ON CONVECTIVE INSTABILITY OF A ROTATING FLUID WITH A HORIZONTAL TEMPERATURE CONTRAST

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ABSTRACT

A simple expression of the criterion for the onset of symmetrical thermal convection of a rotating fluid subject to a horizontal temperature contrast is obtained. It is shown that both the rotation and a positive static stability act to inhibit convection. The extent of these inhibitions depends upon the nondimensional parameters $T = 4\Omega^2 d^4 \nu^{-2}$ and $R = gd^4\kappa^{-1}\nu^{-1}\alpha d\theta_0/dz$, where d denotes the depth of the layer, Ω the rotation rate, α the thermal expansion coefficient, κ the thermometric conductivity, ν the kinematic viscosity, and $d\theta_0/dz$ the vertical temperature gradient. For given values of T and R, the onset of convection requires the nondimensional parameter $Q = gd^4\kappa^{-1}\nu^{-1}\alpha d\theta_0/dr$ to be higher than a critical value Q_c, and furthermore there is a most favorable cell size l_c for which the required value of Q is lowest. Q_c generally increases with T and also increases with l for l larger than l_c. When T is very large, the parameter $K_c = 4dQ_c(lT)^{-1}$, which is the product of a Rossby number and the Prandtl number ν/κ, approaches a constant, suggesting that the onset of such convection in the atmosphere and in the ocean may be determined by this product. In order to examine the possible difference of conditions in systems such as the atmosphere and ocean on the one hand and that in model experiments on the other, two cases are discussed: (a) when the horizontal extent a of the fluid is infinitely large compared with its depth d, and (b) when $a = d$.

INTRODUCTION

The most important factor that keeps the atmosphere in motion is differential heating, particularly that in horizontal directions, since the vertical stratification of the atmosphere is normally stable. Another factor that controls the nature of large scale motions in both atmosphere and ocean is the earth's rotation. In this paper we shall discuss the effects of these two factors. However, because of the complex nature of the motions of these fluid systems, we shall limit ourselves here to a discussion of symmetric thermal convection and leave other types of motion for discussion at a later date.

Some problems concerning symmetric thermal convection in a thin layer of fluid have been discussed in another paper (Kuo, 1954),

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where the relevant equations have been solved in an exact manner. However, the frequency equation derived there is so complicated that the desired information can be obtained only through tedious computations. In this paper we shall discuss symmetric convection by expanding the solution in a double Fourier series with a view to finding a simple approximate expression of the stability criterion for the onset of such motion. Such an approximate method is also useful in dealing with the more complicated asymmetric motions.

Since this type of thermal convection is of particular interest to meteorologists, who commonly believe that the general circulation of the atmosphere is maintained by such a large thermal convection, many attempts have been made to derive such motion from the hydrodynamic equations [for example by Oberbeck (1888), Prandtl (1939), Arakawa (1940) and Davies (1953)]. In most of these efforts the authors assumed fundamentally that the temperature distribution in the fluid is not affected by motion and that this unchanging mean temperature distribution results in an unchanging buoyancy force which drives the symmetric convection continuously. Thus, the motion becomes a forced motion which must therefore be present at all times.

However, a closer scrutiny of these assumptions shows that neither of them are adequate when applied to a fluid system such as the atmosphere. First, in such a shallow layer of fluid with so large a horizontal extent, it is difficult to conceive that the motion of a particle is determined by the mean temperature at points very far away. It must depend primarily upon the temperature or density distribution in its immediate surroundings, i.e., upon the local temperature anomaly. Second, more heat is transported by the motion than by heat conduction, therefore the local temperature anomaly depends directly on the motion. Thus the motion must be treated as a natural motion, in which the velocity field and temperature field are mutually dependent and must be determined simultaneously.

Since the basic concept of the present treatment is quite different from that used by others, it seems worthwhile to dwell again on some of the physical considerations before attacking the problem mathematically. As noted before, we consider the rotation and the differential heating as the two most important factors that determine the motion, the main function of the latter being to produce and to maintain a temperature and thereby a density contrast. It is well known that both rotation and stable stratification tend to suppress the motion whereas the horizontal temperature contrast has the direct effect of increasing the motion by way of the buoyancy force it introduces. Therefore the occurrence of different types of motion under different conditions must have its explanation in the adjustment of the motion to two different effects, the inhibiting effect of rotation and stable stratification and the motion-producing effect of horizontal temperature contrast. Only when the temperature contrast is large and the rotation rate small can a symmetrical direct thermal convection take place. When the rotation rate is high and the temperature contrast relatively small, the fluid particles are not capable of following such a direct circulation; the motion must then be asymmetrical and more horizontal, hence it must appear as waves in the zonal direction, as shown by Fultz (1951) and Hide (1953).

The problem may also be elucidated by another consideration. Since heat is supplied by a heat source and is removed by a cold source, the motion may be considered a necessity in order to transport heat in addition to heat conduction. Since the total kinetic energy that can be developed in such a fluid system from an initial state of relative rest is determined roughly by the total available potential energy or by the temperature contrast, it may be considered as given by the heating. However, the efficiency of different types of motion in transporting heat is quite different, even when the amount of total kinetic energy is the same, because only the cross-isothermal velocity component can produce a net transfer of heat. Therefore the efficiency of the motion in transporting heat depends upon the nature of the motion which is much affected by rotation. If the rotation rate is not too small and if the motion appears in the form of symmetric convection, then the ratio between the zonal velocity and the meridional velocity is roughly of the order $T^{1/3}$ (Kuo, 1954) while the vertical velocity is comparable with the meridional velocity. Therefore most of the total kinetic energy is in the zonal kinetic energy. On the other hand, the meridional velocity becomes comparable with the zonal velocity in the asymmetric motions, particularly when the wave number is large. Thus, large symmetric convection is far less efficient in transporting heat than the asymmetric motions with larger wave numbers, when the rotation rate is high. Since the net amount of heat transported is proportional to the temperature gradient, symmetric motion must occur together with a larger temperature gradient.

In this paper we shall discuss the stability criterion for the onset of symmetric convection in two different cases. In case (a) we consider the depth of fluid much smaller than the horizontal dimension so that the lateral boundary conditions can be replaced by a harmonic requirement. In case (b) we assume the depth of the layer equal to its horizontal extent. As in the other paper (Kuo, 1954), we consider that a temperature contrast in the radial direction is introduced and is later maintained to a degree within the fluid by heat conduction. At the initial moment we may assume that the fluid is in purely zonal motion which balances the radial pressure gradient due to the mean temperature contrast, or, more simply, we may assume that the fluid is initially at relative rest. The latter case can be realized by imagining that the finite horizontal temperature contrast is produced suddenly or that the fluid is in a highly viscous state before the initial moment. Our problem is to determine the minimum horizontal temperature contrasts that are required for the onset of symmetric motion at different rates of rotation. The criterion thus obtained may also be the one that marks the transition from the low to the high-rotation regime with the lowest wave number, if the state of solid rotation is to be disturbed at all.

EQUATIONS OF THE PROBLEM

At the initial moment the fluid is at relative rest or in a purely zonal motion with velocity v_0, and a variable mean temperature θ_0 has been produced and maintained in the fluid by external heating so that θ_0 satisfies the steady state heat conduction equation

$$\nabla^2\theta_0 = 0\,, \tag{1}$$

where ∇^2 stands for the standard Laplacian operator.

As is permissible in this problem, we allow for the variation of density due to thermal expansion only in so far as it modifies gravity and introduces a buoyancy force. Thus ρ, which occurs as a factor of g, is replaced by

$$\rho = \bar{\rho}\,(1 - \alpha\Delta\theta)\,, \tag{2}$$

where α denotes the coefficient of thermal expansion, $\bar{\rho}$ the density corresponding to a mean temperature $\bar{\theta}_0$, and $\Delta\theta$ the deviation of the local temperature from θ_0. Where ρ occurs elsewhere in the equations of motion we regard it as a constant equal to $\bar{\rho}$ and we consider the fluid as incompressible during the motion.

We denote the velocity components in the radial, zonal and vertical directions by u, v and w, the departures of local pressure and temperature from the initial mean values p_0 and θ_0 by p and θ, and we assume that all these quantities are independent of the longitude and are small; then the linearized equations of motion, continuity and heat transfer in cylindrical co-ordinates r, ϕ, and z are

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u - \frac{u}{r^2}\right) \tag{3}$$

$$\frac{\partial v}{\partial t} + 2\Omega u + w\frac{\partial v_0}{\partial z} = \nu\left(\nabla^2 v - \frac{v}{r^2}\right) \tag{4}$$

$$\frac{\partial w}{\partial t} = g\alpha\theta - \frac{1}{\bar{\rho}}\frac{\partial p}{\partial z} + \nu\nabla^2 w \tag{5}$$

$$\frac{1}{r}\frac{\partial ru}{\partial r} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

and

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta_0}{\partial r} + w\frac{\partial \theta_0}{\partial z} = k\nabla^2\theta\,. \tag{7}$$

Here ∇^2 stands for the operator $\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ and ν and κ are kinematic viscosity and thermometric conductivity. In equations (3) and (4) we have neglected $\partial v_0/\partial r$ and v_0/r against 2Ω so that the possibility of having inertial instability is eliminated. Since v_0 satisfies the steady state hydrodynamic equations, its vertical variation is determined approximately by the thermal-geostrophic-wind relation

$$\frac{\partial v_0}{\partial z} = \frac{g\alpha}{2\Omega}\frac{\partial \theta_0}{\partial r}\,. \tag{8}$$

Eliminating u from (4) and (6) and neglecting w/r against $\partial w/\partial r$ we obtain

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\zeta = 2\Omega\frac{\partial w}{\partial z} - \frac{\partial v_0}{\partial z}\frac{\partial w}{\partial r}\,, \tag{9}$$

where $\zeta = r^{-1}\partial(rv)/\partial r$ is the vertical component of the relative vorticity. Multiplying (3) by r, then differentiating with respect to r and z, and making use of the continuity equation we get

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\frac{\partial^2 w}{\partial z^2} = -\,2\Omega\frac{\partial \zeta}{\partial z} + \frac{1}{\bar{\rho}r}\frac{\partial}{\partial r}\left(r\frac{\partial^2 p}{\partial r\partial z}\right). \tag{10}$$

Applying the operator $\partial^2/\partial r^2 + r^{-1}\partial/\partial r$ to (5) and combining with (10) we obtain

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w = -\,2\Omega\frac{\partial \zeta}{\partial z} + \frac{g\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right). \tag{11}$$

Eliminating ζ from (9) and (11) and making use of (8) we arrive at the following equation:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)^2\nabla^2 w + 4\Omega^2\frac{\partial^2 w}{\partial z^2} - g\alpha\frac{\partial \theta_0}{\partial r}\frac{\partial^2 w}{\partial r\partial z} - g\alpha\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \theta}{\partial r}\right) = 0\,. \tag{12}$$

In this study we shall find only the lowest mean temperature contrast required for the onset of the symmetric motion. This contrast must correspond to the slowest rate of development, and therefore it can be found by letting $\partial/\partial t$ approach zero, since the developing disturbances are not oscillatory (see Kuo, 1954). We shall therefore put $\partial/\partial t = 0$ in the preceding equations. The elimination of θ between (12) and (7) and use of the continuity equation then give

$$\nabla^6 w + \frac{4\Omega^2}{\nu^2}\frac{\partial^2 w}{\partial z^2} - \frac{g\alpha}{\kappa\nu}\left(1+\frac{\kappa}{\nu}\right)\frac{\partial\theta_0}{\partial r}\frac{\partial^2 w}{\partial r\partial z} + \frac{g\alpha}{\kappa\nu}\frac{\partial\theta_0}{\partial z}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = 0 . \quad (13)$$

The second term represents the effect of rotation while the last two terms represent the effects of the horizontal and vertical temperature contrasts respectively. Note that if there is no mean temperature contrast in the horizontal directions, this equation then reduces to the equation discussed by Chandrasekhar (1953) if we replace $\partial^2/\partial r^2 + r^{-1}\partial/\partial r$ by the ordinary two-dimensional Laplacian operator. The presence of the term with the factor $\partial\theta_0/\partial r$ makes this equation more difficult to solve, because the method of separation of variables is not applicable. Since we are more concerned with the effects of different physical factors, we shall introduce some approximations by neglecting certain effects which are due to the particular geometry of the cylindrical vessel. Thus we shall replace $\partial^2/\partial r^2 + r^{-1}\partial/\partial r$ by $\partial^2/\partial r^2$ in the Laplacian operator. Since the side boundary conditions have to be considered, it is more convenient to use the stream function ψ as the dependent variable, which, according to the approximation discussed above, may be defined by

$$u = -\frac{\partial\psi}{\partial z}, \quad w = \frac{\partial\psi}{\partial r}. \quad (14)$$

The differential equation for ψ is then the same as (13), namely

$$\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right)^3\psi + \frac{4\Omega^2}{\nu^2}\frac{\partial^2\psi}{\partial z^2} - \frac{g\alpha}{\kappa\nu}\left(1+\frac{\kappa}{\nu}\right)\frac{\partial\theta_0}{\partial r}\frac{\partial^2\psi}{\partial r\partial z} + \frac{g\alpha}{\kappa\nu}\frac{\partial\theta_0}{\partial z}\frac{\partial^2\psi}{\partial r^2} = 0 . \quad (15)$$

In seeking solutions of this equation we must satisfy certain boundary conditions. Certainly the normal velocity components along the boundaries must vanish, which requires that ψ be constant along the surfaces $z = 0, z = d, r = 0$ and $r = a$. We denote this stream line by $\psi = 0$. If these boundaries are rigid, then the tangential velocity components must also vanish, which requires that the normal derivative $\partial\psi/\partial n$ and also $\partial^2\psi/\partial r\partial z$ vanish because of the continuity equation. On the other hand, tangential stresses vanish on a free surface, which requires that the second normal derivative $\partial^2\psi/\partial n^2$ be zero. Since the normal velocity vanishes at all points on the boundaries, this latter condition is equivalent to $\nabla^2\psi = 0$. Additional boundary conditions follow from the basic equations. For example, the elimination of p from (3) and (5) gives

$$\nu\nabla^4\psi - 2\Omega\frac{\partial v}{\partial z} = -g\alpha\frac{\partial\theta}{\partial r}. \tag{16}$$

On a horizontal free surface $\partial v/\partial z$ vanishes. If the radial temperature contrast is kept constant on such a surface so that $\partial\theta/\partial r$ is to vanish, then $\Delta^4\psi$ must vanish. For simplicity, we shall treat all boundaries as free surfaces and shall require that $\Delta^4\psi$ vanish on each of them.[2] Therefore our boundary conditions are

$$\psi = 0,\ \nabla^2\psi = 0 \text{ and } \nabla^4\psi = 0 \tag{17}$$

on the boundaries $z = 0, z = d, r = 0, r = a$.

We now express the solution of (15) in double Fourier series[3]:

$$\psi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi r}{a}\sin\frac{n\pi z}{d}, \tag{18}$$

which satisfies the boundary conditions (17). Then the problem is to determine the coefficients A_{mn} so as to make (18) a solution of (15). If we substitute (18) in (15) and change m to t and n to s, we obtain

$$\begin{aligned}\sum_{t=1}^{\infty}\sum_{s=1}^{\infty} A_{ts}\sin\frac{t\pi r}{a}\sin\frac{s\pi z}{d}\left\{\left[\left(\frac{t\pi}{a}\right)^2 + \left(\frac{s\pi}{d}\right)^2\right]^3 + \frac{4\Omega^2}{\nu^2}\left(\frac{s\pi}{d}\right)^2\right.\\ \left. + \frac{g\alpha}{\kappa\nu}\frac{\partial\theta_0}{\partial z}\left(\frac{t\pi}{a}\right)^2\right\} + \frac{g\alpha}{\kappa\nu}\left(1 + \frac{\kappa}{\nu}\right)\frac{\partial\theta_0}{\partial r}\\ \cdot\sum_{t=1}^{\infty}\sum_{s=1}^{\infty} A_{ts}\cdot\frac{ts\pi^2}{ad}\cdot\cos\frac{t\pi r}{a}\cos\frac{s\pi z}{d} = 0.\end{aligned} \tag{19}$$

[2] More realistic boundary conditions can be obtained from (16). However, it is generally impossible to express these conditions in terms of ψ alone.

[3] This method is widely used in the field of elasticity, where the equations dealt with are generally of a lower order. For example; see Seynel, 1933; Trefftz and Willers, 1936.

If we multiply this equation by sin $(m\pi r a^{-1}) \cdot \sin (n\pi z d^{-1})$ and then integrate over the entire meridional cross-section, and if we make use of the relations

$$\begin{aligned} \int_0^a \sin^2 \frac{m\pi r}{a}\, dr &= \frac{a}{2}, \\ \int_0^a \sin \frac{m\pi r}{a} \cos \frac{t\pi r}{a}\, dr &= \frac{2a}{\pi} \frac{m}{m^2 - t^2} \text{ when } m \pm t \text{ is odd} \\ &= 0 \text{ when } m \pm t \text{ is even} \end{aligned} \tag{20}$$

and of two similar relations for the z integrations, we obtain

$$\left\{\frac{n^2}{\pi^4} T + \frac{m^2}{\pi^4} \cdot \frac{d^2}{a^2} R + d^6 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{d}\right)^2\right]^3\right\} A_{mn} + mn \frac{d}{a} \cdot C \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} A_{ts} \cdot \frac{t}{m^2 - t^2} \cdot \frac{s}{n^2 - s^2} = 0\,, \tag{21}$$

where the summations over t and s are for odd $m \pm t$ and $n \pm s$; the even terms vanish. In this equation we have

$$T = \frac{4\Omega^2}{\nu^2} d^4\,, \quad R = \frac{g\alpha}{\kappa\nu} d^4 \frac{\partial \theta_0}{\partial z}\,, \quad C = \frac{16}{\pi^6} \frac{g\alpha}{\kappa\nu} \left(1 + \frac{\kappa}{\nu}\right) d^4 \frac{\partial \theta_0}{\partial r}. \tag{22}$$

Thus T is the square of a Reynolds number in terms of which the effect of rotation is measured, sometimes called the Taylor number; $-R$ is the Rayleigh number which plays an important part in the problem of thermal convection produced by a vertical temperature contrast; and C is a parameter representing the effect of the horizontal temperature contrast. We now put eq. (21) in the following simple form:

$$A_{mn} \cdot \frac{\varphi(m,n)}{\beta mn} + C \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} A_{ts} \cdot \frac{t}{m^2 - t^2} \cdot \frac{s}{n^2 - s^2} = 0\,, \tag{23}$$

where $\varphi(m,n) = n^2 T' + m^2\beta^2 R' + (n^2 + m^2\beta^2)^3$, with $T' = \pi^{-4}T$, $R' = \pi^{-4}R$ and $\beta = da^{-1}$.

An examination of eq. (21) shows that there are two mutually independent systems of equation; one system is for the unknowns A_{mn} with $m + n$ even, the other is for the unknowns with odd $m + n$. Thus, if $m + n$ is even, we must have both m and n even or both must be odd. Both t and s are then odd or even, since only terms with odd $m \pm t$ and $n \pm s$ occur. On the other hand, if the sum of m and n is odd, then one must be even while the other must be odd, hence the sum of t and s is also odd. In each of these two systems the number of the unknowns and the number of equations is infinite. Since these are homogeneous equations, the determinant Δ formed by the coefficients of these unknowns must be zero. The equation $\Delta = 0$ gives the required critical values of C for given values of T and R.

However, this procedure of computing the critical values can be carried out only when Δ converges rapidly, which requires that there be only a finite number of A_{mn} so that these unknowns can be computed to a sufficient degree of accuracy by neglecting all the other unknowns in the system. This is equivalent to saying that solution (18) can be approximated by a finite number of terms. The mathematical problem is to make this series representation converge rapidly, which requires different arrangements for different cases.

CASE A. A THIN LAYER OF FLUID WITH LARGE HORIZONTAL EXTENT

In this case we assume that the horizontal dimension a is infinitely large compared with the depth d, so that the length ratio $\beta = d/a \rightarrow 0$. This case has been discussed by an exact method (Kuo, 1954). In order to express the results more clearly, we shall derive an approximate formula for the critical temperature contrast by the present method. For this case a finite m represents an infinitely large horizontal wave length; the critical C thus obtained is evidently infinity. To have a finite horizontal wave length, m and t must be of the same order of magnitude as the length ratio $\beta^{-1} = a/d$. When β^{-1} approaches infinity, m and t must also approach infinity, so that βm approaches a finite value λ. Thus we put

$$m = \frac{\lambda}{\beta} + m_1 = p + m_1 \qquad t = p + t_1 \tag{24}$$

where m_1 and t_1 are finite integers with zero included and λ is a finite number so that $\lambda\beta^{-1} = p$ is an integer. When $\beta \rightarrow 0$, we have $p >> m_1$ and $p >> t_1$ and therefore

$$\frac{t}{m^2 - t^2} \doteq \tfrac{1}{2}\frac{1}{m_1 - t_1}, \qquad l = \frac{a}{m} = \frac{d}{\lambda + \beta m_1} \doteq \frac{d}{\lambda},$$

where l is the horizontal half wave length. The function $\varphi(m,n)$ is then given by

$$\varphi(m,n) \doteq n^2T' + \lambda^2R' + (n^2 + \lambda^2)^3 = \varphi(\lambda,n)\,.$$

Thus equation (23) becomes

$$A_{p+m_1,n}\frac{\varphi(\lambda,n)}{\lambda n} + \tfrac{1}{2}C\sum_{t_1=-\infty}^{\infty}\sum_{s=1}^{\infty}A_{p+t_1,s}\cdot\frac{1}{m_1-t_1}\cdot\frac{s}{n^2-s^2} = 0\,. \quad (25)$$

Here we must determine the characteristic value of C by solving the determinant of this system of equations, which contains two mutually independent systems, one with even $m_1 + n$ and even $t_1 + s$, another with odd $m_1 + n$ and odd $t_1 + s$. For the present case these two systems give the same characteristic value of C. We shall use the second system, with odd $m_1 + n$ and odd $t_1 + s$. We then have the following relations

$$A_{p+m_1,n} = +A_{p-m_1,n} \text{ when } n \text{ is odd,}$$

$$A_{p+m_1,n} = -A_{p-m_1,n} \text{ when } n \text{ is even.} \quad (26)$$

We may divide the unknowns $A_{p+m_1,n}$ and $A_{p+t_1,s}$ into two different classes, one class with even n or s and one class with odd n or s. Thus, if $A_{p+m_1,n}$ belongs to one class, then all the other unknowns $A_{p+t_1,s}$ belong to the other class, because if s is odd then n must be even, and vice versa. Since the $A_{p+t_1,s}$ always occur with C, we may take $CA_{p+t_1,s}$ as the unknowns and write equation (25) in the following form

$$\frac{2\varphi(\lambda,n)}{\lambda\cdot n}A_{p+m_1,n} + \sum_{t_1=-\infty}^{\infty}\sum_{s=1}^{\infty}CA_{p+t_1,s}\frac{1}{m_1-t_1}\frac{s}{n^2-s^2} = 0\,. \quad (27)$$

In this way C occurs in the form of C^{-2} in some coefficients and does not occur in the others. We shall compute the determinant of this system of equations by using only the first fourteen, letting m_1 and t_1 take the values of the nine integers from -4 to $+4$ and letting n and s take the values from 0 to 3. However, because of the relations (26), we actually have eight equations, of which the coefficients of the eight unknowns, written in the form of (27), are given in Table I. It is seen that C^{-2} occurs only in two of the diagonal coefficients, hence the determinant is a quadratic equation in C^2.

We note that the sixth order determinant Δ_6, formed by the first six coefficients, is a diagonal determinant; therefore the eighth order determinant formed by the coefficients given in Table I can be reduced to the product of Δ_6 and a second order determinant Δ_2. Thus, multiplying the first row by $-\lambda/3\varphi(\lambda,1)$, the second row by $9\lambda/5\varphi(\lambda,3)$, the third row by $2\lambda/9\varphi(\lambda,1)$, the fourth row by $-6\lambda/5\varphi(\lambda,3)$, the fifth row by $2\lambda/45\varphi(\lambda,1)$ and the sixth row by $-6\lambda/25\varphi(\lambda,3)$, and by adding to the seventh row, all of the first six elements of the seventh row are reduced to zero while the last two elements of this row become

Table I The coefficients of the unknowns of equation (27).

$(p+m_i)\cdot n$ \ $A_{p+t,s}$	$A_{p,1}$	$A_{p,3}$	$A_{p+2,1}$	$A_{p+2,3}$	$A_{p+4,1}$	$A_{p+4,3}$	$CA_{p+1,2}$	$CA_{p+3,2}$
$p\cdot 1$	$\frac{\varphi(\lambda,1)}{\lambda\cdot 1}$	0	0	0	0	0	$\frac{2}{3}$	$\frac{2}{9}$
$p\cdot 3$	0	$\frac{\varphi(\lambda,3)}{\lambda\cdot 3}$	0	0	0	0	$-\frac{2}{5}$	$-\frac{2}{15}$
$(p+2)\cdot 1$	0	0	$\frac{\varphi(\lambda,1)}{\lambda\cdot 1}$	0	0	0	$-\frac{2}{9}$	$\frac{2}{5}$
$(p+2)\cdot 3$	0	0	0	$\frac{\varphi(\lambda,3)}{\lambda\cdot 3}$	0	0	$\frac{2}{15}$	$-\frac{6}{25}$
$(p+4)\cdot 1$	0	0	0	0	$\frac{\varphi(\lambda,1)}{\lambda\cdot 1}$	0	$-\frac{2}{45}$	$-\frac{2}{7}$
$(p+4)\cdot 3$	0	0	0	0	0	$\frac{\varphi(\lambda,3)}{\lambda\cdot 3}$	$\frac{2}{75}$	$\frac{6}{35}$
$(p+1)\cdot 2$	$\frac{1}{3}$	$-\frac{3}{5}$	$-\frac{2}{9}$	$\frac{2}{5}$	$-\frac{2}{45}$	$\frac{2}{25}$	$\frac{\varphi(\lambda,2)}{\lambda}C^{-2}$	0
$(p+3)\cdot 2$	$\frac{1}{9}$	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{18}{25}$	$-\frac{2}{7}$	$\frac{18}{35}$	0	$\frac{\varphi(\lambda,2)}{\lambda}C^{-2}$

$$a_{77} = \frac{\varphi_2}{\lambda C^2} - \frac{554\lambda}{2025\varphi_1} - \frac{554\lambda}{625\varphi_3}, \qquad a_{78} = \frac{2\lambda}{945\varphi_1} + \frac{6\lambda}{875\varphi_3},$$

where φ_n is written for $\varphi(\lambda,n)$. In a similar manner, we multiply the first six rows by $-\lambda/9\varphi_1$, $3\lambda/5\varphi_3$, $-2\lambda/5\varphi_1$, $54\lambda/25\varphi_3$, $2\lambda/7\varphi_1$ and $-54\lambda/35\varphi_3$, respectively, and by adding to the eighth row, the first six elements of this row are then reduced to zero while the last two elements become

$$a_{87} = \frac{2\lambda}{35}\left(\frac{1}{27\varphi_1} + \frac{3}{25\varphi_3}\right), \quad a_{88} = \frac{\varphi_2}{\lambda C^2} - \frac{26426\lambda}{1225}\left(\frac{1}{27\varphi_1} + \frac{3}{25\varphi_3}\right).$$

In this way we find $\Delta = \Delta_6 \cdot \Delta_2$, where Δ_2 is the second order determinant formed by a_{77}, a_{78}, a_{87} and a_{88}. Since Δ_6 is always positive, we must have Δ_2 equal to zero, i.e.,

$$\Delta_2 = \begin{vmatrix} a_{77} & a_{78} \\ a_{87} & a_{88} \end{vmatrix} = 0\,. \tag{28}$$

This gives a quadratic equation in $\chi = \varphi_2 C^{-2}/2\lambda$,

$$x^2 - \frac{26786}{3675}bx + \frac{3659776}{275625}b^2 = 0\,, \tag{29}$$

where $b = \dfrac{\lambda}{27\varphi_1} + \dfrac{3\lambda}{25\varphi_3}$. The two roots are $\chi_1 = 3.701b$ and $\chi_2 = 3.585b$ respectively.[4] The former gives the lower value of C^2 and

[4] If only the first seven equations in table I are used, the determinant reduces to $a_{77} = 0$ which gives $x = 3{\cdot}6933b$. The value of C so obtained is only slightly higher than that given by $\Delta_2 = 0$.

is therefore the one we want. From χ_1 we find that the critical values of C^2 are given by

$$C^2 = \frac{675}{7.402} \cdot \frac{1}{\lambda^2} \cdot \frac{\varphi_1 \cdot \varphi_2 \cdot \varphi_3}{81\varphi_1 + 25\varphi_3}, \tag{30}$$

where $\varphi_n = n^2T' + \lambda^2R' + (n^2 + \lambda^2)^3$. From this formula we see that, as $T' \to \infty$, C becomes proportional to T' and inversely proportional to λ. The asymptotic value of C is

$$\lim_{T' \to \infty} C = 3.28 \frac{T'}{\lambda} = 3.28 \frac{l}{d} T'. \tag{31}$$

This asymptotic value is approached rapidly as $T(= \pi^4T')$ becomes larger than 5×10^3. Thus, for fairly large Taylor numbers, the critical temperature contrast required to produce a symmetric convection is proportional to the square of the rotation rate as well as to the horizontal dimension of the convection.

Formula (30) shows that, for any given values of T' and R, there is a particular value of $\lambda = \lambda_c$ for which C^2 is a minimum. This λ_c is the root of the equation

$$\frac{dC^2}{d\lambda} = 0. \tag{32}$$

The value of C corresponding to this value of λ is an absolute minimum below which no symmetric motion is possible. We denote this C by C_m. When the value of C reaches the value C_m, symmetric convection with the horizontal dimension $l_c = d \cdot \lambda_c^{-1}$ will appear. When C is above C_m, larger and smaller convective cells may also be produced. The values of λ_c for different Taylor numbers and zero R are given in Table II, together with the minimum horizontal temperature contrast, represented by the minimum value of the parameter Q, given by

$$Q_c = \frac{\pi^6}{16} C = \left(1 + \frac{k}{\nu}\right) \frac{gd^4}{k\nu} \alpha \frac{\partial \theta_0}{\partial r} = 0.5968\, \pi^6 \frac{l}{d} \left(\frac{\varphi_1 \cdot \varphi_2 \cdot \varphi_3}{81\varphi_1 + 25\varphi_3}\right)^{\frac{1}{2}}. \tag{33}$$

The variations of Q_{cm} as a function of T are illustrated in curve a of Fig. 1.

The minimum values of Q_c in Table II, given by formula (33), correspond to $\lambda = \lambda_c$ and are to be compared with values obtained from the exact solutions (Kuo, 1954). As expected, these values are slightly higher than those obtained from the exact solutions. However, the difference is small for small values of T and is less than 1% for T smaller than 10^3; the difference is only 6.7% for $T = 10^6$. In view of the simplicity of this formula and the ease with which the

Table II. Minimum critical values of Q, P and K corresponding to $\lambda = \lambda_c = d/l_c$, for the case when the vertical stability is zero.

T	λ_c	Q_c	P_c	K_c
0	0.5868	2,747	6,448	∞
10	0.5990	2,810	6,733	673.30
50	0.6395	3,039	7,774	155.50
10^2	0.6791	3,291	8,940	89.40
5×10^2	0.8507	4,766	16,218	32.40
1.0×10^3	0.9644	6,146	23,709	23.70
2.5×10^3	1.1576	9,391	43,484	17.40
5.0×10^3	1.3390	13,736	73,570	14.71
1.0×10^4	1.5530	20,975	130,297	13.03
2.5×10^4	1.8870	38,714	292,213	11.69
5.0×10^4	2.1860	63,500	556,000	11.12
1.0×10^5	2.5094	106,449	1,069,000	10.69
5.0×10^5	3.4360	371,000	5,090,000	10.18
1.0×10^6	3.8990	646,500	10,070,000	10.07
∞		∞	∞	8.10

different physical factors can be discussed, including those involving the vertical stability, and in view of the enormous computation required to obtain information from the exact solutions, we may consider this approximate result quite satisfactory.

Since the critical temperature contrast becomes proportional to the horizontal dimension l for large values of T and $l > l_c$, it is more convenient to use the parameter $p = 4dl^{-1}Q$ instead of Q, especially for large values of T since p_c is nearly independent of l. The values of p_c corresponding to $\lambda \leqslant \lambda_c$ are also given in Table II and are plotted against T in Fig. 2. These values and the graphs show that Q_{cm}, p_c and λ_c increase as T increases, and that, as $T \rightarrow \infty$, $Q_{cm} \alpha T^{0.79}$ and $\lambda_c \alpha T^{0.21}$, so that p_c becomes directly proportional to T. Thus, when the rotation rate is high, not only is a larger horizontal temperature contrast required to produce symmetric convection but the motion also tends to break up into cells of smaller horizontal dimensions.

In Fig. 2, the parameters p and T obtained from experimental data on symmetric convection, supplied by Dr. Fultz (as yet unpublished) and Miss Sabin (1954), have also been entered. The theoretical curve for $a = d$, which will be derived in the following, is also plotted in Fig. 2.

Since p_c becomes proportional to T and varies but little with l for large values of T and $l > l_c$, its ratio to T approaches a constant as T increases. We denote this ratio by K_c, which is given by

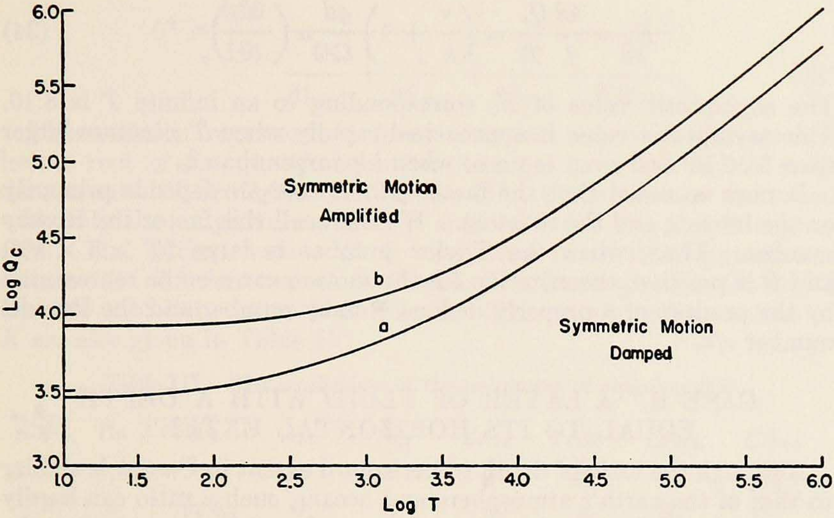

Figure 1. Variation of the critical value of Q for the onset of symmetric convection as a function of T for two cases: (a) when the horizontal extent of the fluid is infinitely large as compared with depth; (b) when the horizontal extent is equal to depth.

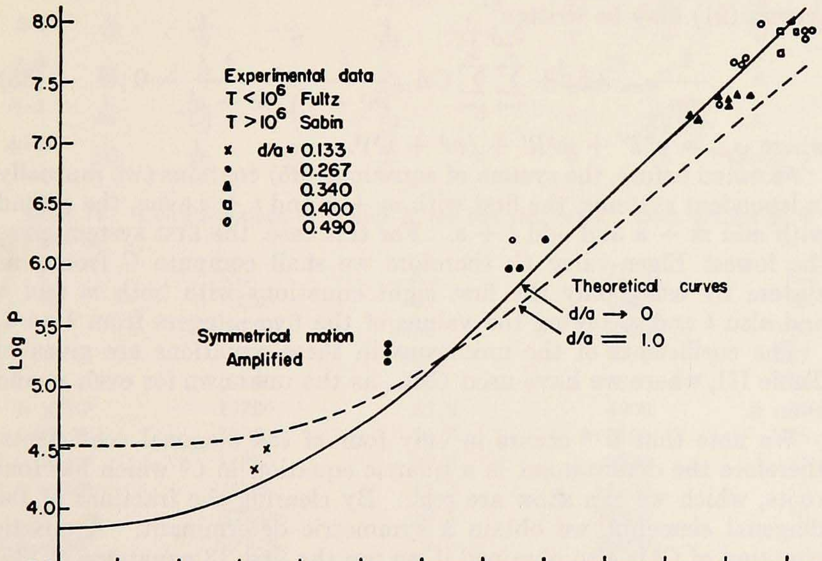

Figure 2. The variation of the parameter p as a function of T. The experimental data are obtained from Dr. Fultz' and Miss Sabin's (1954) observations on the fully developed symmetric motions.

$$K_c = \frac{4d}{l}\frac{Q_c}{T} = \left(\frac{\nu}{k} + 1\right)\frac{gd}{\Omega^2 l}\alpha\left(\frac{\partial\theta_0}{\partial r}\right)_c. \tag{34}$$

The asymptotic value of K_c corresponding to an infinite T is 8.10. This asymptotic value is approached rapidly when T becomes larger than 5×10^3 and even more so when l is larger than l_c.

It may be noted that the factor $gdl^{-1}\Omega^{-2}\alpha\partial\theta_0/\partial r$ depends primarily on the heating and the rotation. We shall call this factor the Rossby number. Thus, when the Taylor number is large ($T > 5 \times 10^3$) and R is positive, the criterion for the motion can also be represented by the product of a properly defined Rossby number and the Prandtl number ν/k.

CASE B. A LAYER OF FLUID WITH A DEPTH EQUAL TO ITS HORIZONTAL EXTENT

Although the ratio of depth to horizontal extent in Case A is similar to that of the earth's atmosphere and oceans, such a ratio can hardly be reproduced in a model experiment. In order to examine the effect of this ratio on the motion, we shall discuss in this section another case with depth d equal to horizontal extent a. For this case, equation system (21) may be written

$$\frac{1}{mn}\varphi_{mn} A_{mn} + \sum_{t=1}^{\infty}\sum_{s=1}^{\infty} CA_{ts}\frac{t}{m^2 - t^2}\cdot\frac{s}{n^2 - s^2} = 0\,, \tag{35}$$

where $\varphi_{mn} = n^2T' + m^2R' + (m^2 + n^2)^3$.

As noted before, the system of equations (35) contains two mutually independent systems, the first with $m + n$ and $t + s$ even, the second with odd $m + n$ and odd $t + s$. For this case, the first system gives the lowest Eigen-value C; therefore we shall compute C from this system by using only the first eight equations with both m and n and also t and s, taking the values of the five integers from 1 to 4.

The coefficients of the unknowns in these equations are given in Table III, where we have used CA_{mn} as the unknown for even m and even n.

We note that C^{-2} occurs in only four of the diagonal coefficients; therefore the determinant is a quartic equation in C^2 which has four roots, which we can show are real. By clearing the fractions of the diagonal elements, we obtain a symmetric determinant. A quartic equation of C^2 is also obtained if we use the first 13 equations of (35) and take integers for m and n as well as t and s from 1 to 5. On the other hand, if we use only the first five equations by taking m and n and t and s from 1 to 3, then the determinant reduces to a linear equation in C^2 which is given by

$$C^2 = \frac{\varphi_{22}}{16} \cdot \frac{1}{\dfrac{1}{81\varphi_{11}} + \dfrac{1}{25\varphi_{13}} + \dfrac{1}{25\varphi_{31}} + \dfrac{81}{625\varphi_{33}}}. \tag{36}$$

This solution represents a fairly accurate approximation to the lowest root of the quartic equation obtained from the eighth order determinant of the system in Table III. Since we are concerned with only the rough values of C, it suffices to use this simple expression. The critical values of Q computed from this approximate solution for different values of T and $R = 0$ are given in Table IV and are ploted in curve b of Fig. 1. The corresponding values of K are also given in Table IV.

Table III. The coefficients of the unknowns of equation (35).

$m \cdot n$ \ A_{ts}	A_{11}	A_{13}	A_{31}	A_{33}	CA_{22}	CA_{24}	CA_{42}	CA_{44}
$1 \cdot 1$	$\frac{1}{1\cdot 1}\varphi_{11}$	0	0	0	$\frac{4}{9}$	$\frac{8}{45}$	$\frac{8}{45}$	$\frac{16}{225}$
$1 \cdot 3$	0	$\frac{1}{1\cdot 3}\varphi_{13}$	0	0	$-\frac{4}{15}$	$\frac{8}{21}$	$-\frac{8}{75}$	$\frac{16}{105}$
$3 \cdot 1$	0	0	$\frac{1}{3\cdot 1}\varphi_{31}$	0	$-\frac{4}{15}$	$-\frac{8}{75}$	$\frac{8}{21}$	$\frac{16}{105}$
$3 \cdot 3$	0	0	0	$\frac{1}{3\cdot 3}\varphi_{33}$	$\frac{4}{25}$	$-\frac{8}{35}$	$-\frac{8}{35}$	$\frac{16}{49}$
$2 \cdot 2$	$\frac{1}{9}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{9}{25}$	$\frac{1}{2\cdot 2}\varphi_{22}C^{-2}$	0	0	0
$2 \cdot 4$	$\frac{1}{45}$	$\frac{1}{7}$	$-\frac{1}{25}$	$-\frac{9}{35}$	0	$\frac{1}{2\cdot 4}\varphi_{24}C^{-2}$	0	0
$4 \cdot 2$	$\frac{1}{45}$	$-\frac{1}{25}$	$\frac{1}{7}$	$-\frac{9}{35}$	0	0	$\frac{1}{4\cdot 2}\varphi_{42}C^{-2}$	0
$4 \cdot 4$	$\frac{1}{225}$	$\frac{1}{35}$	$\frac{1}{35}$	$\frac{9}{49}$	0	0	0	$\frac{1}{4\cdot 4}\varphi_{44}C^{-2}$

Table IV. Critical values of Q and K for the two cases. Both are for $R = 0$ and $l = d$.

T	Q_{cb}	K_{cb}	Q_{ca}	K_{ca}
0	8320	∞	3580	∞
10	8350	3350	3620	1450
50	8590	688	3710	297
10^2	8850	355	3850	154
5×10^2	10700	85.6	4900	39.2
10^3	12650	50.7	6150	24.6
2.5×10^3	17350	27.8	9620	15.4
5.0×10^3	24100	19.3	15100	12.1
10^4	33500	13.5	26600	10.6
2.5×10^4	59000	9.5	56500	9.1
5.0×10^4	95000	7.6	107000	8.6
10^5	165000	6.6	208000	8.3
5×10^5	660000	5.3	1015000	8.1
10^6	1270000	5.1	2024000	8.1
∞	∞	4.8	∞	8.1

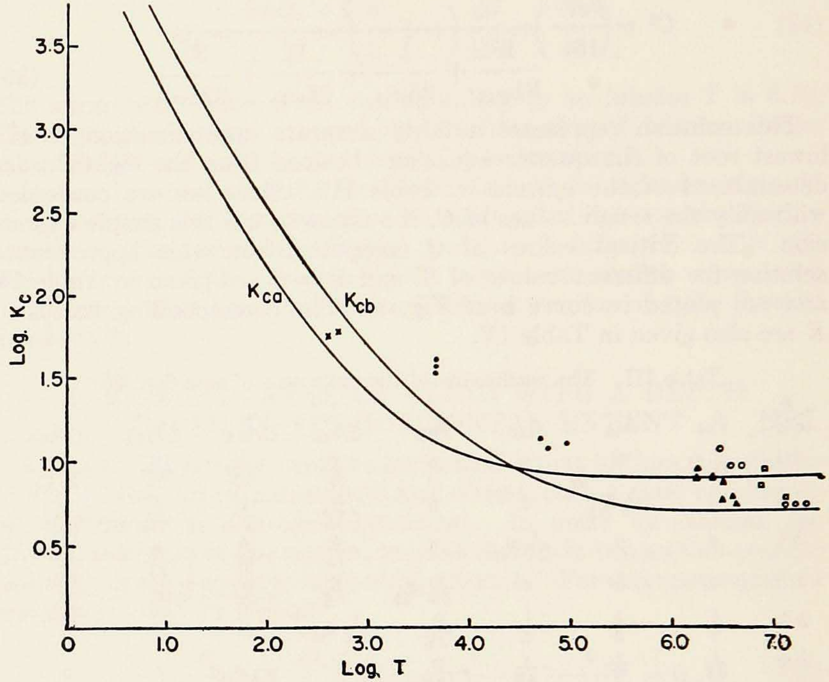

Figure 3. The variation of the critical value of K as a function of T for the symmetric convection with $l = d$. K_{ca} is for case A, K_{cb} for case B. Experimental data are the same as in Fig. 2.

As expected, the critical values of Q for this case, as given by Q_{cb} in Table IV are 2 to 3 times larger than those in Table II. This is due partly to the effect of the lateral boundaries and partly to the horizontal scale of the motion. In order to get a clearer idea of the effect of the ratio of depth to lateral extent, we may also compute the critical values of Q and K from eq. (33) by putting $l = d$. These values are given in the last two columns of Table IV as Q_{ca} and K_{ca}. Here, for values of T smaller than 2.5×10^4, Q_{cb} is larger than Q_{ca}, indicating the preponderance of the boundary effect. On the other hand, for values of T larger than 2.5×10^4, Qcb becomes smaller than Q_{ca}, indicating the difficulty of producing large scale symmetric convection in a shallow layer of fluid. We also note that the asymptotic value of K_c for case B is 4.8 as compared with the asymptotic value of 8.1 for case A. Since the ratio used in most experiments is generally between the limits of these two cases, we may take the asymptotic value of K_c to be within the limits 4.8 and 8.1. The variations of K_{ca} and K_{cb} are represented in Fig. 3.

SUMMARY OF RESULTS

The results show that the onset of symmetric convection in each case requires that the nondimensional parameter Q be above a certain critical value Q_c which increases with the rotation. Therefore, the effect of rotation is to inhibit the convection, the extent of which depends on the nondimensional parameter T. The effect of the vertical temperature distribution, represented by the parameter R, is to inhibit convection if the temperature increases upward ($R > 0$) and to facilitate convection if the temperature decreases upward ($R < 0$). In applying the results to symmetric convection in the atmosphere, the vertical temperature gradient should be replaced by the vertical gradient of the potential temperature which normally increases upward. This effect is included in equations (33) and (36). However, since it is similar to the effect of rotation, we may also consider that it is included in the effect of rotation by modifying the value of T.

For case A, the lowest critical value of Q for any given values of T and R is associated with a certain horizontal scale l_c which decreases as T increases. When T is large ($T > 5 \times 10^3$), Q_c rapidly becomes proportional to $T^{0.79}$ and to the horizontal dimension l of the cell for all values of l larger than $l_c (p_c = 4dl^{-1}Q_c\alpha T)$; thus the parameter $K_c = 4dl^{-1}Q_cT^{-1}$ rapidly approaches a constant value, showing that the onset of large symmetric convection in a shallow layer of fluid such as the atmosphere and the oceans is determined by the parameter K. This parameter may be expressed as the product of a Rossby number and Prandtl number. The asymptotic value of K for case A is 8.1.

For case B, the critical values of Q are generally 2 to 3 times larger than the minimum values obtained in case A. However, the values of Q obtained from eq. (33) by putting $l = d$ become larger than those of case B when T is larger than 2.5×10^4, showing that it is more difficult to produce large symmetric convective motions in a thin layer of fluid when the rotation rate is great.

The results obtained in this investigation may have many applications in both atmosphere and oceans, which are more nearly represented by case A. Thus we may find the minimum horizontal temperature gradient required for the onset of symmetric convection with a certain vertical dimension d at any latitude, or we may compute the maximum horizontal scale of the motion when the horizontal temperature gradient is given. A rough computation shows that meridional cells of the horizontal dimensional of 10 to 20 latitude degrees may exist near to the equator but not in middle and higher latitudes.

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