STEADY FLOW IN A FRICTIONLESS HOMOGENEOUS OCEAN[1]

By

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ABSTRACT

A mathematical model is developed to study the free (frictionless) steady horizontal flow which can occur in a homogeneous ocean of constant depth. The flow satisfies the dynamic constraint that the vertical component of absolute vorticity is constant along a streamline. The conclusion is reached that in an enclosed ocean a free steady circulation cannot have any slow broad eastward currents. The eastward currents must occur as narrow streams of high velocity and high relative vorticity. Intensified currents are present along the eastern and western coasts.

The theory which is developed for the homogeneous ocean of constant depth can be applied to the two-layer ocean if the horizontal divergence of flow is negligible. If the horizontal divergence is not negligible, then the intensification of poleward currents is more pronounced and that of equatorward currents less pronounced as compared with the homogeneous ocean.

INTRODUCTION

The theoretical models of the wind-induced oceanic circulation developed by Stommel (1948), Munk (1950), and others have contributed a great deal to the understanding of some large-scale features of the circulation in the major oceans, but these models are limited by the fact that they are based on linearized equations and on the artificial concept of eddy viscosity. It is possible that significant features of oceanic circulation cannot be simulated by a model in which the nonlinear terms, i. e., the relative-acceleration terms, are neglected. Attempts to improve and extend these models are hindered by the analytical complexity of a model which includes both relative accelerations and dissipative effects.

In the present paper, simple types of ideal fluid motion are studied in order to describe characteristic features of steady free circulation in an enclosed ocean. By considering an ideal fluid, the analytical difficulties presented by the nonlinear terms in the hydrodynamic equations

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are avoided. It is possible to show that some features of oceanic circulation, e.g., the westward intensification of the circulation, exist in the absence of friction.

The simplest model that includes accelerations in the flow is a frictionless homogeneous constant-depth ocean in which the water moves horizontally in the absence of driving forces. This model does not contain any mechanism by which free flow can be established from a state of rest, but it does contain certain dynamic restrictions which must be satisfied by an existing free flow. For steady flow, the most important constraint is that the vertical component of absolute vorticity is constant along a streamline. This constraint on the absolute vorticity is stringent, because the change of the Coriolis parameter over a few degrees of latitude is, over most of the real ocean, about two orders of magnitude greater than the relative vorticity of the flow. It is apparent, therefore, that the possible types of free flow are severely restricted in the ocean.

Rossby (1936) has considered the dynamic constraint on the vertical component of absolute vorticity in his study of the Gulf Stream, but his analysis did not extend to the entire ocean. In the study of free atmospheric flow, there has been more progress and some attempts have been made to adapt the results of these studies to circulation in the oceans. Høiland (1950) found particular examples of free flow which he applied to the ocean. Although these examples agree qualitatively with the free flows discussed in the present paper, Høiland fails to point out the pronounced characteristics that the free flow would exhibit.

ASSUMPTIONS AND NOTATION

The ocean model chosen in this analysis consists of a homogeneous body of water of constant depth bounded by smooth vertical lateral boundaries. The motion is assumed to be horizontal and uniform with depth. The effects of thermohaline structure, compressibility, and frictional forces are neglected everywhere. Vertical motions and accelerations are also neglected so that the pressure is simply determined by the hydrostatic equation.

The unit vector $\mathbf{k}$ is directed vertically upward, and z is the vertical distance measured upward from a level undisturbed ocean surface. The ocean surface in the model is at $z = \eta$. The horizontal velocity is designated by $\mathbf{v}$ and the stream function of the horizontal circulation by ψ. The vertical component of absolute vorticity ζ_a equals $f + \zeta$, where f is the Coriolis parameter and ζ is the vertical component of relative vorticity. The quantity Q is introduced for $p/\rho + \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) + gz$, where p is pressure, ρ density, and g acceleration of gravity.

EQUATIONS OF MOTION

The equations of motion, consistent with the assumptions made above, may be written

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} + f(\mathbf{k}\times\mathbf{v}) = -\nabla p/\rho\,, \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0\,, \tag{2}$$

$$p = \rho g(\eta - z)\,, \tag{3}$$

where the differential vector operator ∇ refers to the horizontal coordinates only.

Equation (1) may be written $\zeta_a(\mathbf{k}\times\mathbf{v}) = -\nabla Q$ or

$$\zeta_a\mathbf{v} = -\nabla\times(Q\mathbf{k})\,, \tag{4}$$

where

$$Q = g\eta + \tfrac{1}{2}(\mathbf{v}\cdot\mathbf{v})\,. \tag{5}$$

The velocity stream function ψ is defined by the equation

$$\mathbf{v} = \nabla\times(\psi\mathbf{k})\,. \tag{6}$$

Elimination of $\mathbf{v}$ from equation (4) yields

$$\zeta_a\nabla\psi = -\nabla Q\,, \tag{7}$$

so that

$$Q = Q(\psi)\,, \tag{8}$$

$$\zeta_a = -dQ/d\psi\,. \tag{9}$$

Equation (9) is a statement of the well known result that the vertical component of absolute vorticity is constant along a streamline. This equation expresses the basic dynamic constraint to be satisfied by the steady free motion in a homogeneous ocean.

DIMENSIONAL ANALYSIS

The following dimensional constants are chosen as being representative of a typical ocean:

L_0 = 2000 km, a typical horizontal length;
f_0 = 10^{-4}, a typical value of the Coriolis parameter,
η_0 = 100 cm, a typical displacement of the free surface from the undisturbed level $z = 0$;
g = 10^3 cm sec^{-2}, gravity.

The equations are converted into nondimensional form by introducing the nondimensional variables $\mathbf{v}' = \mathbf{v}/U$, where U is a characteristic velocity, $\psi' = \psi/UL_0$, $Q' = Q/g_0$, $\zeta_a' = \zeta_a/f_0$, $f' = f/f'_0$, $\zeta' = \zeta/(U/L_0)$, and $\nabla' = L_0\nabla$, the nondimensional differential operator.

By choosing $U = g\eta_0/f_0L_0 = 5$ cm sec^{-1} and defining $\delta = U/f_0L_0 = 2.5 \times 10^{-4}$, the nondimensional form of (6), (8), and (9) may be written

$$\mathbf{v}' = \nabla' \times (\psi'\mathbf{k}) , \tag{10}$$

$$Q' = Q'(\psi') = \eta' + \tfrac{1}{2}\,\delta(\mathbf{v}'\cdot\mathbf{v}') , \tag{11}$$

$$\zeta_a' = -\,dQ'/d\psi' = f' + \delta\zeta' = f' - \delta\nabla'^2\psi' . \tag{12}$$

THE VORTICITY EQUATION

The vorticity equation (12) contains two unknown functions, ψ' and ζ_a', whereas (11) contains three unknown functions, ψ', Q', and η'. Therefore the motion is restricted only by (12), because η' can always be chosen to satisfy (11) once ψ' and ζ_a' are determined. Equation (12) does not determine the motion uniquely, because the vertical component of absolute vorticity can be any function of the stream function.

It is evident from (12) that, unless relative vorticities comparable with the Coriolis parameter (i.e., $\zeta' \sim 1/\delta$) are present in the flow, the stream function will be a function of f' only and the flow will be along latitude circles only. It follows, therefore, that in the vicinity of meridional boundaries where the flow must be across latitude circles, the relative vorticity and hence the velocity must be much larger than the average for the ocean. It is not possible to have in an enclosed ocean a steady free circulation which does not contain relatively large velocities unless the circulation is confined to a narrow range of latitude. For certain choices of $\zeta_a'(\psi')$, the circulation cannot be slow, e.g., $\zeta_a' =$ constant. In order to obtain a steady free circulation that is slow over most of the ocean, restrictions must be imposed upon the choice of $\zeta_a'(\psi')$. The nature of these restrictions is illustrated by considering a simple example.

A SIMPLE EXAMPLE OF FREE FLOW

A rectangular nondimensional coordinate system x', y' is introduced with the positive x'-axis eastward and the positive y'-axis northward. A rectangular region with sides a', b' is chosen to represent the ocean. The function $\zeta_a'(\psi')$ is assumed to be a linear function of ψ', so that (12) becomes

$$f' - \delta\nabla'^2\psi' = c_0 + c_1\psi' , \tag{13}$$

where c_0 and c_1 are constants.

The boundary condition on ψ' is that the ocean boundary is a streamline. This condition is satisfied by requiring the stream function to be zero along the ocean boundary.

The Coriolis parameter is assumed to be a linear function of y' of the form $f' = f_0' + \beta y'$.

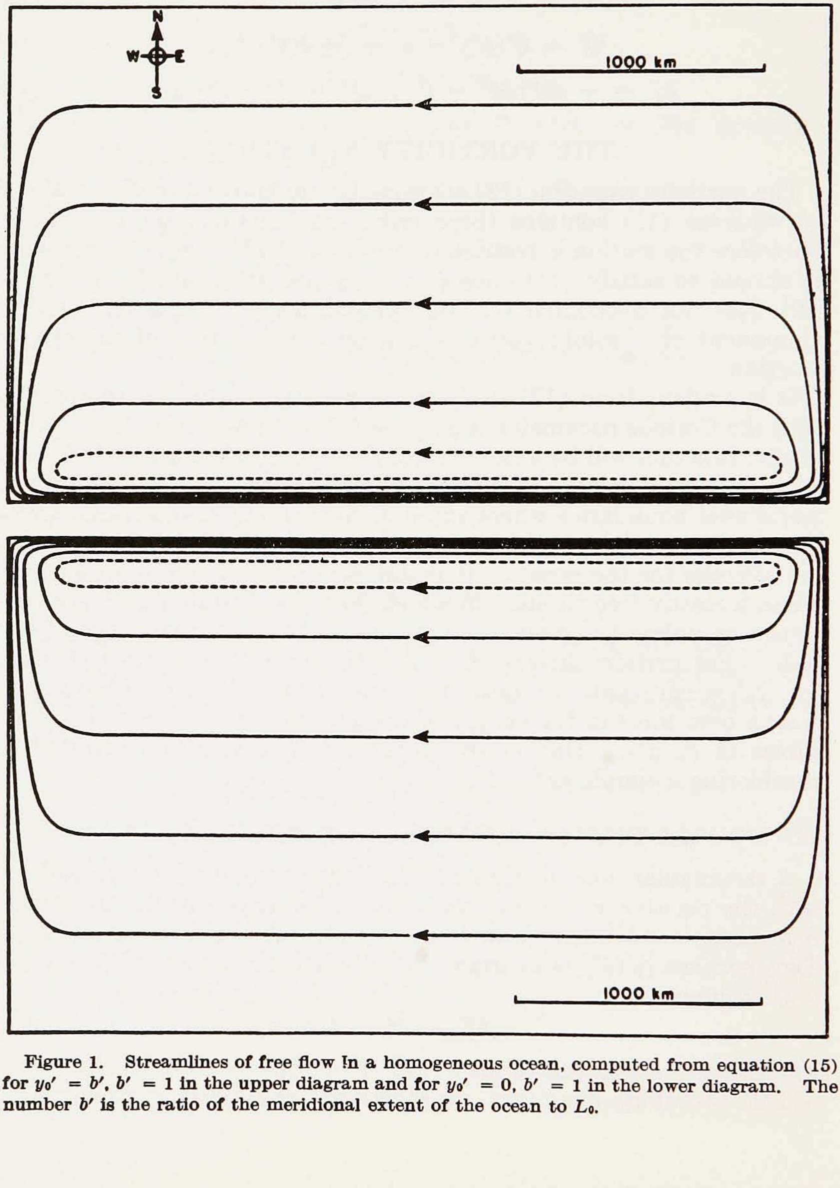

Figure 1. Streamlines of free flow in a homogeneous ocean, computed from equation (15) for $y_0' = b'$, $b' = 1$ in the upper diagram and for $y_0' = 0$, $b' = 1$ in the lower diagram. The number b' is the ratio of the meridional extent of the ocean to L_0.

By defining u_0' and y_0' so that $u_0' = \beta/c_1$ and $y_0' = (c_0 - f_0')/c_1$, equation (13) may be written

$$(u_0'\delta/\beta)\nabla'^2\psi' + \psi' = u_0'(y' - y_0') . \tag{14}$$

In equation (14), u_0' is a nondimensional velocity, δ a ratio of relative vorticity to the Coriolis parameter, and β the nondimensional northward variation of the Coriolis parameter.

Wherever the relative vorticity is negligible, the velocity is given by u_0'; eastward if u_0' is positive and westward if negative. A study of (14) reveals (a) that the relative vorticity in an enclosed ocean can be negligible only if u_0' is negative and (b) that all eastward currents

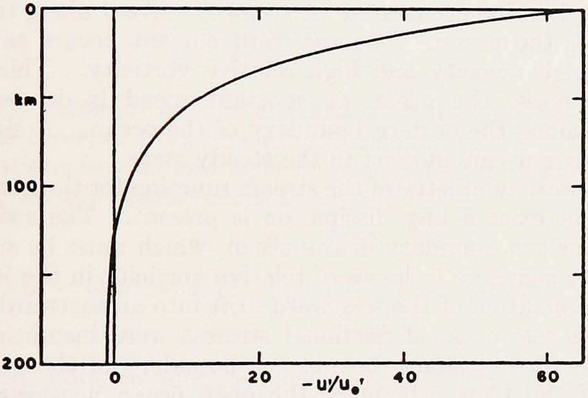

Figure 2. Velocity distribution across the eastward jet, computed for $y_0' = 0$, $b' = 1$, and $L_0 = 2000$ km. The number b' is the ratio of the meridional extent of the ocean to L_0.

must occur as narrow streams of high velocity and high relative vorticity.

Although the exact solution of (14) can be obtained, the simpler approximate solution obtained by boundary-layer methods is more convenient. The boundary-layer solution may be written

$$\psi' = u_0'\{(y' - y_0') + y_0'\exp(- y'/\varepsilon) - (b' - y_0')\exp[- (b' - y')/\varepsilon]\} \times \{1 - \exp(- x'/\varepsilon) - \exp[- (a' - x')/\varepsilon]\} , \tag{15}$$

where $\varepsilon = (|u_0'\delta/\beta|)^{\frac{1}{2}}$. This solution is valid so long as $b'/\varepsilon \gg 1$ and $a'/\varepsilon \gg 1$.

In Fig. 1, streamlines of the free flow are shown for $y_0' = b'$, $b' = 1$ in the upper diagram and for $y_0' = 0$, $b' = 1$ in the lower diagram. These diagrams may be joined to give a symmetrical jet along the middle of the ocean, flanked by broad westward currents to the north and south. The velocity distribution across the eastward jet for $y_0' = 0$, $b' = 1$ is computed from the equation

$$u'/u_0' = 1 - (1/\varepsilon) \exp [- (1 - y')/\varepsilon]$$

and is shown in Fig. 2. All the figures are based on computations for which $\varepsilon^2 = 2.5 \times 10^{-4}$.

DISCUSSION

It is apparent from the preceding analysis that the free steady circulation in an enclosed ocean has distinctive features due to the variation of Coriolis parameter with latitude. The circulation, characterized by a slow westward drift over the major portion of the ocean, is accelerated and concentrated into a swift current along the western boundary of the ocean. The eastward current occurs as a narrow stream of high velocity and high relative vorticity. This eastward jet which crosses the ocean at constant speed is decelerated and spread out along the eastern boundary of the ocean. A slow, broad, eastward current cannot exist in the steady state.

The east-west symmetry of the stream function for the homogeneous ocean cannot exist if any dissipation is present. The swift current along the eastern boundary of the ocean, which must be supplied by a jet, is very sensitive to losses of relative vorticity in the jet.

The concentration of the westward drift into an eastward jet would occur in the real ocean if frictional stresses were insufficient to decelerate the current along the western boundary of the ocean. The presence of the Gulf Stream in the open ocean northeast of Cape Hatteras suggests that a considerable portion of the momentum and relative vorticity of the Stream actually reaches the open ocean and is dissipated there.

EXTENSION TO THE TWO-LAYER OCEAN

Most of the real ocean has a more or less clearly defined upper layer which contains the relatively strong motion and which represents the portion of the ocean influenced by wind stress. Below this layer there is a much larger body of water of greater density which shows little motion and little response to the prevailing atmospheric state. This picture of the ocean has often been represented in theoretical considerations, in idealized form, by the two-layer ocean consisting of two homogeneous layers of water of different densities with the motion confined entirely to the upper less dense layer.

The two-layer ocean in its undisturbed state may be completely specified by giving the lateral boundaries, the densities ρ_0 and ρ_1 of the upper and lower layers respectively, and the volume V_0 of the upper layer. In order to adapt the analysis of the homogeneous ocean to the two-layer ocean, the additional quantities η_1, h, ψ, σ, γ are introduced and defined as follows: $z = \eta_1$, the equation for the interface between the two layers; $h = \eta + \eta_1$, the thickness of the upper layer; ψ, the stream function of volume transport; $\sigma = \nabla \cdot \mathbf{v}$, the divergence of the horizontal velocity; $\gamma = \rho_0/(\rho_1 - \rho_0)$.

The pressure in the lower layer is given by

$$p = \rho_0 g(\eta + \eta_1) - \rho_1 g(z + \eta_1) \,. \tag{16}$$

The condition for no motion in the lower layer requires the horizontal pressure gradient to be zero everywhere in the lower layer, so that

$$\begin{aligned} \nabla\eta_1 &= \gamma\nabla\eta \,, \\ \eta_1 &= \gamma\eta + h_0 \,, \\ h &= (1 + \gamma)\eta + h_0 \,. \end{aligned} \tag{17}$$

The arbitrary constant h_0 is determined from the specified volume V_0 of the upper layer, or from an equivalent condition.

The equations for the two-layer ocean, analogous to equations (5), (6), and (9) for the homogeneous ocean, are

$$\mathbf{v}h = \nabla \times (\psi\mathbf{k}) \,, \tag{18}$$

$$Q = Q(\psi) = g(h - h_0)/(1 + \gamma) + \tfrac{1}{2}(\mathbf{v} \cdot \mathbf{v}) \,, \tag{19}$$

$$\zeta_a'/h = -\, dQ/d\psi \,. \tag{20}$$

The corresponding nondimensional equations for the two-layer ocean are formed by introducing the additional nondimensional variables h', ψ', σ', defined as $h' = h/h_0$, $\psi' = \psi/Uh_0L_0$, $\sigma' = \sigma/(U/L_0)$, and the nondimesnional parameter α, defined as $\alpha = (1 + \gamma)\eta_0/h_0$.

The nondimensional equations which form the dynamic constraints for free motion in the two-layer ocean are

$$\zeta_a'/h' = -\, dQ'/d\psi' \,, \tag{21}$$

$$Q'(\psi') = (1/\alpha)(h' - 1) + \tfrac{1}{2}\delta(\mathbf{v}' \cdot \mathbf{v}') \,, \tag{22}$$

where $\zeta_a' = f' + \delta\zeta'$ and $h' = 1 + \alpha\eta'$.

Equations (21) and (22) contain three unknown functions each and must be solved simultaneously. Thus, in the two-layer ocean, (22) is restrictive in contrast with (11) which is not restrictive. As in the homogeneous ocean, the circulation is not uniquely determined, because ζ_a'/h' can be any function of ψ'.

No solutions of the equations for the two-layer ocean have been obtained. The circulation in the upper layer will resemble the circulation in the homogeneous ocean if the divergence σ' can be neglected.

The divergence is small if the change in thickness of the upper layer along a streamline is small compared with the total thickness of the upper layer, i.e., if α is small compared with unity.

In high-velocity regions of the two-layer ocean the thickness of the upper layer is reduced, so that the absolute vorticity is smaller in magnitude than the absolute vorticity in a corresponding region of the homogeneous ocean. For a two-layer ocean located in the northern hemisphere, the relative vorticity must be less than the relative vorticity in the homogeneous ocean for corresponding flows. Therefore, the intensification of a current flowing toward the pole is more pronounced and that of a current flowing toward the equator is less pronounced than the intensification of a corresponding current in the homogeneous ocean.

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