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THEORETICAL PROOF OF THE EXISTENCE OF CHARACTERISTIC DIFFUSE LIGHT IN NATURAL WATERS[1]

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ABSTRACT

This paper develops a mathematical model for the radiance distribution of light penetrating a homogeneous hydrosol on the basis of the general theory of radiative transfer. It is proved that the radiance distribution approaches an asymptotic pattern at great depths. This is in accord with previous field measurements of the directional patterns in underwater light and with L. V. Whitney's conjecture that there is at some depth in natural waters a characteristic diffuse light symmetrically distributed around the vertical. The angular form of this equilibrium light pattern is derived in terms of the mathematical model presented.

INTRODUCTION

Recent experimental evidence (Tyler, 1958) forms the basis for fresh support of the long-standing conjecture that the radiance distribution about a point in an optically deep natural hydrosol approaches, with increasing depth, a characteristic form which is independent of the external lighting conditions and of the optical state of the medium's surface and which depends only on the inherent optical

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properties of the medium. This conjecture was apparently given its first definitive formulation by Whitney (1941a, 1941b), who referred to the asymptotic radiance distribution as *characteristic diffuse light.* (Some early experimental evidence for this conjecture is cited in Whitney's papers.) In this note we complement experimental evidence in favor of this conjecture by supplying a simple mathematical proof of the existence of characteristic diffuse light in all homogeneous optically deep natural waters. The paper concludes with a derivation of the integral equation governing the angular structure of characteristic diffuse light as well as a brief discussion of an interesting and tractable example for the case of isotropic scattering.

We note in passing that the applicability of Whitney's hypothesis has been widened considerably since its formulation. The problem of a limiting angular distribution has since been encountered in modern neutron transport theory, basically as an abstract mathematical problem rather than as an experimental phenomenon. A similar type of problem has long been extant in astrophysical radiative transfer. A general proof of the existence of an asymptotic radiance distribution which covers all of these contexts recently has been devised (Preisendorfer, 1958a).

However, the hypothesis still retains its greatest usefulness in the context of geophysical optics. For in this field, unlike the others mentioned above, the trend to a characteristic limiting form is a directly observable phenomenon. Furthermore, the existence of such a form is of inestimable importance to all experimental research work dealing with the determination of optical properties of natural waters. In many important instances, knowledge that an asymptotic radiance distribution exists will obviate the necessity of experimental probings to extremely large depths; for such knowledge will allow, by means of relatively simple formulae, accurate prediction of the geometrical structure of the light field in great-depth ranges. Some of these practical consequences of the asymptotic radiance hypothesis have been formulated recently (Preisendorfer 1958b).

PHYSICAL BACKGROUND OF THE METHOD OF PROOF

The argument used by Whitney in establishing experimental evidence for the asymptotic radiance hypothesis went basically as follows: he showed that when experimentally obtained plots of radiance distributions at various large depths were blown up to the same size (more precisely, the zenith readings were normalized to a common value), they formed a set of nearly congruent figures. Now, an interesting feature of such distributions is that they assume the same *shape* and decrease in size with increasing depth at nearly the same exponential rate. This fact can be stated precisely as follows:[2]

$$N(z, \theta, \phi) = g(\theta, \phi)e^{-kz}. \quad (1)$$

From this we see that the asymptotic radiance hypothesis is equivalent to the statement that *the directional and depth dependence of radiance distributions multiplicatively uncouple at great depths.* That is, the radiance function N may be represented as the product of two functions: g which gives the shape or directional structure common to all distributions and an exponential function which gives the depth dependence of the distributions.

Each factor on the right-hand side of (1) has special physical significance. The function g evidentally defines the angular form of characteristic diffuse light. The exponent k of the exponential function has the following interesting interpretation.

Define *scalar irradiance* $h(z)$ at depth z as follows:

$$h(z) = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} N(z, \theta, \phi) \sin \theta \, d\theta \, d\phi. \quad (2)$$

The quantity $h(z)$ is then a measure of the volume density of radiant energy at depth z. Measurements of $h(z)$ over the years in many hydrosols have shown that $h(z)$ varies essentially in an exponential manner with depth. That is, semilog plots of $h(z)$ *vs* depth show an unmistakable trend toward linearity as depth increases. In any event, $h(z)$ may be accurately represented by a general formula of the type

$$h(z) = h(o) \exp\left\{ -\int_0^z k(z') \, dz' \right\}, \quad (3)$$

where $k(z)$ is the negative logarithmic derivative of $h(z)$. (Here and below, a primed symbol refers to a dummy variable of integration.) As depth increases, experimental evidence gathered in the field indicates that $k(z)$ approaches a constant value. Denote this limit value by k_∞. Now, assuming that an asymptotic radiance distribution is approached by the radiance distributions in a particular body of water, we see from (1), (2) and (3) that

$$h(z) = h(z_0)e^{-k_\infty(z-z_0)} = e^{-kz} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} g(\theta, \phi) \sin \theta \, d\theta \, d\phi, \quad (4)$$

[2] It is implicit in the definition of radiance that it applies to an arbitrary but fixed wavelength of radiant flux. This is also true of all the other radiometric quantities used in this note.

where z_0 is the depth below which we may assume that $k(z) = k_\infty$. From this we conclude that

$$k = k_\infty. \tag{5}$$

Hence, under the above assumption (1) we see that, at great depths, the size of a radiance distribution plot decreases exponentially with increasing depth and that the rate of this decrease is precisely that of scalar irradiance (or energy density).

The close connection between the depth dependence of scalar irradiance and that of the radiance distributions, as summarized in (5), suggests the following mode of representation of the radiance distributions for any depth: Define, for each direction (θ, ϕ),

$$K(z, \theta, \phi) = \frac{-1}{N(z, \theta, \phi)} \frac{dN(z, \theta, \phi)}{dz}. \tag{6}$$

Then, in analogy to (3), $N(z, \theta, \phi)$ at any depth z may be represented exactly by

$$N(z, \theta, \phi) = N(0, \theta, \phi) \exp\left\{- \int_0^z K(z', \theta, \phi)\, dz'\right\}. \tag{7}$$

Now suppose there is some depth z_0 below which we have $K(z, \theta, \phi) = k_\infty$ for all directions (θ, ϕ). Then (7) may be written

$$\begin{aligned} N(z, \theta, \phi) &= N(0, \theta, \phi) \exp\left\{- \int_0^{z_0} K(z', \theta, \phi)\, dz' - \int_{z_0}^{z} K(z', \theta, \phi)\, dz'\right\} \\ &= N(z_0, \theta, \phi) \exp\left\{- k_\infty(z - z_0)\right\}. \end{aligned}$$

If we set

$$g(\theta, \phi) = N(z_0, \theta, \phi) \exp\left\{k_\infty z_0\right\},$$

then we may write

$$N(z, \theta, \phi) = g(\theta, \phi)\, e^{-k_\infty z}, \tag{8}$$

for all depths z below z_0.

The similarity between (1) and (8) is unmistakable and it points out a method of attack we may follow in order to prove the asymptotic radiance hypothesis: we must show that the quantities $K(z, \theta, \phi)$ approach a limit as depth is increased and that this limit is independent of the directions (θ, ϕ). Furthermore, this limit, in accordance with the preceding discussion, should be none other than the limit k_∞ of $k(z)$, as defined in (3).

THE PROOF

We make use of the equation of transfer for radiance:

$$\frac{dN(z, \theta, \phi)}{dr} = -\alpha N(z, \theta, \phi) + N_*(z, \theta, \phi), \tag{9}$$

where

$$N_*(z, \theta, \phi) = \int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi} \sigma(\theta, \phi; \theta', \phi')\, N(z, \theta', \phi') \sin \theta' \, d\theta' \, d\phi' \tag{10}$$

defines the path function N_*; σ is the volume scattering function (which governs the law of scattering in the water) and α is the volume attenuation coefficient. The formal solution of (9) is readily obtained:

$$N(z, \theta, \phi) = N^0(z, \theta, \phi) + \int_0^r N_*(z', \theta, \phi)\, e^{-\alpha(r-r')}\, dr'. \tag{11}$$

The first term represents the component of N consisting of unscattered light. The second represents the space light over the path of length r

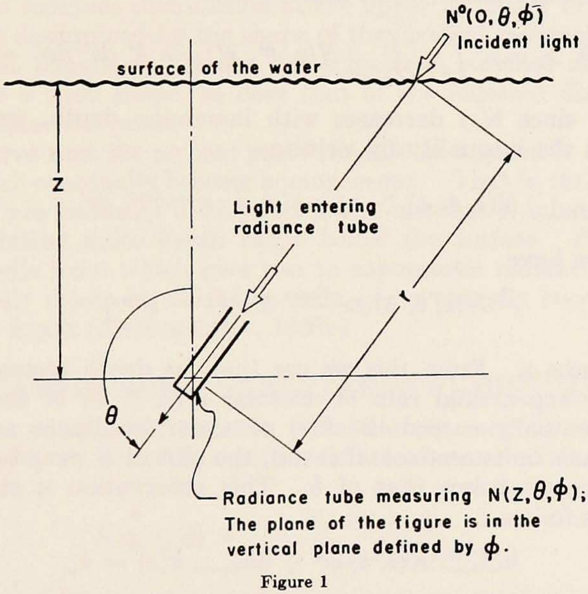

Figure 1

(Fig. 1) which has been generated by light scattered into the path of sight all along its extent. The formal solution (11) has been written for a general downward direction of flow of light (see Fig. 1) so that $N^0(z, \theta, \phi)$ is interpreted as the directly transmitted light from the upper boundary of the medium and is of the form

$$N^0(z, \theta, \phi) = N^0(0, \theta, \phi)\, e^{-\alpha r},$$

where

$$-r \cos \theta = z.$$

We now turn eq. (11) into a useful inequality by means of the following three steps:

First, since $N(z, \theta, \phi)$ clearly exceeds its spacelight component at all depths, we write

$$N(z, \theta, \phi) > \int_0^r N_*(z', \theta, \phi)\, e^{-\alpha(r-r')}\, dr'.$$

Second, using the definition of N_*, we strengthen the inequality when we write

$$N(z, \theta, \phi) > \sigma_{\min} \int_0^r h(z')\, e^{-\alpha(r-r')}\, dr',$$

where $\sigma_{\min}$ is the minimum value of the volume scattering function; that is, we have used (10) to deduce that

$$N_*(z, \theta, \phi) > \sigma_{\min} \int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi} N(z, \theta', \phi') \sin \theta'\, d\theta'\, d\phi' = \sigma_{\min} h(z).$$

Finally, since $h(z)$ decreases with increasing depth, we certainly strengthen the inequality by writing

$$N(z, \theta, \phi) > \sigma_{\min}\, h(z) \int^r e^{-\alpha(r-r')}\, dr'.$$

That is, we have

$$N(z, \theta, \phi) > \frac{\sigma_{\min}}{\alpha}\, h(z)(1 - e^{-\alpha r}) \tag{12}$$

for all depths z. From this we see that, as depth increases indefinitely, the exponential rate of decrease $K(z, \theta, \phi)$ of the radiance cannot eventually exceed the $k(z)$ of scalar irradiance and remain larger by any finite amount; if it did, the plot of N would eventually fall and remain below that of h. This observation is stated symbolically as follows:

$$\lim_{z\to\infty} K(z, \theta, \phi) \leqslant \lim_{z\to\infty} k(z) = k_\infty \tag{13}$$

for all downward directions (θ, ϕ). We now show that strict equality must hold in (13). We achieve this by initially assuming the contrary; that is, we assume that there is a nonzero solid angle Ω_0 of directions over which

$$\lim_{z\to\infty} K(z, \theta, \phi) \leqslant k_\infty - \epsilon,$$

where ϵ is *any* small positive number. Then it is clear that the radiances in this set of directions decrease at a definitely smaller rate than scalar irradiance; so much smaller in fact that, by our assumption, for some depth z_1 we must have

$$\int_{\Omega_0} N(z_1, \theta, \phi) \sin \theta \, d\theta \, d\phi > h(z_1).$$

However, this conclusion clearly contradicts (2) since a part cannot exceed the whole. We have reached a contradiction which leaves only one other possibility:

$$\lim_{z \to \infty} K(z, \theta, \phi) = k_\infty \tag{14}$$

for all downward directions (θ, ϕ). In the light of the preceding discussion [*cf* (8)], this means that the shape of the radiance distributions impinging on the upper boundaries of deep layers of water eventually assume a fixed form. But it is known that the shape of the *reflected* radiance distribution at the upper boundary of a scattering layer is determined by the shape of the incident radiance distribution at that boundary. Hence if the incident radiance distribution approaches a fixed shape, so does that of the reflected distribution. This completes the proof.

We observe that the present proof can also be applied in all natural waters which eventually become homogeneous. That is, the preceding arguments are basically unchanged if the medium is inhomogeneous over any initial finite depth range below the surface. Even more general media exist which give rise to asymptotic radiance distributions, namely media in which the ratio σ/α eventually becomes independent of depth (Preisendorfer, 1958a).

THE EQUATION FOR CHARACTERISTIC DIFFUSE LIGHT

Using the equation of transfer, definition (6), and the relation between z and r, we write the equation of transfer in the following general form:

$$N(z, \theta, \phi) = \frac{N_* \, (z, \phi, \theta)}{\alpha + K(z, \theta, \phi) \cos \theta} . \tag{15}$$

From (14) and (8) we see that the limiting form of (15) (as depth increases indefinitely) is

$$g(\theta, \phi) = \frac{\int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi} \sigma(\theta, \phi; \theta', \phi') \, g(\theta', \phi') \sin \theta' \, d\theta' \, d\phi'}{\alpha + k_\infty \cos \theta} ; \tag{16}$$

this is the equation governing the angular form of characteristic diffuse light. It is a property of equations of the type shown in (16) that the function g is independent of ϕ for all real physical situations. Thus characteristic diffuse light is always represented by a surface of revolution whose axis of symmetry is vertical.

The theory of the solution of such equations as (16) is fairly well understood (*e.g.*, see Davison, 1957). The present note, therefore, will not discuss (16) in any detail. However, there is one simple special case which is immediately solved and which can shed much light on some of the salient details of the structure of the asymptotic radiance distributions. This is the case of isotropic scattering, where the volume scattering function σ is independent of direction and has the form

$$\sigma(\theta, \phi; \theta', \phi') = \frac{s}{4\pi}, \tag{17}$$

where s is the total scattering coefficient.

To observe the resulting structure of the asymptotic radiance distribution it is convenient in the present case to turn to (15). With assumption (17) and definitions (2) and (10), we have

$$N(z, \theta, \phi) = \frac{s}{4\pi} \frac{h(z)}{\alpha + K(z, \theta, \phi) \cos \theta},$$

which at great depths approaches the form

$$N(z, \theta, \phi) = \frac{1}{4\pi}\left(\frac{s}{\alpha}\right) \frac{h(z_0)\; e^{-k_\infty(z-z_0)}}{1 + \left(\frac{k_\infty}{\alpha}\right) \cos \theta}. \tag{18}$$

Here z_0 is the depth below which $h(z)$ decreases exponentially with increasing depth. Comparing (18) with (8), we see that for the present case

$$g(\theta, \phi) = \frac{1}{4\pi}\left(\frac{s}{\alpha}\right) \frac{h(z_0)\; e^{k_\infty z_0}}{1 + \left(\frac{k_\infty}{\alpha}\right) \cos \theta}. \tag{19}$$

We have written (19) in the indicated form to point up the following geometric fact: A polar plot of $g(\theta, \phi)$ is generally a prolate ellipsoid of revolution with vertical axis and of eccentricity k_∞/α. When there is no absorption in the medium, it is easy to deduce then that $k_\infty = 0$ and that characteristic diffuse light is represented by a sphere. On the other hand, if there is little scattering as compared to absorption, the figure assumes a narrow pencil-like shape. In the limit of no scattering, k_∞ approaches α, and the figure degenerates into a vertical line.

The structure of the expression in (18) is related to the limiting form of a simple model for the radiance distribution discussed elsewhere (Preisendorfer, 1958c) and to a formula derived by Poole (1945). We conclude with the observation that (19) predicts a different limiting ratio of horizontal to upward radiance than that derived by Whitney (1941a) under the same circumstances (*i.e.*, isotropic scattering). Instead of the ratio 2:1, as suggested by Whitney, the present formula yields

$$\frac{g(\pi/2, \phi)}{g(0, \phi)} = 1 + \left(\frac{k_\infty}{\alpha}\right) \leqslant 2. \tag{20}$$

In other words, the ratio in (20) is not a fixed magnitude but depends on the optical properties of the medium in the manner shown.

The distribution (19) can serve as a convenient standard reference distribution against which experimentally determined radiance distributions can be compared. The amount of departure of the experimental distributions from (19) would then serve as a measure of the anisotropy of scattering in the real medium.

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