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A THEORY OF TIDAL MIXING IN A "VERTICALLY HOMOGENEOUS" ESTUARY[1]

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Abstract. An equation is derived for the distribution of a solute in an estuary where river flow and tides are predominant factors and where the dynamics and mixing can be described by a simple physical model. This model is subsequently generalized in various ways and it is found that the basic form of the diffusion equation is unchanged. An equation is derived which allows for time dependence of the various quantities involved and for introduction of a solute into the estuary by an external agent. Some solutions of this equation are studied.

Introduction. In this study we have investigated the distribution of salt or any other solute in estuaries as influenced by tides and river flow to the exclusion of all other influencing factors. Further, we have confined our attention to estuaries which are "essentially homogeneous vertically." Thus, if measurements of the concentration of a solute are made by sampling an entire transverse cross-section of an estuary (i.e., a vertical section perpendicular to the direction of river flow) at any time during one or more tidal cycles (during which time the conditions of river flow and tidal amplitude are assumed to be unchanged), then the concentration values so obtained will differ from one another by a smaller order of magnitude than will values obtained

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by taking measurements at positions separated by distances of the order of the tidal excursion along the length of the estuary.

A comprehensive theoretical solution of this problem should permit us to predict the distribution of any solute within an estuary from a knowledge of river flow and tides, possibly including the introduction or removal of any solute by means of an external source or sink, as when a pollutant is discharged by an industrial plant. A complete theory would require detailed study of velocity distribution and mixing processes. However, at our present stage of development, such a program is not feasible, hence we must content ourselves with a more macroscopic approach, i.e., one which aims at predicting only some convenient average variation of solute concentration as a function of position along the estuary and possibly of time. Ketchum (1951) has used the concept of tidal prism in his study and Arons and Stommel (1951) have postulated a diffusion equation. In this investigation an attempt is made to derive rather than postulate diffusion equations by assuming that the mixing process can be described in terms of simple physical models.

Assume that a channel is divided into two regions; a lower one with cross-sectional area $b_1(x,t)$, salinity $s_1(x,t)$ (dimensions ML^{-3}) and velocity $u_1(x,t)$; an upper one with cross-sectional area, salinity and velocity $b_2(x,t)$, $s_2(x,t)$, $u_2(x,t)$ respectively; x is the distance co-ordinate along the length of the channel, and u_1, u_2 are the tidal velocities. Actually we need not envisage such a clear-cut division between the layers; they may be interspersed. We need to assume only that the channel contains fluids of two kinds, one with salinity s_1 moving with velocity u_1, the other with salinity s_2 moving with velocity u_2; finally we assume that the total cross-sectional area of the fluid of salinity s_1 is b_1 and that that of salinity s_2 is b_2. For convenience, however, we shall speak of the channel as if it contained two distinct layers.

The tides cause upstream transport of the solute in the upper layer and an increase in the concentration difference between the two layers as a result of their relative velocity. Vertical mixing then carries some of the solute into the lower layer. It is assumed that these processes go on simultaneously in a more realistic estuary, but in our first model we shall assume that vertical mixing takes place instantaneously at high and low tide only, and further, that the mixing at each of these times is complete, i.e., that the salinities of the upper and lower layers are equal following each mixing. For this simplified model the concept of "essentially homogeneous vertically" applies only for a short time after mixing. However, if mixing takes place continuously or at many stages (see later discussion), then the model is consistent at all times.

Kinetic Considerations. First we assume that the channel is of constant width. For the continuity of fluid in the lower layer we have

$$\frac{\partial b_1}{\partial t} + \frac{\partial (b_1 u_1)}{\partial x} = 0, \tag{1}$$

for that in the upper layer

$$\frac{\partial b_2}{\partial t} + \frac{\partial (b_2 u_2)}{\partial x} = 0. \tag{2}$$

Although we assume two distinct layers, each with its own velocity, an exchange of salt between the two layers is permissible; thus the equation for salt continuity for the total channel is:

$$\frac{\partial (b_1 s_1 + b_2 s_2)}{\partial t} + \frac{\partial (b_1 s_1 u_1 + b_2 s_2 u_2)}{\partial x} = 0. \tag{3}$$

Substituting (1) and (2) in (3) we have

$$b_1 \left[\frac{\partial s_1}{\partial t} + u_1 \frac{\partial s_1}{\partial x} \right] + b_2 \left[\frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x} \right] = 0. \tag{4}$$

Note that in (1), (2), (3) and (4) we have assumed that the fluid in each layer moves with uniform velocity over the entire depth of the layer. If we assume instead that it moves over partial depths b_1', b_2', then in (1), (2), (3) and (4), wherever b_1 and b_2 multiply u_1 or u_2, they must be replaced by b_1' and b_2' respectively. We will assume, however, that $b_1' = b_1$, $b_2' = b_2$.

For our first specific case,

$$u_1 = 0, \qquad b_1 = a, \qquad b_2 = b + \eta, \qquad \eta = \zeta \sin \omega t, \tag{5}$$

where a, b, ζ and ω are positive constants. The equation $b_2 = b + \eta$ assumes that the tidal wave-length is large compared with the length of the estuary so that the level of the estuary rises uniformly over its length. Using equation (2) and expressions (5), u_2 is determined:

$$\frac{\partial b_2}{\partial t} + \frac{\partial (b_2 u_2)}{\partial x} = \zeta\omega \cos \omega t + \frac{\partial [(b + \eta) u_2]}{\partial x} = 0.$$

Integrating,

$$u_2 = - \frac{\zeta x \omega \cos \omega t}{b + \zeta \sin \omega t}, \tag{6}$$

where we have set the arbitrary function of t equal to zero so that $u_2(0,t) = 0$ for all t. The region of validity of the expressions for η

and u_2 probably end for some positive value of x. Equation (6) gives the velocity $u_2(x,t)$ of a particle in terms of its position x and t. The position of the particle $l_2(t)$ as a function of time may be found by substituting l_2 for x and $l_2'(t)$ for u_2 in (6) and integrating with respect to t, giving

$$l_2(t) = \frac{l_2(t_0)(b + \zeta \sin \omega t_0)}{b + \zeta \sin \omega t}, \tag{7}$$

where $l_2(t_0)$ is the position of the particle at some fixed time t_0.

Our object is to calculate the flux of salt past a given cross-section during a tidal period due to the action of the tides. Since our ultimate goal is to derive a differential equation for salinity as a function of position along the estuary we shall want to describe the flux in terms of the salinity. Such an equation will express the fact that in an estuary where conditions do not change from one tidal cycle to the next the upstream flux of salt due to tides must be exactly balanced by the downstream flux due to the river.

In our first model, the salinity of the upper layer, as measured in a frame of reference moving with the velocity of the upper layer, is constant, except when $\omega t = \pi/2, 3\pi/2, \cdots$, hence

$$\frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x} = 0 \tag{8}$$

except when $\omega t = \pi/2, 3\pi/2, \cdots$.

From (8), in particular,

$$s_2\left(x, \frac{(4n+1)\pi}{2\omega} - 0\right) = s_2\left(x + \xi_2, \frac{(4n-1)\pi}{2\omega} + 0\right),$$

$$n = 0, \quad \pm 1, \quad \pm 2, \quad \cdots \tag{9}$$

where, from (7)

$$x + \xi_2 = \frac{x(b + \zeta)}{b - \zeta}. \tag{10}$$

That is, the salinity of the upper layer at x just before mixing at high tide is equal to the salinity of the upper layer at $x + \xi_2$ farther downstream, just after the previous mixing at low tide, ξ_2 being the distance traveled from low to high tide by that section of the upper layer which is at x at high tide. Similarly,

$$s_2\left(x, \frac{(4n+3)\pi}{2\omega} - 0\right) = s_2\left(x - \xi_1, \frac{(4n+1)\pi}{2\omega} + 0\right),$$

$$n = 0, \quad \pm 1, \quad \pm 2, \quad \cdots \tag{11}$$

where, from (7)

$$x - \xi_1 = \frac{x(b - \zeta)}{b + \zeta}, \tag{12}$$

ξ_1 being the distance traveled from high to low tide by that section of the upper layer which is at x at low tide.

Since the lower layer is assumed to be stationary, we have

$$s_1\left(x, \frac{(4n + 1)\pi}{2\omega} - 0\right) = s_1\left(x, \frac{(4n - 1)\pi}{2\omega} + 0\right), \tag{13}$$

$$s_1\left(x, \frac{(4n + 3)\pi}{2\omega} - 0\right) = s_1\left(x, \frac{(4n + 1)\pi}{2\omega} + 0\right), \tag{14}$$

$$n = 0, \quad \pm 1, \quad \pm 2,$$

Further, from the assumption of complete mixing after high and low tides, we have

$$s_1\left(x, \frac{(4n + 1)\pi}{2\omega} + 0\right) = s_2\left(x, \frac{(4n + 1)\pi}{2\omega} + 0\right) \tag{15}$$

$$s_1\left(x, \frac{(4n + 3)\pi}{2\omega} + 0\right) = s_2\left(x, \frac{(4n + 3)\pi}{2\omega} + 0\right) \tag{16}$$

$$n = 0, \quad \pm 1, \quad \pm 2,$$

At this point we leave for a moment our rather formalized derivation and derive quite simply the flux over a tidal period by properly neglecting terms of relative order ζ/b. We shall not attempt to justify the particular approximations that are made therein, since a more rigorous derivation will be given subsequently.

Derivation of Diffusion Equation from Elementary Physical Considerations. Let us begin with conditions just after high tide mixing and consider the transport of salt during a tidal cycle across the section at x. During ebb tide an upper layer of water of length ξ_1 (see Fig. 1) and of cross-sectional area $b = \zeta \sim b$ moves downstream past x. Since no mixing takes place, this volume carries all of its salt content past section x. At low tide it instantaneously mixes with the more saline water below and thus acquires salt. During flood tide that same volume of water (now of cross-sectional area $b - \zeta \sim b$ and of length ξ_2) moves upstream across section x, so that at high tide this same volume then occupies the same position as it did at the beginning of the cycle, but, having acquired additional salt at low tide, it now has a higher salinity than it had in the beginning. Conversely, the lower

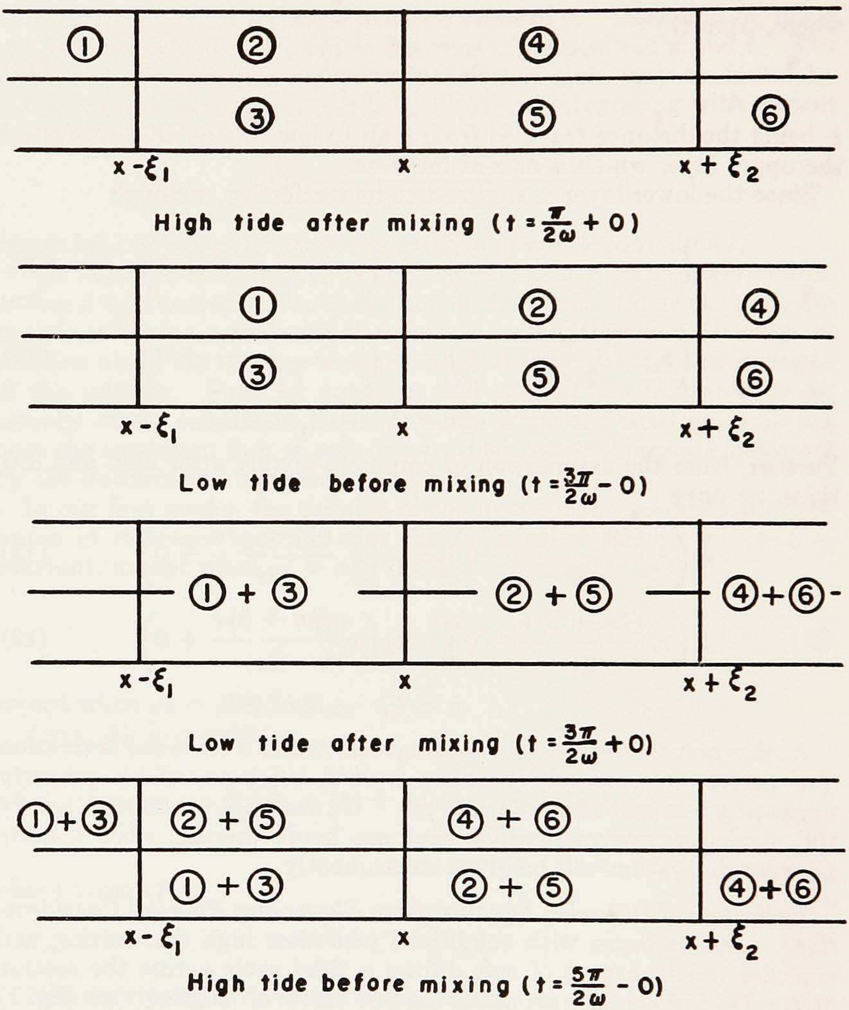

Figure 1. Schematic diagram showing motion of water due to tides and mixing.

layer upstream of x, having lost some salt to the upper layer during low tide mixing, is now less saline than it was at the beginning of the cycle. Hence, when mixing takes place at high tide, the upper layer yields salt to the lower one. With the cycle completed, the net result is an upstream transport or flux of a certain amount of salt past section x. We will now express this process in mathematical terms.

Let

$$s_2\left(x, \frac{(4n+1)\pi}{2\omega} + 0\right) = S_H(x), \tag{17}$$

$$s_2\left(x, \frac{(4n+3)\pi}{2\omega} + 0\right) = S_L(x). \tag{18}$$

For

$$\zeta \ll b, \quad \xi_1 \sim \xi_2 \sim \int_{\pi/2\omega}^{3\pi/2\omega} u_2 dt \sim \int_{\pi/2\omega}^{3\pi/2\omega} -\frac{\zeta x\omega}{b}\cos\omega t\, dt = \frac{2x\zeta}{b}$$

from (6). Thus we may call

$$\xi = \frac{2x\zeta}{b} \tag{19}$$

the approximate amplitude of tidal displacement. [Note that, if we neglect terms of higher order in ζ/b in (10) and (12), we have $\xi_1 \sim \xi_2 \sim \xi$.] Then, using (11), (14), (16), (17) and (18) we may write the equation for salt continuity at low tide as

$$aS_H(x) + bS_H(x-\xi) = (a+b)S_L(x) \tag{20}$$

and for that at high tide as

$$aS_L(x) + bS_L(x+\xi) = (a+b)S_H(x), \tag{21}$$

in which we have substituted ξ for both ξ_1 and ξ_2 and have neglected terms of relative order ζ/b [cf. (20) with (33)]. Now the salt transport across a section due to tides is approximately

$$b\int_0^{\xi} S_L(x+l)dl \sim b\xi S_L(x+\xi/2)$$

upstream, during flood tide, and

$$b\int_0^{\xi} S_H(x-l)dl \sim b\xi S_H(x-\xi/2)$$

downstream, during ebb tide, so that the net flux upstream during a tidal cycle across a section at x is

$$F = b\xi\left[S_L\left(x+\frac{\xi}{2}\right) - S_H\left(x-\frac{\xi}{2}\right)\right] \tag{22}$$

Then, for $\zeta/b \ll 1$, we may write (20) and (21) as

$$aS_H\left(x+\frac{\xi}{2}\right) + bS_H\left(x-\frac{\xi}{2}\right) = (a+b)S_L\left(x+\frac{\xi}{2}\right) \tag{20'}$$

$$aS_L\left(x - \frac{\xi}{2}\right) + bS_L\left(x + \frac{\xi}{2}\right) = (a + b)S_H\left(x - \frac{\xi}{2}\right). \quad (21')$$

Substituting (20′) in (22) to obtain an equation in S_H alone, we have

$$F = \frac{ab\xi}{a + b}\left[S_H\left(x + \frac{\xi}{2}\right) - S_H\left(x - \frac{\xi}{2}\right)\right] \quad (23)$$

$$= \frac{ab\xi^2}{a + b} S_H'(x)$$

plus terms of higher order in ξ and hence of higher order in ζ/b. This result is identical with that given later in (36), since from (19) $\xi = 2x\zeta/b$ and since the constant c in (36) is finally set equal to zero. The difference in sign occurs because F in (23) is the net flux upstream whereas F in (36) is the net flux downstream.

Similarly, substituting (21′) in (22) we obtain an equation comparable to (23) but with S_L replacing S_H. Thus if we call

$$\bar{s}(x) \equiv \frac{S_H(x) + S_L(x)}{2} \quad (24)$$

the average salinity during a tidal cycle, then

$$F = \frac{ab\xi^2}{a + b}\,\bar{s}'(x). \quad (25)$$

Since we assume that there is no net increase of salt over a tidal cycle, this net flux upstream must equal the net flux downstream due to river flow, so that

$$R_c\bar{s} = \frac{ab\xi^2}{a + b}\frac{d\bar{s}}{dx}, \quad (26)$$

where R_c is the river flow per tidal cycle. Equation (26) is the integrated form of the diffusion equation

$$R_c\frac{d\bar{s}}{dx} = \frac{d}{dx}\left(\frac{ab\xi^2}{a + b}\frac{d\bar{s}}{dx}\right) \quad (27)$$

that is obtained again in (39). By using (19), Eq. (26) may be integrated as in Eqs. (39) to (44).

Calculation of Salt Flux. Although this direct derivation of flux and salinity is the most satisfactory procedure to demonstrate the origin of the various terms in (26), a more formal derivation to exhibit the terms that are to be neglected seemed desirable when generalizing the various factors that influence the final salinity. Therefore we turn to the formal computation of the salt flux $F(x)$ at a fixed section x over a tidal period due to tides and small scale mixing alone:

$$F(x) = \int_{\pi/2\omega-0}^{5\pi/2\omega-0} b_2 u_2 s_2 \, dt \, . \tag{28}$$

Here F is the net flux downstream. In the integrand, x is held constant when integrating over t. Since we wish to speak of mixing or salt transfer between the two layers "at" high or low tide, i.e., when $t = (4n \pm 1)\pi/2\omega$ ($n = 0, \pm 1, \pm 2, \cdots$), it is convenient to have these points either definitely within or definitely outside of the range of integration, hence we have the "-0" in the limits of integration. For this same reason we may omit the intervals $\pi/2\omega - 0 < t < \pi/2\omega + 0$ and $3\pi/2\omega - 0 < t < 3\pi/2\omega + 0$ and write

$$F(x) = \int_{\pi/2\omega+0}^{3\pi/2\omega-0} f(x, t) \, dt + \int_{3\pi/2\omega+0}^{5\pi/2\omega-0} f(x, t) \, dt \, , \tag{28'}$$

where $f(x,t) = b_2 u_2 s_2$. We now differentiate (28′) with respect to x and note that, from Eq. (2),

$$\frac{\partial f}{\partial x} = b_2 \left(\frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x} \right) - \frac{\partial (b_2 s_2)}{\partial t} \, .$$

However, in either of the intervals over which we integrate in (28′), $\partial s_2/\partial t + u_2 \, \partial s_2/\partial x = 0$, so that, in view of (5),

$$\begin{aligned} \frac{dF}{dx} &= -\int_{\pi/2\omega+0}^{3\pi/2\omega-0} \frac{\partial (b_2 s_2)}{\partial t} \, dt - \int_{3\pi/2\omega+0}^{5\pi/2\omega-0} \frac{\partial (b_2 s_2)}{dt} \, dt \, , \\ &= (b + \zeta) \, s_2 \left(x, \frac{\pi}{2\omega} + 0 \right) - (b - \zeta) \, s_2 \left(x, \frac{3\pi}{2\omega} - 0 \right) , \qquad (29) \\ &\quad + (b - \zeta) \, s_2 \left(x, \frac{3\pi}{2\omega} + 0 \right) - (b + \zeta) \, s_2 \left(x, \frac{5\pi}{2\omega} - 0 \right) . \end{aligned}$$

We now express s_2 at $t = 3\pi/2\omega - 0$, $3\pi/2\omega + 0$, and $5\pi/2\omega - 0$ in terms of s_2 at $t = \pi/2\omega + 0$. From (11),

$$s_2 \left(x, \frac{3\pi}{2\omega} - 0 \right) = s_2 \left(x - \xi_1, \frac{\pi}{2\omega} + 0 \right) . \tag{30}$$

In order to express s_2 at $3\pi/2\omega + 0$ and $5\pi/2\omega - 0$ in terms of s_2 at

$\pi/2\omega + 0$, we need the equation of salt continuity at the time of mixing, which may be written as

$$as_1\left(x, \frac{(4n+1)\pi}{2\omega} - 0\right) + (b + \zeta)\, s_2\left(x, \frac{(4n+1)\pi}{2\omega} - 0\right)$$
$$= as_1\left(x, \frac{(4n+1)\pi}{2\omega} + 0\right) + (b + \zeta)\, s_2\left(x, \frac{(4n+1)\pi}{2\omega} + 0\right) \quad (31)$$

for high tide, and

$$as_1\left(x, \frac{(4n+3)\pi}{2\omega} - 0\right) + (b - \zeta)\, s_2\left(x, \frac{(4n+3)\pi}{2\omega} - 0\right)$$
$$= as_1\left(x, \frac{(4n+3)\pi}{2\omega} + 0\right) + (b - \zeta)\, s_2\left(x, \frac{(4n+3)\pi}{2\omega} + 0\right) \quad (32)$$

for low tide. Substituting (11), (14) and (16) in (32) we have

$$s_2\left(x, \frac{3\pi}{2\omega} + 0\right)$$
$$= \frac{1}{a + b + \zeta}\left[as_2\left(x, \frac{\pi}{2\omega} + 0\right) + (b - \zeta)\, s_2\left(x - \xi_1, \frac{\pi}{2\omega} + 0\right)\right] \quad (33)$$

and from (9) and (33) we have

$$s_2\left(x, \frac{5\pi}{2\omega} - 0\right)$$
$$= \frac{1}{a + b + \zeta}\left[as_2\left(x + \xi_2, \frac{\pi}{2\omega} + 0\right) + (b - \zeta)\, s_2\left(x, \frac{\pi}{2\omega} + 0\right)\right]. \quad (34)$$

Note that the argument of the term multiplying $(b - \zeta)$ in (34) is now the high tide position of that section of the upper layer which is found at $x + \xi_2$ at low tide, i.e., x, not $x + \xi_2 - \xi_1$. Substituting (10), (12), (30), (33) and (34) in (29), we expand the right-hand side of (29) in a power series in $\zeta/b \ll 1$ and keep only the terms of lowest order in ζ/b:

$$\frac{dF}{dx} = -\frac{4\zeta^2}{b^2}\frac{ab}{a + b}\frac{\partial}{\partial x}\left\{x^2 \frac{\partial s_2\left(x, \dfrac{\pi}{2\omega} + 0\right)}{\partial x}\right\}. \quad (35)$$

The flux of salt at a section at x over the tidal period $\pi/2\omega - 0 \le t \le 5\pi/2\omega - 0$ is therefore

$$F = -\frac{4\zeta^2}{b^2}\,\frac{ab}{a+b}\,x^2\,\frac{\partial s_2\left(x, \dfrac{\pi}{2\omega}+0\right)}{\partial x} + c\,. \tag{36}$$

We now compute the flux due to river flow (the third factor which determines the salt distribution) and add this to the flux produced by the tides given in (36) so that there is no net flux of salt over a tidal period. The flux due to the river is

$$G = \int_{\pi/2\omega-0}^{5\pi/2\omega-0} Rs^*(x, t)\,dt\,. \tag{37}$$

Here R, the river flow across any section, has the dimensions of L^3T^{-1} and s^* is related to s_1 and s_2 as follows:

If the river produced a flow in each of the layers that was proportional to the cross-section of the layer, then we might define $s^* = [as_1 + (b + \eta)s_2]/(a + b + \eta)$. If the river acted over a cross-section $A(x,t)$ of the lower layer and over a cross-section $B(x,t)$ of the upper layer, then we would have $s^* = (As_1 + Bs_2)/(A + B)$. However, over a tidal cycle, the net flux produced by tides and small scale mixing and by river flow is zero, so that the salinity of each of the layers is periodic. Further, all variations of salinity in either layer during a period are $0(\zeta/b)$. Thus, (37) becomes

$$G = \frac{2\pi}{\omega} R\left[s_2\left(x, \frac{\pi}{2\omega}+0\right) + 0\left(\frac{\zeta}{b}\right)\right]. \tag{38}$$

Derivation of Salinity. In adding the flux due to tides and small scale mixing to that due to river flow we neglect the terms of $0(\zeta/b)$ in (38). Noting that the time variation of salinity in either layer is $0(\zeta/b)$, we may again neglect terms of $0(\zeta/b)$ and replace $s_2(x, \pi/2\omega + 0)$ in (36) and (38) by $s(x)$, which may be interpreted as the salinity at x to within terms of relative order ζ/b at any time. Adding F and G in (36) and (38) and setting their sum equal to zero we then have

$$Dx^2\frac{ds}{dx} - c = Rs\,, \tag{39}$$

where

$$D = \frac{4\zeta^2}{b^2}\left(\frac{ab}{a+b}\right)\frac{\omega}{2\pi}, \tag{40}$$

the solution of which is

$$s(x) = -\frac{c}{R} + \frac{\omega}{2\pi R}\, e^{\frac{2\pi R}{\omega}\, c' - \frac{k}{x}}, \tag{41}$$

where

$$k = R/D. \tag{42}$$

We now set $c = 0$ so that $s(0) = 0$, which is consistent with the fact that $u_2(0,t) = 0$ for all t so that no salt can get upstream of the section at $x = 0$.

The constant c' is determined by stipulating the salinity at some particular point, $x = L$, giving

$$s(x) = s(L)e^{k\left(\frac{1}{L}-\frac{1}{x}\right)}. \tag{43}$$

In particular, if we denote the ocean salinity by σ and define the length of the estuary L as the distance between the point at which $s(x) = 0$ and $s(x) = \sigma$, then

$$\frac{s(x)}{\sigma} = e^{\frac{k}{L}\left(1-\frac{L}{x}\right)}. \tag{44}$$

Note that Arons and Stommel (1951), who arrive at this same result by using dimensional analysis, call k/L the flushing number. In terms of the notation used here, their flushing number equals $R(a + b)/2B\zeta^2\omega L$, where B is a constant which remains undetermined in their paper. It arises from their assumption that the eddy diffusivity A of the salt transfer equation $\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(A\frac{\partial s}{\partial x}\right)$, which they postulate as a basis for their considerations, is given by $A = 2B\xi_0 U_0$, $2\xi_0$ being the total excursion, over a tidal period, of a particle due to tides, U_0 the amplitude of the tidal velocity, and u the river velocity. Equating their flushing number to k/L [see Eqs. (40), (42)], we have $B = a/\pi b$, which relates B to the parameters of this paper.

Calculation of Flux for Generalized Mixing Process. We shall now consider the various ways in which the described model may be modified so that it may correspond more closely with natural conditions, and we will also investigate the effect of such modifications on salinity distribution.

First, consider a model in which the dynamics are those previously described, but instead of assuming that mixing takes place at high and low tide only ($t = \pi/2\omega$ and $t = 3\pi/2\omega$) assume that there is mixing at $t = \pi/2\omega$, π/ω, $3\pi/2\omega$, $2\pi/\omega$. Moreover, instead of requiring complete mixing each time we shall require only that there be an interchange of an identical quantity of fluid between top and bottom layers at each mixing. Making appropriate modifications for these new requirements but following essentially the procedure given for the simplified model (for further details, see Maximon and Morgan, 1953: 23–33), we obtain an equation for dF/dx which is identical with (35) except that the right-hand side is multiplied by a constant which depends on the degree of mixing at each of the mixing times and which is always between zero and one, the value one being achieved only when mixing is complete and occurs at high and low tide exclusively. If a and b can be obtained from observations, and if the observed salinity be compared with the salinity predicted by this theoretical model, then it should be possible to obtain not only an estimate of this constant but an idea of where during the tidal cycle most of the mixing takes place, large values of the constant corresponding to a state in which most of the mixing takes place at high and low tide.

Extension of Analysis to Channel of Varying Cross-Section. Next, consider an extension of the original model in which we again assume that complete mixing occurs only at high and low tides. However, the cross-sectional area of the stationary layer is now $a(x)$ instead of constant a, the average area of the moving layer is $b(x)$, and the time-varying part of the moving layer is $\zeta(x) \sin \omega t$. Following essentially the calculations used in the first model (see again Maximon and Morgan, 1953: 33–38), the salinity, formerly given by (43), is now expressed by the following equation:

$$s(x) = s(L)e^{\int_L^x k(x')dx'} \tag{45}$$

where

$$k(x) = \frac{\pi R}{2\omega} \frac{a(x) + b(x)}{a(x) \cdot b(x)} \left[\frac{b(x)}{\int_0^x \zeta(x')dx'} \right]^2 . \tag{46}$$

Thus equation (45) is identical to (43) if $a(x)$, $b(x)$ and $\zeta(x)$ are constant.

Extension of Analysis to More General Kinetic Conditions. Finally, consider a model which is identical to the first (a,b,ζ constant, complete mixing at high and low tide and no mixing at other times) except for the assumption that the bottom layer has a periodic velocity

$$u_1(t) = - U \cos (\omega t + \varphi). \tag{47}$$

The appropriate calculations following the original derivation give (see Maximon and Morgan, 1953: 38–42)

$$s(x) = s(L)e^{k \left(\frac{1}{L - x_0} - \frac{1}{x - x_0} \right)} , \tag{48}$$

where

$$x_0 = \frac{bU}{\zeta\omega} \cos \varphi . \tag{49}$$

Diffusion Equation in the Non-Steady Case and in the Presence of a Solute Source Distribution. Equation (39) expresses a balance between upstream and downstream flux in an estuary where conditions do not change from cycle to cycle and where there are no solute sources. If we allow both time variation and external sources, and if we assume that the mixing model is essentially unchanged, then the following equation for the conservation of solute may be derived in place of (39):

$$(a + b) \frac{\partial s}{\partial t} + R(t) \frac{\partial s}{\partial x} = D \frac{\partial}{\partial x} \left(x^2 \frac{\partial s}{\partial x} \right) + q(x,t) . \tag{50}$$

Here R now depends on time and $q(x,t)$ denotes the mass of solute introduced into the estuary per unit length of estuary and per unit time (see Maximon and Morgan, 1953: 42–45).

Some Solutions of the Diffusion Equation. In this section we consider several solutions of (50). In the first solution, s is the salinity concentration in the estuary, R is a function of time, and $q(x,t) = 0$; in succeeding solutions s is the concentration of an externally introduced solute, R is constant, and q is assumed to be dependent on x only.

(a) We assume that the time variation of R is sufficiently small so that $(a + b)\, \partial s/\partial t$ is small relative to the other terms in (50); hence we use successive approximations in the solution of (50). Boundary conditions for each of the successive approximations of s are that the salinity at the mouth of the estuary be σ, a constant, and that the salinity at some point upstream be zero. Since river discharge is a function of time, we permit the distance L from this point to the mouth (the effective length of the estuary) to vary with time, in which case it is convenient to choose our co-ordinate system with origin at the mouth of the estuary. However, with this shift in the origin, the term x^2 in (50), which is proportional to the square of the distance moved by a particle due to tides, must be replaced by $(L + x)^2$, so that we now have

$$c \frac{\partial s}{\partial t} + R \frac{\partial s}{\partial x} = D \frac{\partial}{\partial x} \left[(L + x)^2 \frac{\partial s}{\partial x} \right], \qquad c = a + b, \tag{51}$$

in which R and L are assumed to be known functions of time. Note that, just as for constant R, L is a constant length that must be assumed known in the analysis, so $L(t)$ must be known when R varies.

Also, it must be assumed that R and L do not vary significantly over a tidal period in order that the derivation which leads to (50) remain valid. Physically, this appears to be a reasonable assumption.

We now define successive approximations $s_0, s_1, \cdots$ to the solution of (51) by

$$D\frac{\partial}{\partial x}\left[(L+x)^2\frac{\partial s_0}{\partial x}\right] = R\frac{\partial s_0}{\partial x}, \tag{52}$$

$$D\frac{\partial}{\partial x}\left[(L+x)^2\frac{\partial s_n}{\partial x}\right] = R\frac{\partial s_n}{\partial x} + c\frac{\partial s_{n-1}}{\partial t}, \quad n = 1, 2, \cdots, \tag{52'}$$

where $s_n(x,t)$ $(n = 0, 1, 2, \cdots)$ are assumed to satisfy the boundary conditions

$$s_n(0,t) = \sigma \qquad \text{for all } t, \tag{53}$$

$$s_n(-L(t),t) = 0 \qquad \text{for all } t. \tag{54}$$

Integrating (52) subject to (53) and (54) we obtain

$$s_0(x,t) = \sigma e^{\frac{R}{D}\left(\frac{1}{L}-\frac{1}{L+x}\right)}. \tag{55}$$

Similarly, integrating (52′) with $n = 1$, subject to (53) and (54), and using (55), we obtain after some manipulation,

$$s_1(x,t) = s_0(x,t)\left\{1 + \frac{cR'}{DR}\log\left|\frac{L}{L+x}\right| + \frac{cL'}{D}\left(\frac{1}{L}-\frac{1}{L+x}\right) + \frac{cR'}{DR}\left(1+\frac{\alpha R}{D}\right)\left[M\left(\frac{R}{D(L+x)}\right) - M\left(\frac{R}{DL}\right)\right]\right\}, \tag{56}$$

where

$$M(x) \equiv e^x\int_x^\infty \frac{e^{-y}}{y}\,dy \qquad \text{and} \qquad \alpha = \frac{\left(\frac{R}{L}\right)'}{R'}. \tag{56'}$$

Here $s_1(x,t)$ represents the first approximation beyond the quasi-steady solution $s_0(x,t)$ given by (55). The solution is given here in the hope that it will be tested by other investigators who have available the data required for the solution of (56). Values of the function

$$-Ei(-x) \equiv \int_x^\infty \frac{e^{-y}}{y}\,dy$$

have already been tabulated by the Federal Works Agency Works Projects Administration (1940).

For the case in which the mean cross-sectional area c is a function of x, the formal procedure leading to s_0 and s_1 is identical with that just given, although the integrations are more cumbersome. The term $D \frac{\partial}{\partial x}\left(x^2 \frac{\partial s}{\partial x}\right)$ in (50) must be modified slightly and may be obtained from (46).

(b) Considering the case in which R is constant and in which $q(x,t) = q(x)$, so that $s(x,t) = s(x)$, (50) then becomes

$$Dx^2s'' + (2Dx - R)s' = - q(x), \tag{57}$$

where primes denote differentiation with respect to x. Using the method of variation of parameters, the general solution of (57) is

$$s(x) = \frac{1}{R}\int_0^x q(x)dx + \frac{1}{R} e^{\frac{-R}{Dx}} \int_x^L e^{\frac{R}{Dx}} q(x)dx + c_1 + c_2 e^{\frac{-R}{Dx}}, \tag{58}$$

where c_1 and c_2 are arbitrary constants.

Consider now two special cases, namely

(A) $q(x) = Q\delta(x - x_0)$, in which Q is a constant, $0 < x_0 < L$. This choice for $q(x)$ should apply to the introduction of some solute at one point ($x = x_0$) in the estuary. Q is then the rate of discharge of this solute into the estuary (dimensions MT^{-1}).

(B) $q(x) = q$, in which q is constant over the range $0 < x < L$. This choice for $q(x)$, which corresponds to the introduction of a solute uniformly over the entire effective length of the estuary, might be applicable in the investigation of ground seepage. The total rate of discharge of the solute into the estuary over the range $0 < x < L$ is then qL.

We impose the boundary conditions $s(0) = s(L) = 0$; $s(0)$ is set equal to zero, since there is no mechanism to carry the solute upstream of the point $x = 0$, at which point the motion in the estuary due to tides is zero; and $s(L)$ is set equal to zero because of the essentially infinite reservoir provided by the ocean at the mouth of the estuary. For case (A),

$$s(x) = \frac{Q}{R}\left[e^{\frac{R}{D}\left(\frac{1}{x_0} - \frac{1}{x}\right)} - e^{\frac{R}{D}\left(\frac{1}{L} - \frac{1}{x}\right)}\right], \qquad 0 \leq x \leq x_0$$

$$s(x) = \frac{Q}{R}\left[1 - e^{\frac{R}{D}\left(\frac{1}{L} - \frac{1}{x}\right)}\right], \qquad x_0 \leq x \leq L. \tag{59}$$

For case (B),

$$s(x) = \frac{qx}{R} - \frac{qL}{R} e^{\frac{R}{D}\left(\frac{1}{L} - \frac{1}{x}\right)} + \frac{q}{R} e^{\frac{-R}{Dx}} \int_x^L e^{\frac{R}{Dx}} dx \,. \tag{60}$$

If we let $y = R/Dx$ in the integral in (60) and integrate by parts, we obtain

$$s(x) = \frac{q}{D} e^{\frac{-R}{Dx}} \int_{R/DL}^{R/Dx} \frac{e^y}{y} dy \,. \tag{61}$$

We now compare the solute distributions that result for $q(x)$ as given in (A) and (B) respectively, assuming that the rates of discharge of solute throughout the entire effective length of the estuary are equal for the two cases, i.e., $qL = Q$. In Fig. 2, Rs/Q as a function of x/L is plotted for $R/DL = 0.8$, the value given for the Raritan River by Arons and Stommel (1951). Their constant F, the flushing number, is equal to R/DL in the notation used here. Values of the integral appearing in (61) may be found in Tables of Sine, Cosine and Exponential Integrals (1940).

Fig. 2 presents two curves for the case in which $q(x) = Q\delta(x - x_0)$;

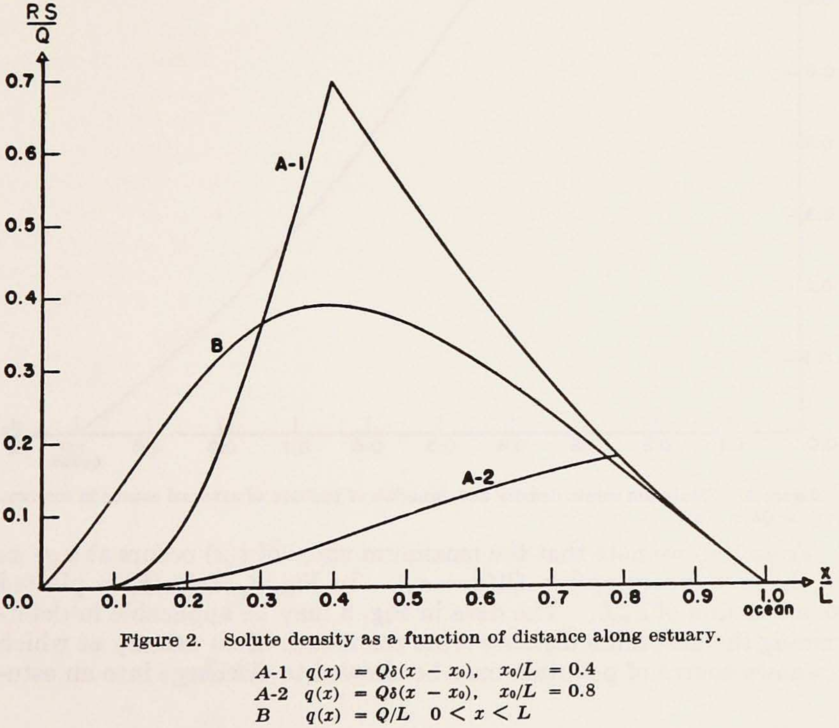

Figure 2. Solute density as a function of distance along estuary.

A-1 $q(x) = Q\delta(x - x_0)$, $x_0/L = 0.4$
A-2 $q(x) = Q\delta(x - x_0)$, $x_0/L = 0.8$
B $q(x) = Q/L \quad 0 < x < L$

in one $x_0/L = 0.4$ and in the other $x_0/L = 0.8$; note in particular that the solute density is appreciably greater over most of the estuary for the former. It is hoped that this sensitivity of the solute density upstream of the external source to the position of the source may be used to check experimentally the validity of the solute diffusion equation (50).

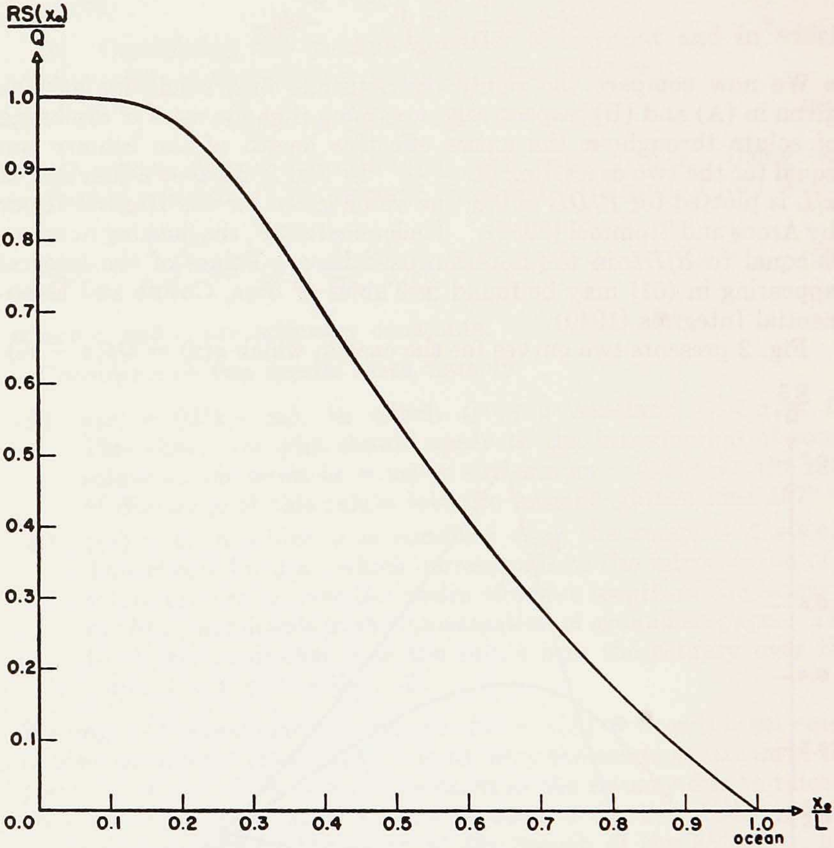

Figure 3. Maximum solute density as a function of position of external source in estuary. $q(x) = Q\delta(x - x_0)$

From (59) we note that the maximum value of $s(x)$ occurs at $x = x_0$ in the case where $q(x) = Q\delta(x - x_0)$. In Fig. 3, $Rs(x_0)/Q$ is plotted as a function of x_0/L. The data in Fig. 3 may be applicable in determining the maximum distance from the mouth of an estuary at which a known source of pollution may be allowed to discharge into an estuary if specified limits are placed on the permissible density of pollution in the estuary.

REFERENCES

B. H. Ketchum.

1951. The exchanges of fresh and salt waters in tidal estuaries. J. Mar. Res., *10* (1): 18–38.

A. B. Arons and Henry Stommel.

1951. A mixing length theory of tidal flushing. Trans. Amer. geophys. Un., *32* (3): 419–421.

1940. Tables of Sine, Cosine and Exponential Integrals, Vols. I and II. Prepared by the Federal Works Agency, Work Projects Admin. for the City of N. Y. as a report of Official Project No. 765-97-3-10.

L. C. Maximon and G. W. Morgan.

1953. A theory of tidal mixing in an essentially vertically homogeneous estuary. Tech. Rep. No. 2. Nonr-562(02), (NR-083-067).

L. C. Maximon.

1953. Some solutions of the solute diffusion equation for an essentially vertically homogeneous estuary. Tech. Rep. No. 3. Nonr-562(02), (NR-083-067).