A METHOD FOR DETERMINING MEAN LONGITUDINAL VELOCITIES IN A COASTAL PLAIN ESTUARY[1]

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ABSTRACT

It is shown that the longitudinal component of the mean velocity in a coastal plain estuary may be computed by indirect methods. To do this, use is made of the lateral and longitudinal components of the equation of motion, the tidal velocity amplitudes, and a deduced relationship between the vertical and lateral eddy stresses. The method is evaluated for a station in the James River estuary. The resultant computed velocities agree quantitatively with the corresponding observed velocities.

Consider a coastal plain estuary in which there exists a mean circulation pattern similar to that shown schematically in Fig. 1. Here the two-dimensional flow pattern consists of a seaward directed flow in the upper portion of the water column and a landward directed flow in the lower portion. This estuarine system has been studied in detail by Pritchard (1952, 1954, 1956), who showed that:

a) the salt balance is maintained primarily by a longitudinal advective salt flux and a vertical nonadvective salt flux,
b) the mean field acceleration terms are small compared with other terms in the equation of motion, and
c) the horizontal and lateral components of the mean equation of motion are given essentially by balances between the forces of pressure, eddy friction, and tidal acceleration; and pressure, eddy friction, and Coriolis, respectively. Thus,

$$U_0 \frac{\partial U_0}{\partial x_1} = - \left\langle \alpha \frac{\partial p}{\partial x_1} \right\rangle_0 + b_0 - \frac{\partial}{\partial x_2} \langle v_2' v_1' \rangle, \tag{1}$$

and

$$0 = - \left\langle \alpha \frac{\partial p}{\partial x_3} \right\rangle_0 + b_0' + f\bar{v}_1 - \frac{\partial}{\partial x_2} \langle v_2' v_3' \rangle. \tag{2}$$

A left-handed co-ordinate system is used in which the x_1, x_2, and x_3-axes are directed horizontally seaward, vertically downward, and

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laterally across the estuary, respectively. The symbols used in (1) and (2) have the following meanings:

$\langle\ \rangle$ designates a time mean taken over one or more tidal cycles. (A superscript bar has the same meaning.)

$-\left\langle \alpha \dfrac{\partial p}{\partial x_1} \right\rangle_0,\ -\left\langle \alpha \dfrac{\partial p}{\partial x_3} \right\rangle_0$ denote the longitudinal and lateral components, respectively, of the mean pressure force at any depth x_2 relative to the pressure force at the surface.

b_0, b_0' are the longitudinal and lateral components, respectively, of the mean pressure force at the surface.

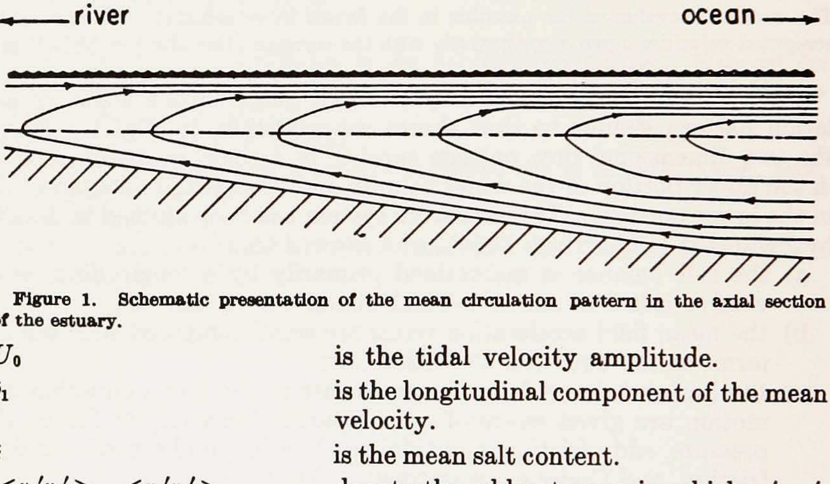


Figure 1. Schematic presentation of the mean circulation pattern in the axial section of the estuary.

U_0 is the tidal velocity amplitude.

$\bar{v}_1$ is the longitudinal component of the mean velocity.

$\bar{s}$ is the mean salt content.

$\langle v_2'v_1' \rangle,\ \langle v_2'v_3' \rangle$ denote the eddy stresses in which v_1', v_2', and v_3' are random deviations from the mean velocities $\bar{v}_1$, $\bar{v}_2$, and $\bar{v}_3$.

R is the volume rate of freshwater inflow from the river.

w, h are the width and depth, respectively, of the estuary.

It will be demonstrated through momentum equations (1) and (2) that the mean longitudinal velocity $\bar{v}_1$ as a function of depth can be determined if we know the mean distributions of temperature and salinity, the variation of the tidal velocity amplitude along the estuary, the relationship between the eddy stresses $\langle v_2'v_1' \rangle$ and $\langle v_2'v_3' \rangle$, and the magnitudes of these stresses near the bottom of the estuary.

Before proceeding with the formal determination of $\bar{v}_1$ let us consider the evaluation of (1) and (2) for the eddy stresses $<v_2'v_1'>$ and $<v_2'v_3'>$ as well as the evidence that suggests a relationship between these terms.

Since the pressure force components $-<\alpha \dfrac{\partial p}{\partial x_1}>_0$ and $-<\alpha \dfrac{\partial p}{\partial x_3}>_0$

can be determined as functions of depth from the distributions of temperature and salinity, it is possible to solve (1) and (2) for $<v_2'v_1'>$ and $<v_2'v_3'>$ providing two boundary values of the stress terms are known. It is assumed here that the wind stress on the surface will give one boundary value of the stress terms. The boundary condition employed for evaluation of the bottom stress stems from two sources: (1) from velocity measurements made by Lesser (1951) near the bottom at several open ocean locations which showed that in the lowest meter of water the velocity-depth variation was logarithmic and that the Prandtl-von Karman boundary layer theory was applicable; and (2) from our measurements in the James River estuary which showed that the region of rapid velocity decrease begins about one meter from the bottom, with the vertical gradient of the mean velocity going to zero at this depth. Pritchard (1955) evaluated eq. (1) for the stress term $<v_2'v_1'>$ and found it equal to zero at one meter from the bottom. In view of Lesser's and Pritchard's results we make the assumptions that the surface values of $<v_2'v_1'>$ and $<v_2'v_3'>$ are given by the appropriate component of the surface wind stress and that these terms pass through the value zero at a distance of one meter from the bottom.

Using these two boundary conditions, the eddy stresses $<v_2'v_1'>$ and $<v_2'v_3'>$ were obtained from (1) and (2) for a station located in the James River estuary. Mean values of temperature, salinity, and velocity observations taken during the period 18–23 June 1950 were employed in the evaluation. The computed values of $<v_2'v_1'>$ and $-<v_2'v_3'>$ are shown as functions of depth in Fig. 2. Also shown is $<v_2'v_1'>$ multiplied by a constant (0.56). The similarity between the latter curve and that for $-<v_2'v_3'>$ suggests the relationship

$$<v_2'v_3'> = \eta <v_2'v_1'>, \tag{3}$$

where η is a constant.

For natural turbulent systems there exists little independent observational support of the hypothesis expressed by eq. (3). However, Fleagle and Badgley (1952) have measured the stress terms $<v_2'v_1'>$ and $<v_2'v_3'>$ in the atmosphere at an elevation of two meters. Their results yield a value of η equal to -0.45. As will be shown subse-

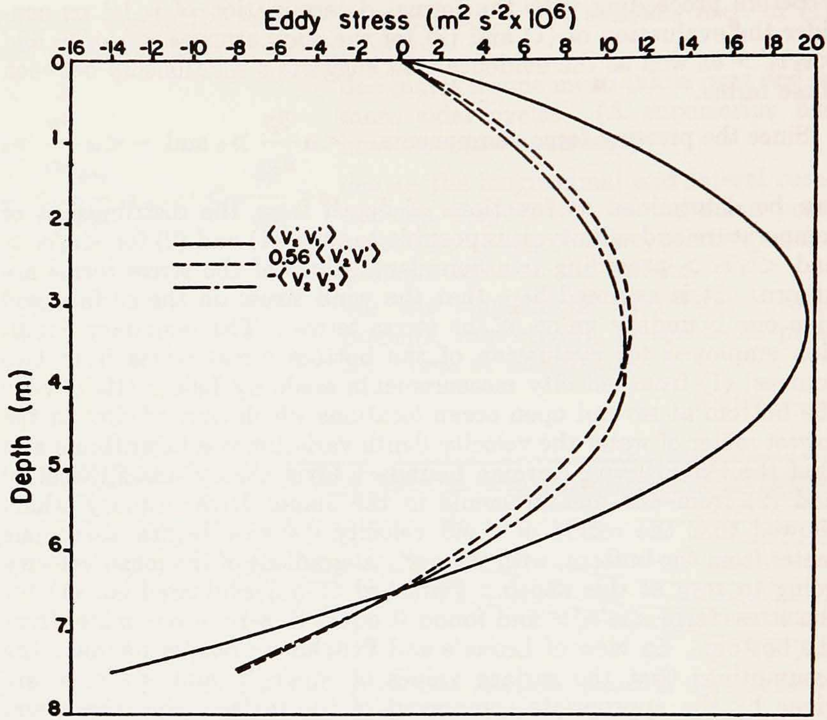


Figure 2. Computed values of $<v_2'v_1'>$, $0.56<v_2'v_1'>$ and $-<v_2'v_3'>$ as functions of depth.

quently, this value of η compares favorably with that calculated from the James River data.

Determination of $\bar{v}_1$ as a Function of Depth. A step-wise procedure by means of which the mean longitudinal velocity $\bar{v}_1$ can be determined will now be presented.

Substitution of eq. (3) into eq. (2) gives

$$0 = - < \alpha \frac{\partial p}{\partial x_3} >_0 + b_0' + f\bar{v}_1 - \eta \frac{\partial}{\partial x_2} < v_2'v_1' > . \tag{4}$$

Applying the equation of continuity in the form

$$\int_0^h w\bar{v}_1 dx_2 = R \tag{5}$$

to eq. (4) yields

(6) $$0 = -\int_0^h w < \alpha \frac{\partial p}{\partial x_3} >_0 dx_2 + b_0' \int_0^h w dx_2 + f \cdot R - \eta \int_0^h w \frac{\partial}{\partial x_2} < v_2' v_1' > dx_2 \, .$$

Similarly, applying the concept of salt conservation in the form

(7) $$\int_0^h w \bar{v}_1 \bar{s} dx_2 = 0$$

to eq. (4) yields

(8) $$0 = -\int_0^h w \bar{s} < \alpha \frac{\partial p}{\partial x_3} >_0 dx_2 + b_0' \int_0^h w \bar{s} dx_2 - \eta \int_0^h w \bar{s} \frac{\partial}{\partial x_2} < v_2' v_1' > dx_2 \, .$$

Equations (1), (4), (6), and (8) are utilized as follows:

a) $< \alpha \frac{\partial p}{\partial x_1} >_0$ and $< \alpha \frac{\partial p}{\partial x_3} >_0$ are evaluated from the observed temperature and salinity fields.

b) $U_0 \frac{\partial U_0}{\partial x_1}$ can be evaluated from the Coast and Geodetic Survey tidal current tables.

c) Using the given boundary values of $< v_2' v_1' >$, eq. (1) can be expressed as

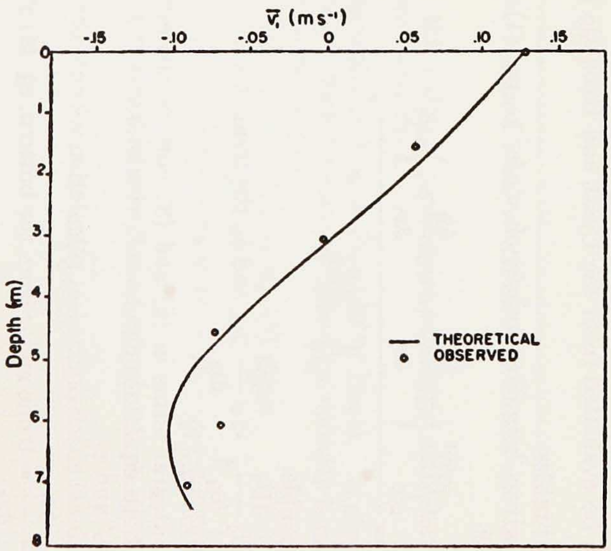
(9) $$b_0 = \frac{\int_0^{h-1} \left\{ U_0 \frac{\partial U_0}{\partial x_1} + < \alpha \frac{\partial p}{\partial x_1} >_0 \right\} dx_2}{\int_0^{h-1} w dx_2} ,$$

hence b_0 is determinable.

d) Knowing $U_0 \frac{\partial U_0}{\partial x_1}$, $< \alpha \frac{\partial p}{\partial x_1} >_0$, and b_0, the term $< v_2' v_1' >$ can be obtained from eq. (1).

e) All of the integral terms in (6) and (8) now can be evaluated and hence both equations can be solved simultaneously for the constants b_0' and η.

f) As an additional check on the last calculation we note that, since $< v_2' v_1' > = \frac{\partial \bar{v}_1}{\partial x_2} = 0$ one meter from the bottom, eq. (4) yields



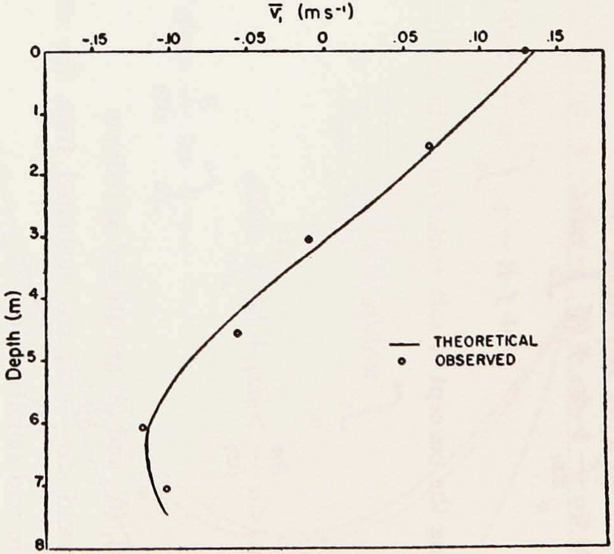


Figure 3a (left). Theoretical and observed values of the mean longitudinal velocity as a function of depth for the period 18–23 June 1950 at Station J-17.

Figure 3b (right). Theoretical and observed values of the mean longitudinal velocity as a function of depth for the period 26 June–7 July 1950 at Station J-17.

$$(10) \qquad \eta = \frac{\left[\dfrac{\partial}{\partial x_2} < \alpha \dfrac{\partial p}{\partial x_3} >_0\right]_{h-1}}{\left[\dfrac{\partial^2}{\partial x_2^2} <v_2'v_1'>\right]_{h-1}}.$$

g) Eq. (4) now can be solved for $\bar{v}_1$, the required quantity.

Serial observations of temperature and salinity obtained in the James River estuary for periods 18–23 June 1950 and 26 June–7 July 1950 were used in solving for $\bar{v}_1$ in the described manner. Figs. 3a and 3b show the calculated velocities as a function of depth for both periods. Also shown are the corresponding velocities observed at six depths. It is seen that agreement between the theoretical and observed velocities is quite good. Computed values of η and b_0' for two periods at Station J-17 in the James River estuary are as follows:

Period	η (*eqs. 6, 8*)	η (*eq. 10*)	b_0' $m \cdot s^{-2} \times 10^6$
18–23 June 1950	−0.43	−0.32	−16.54
26 June–7 July 1950	−0.38	−0.33	−17.32

The values of η were obtained from the simultaneous solutions of (6) and (8) and were determined also by (10). It is interesting to note that the average value of η obtained from the simultaneous solution of (6) and (8) compares favorably with the value −0.45 obtained from the data of Fleagle and Badgley.

An example of the procedure followed in computing $\bar{v}_1$ from eq. (4), along with the pertinent data, is given in Tables I–IV. Thus:

Table I evolves the lateral component of the mean pressure force, $<\alpha \dfrac{\partial p}{\partial x_3}>_0$;

Table II the gradient of the vertical eddy stress, $\dfrac{\partial}{\partial x_2} <v_2'v_1'>$;

Table III the parameters η and b_0'; and

Table IV shows finally the evaluation of eq. (4) for the mean longitudinal velocity $\bar{v}_1$ (Fig. 3a).

TABLE I. EVALUATION OF $-<\alpha \frac{\partial p}{\partial x_2}>_0$ FROM THE MASS FIELD FOR THE PERIOD 18–23 JUNE 1950 AT STATION J–17 IN THE JAMES RIVER ESTUARY.

	Station J-17 A			Station J-17 B			Station J-17	
Depth m	$\bar{S}$ $gm^{-1} \times 10^3$	T °C	$(\Delta D)_0$ m^2s^{-2}	$\bar{S}$ $gm^{-1} \times 10^3$	T °C	$(\Delta D)_0$ m^2s^{-2}	$\frac{\partial}{\partial x_2}(\Delta D)_0$ $ms^{-2} \times 10^5$	$-<\alpha \frac{\partial p}{\partial x_2}>_0$ $ms^{-2} \times 10^6$
0.0			0.00000			0.00000	0.00	
	10.58	24.7		11.41	24.7			1.14
0.5			0.01112			0.01082	2.29	
	10.66	24.6		11.53	24.6			3.47
1.0			0.02220			0.02159	4.66	
	10.68	24.5		11.60	24.5			5.99
1.5			0.03326			0.03230	7.33	
	10.68	24.4		11.64	24.4			8.66
2.0			0.04431			0.04300	10.00	
	10.70	24.3		11.69	24.3			11.37
2.5			0.05534			0.05367	12.75	
	10.83	24.2		11.85	24.2			14.20
3.0			0.06631			0.06426	15.65	
	11.52	24.1		12.54	24.1			17.10
3.5			0.07701			0.07458	18.55	
	11.86	24.0		12.85	24.0			19.96
4.0			0.08757			0.08477	21.37	
	12.98	23.9		13.92	23.9			22.71
4.5			0.09770			0.09455	24.05	
	13.26	23.8		14.10	23.8			25.22
5.0			0.10771			0.10425	26.41	
	13.45	23.8		14.15	23.8			27.36
5.5			0.11765			0.11394	28.32	
	13.63	23.7		14.16	23.7			29.04
6.0			0.12751			0.12361	29.77	
	13.84	23.7		14.24	23.7			30.26
6.5			0.13728			0.13325	30.76	
	14.23	23.6		14.30	23.6			30.84
7.0			0.14691			0.14286	30.92	
	14.57	23.6		14.57	23.6			30.92
7.5			0.15641			0.15236	30.92	

1. When looking seaward, Stations J–17A and J–17B are laterally to the right and left, respectively, of Station J–17. The distance separating J–17A and J–17B is 1.31×10^3 m.

2. $[(\Delta D)_0]_{x_2} = \int_0^{z_2} \delta dp$, where $\delta = f(s_1 r_1 x_2)$.

3. $\left(\frac{\alpha}{10} \frac{\partial p}{\partial x_2}\right) = \frac{\partial}{\partial x_2} (\Delta D)$.

TABLE II. EVALUATION OF $\frac{\partial}{\partial x_2} <v_2'v_1'>$ AS A FUNCTION OF DEPTH FOR THE PERIOD 18–23 JUNE 1950 AT STATION J–17 IN THE JAMES RIVER ESTUARY

Depth m	$<\alpha \frac{\partial p}{\partial x_1}>_0$ $ms^{-2} \times 10^6$	$<\alpha \frac{\partial p}{\partial x_1}>_0 + U_0 \frac{\partial U_0}{\partial x_1}$ $ms^{-2} \times 10^6$	$-<\alpha \frac{\partial p}{\partial x_2}>$ $ms^2 \times 10^6$	$\frac{\partial}{\partial x^2} <v_2'v_1'>$ $ms^{-2} \times 10^6$	$<v_2'v_1'>$ $m^2s^{-2} \times 10^6$
0					0.00
	0.89	3.86	14.17	11.20	
.5					5.60
	2.75	5.72	12.31	9.34	
1.0					10.27
	4.63	7.60	10.43	7.46	
1.5					14.00
	6.50	9.47	8.56	5.59	
2.0					16.80
	8.36	11.33	6.70	3.73	
2.5					18.66
	10.22	13.19	4.84	1.87	
3.0					19.60
	12.09	15.06	2.97	0.00	
3.5					19.60
	13.96	16.93	1.10	−1.87	
4.0					18.66
	15.83	18.80	−0.77	−3.74	
4.5					16.79
	17.68	20.65	−2.62	−5.59	
5.0					14.00
	19.57	22.54	−4.51	−7.48	
5.5					10.26
	21.41	24.38	−6.36	−9.32	
6.0					5.60
	23.28	26.25	−8.22	−11.19	
6.5					0.00
	25.11	28.08	−10.05	−13.02	
7.0					−6.51
	26.96	29.93	−11.90	−14.87	
7.5					−13.95

1. $U_0 \frac{\partial U_0}{\partial x_1} = 2.97 \times 10^{-6}\, ms^{-2}$.

2. $b_0 = \frac{\sum_{0}^{h=1} \left[<\alpha \frac{\partial p}{\partial x_1}>_0 + U_0 \frac{\partial U_0}{\partial x_1} \right] \alpha x_2}{(h-1)} = \frac{195.78 \times 10^{-6}}{13} = 15.06 \times 10^{-6}\, ms^{-2}$.

3. Assume $<v_2'v_1'>_0 = <v_2'v_1'>_{h-1} = 0$.

TABLE III. EVALUATION OF EQUATIONS (6) AND (8) FOR b' AND η FOR THE PERIOD 18–23 JUNE 1950 AT STATION J–17 IN THE JAMES RIVER ESTUARY

Depth m	$\bar{s}$ $gm^{-3} \times 10^3$	w $m^3 \times 10^3$	$w\bar{s}$ $gm^{-1} \times 10^6$	$\frac{\partial}{\partial x_3} <v_3'v_1'>$ $ms^{-2} \times 10^6$	$-<\alpha \frac{\partial p}{\partial x_3}>_0$ $ms^{-2} \times 10^6$	$-w<\alpha \frac{\partial p}{\partial x_3}>_0$ $m^2s^{-2} \times 10^3$	$w\bar{s}<\alpha \frac{\partial p}{\partial x_3}>_0$ $gm^{-1}s^{-2} \times 10^{-2}$	$-w \frac{\partial}{\partial x_3} <v_3'v_1'>$ $m^3s^{-2} \times 10^3$	$-w\bar{s} \frac{\partial}{\partial x_3} <v_3'v_1'>$ $gm^{-1}s^{-2} \times 10^{-2}$
0									
	10.93	3.03	33.12	11.20	1.14	3.45	0.38	−33.94	−3.71
.5									
	11.03	3.03	33.42	9.34	3.47	10.51	1.16	−28.30	−3.12
1.0									
	11.07	3.03	33.54	7.46	5.99	18.15	2.01	−22.60	−2.50
1.5									
	11.10	3.03	33.63	5.59	8.66	26.24	2.91	−16.94	−1.88
2.0									
	11.13	3.01	33.50	3.73	11.37	34.22	3.81	−11.23	−1.25
2.5									
	11.27	2.48	27.95	1.87	14.20	35.22	3.97	−4.64	−0.52
3.0									
	11.96	2.24	26.79	0.00	17.10	38.30	4.58	0.00	0.00
3.5									
	12.28	2.10	25.79	−1.87	19.96	41.92	5.15	3.93	0.48
4.0									
	13.36	1.91	25.92	−3.74	22.71	43.38	5.80	7.14	0.95
4.5									
	13.60	1.71	23.26	−5.59	25.22	43.13	5.87	9.56	1.30
5.0									
	13.72	1.59	21.81	−7.48	27.36	43.50	5.97	11.89	1.63
5.5									
	13.83	1.56	21.57	−9.32	29.04	45.30	6.26	14.54	2.01
6.0									
	14.02	1.49	20.98	−11.19	30.26	45.09	6.32	16.67	2.34
6.5									
	14.28	1.39	19.85	−13.02	30.84	42.87	6.12	18.10	2.58
7.0									
	14.57	1.28	17.92	−14.87	30.92	38.03	5.54	18.29	2.66
7.5									

Calculations for b_0' *and* η:

From Equations (6) and (8):

$$0 = 254.65 \times 10^{-3} + 16.42 \times 10^3 b_0' + 10.98 \times 10^{-3} + 8.77 \times 10^{-3},$$

$$0 = 32.93 \times 10^2 + 199.28 \times 10^6 b_0' - 0.49 \times 10^2,$$

$$\therefore b_0' = -16.54 \times 10^{-6} ms^{-2}; \ \eta = -0.43.$$

TABLE IV. EVALUATION OF EQ. (4) FOR THE MEAN LONGITUDINAL VELOCITY $\bar{v}_1$ FOR THE PERIOD 18–23 JUNE 1950 AT STATION J–17 IN THE JAMES RIVER ESTUARY.

Depth	$<\alpha \frac{\partial p}{\partial x_1}>$	$\eta \frac{\partial}{\partial x_3} <v_3'v_1'>$	$f_1 \bar{v}_1$	$\bar{v}_1$	$\bar{v}_1$ *observed*
m	$ms^{-2} \times 10^6$	$ms^{-2} \times 10^6$	$ms^{-2} \times 10^6$	ms^{-1}	ms^{-1}
0.0					
	15.40	−4.82	10.58	0.121	0.116
0.5					
	13.07	>4.02	9.05	0.103	0.092
1.0					
	10.55	−3.21	7.34	0.084	0.071
1.5					
	7.88	−2.40	5.48	0.062	0.054
2.0					
	5.17	−1.60	3.57	0.041	0.035
2.5					
	2.34	−0.80	1.54	0.018	0.012
3.0					
	−0.56	0.00	−0.56	−0.006	−0.014
3.5					
	−3.42	0.82	−2.62	−0.030	−0.038
4.0					
	−6.17	1.60	−4.57	−0.052	−0.056
4.5					
	−8.68	2.41	−6.27	−0.071	−0.065
5.0					
	−10.82	3.22	−7.60	−0.087	−0.070
5.5					
	−12.50	4.02	−8.48	−0.097	−0.074
6.0					
	−13.72	4.82	−8.90	−0.101	−0.076
6.5					
	−14.30	5.60	−8.70	−0.099	−0.076
7.0					
	−14.38	6.39	−7.99	−0.091	−0.068
7.5					

1. $b_0' = -16.54 \times 10^{-6}\ ms^{-2}$.
2. $\eta = -0.43$.

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