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THE TRANSIENT DEVELOPMENT OF A LEE WAVE[1]

By

M. G. WURTELE

Massachusetts Institute of Technology

ABSTRACT

A stream with an initially undisturbed free surface is set impulsively in motion and a (small) pressure perturbation is applied to the surface. The form of the free surface as a function of time is examined in detail up to the point where the disturbance is sensibly that of a steady-state lee wave.

INTRODUCTION

The theoretical problem of waves in the lee of a surface disturbance was first attacked by Lord Rayleigh (1883). He pointed out that, if a steady state is assumed, the problem is indeterminate in that an infinite number of solutions exist. Stationary free waves of any amplitude are solutions; and in a fluid of infinite depth it is always possible to assign a wave length such that the free wave will move with velocity equal and opposite to that of the current and so remain stationary relative to the point of disturbance.

To specify a unique solution, Rayleigh employed the much used device of a small fictitious dissipative force proportional to the relative velocity. This has the effect of prohibiting any upstream wave (except one with infinite amplitude at infinity), hence the only physically acceptable solution is the lee wave.

Lord Kelvin (1886), in his investigation of the slightly different problem of irregularities in the bed of a stream, used a different method.

[1] This paper was presented at the Johns Hopkins Symposium on the Application of Experimental Models to Geophysical Research, September 1–4, 1953.

After obtaining a solution which satisfied the boundary conditions, he added free waves with just such amplitudes as would render the upstream disturbance monotonically decreasing at large distances. The result is exactly that obtained by the method of Rayleigh when Kelvin's method is applied to this problem, as shown by Höiland (1951).

The indeterminateness of the problem persisted *a fortiori* in the more complex models, as in that of Lyra (1943), with continuously distributed stability. However, recently Höiland (1951) has shown that if the stream is allowed to begin from a state of rest with an undisturbed surface, the disturbance, after sufficient time has elapsed, will be sensibly the same as the steady-state lee wave of Rayleigh and Kelvin. Höiland's explanation of the lee wave has considerable intuitive appeal, and it is the purpose of this paper to show in some detail the transient development of such a wave by using Rayleigh's original model.

THE PROBLEM AND ITS FORMAL SOLUTION

Consider a homogeneous incompressible fluid stream which is bounded on top by a free surface but which is infinite in depth and breadth. We fix a co-ordinate system in the stream so that the fluid is moving with constant speed U in the direction of the x-axis. The z-axis will be taken increasing upward, with origin in the free surface. If a disturbing pressure of sinusoidal form, $p = p_0 e^{ikx}$, is applied to the surface of the stream, the resulting motion may be determined. The disturbance, assumed to be of small amplitude, may be represented by a velocity potential ϕ and a stream function ψ, both of which satisfy Laplace's equation. Since the current is uniform, the vertical displacement ζ satisfies the same equation. We may thus discuss these quantities in terms of their amplitudes ϕ_0, ψ_0, ζ_0 at the surface:

$$\phi = \phi_0 e^{kz + ikx}, \qquad \psi = \psi_0 e^{kz + ikx}, \qquad \zeta = \zeta_0 e^{kz + ikx}. \quad (2.1)$$

For steady conditions, therefore, the dynamic condition at the surface reduces to $\rho(kU^2 - g)\zeta_0 = p_0$.

If we employ Rayleigh's notation, $\kappa = g/U^2$, we have the result in his form,

$$\zeta_0 = \frac{p_0}{\rho k} \cdot \frac{e^{ikx}}{U^2 - C^2} = \frac{p_0}{\rho U^2} \frac{e^{ikx}}{k - \kappa}, \quad (2.2)$$

where $C^2 = g/k$.

We may now consider the time-dependent problem. The initial state of the system is specified and the assumption of steady state is relinquished; the problem, then, is to determine the motion of the system at any future time. A simple rule may be given (Wurtele, 1953) for the construction of the required solution. There are two possible free waves of the system, that is, waves which move so that their corresponding perturbation pressures at the free surface are zero. These waves have the Stokesian velocities $\pm C$, and they may be added with arbitrary amplitudes to the stationary solution (2.2), which is needed to satisfy the boundary conditions. The time-dependent solution may be written

$$\zeta_0 \frac{p_0}{k(U^2 - C^2)} \{ \exp ikx - \alpha \exp ik[x - (U - C)t] \\ - \beta \exp ik[x - (U + C)t] \} . \tag{2.4}$$

This solution may be verified mathematically by elimination of the velocity potential between the dynamic surface condition for non-steady flow,

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} = g \zeta + \frac{1}{\rho} p ,$$

and the identity,

$$- \frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial x}$$

with the use of (2.1). The resulting equation is

$$\frac{\partial^2 \zeta_0}{\partial t^2} + 2iUk \frac{\partial \zeta_0}{\partial t} - k^2(U^2 - C^2)\zeta_0 = \frac{kp_0}{\rho} , \tag{2.5}$$

of which (2.4) is the general solution. Of course the constants α and β are to be determined by the given initial state of the system, more specifically by the vertical displacement and vertical velocity of the free surface at $t = 0$. For present purposes, we shall select these values so as to exhibit the simplest possible pattern of development of the disturbance. At the initial moment we specify that the free surface shall be undisturbed

$$\zeta_0 = 0 \text{ at } t = 0 \tag{2.6}$$

but that the motion of the free surface is a simple sinusoidal pattern

$$\frac{\partial \zeta_0}{\partial t} = \frac{ip_0 e^{ikx}}{\rho(U + C)} \text{ at } t = 0 . \tag{2.7}$$

If these two conditions are imposed upon the general solution (2.4), we find that $\alpha = 1, \beta = 0$; there the solution of the particular physical problem proposed is

$$\zeta_0 = \frac{p_0}{k(U^2 - C^2)} \{ \exp ikx - \exp ik[x - (U - C)t] \} . \tag{2.8}$$

The wave moving forwards (relative to the fluid) is not excited in this system owing to the particular initial state specified. But the physical mechanism of the development is made more evident thereby, since, as we shall see, this advancing wave does not contribute to the establishment of the lee wave.

Following Rayleigh, we may consider the applied pressure as concentrated in a narrow band at the origin so that it may be represented mathematically as a delta function. The Fourier-component amplitudes are therefore equal for all waves, and p_0 may be treated as constant. The solution may be written as

$$\zeta_0 = \frac{p_0}{\rho}\int_0^\infty \{\exp ikx - \exp ik[x - (U - C)t]\}\frac{dk}{k(U^2 - C^2)}. \quad (2.9)$$

It can easily be shown by methods developed for a similar model (Wurtele, 1953) that (2.9) approaches the classical steady-state solution as $t \rightarrow \infty$. In the next sections the transient state will be examined and interpreted.

EVALUATION OF THE INTEGRAL

For the purpose of evaluation, the integral (2.9) should be written in terms of the fundamental wave number κ. If we take κ^{-1} as the unit of length, the integral is made nondimensional; and if we absorb the constant factor $p_0/\rho U^2$ into ζ_0 and take the real part of (2.9), we have

$$\zeta_0 = \int_0^\infty \{\cos kx - \cos[kx - Ut(k - k^{\frac{1}{2}})]\}\frac{dk}{k - 1} \quad (3.1a)$$

$$= P\int_0^\infty \cos kx \frac{dk}{k-1} - P\int_0^\infty \cos[kx - Ut(k - k^{\frac{1}{2}})]\frac{dk}{k-1}$$

$$= I_1 - I_2 \quad (3.1b)$$

where the integrals in (3.1b) take Cauchy principle values at the point $k = 1$. The integral I_1 is a solution of the corresponding stationary problem. It may be expressed in terms of tabulated functions as follows:

$$P\int_0^\infty \cos kx \frac{dk}{k-1} = \cos x\, P\int_{-1}^\infty \cos xy \frac{dy}{y} - \sin x\, P\int_{-1}^\infty \sin xy \frac{dy}{y}$$

$$= -\cos x\, Ci(x) - \sin|x| \,[\tfrac{1}{2}\pi + Si(|x|)], \quad (3.2)$$

where the sine and cosine integrals have their usual notations,

$$Ci(x) = -\int_x^\infty \cos y \frac{dy}{y}, \qquad Si(x) = \int_0^x \sin y \frac{dy}{y}.$$

Using the asymptotic expansions of Ci and Si, we may write

$$I_1 = -\pi \sin |x| + 0(1/x). \tag{3.2a}$$

The second integral I_2 in (3.1b) is time-dependent and less easily evaluated. We may write it as

$$I_2 = Re\, P \int_0^\infty \exp i(kx' - n't) \frac{dk}{k-1}, \tag{3.3}$$

where we have introduced the co-ordinate moving with the current, $x' = x - Ut$, and the corresponding frequency,

$$n' = -Uk^{\frac{1}{2}}. \tag{3.4}$$

We may eliminate the radical in the exponent by writing

$$I_2 = 2\, Re\, P \int_0^\infty \exp if(y) \frac{ydy}{y^2 - 1}, \tag{3.5}$$

where $f(y) = x'y^2 + Uty$.
There may be two neighborhoods on the path of integration which contribute significantly to the integral. The first is the pole,

$$y = 1, \tag{3.6}$$

which is on the path for all values of the parameters. The second is the saddle-point, given by $df/dy = 0$,

$$y_0 = \tfrac{1}{2}(1 - x/Ut)^{-1}, \tag{3.7}$$

which can lie on the path of integration only if $x < Ut$. So long as these two points are separated by an appropriate distance, the integral can be evaluated by indenting the contour at the pole and by deforming the path of integration so it will traverse the saddle-point in a path of steepest descent. When the parameters assume certain values, specifically when $Ut \doteq 2x$, the saddle-point approaches and then passes over the pole. In this region special methods must be used. So that the essential features of the wave development do not become obscured, we assume that the system has been in motion long enough for the dispersion to be significant, that is, $Ut \gg 1$.

The stages of the wave development are then as follows.

(*i*) $Ut < x$. As in all models involving an incompressible fluid, a disturbance appears instantaneously throughout the fluid. In general also, since the fluid is infinite in depth, the forward-moving waves will always show dispersive effects in advance of the point Ut. However, under the assumed initial conditions, only the backward-moving waves are excited, and the group velocity corresponding to the frequency (3.4) will be negative: $C_g' = dn'/dk = -\frac{1}{2}Uk^{-\frac{1}{2}}$. The component waves cannot reinforce each other except in the region $x' = C_g't$, that is, $x = Ut(1 - \frac{1}{2}k^{-\frac{1}{2}})$; therefore, in the region $Ut < x$ we may expect completely negligible contributions from (3.1). To show this formally, we note that I_2 may be evaluated

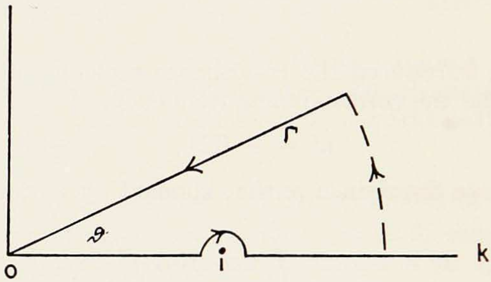

Figure 1. Path for evaluation of integral I_2 when $Ut < x$.

along the path Γ (Fig. 1) and that the angle ϑ is determined by the relative magnitudes of x' and Ut. The integral is thus expressed as a residue term and an asymptotic term:

$$I_2 = -\pi \sin x + 2\, Re \int_0^\infty \exp\{-x' \sin 2\vartheta\, y^2 - Ut \sin\vartheta\, y$$

$$i\,[x' \cos 2\vartheta\, y^2 + Ut \cos\vartheta\, y]\}\,\frac{y dy}{y^2 - e^{-2i\vartheta}}.$$

The asymptotic term is $0(1/x')$ or $0(1/Ut)$, whichever is smaller; since both of these are negligible compared to the amplitude of the lee wave, using (3.2a) we may write for this region

$$\zeta_0 = 0(1/x') \text{ or } 0(1/Ut)\,. \tag{3.8}$$

(*ii*)[2] When $Ut > x$, the dispersive effects of the backward-moving waves first appear. When $\frac{1}{2}Ut \ll x < Ut$, the classical method of

[2] Readers concerned with the result rather than with its derivation may omit this discussion without loss and skip to Section (iii).

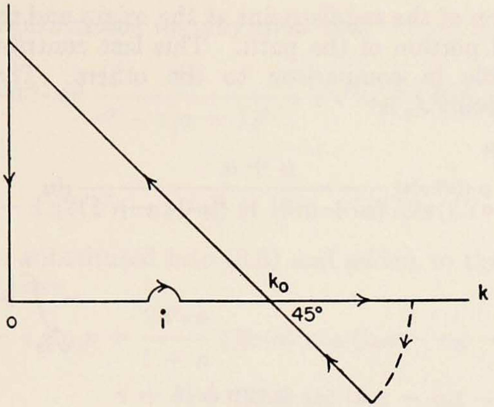

Figure 2. Path for evaluation of integral I_2 when $\frac{1}{2}Ut < x < Ut$.

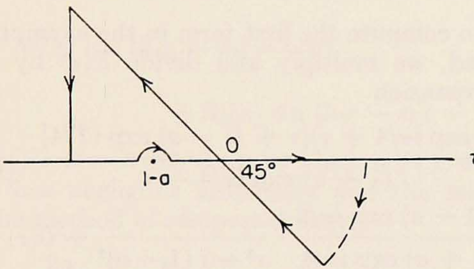

Figure 3. Path for evaluation of integral J when $\frac{1}{2}Ut < x < Ut$.

steepest descent is applicable, but if we are to obtain a formula which has validity also as the time increases and as the saddle-point (3.7) approaches the pole (3.6), we must employ the more general method of Pauli (1938) and Ott (1943). The path is now that in Fig. 2. By the substitution $u^2 = y^2 + Uty/(Ut - x)$, we may write (3.5) as

$$I_2 = 2 \, Re \, e^{i(k_0 x - n_0 t)} \, P \, J ,$$

where

$$J = \int_{-a}^{\infty} e^{-i(Ut-x)u^2} \frac{a + u}{(u + a - 1)(u + a + 1)} du$$

and

$$k_0 = \tfrac{1}{4}(1 - x/Ut)^{-2} , \; n_0 = \tfrac{1}{4}U(2x/Ut - 1)(1 - x/Ut)^{-2} , \qquad a = k_0^{\frac{1}{2}} . \quad (3.9)$$

The path is thereby transformed into that shown in Fig. 3. The integral is thus expressible as the sum of the residue term at $u = 1 - a$, the contribution of the saddle-point at the origin and the contribution of the vertical portion of the path. This last contribution is shown to be negligible in comparison to the others. The contribution of the saddle-point J_s is

$$J_s = \int_{-\sqrt{2}a \exp - i\pi/4}^{\infty \exp - i\pi/4} e^{-i(Ut-x)u^2} \frac{u+a}{(u+a-1)(u+a+1)} du$$

$$= \int_{-\sqrt{2}a}^{\infty} e^{-(Ut-x)\tau^2} F(\tau) d\tau ,$$

where

$$F(\tau) = \frac{a \exp i\pi/4 + \tau}{[(1-a) \exp i\pi/4 + \tau][(1+a) \exp i\pi/4 + \tau]} ,$$

It is sufficient to compute the first term in the asymptotic expansion. Toward this end, we multiply and divide $F(\tau)$ by $\tau^2 - i(1-a)^2$, obtaining the expansion

$$F(\tau) = \frac{(a \exp i\pi/4 + \tau)[\tau + U - a) \exp iT/4]}{(1+a) \exp i\pi/4 + \tau} \cdot \frac{1}{\tau^2 - i(1-a)^2}$$

$$= \frac{a(1-a) \exp i\pi/2}{(1+a) \exp i\pi/4} \cdot \frac{1}{\tau^2 - i(1-a)^2} + 0(\tau) .$$

Thus the first term in the asymptotic expansion is

$$J_s \sim e^{i\pi/4} \frac{a(1-a)}{1+a} \int_{-\sqrt{2}a}^{\infty} e^{-(Ut-x)\tau^2} \frac{d\tau}{\tau^2 - i(1-a)^2} .$$

Since $a[2(Ut-x)]^{\frac{1}{2}}$ is large, we may treat the lower limit of this integral as infinite. This integral can be transformed into an expression containing the Fresnel integrals. However, it is strange but true that these functions which appear so frequently in mathematical physics have never been tabulated in sufficient detail for even such a simple study as the present one. The only applicable tabulation known to the author is that by Rosser (1948) of a related pair of functions, $Rr(y)$ and $Ri(y)$, defined by the integrals

$$Rr(y) = \frac{\sqrt{2}}{\pi} \int_0^{\infty} e^{-\frac{\pi}{2}y^2\tau^2} \frac{d\tau}{1+\tau^4} , \qquad Ri(y) = \frac{\sqrt{2}}{\pi} \int_0^{\infty} \tau^2 e^{-\frac{\pi}{2}y^2\tau^2} \frac{d\tau}{1+\tau^4} .$$

By simple transformations we may show that

$$(1 - a) \int_{-\infty}^{\infty} e^{-(Ut-x)\tau^2} \frac{d\tau}{\tau^2 - i(a - 1)^2} = \pi\sqrt{2}e^{-\frac{i\pi}{2}} \{Rr(\alpha) - i\,Ri(\alpha)\} ,$$

where

$$\alpha = (a - 1)[2(Ut - x)/\pi]^{\frac{1}{2}} = (2x - Ut)/[2\pi(Ut - x)]^{\frac{1}{2}} .$$

If this result is substituted into (3.8) and added to the contribution of the pole, we have

$$I_2 = -\pi \sin x + \frac{2^{3/2}\pi a}{1 + a} \{Rr(\alpha) \cos (k_0x - n_0t - \tfrac{1}{4}\pi) + Ri(\alpha) \sin (k_0x - n_0t - \tfrac{1}{4}\pi) \} .$$

The displacement is therefore

$$\zeta_0 = -\frac{2^{3/2}\pi a}{1 + a} \{ Rr(\alpha) \cos (k_0x - n_0t - \tfrac{1}{4}\pi) + Ri(\alpha) \sin (k_0x - n_0t - \tfrac{1}{4}\pi)\} . \quad (3.10)$$

(*iii*) When $\frac{1}{2}Ut \ll x < Ut$, that is, when $a \gg 1$, $\alpha \gg 1$, the sine wave of (3.10) has negligible amplitude and the result[3] is exactly that given by the method of steepest descent,

$$\zeta_0 \doteq \frac{2a}{a^2 - 1}\left[\frac{\pi}{Ut - x}\right]^{\frac{1}{2}} \cos (k_0x - n_0t - \pi/4) . \quad (3.11)$$

These waves begin at x when $Ut = x$ with small wave lengths, small amplitudes, and with phase speed almost equal to the current speed. As time increases at a point they grow in amplitude and wave length, but diminish in phase speed.

(*iv*) As $Ut \to 2x$ and $\alpha \to 0$, the quasi resonance effect discovered by Höiland (1951) augments the amplitude of the sine wave. The result[3] is $\zeta_0 \doteq -\pi \sin (k_0x - n_0t)$.

At the moment $Ut = 2x$, that is, when $a = 1$, $k_0 = 1$, $n_0 = 0$, $\alpha = 0$, we have

$$\zeta_0 = -\pi \sin x . \quad (3.12)$$

[3] The results of (iii) and (iv) may be verified mathematically by substitution into (3.10) of the values of the Rosser functions for large and small arguments, respectively.

$Rr(y) = 1/\pi y - 0(y^{-5})$ $\qquad Rr(y) = \frac{1}{2}\cos \frac{1}{2}\pi y^2 - \frac{1}{2}\sin \frac{1}{2}\pi y^2 + 0(y^3)$

$Ri(y) = 0(y^{-3})$ $\qquad Ri(y) = \frac{1}{2}\cos \frac{1}{2}\pi y^2 + \frac{1}{2}\sin \frac{1}{2}\pi y^2 + 0(y)$

There are no propagating waves at x at this moment, and through this window we can see the stationary lee wave, now developed to one-half its final amplitude. Interpreted differently, the situation at this moment is a result of the fact that the group velocity of the stationary wave ($k_0 = 1$) is exactly $\frac{1}{2}U$.

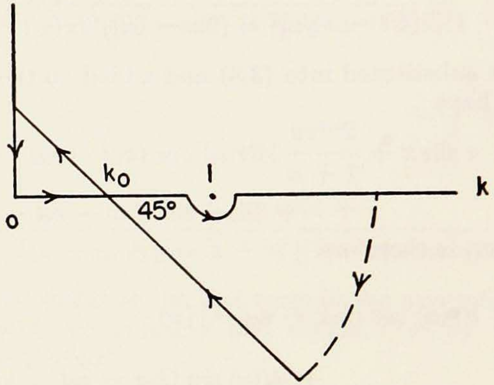

Figure 4. Path for evaluation of integral I_2 when $x < \frac{1}{2}Ut$.

(v) When $0 < x \ll \frac{1}{2}Ut$, we may apply the analysis of section (ii) to the path shown in Fig. 4. The residue term is now of the opposite sign, and since $a < 1$, we have $[(a-1)^2]^{\frac{1}{2}} = 1 - a$. The equation for the displacement corresponding to (3.10) is therefore

$$\zeta_0 = -2\pi \sin x + \frac{2^{3/2}\pi a}{1+a}\{Rr(\alpha)\cos(k_0x - n_0t - \tfrac{1}{2}\pi) + Ri(\alpha)\sin(k_0x - n_0t - \tfrac{1}{2}\pi)\}, \quad (3.13)$$

where now

$$\alpha = (Ut - 2x)/[2n(Ut - x)]^{\frac{1}{2}}.$$

As the stationary lee wave grows in amplitude, the propagating waves die out. The sine wave rapidly becomes negligible in amplitude, leaving the solution

$$\zeta_0 = -2\pi \sin x - \frac{2a}{1-a^2}\left[\frac{\pi}{Ut - x}\right]^{\frac{1}{2}} \cos(k_0x - n_0t - \tfrac{1}{4}\pi). \quad (3.14)$$

This solution remains valid for all subsequent times, but it is seen that the stationary wave becomes rapidly established as the only sensible disturbance. For example, by the time $Ut = 4x$, $a = \frac{2}{3}$, the ratio of the amplitude of the propagating wave to that of the stationary wave is less than $0.4x^{-\frac{1}{2}}$. Thus if x is one stationary wave length from the origin (2π non-dimensional units), this ratio is about 0.16, and the lee wave is sensibly established.

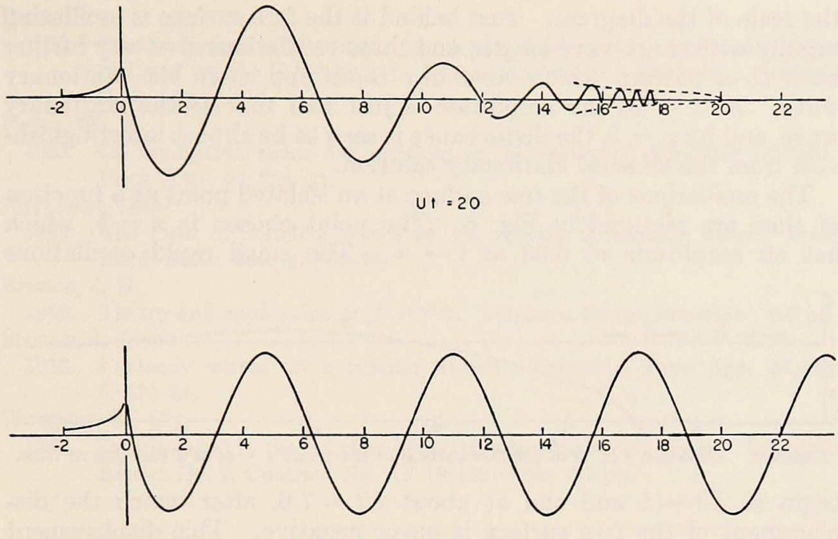

Figure 5. Form of the free surface when $Ut = 20$, with Rayleigh's classical stationary solution for comparison.

(*vi*) When $x < 0$, the analysis of section (v) remains valid. However, in this region the residue term of (3.14) has a different sign whereas that of the stationary integral I_1 in (3.2a) does not. These terms cancel, leaving only the transient term

$$\zeta_0 = -\frac{2a}{1-a^2}\left[\frac{\pi}{Ut-x}\right]^{\frac{1}{2}} \cos\,(k_0x - n_0t - \tfrac{1}{4}\pi)\,. \qquad (3.15)$$

The very long waves have sufficiently high negative group velocities to show dispersive effects in this region, but the group velocity of the stationary wave is always positive. Thus, in this initial-value model, the upstream stationary wave occasions no difficulties because there is no mechanism for its generation. Further, since $a < \frac{1}{2}$ and $Ut - x > Ut$, the amplitudes of the propagating waves are negligible in comparison with the amplitude of the lee wave.

4. *An example computed.* By use of formulas (3.10) and (3.13) a straightforward computation of the elevation of the free surface at any time is possible. Fig. 5 (still in non-dimensional units) shows the system at the moment $Ut = 20$, with the stationary (Rayleigh) solution for comparison. The particle with co-ordinate 20 is therefore the one which was at the origin when the system began from rest. Ahead of it the disturbance exists though it is negligible on

the scale of the diagram. Just behind it the free surface is oscillating rapidly with short wave length, and these oscillations give way further back to a pattern which resembles more and more the stationary wave. At $x = 10$ the amplitude is just half that of the stationary wave, and by $x = 5$, the disturbance is seen to be almost indistinguishable from the classical stationary solution.

The oscillations of the free surface at an isolated point as a function of time are pictured in Fig. 6. The point chosen is $x = 5$, which has an amplitude of 6.03 as $t \rightarrow \infty$. The small rapid oscillations

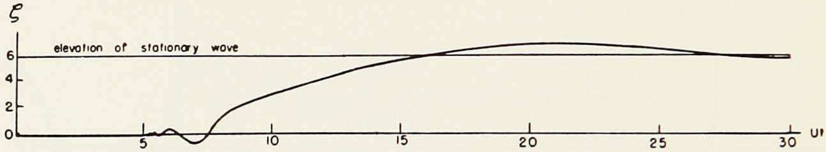

Figure 6. Elevation ζ (5, t) of the free surface at the point $x = 5$, as a function of time.

begin at $Ut = 5$ and end at about $Ut = 7.6$, after which the displacement of the free surface is never negative. This displacement increases steadily until it reaches the stationary value, which occurs at $Ut = 16$. Thereafter, it oscillates about this stationary value with period approaching 8π, but after about $Ut = 20$ it is always within 10% of this value.

NOTE: Since the above paper was presented, two articles, Palm (1953) and Stoker (1953), have appeared on the problem of flow over a corrugated bed. Their models have the advantage of resembling more closely actual flow over an obstacle, but their greater complexity makes it difficult or impossible to examine the transient state of the lee wave in the detail afforded by the model of this paper with a surface pressure disturbance.

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