than for any of the other hypotheses tested.

The evaluation discussed above was made by using the average of data collected during three periods of several days each. The data

from the individual periods show significant differences in vertical
stability as well as in other parameters which enter the computations.
In the following discussion the observed data from each of the three
periods have been used in further testing the suitability of eq. (11) in
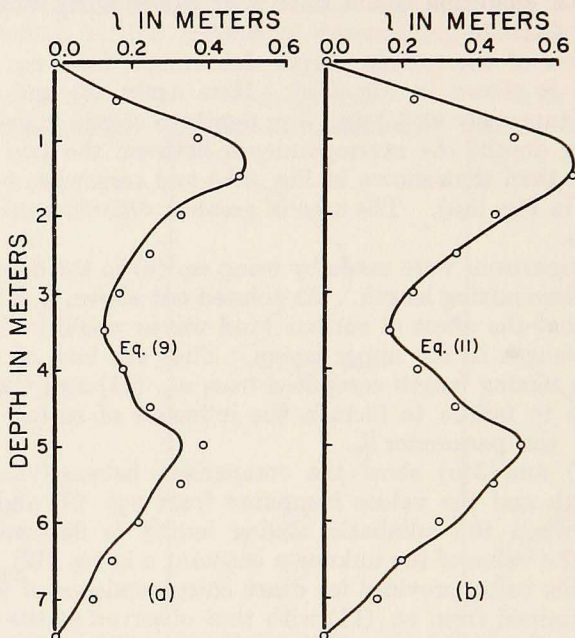defining a mixing length under stable conditions.

Figure 3. Comparison of *observed* mixing length as given by eq. (6), plotted here as discrete points
for each one-half meter depth, with the *theoretical* mixing length given by eqs. (9) and (11), when the
adiabatic mixing length is given by eq. (12).

The proportionality factors η and β should be constant if eq. (11) is
a valid expression for the mixing length under stable conditions.   The
values of these two factors as determined from the combined data
have been used in computing the *observed* and *theoretical* mixing lengths,
using eq. (6) and eq. (11) respectively for each of the three periods.
Results of this comparison are shown in Fig. 4.   The curves for each
of the three periods as computed from eq. (11) show differences, one
from the other, which are also reflected in the *observed* mixing length.
The correspondence of the two lengths for these three periods is quite
good.

Table I summarizes the results of the various comparisons discussed
above. Data available from the James River estuary indicate the
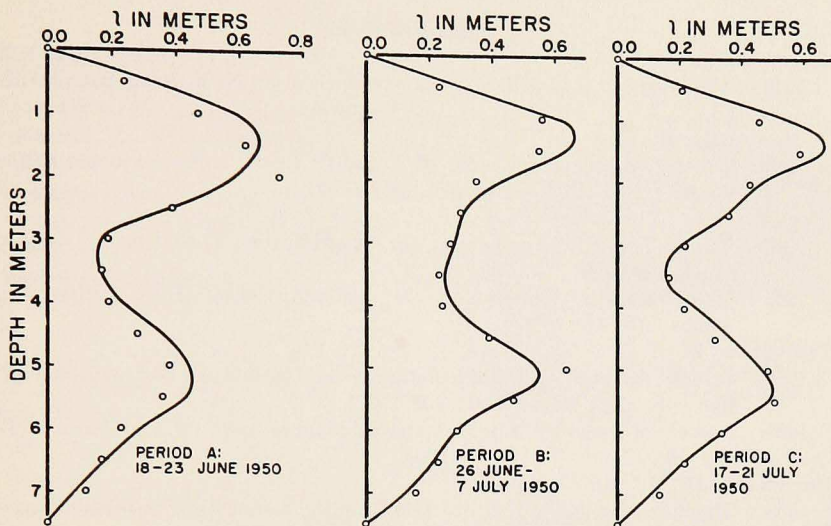following:

Figure 4. Curves are the mixing length computed from eq. (11), using $\beta = 1.65 \times 10^{-3}$. The circled points are the *observed* mixing length determined from eq. (6), using $\eta = 0.253$. The effect of wind waves has been included in the computation of the *theoretical* curve.

(a) The original proposal of Rossby and Montgomery for the mixing length under stable conditions is more compatible with the data than is the later proposal by Holzman.

(b) The relationship given by eq. (11) appears to provide a somewhat better correspondence with the data than that provided by the relationship of Rossby and Montgomery.

(c) The influence of surface wind waves is effective in increasing the adiabatic mixing length in the upper layers.

TABLE I

Hypothesis	η	β	Root mean square percentage difference, observed and theoretical mixing lengths*
Eqs. (8) and (9)	0.253	8.05 $\times 10^{-3}$	29.9
	0.335	14.14 $\times 10^{-3}$	23.4
Eqs. (8) and (10)	0.201	0.412 $\times 10^{-3}$	41.7
Eqs. (8) and (11)	0.253	1.58 $\times 10^{-3}$	28.0
Eqs. (12) and (9)	0.335	16.55 $\times 10^{-3}$	16.1
Eqs. (12) and (11)	0.253	1.65 $\times 10^{-3}$	15.0

* Depths at which agreement between the two lengths was "forced" by virtue of the selection of the coefficients are omitted from this evaluation.

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Note Added in Proof

I am indebted to Dr. R. B. Montgomery for pointing out to me that in the Richardson number the factor $(1/\sigma_t)(\partial\sigma_t/\partial x_2)$ should more correctly be $(1/\rho)(\partial\rho/\partial x_2)$. In the text this change would affect only the values of β, which should be multiplied by the factor 1.47×10^2. Thus the value of $\beta = 1.65 \times 10^{-3}$ should be corrected to $\beta = 0.242$, etc.