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# FORMULAS FOR SOUND VELOCITY IN SEA WATER<sup>1</sup>

By

ROBERT T. BEYER

*Physics Department, Brown University  
Providence 12, Rhode Island*

## ABSTRACT

Recent measurements by Del Grosso show a discrepancy between computed and observed values of sound velocity in sea water. The basis of Kuwahara's formula has been re-examined. The chief cause of the discrepancy lies in the use of a value for the compressibility of pure water that is too large.

## INTRODUCTION

For nearly 15 years, the formulas presented by Kuwahara (1939), or tables derived from them by Stephenson and Woodsmall (1941), have been employed in this country as a basis for computing the velocity of sound in sea water from bathythermographic measurements of temperature, pressure and salinity. These tables have had widespread application in sonar studies, fathometry and, more recently, in sofar work. Similiar tables developed by Matthews (1939) have been employed in Great Britain for similar purposes.

In recent years there have been reported instances of direct sound velocity measurements in which the experimental values of the velocity exceeded Kuwahara's calculated values by several meters per second. The most significant data of this kind are those reported by Weissler and Del Grosso (1951) and by Del Grosso (1952). These observers reported a consistent error in Kuwahara's values of the order of 3-4 m/sec as well as a slightly different expression for the dependence of the sound velocity upon salinity.

Because of the important rôle played by sound velocity in underwater acoustics, it is desirable to re-examine the basis on which the Kuwahara tables were calculated in an effort to determine what velocity values are the most probable.

## KUWAHARA'S CALCULATION

The sound velocity  $V$  can be expressed in terms of the physical properties of the medium by means of the formula

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$$V = \sqrt{\frac{\gamma}{\rho\beta}}, \quad (1)$$

where  $\gamma$  is the ratio of specific heats,  $\rho$  the mean density of the medium, and  $\beta$  the isothermal compressibility. An idea of the magnitudes of the quantities involved is useful. For sea water of salinity 35 ‰, at 0°C and zero depth (atmospheric pressure), the values of the quantities in (1), computed from Kuwahara's formulas, are:

$$\rho = 1.02813 \text{ gm/cm}^3; \gamma = 1.000386; \beta = 46.57 \times 10^{-12} \text{ cm}^2/\text{dyne}; \\ V = 1445.5 \text{ m/sec.} \quad (2)$$

The discrepancies reported by Del Grosso are of the order of 3 m/sec or 0.2%. One can look for such discrepancies in errors involved in the measurement of  $\gamma$ ,  $\rho$ ,  $\beta$  or in the formulation of (1) itself.

The derivation of (1) rests on two assumptions: (a) that the amplitude of the wave is infinitesimal and (b) that there is no dissipation in the medium. If the analysis of Lamb (1925) is followed, one obtains for the velocity  $V_F$  of a wave of finite amplitude:

$$V_F = V + \xi, \quad (3)$$

where  $\xi$  is the particle velocity; for a sinusoidal wave,  $\xi$  varies from  $+\xi_0$  to  $-\xi_0$ . The value of  $\xi_0$  can be related to the average acoustic power  $P$  transmitted per unit area:

$$P = \frac{1}{2}\rho\xi_0^2V \quad (4)$$

or

$$\xi_0 = \sqrt{\frac{2P}{\rho V}}.$$

For a sound wave of 1 watt/cm<sup>2</sup>, a power considerably in excess of that customarily used for velocity measurements, the data of (2) yield

$$\xi_0 = \sqrt{\frac{2 \times 10^7}{1.028 \times 1.445 \times 10^5}} = 12 \text{ cm/sec.} \quad (5)$$

Thus, the maximum deviation of (3) from the simple expression (1) is smaller by an order of magnitude than the discrepancy observed. Furthermore, in a continuous wave the average value of  $V_F$  will be much closer to  $V$ , since  $\xi$  is alternately positive and negative. Only in measurements on the leading edge of a pulse could variations of the order of (5) be observed. One can therefore neglect errors due to finite amplitude.

The error produced by the neglect of the dispersion due to absorptive processes has been computed by Del Grosso (1952) and were found to be less than 3 cm/sec. Hence such dispersion can be safely neglected. The discrepancies must therefore be accounted for by errors in measurement. Since  $V$  is proportional to  $\gamma^{1/2}$ ,  $\gamma^{-1/2}$ ,  $\beta_{-1}$ , it is evident that the measurement for any one of these factors would have to be in error by 0.4% to be responsible for the discrepancy. The mode of calculation of each of these quantities will now be considered.

*Density.* Kuwahara took his density values for sea water from formulas prepared by Knudsen (1901) and Ekman (1908), whose formulas are quite elaborate and will not be reproduced here since they can be found in Kuwahara's paper.<sup>2</sup> An error of 0.4% in the density would correspond to one of 4 units in the third decimal place. Since measurements are commonly made of  $\rho - 1$  rather than of the absolute magnitude of  $\rho$ , it follows that such an error would amount to 4 parts in 28 for the datum in (2). It is highly unlikely, therefore, that the gross discrepancy can be due to an error in the density measurement.

*Ratio of Specific Heats.* The ratio of specific heats  $\gamma$  is also measured in terms of its deviation from unity, usually from the expression

$$\gamma = \frac{1}{1 + \frac{T}{Jc_p} \left( \frac{\partial v}{\partial p} \right)^2}, \quad (6)$$

where  $T$  = absolute temperature,  $J$  = Joule's equivalent,  $c_p$  = specific heat at constant pressure, and  $v$  = specific volume.

As in the case of density, the same argument regarding the gross error applies here but with greater emphasis, since the deviation of  $\gamma$  from unity would have to be a whole order of magnitude greater than it actually is to account for the difference in the Del Grosso and Kuwahara values.

<sup>2</sup> Attention should be called to the fact that, while the expression for the density of pure water ( $1 + \Sigma_i$ ) is given correctly in that paper, the temperature derivative  $\frac{\partial \Sigma_i}{\partial T}$  is given incorrectly and will lead to numerical values inconsistent with the remainder of Kuwahara's paper.

*Isothermal Compressibility.* In view of the foregoing observations, we are left with isothermal compressibility as the only possible source of appreciable error. In fact, this quantity was estimated by Kuwahara to be the least accurate in his formula. Kuwahara took his values for compressibility from a formula devised by Ekman (1908). Ekman measured not only the isothermal compressibility of sea water both on ocean going trips and in the laboratory but also the compressibility of distilled water. His empirical formula was so constructed as to be consistent with the value of the isothermal compressibility of distilled water at 0°C, 1 atmosphere pressure, as deduced from the measurement of Amagat (1893). Ekman's formula was expressed in terms of the mean compressibility  $\mu$ :

$$10^8 \mu = \frac{4886}{1 + 0.000183 p} - [227 - 28.33 t - 0.55 t^2 + 0.004 t^3] \\ + 10^{-3} p [105.5 + 9.50 t - 0.158 t^2] - 1.5 \times 10^{-6} p^2 t \\ - (\sigma_{S00} - 28) [147.3 - 2.72 t + 0.04 t^2 \\ - 10^{-3} p (32.4 - 0.87 t + 0.02 t^2)] 10^{-1} \quad (7) \\ + (\sigma_{S00} - 28) [4.5 - 0.1 t - 10^{-3} p (1.8 - 0.06 t)] 10^{-2},$$

where  $\mu$  is the mean compressibility in reciprocal bars,  $p$  is the pressure in bars ( $10^6$  dynes/cm<sup>2</sup>) above atmospheric pressure, and  $t$  is the temperature in degrees centigrade. The quantity  $\mu$  is related to the isothermal compressibility by the expression

$$\beta = \frac{\mu + p \frac{\partial \mu}{\partial p}}{1 - \mu p}, \quad (8)$$

where

$$\sigma_{S00} = -0.069 + 1.4708 Cl - 0.001570 Cl^2 - 0.000398 Cl^3,$$

and the chlorinity  $Cl$  is given in terms of the salinity  $S$  (both in ‰) by

$$S = 0.030 + 1.8050 Cl.$$

For a 0.2% error in the sound velocity, the error in  $\beta$  must equal  $(0.004) 46.57 \times 10^{-6} \text{ bar}^{-1} = 0.19 \times 10^{-6} \text{ bar}^{-1}$  (or  $0.19 \times 10^{-12} \text{ cm}^2/\text{dyne}$ ). Such an error would be of the same order of magnitude as that estimated by Ekman for the possible error in his formula.

Thus the accuracy of Kuwahara's formula depends critically upon a measurement of the compressibility of sea water made over 45 years ago and upon a measurement of the compressibility of distilled water made over 60 years ago.

## COMPRESSIBILITY MEASUREMENTS

A search of the literature reveals that no direct measurements of the compressibility of sea water have been made since Ekman. Such a measurement, employing more modern techniques, would appear to be desirable.

The compressibility of distilled water has been measured a number of times, in several different ways, since 1893. Before these results are listed, however, it is well to point out how Ekman deduced from Amagat's data a value of  $\beta$  at  $0^{\circ}\text{C}$ , 1 atmosphere (subsequently called  $\beta_0$ ), for distilled water. Amagat measured the specific volume change of distilled water at pressures varying from 25 to 900 atmospheres, and the relative error in these measurements is greater at the lower pressures where the deviation of the specific volume from unity is slight. Therefore, Ekman employed only those values at 150 atmospheres and higher. From these he deduced an empirical relation for the pressure dependence and then extrapolated to 1 atmosphere. This relationship is embodied in the first two terms of (7).

It is interesting to note that use of Amagat's values at the lower pressures would have yielded a higher compressibility value and would have caused the computed values of  $V$  to be even lower than they are, i.e., the difference between computed and observed values would have been increased. Ekman's selection of the values for  $\beta_0$  was therefore a fortunate one.

Some more recent values of  $\beta_0$  are shown in Table I. Tyrer (1913) measured the adiabatic compressibility in the range 1 to 2 atmospheres; a small correction was necessary to obtain  $\beta_0$ . The value attributed to Bridgmann (1913) was obtained by extrapolating his mean compressibilities to atmospheric pressure (as listed in Dorsey, 1940) for the range one to several thousand atmospheres. While the accuracy of this extrapolation is not great, since only four points on the curve were available, the mean compressibility for each pressure range as measured by Bridgmann was less than that measured by Amagat. Hence one would expect the value for  $\beta_0$  from Bridgmann's data to be less than that of Amagat.

TABLE I. ISOTHERMAL COMPRESSIBILITY  $\beta_0$  OF DISTILLED WATER AT  $0^{\circ}\text{C}$ ,  
ATMOSPHERIC PRESSURE

Observer	Year	$\beta_0$ ( $\text{cm}^2 \text{ dynes} \times 10^{12}$ )
Amagat	1893	51.11
Tyrer	1913	49.56
Bridgmann	1913	50.5

No direct measurements have been made of  $\beta_0$  in recent years. However, a number of observers (Hubbard and Loomis, 1928; Randall, 1932; Lagemann, Gilley and McLeroy, 1953) have deduced its value from measurements of sound velocity (see Table II).

TABLE II. SOUND VELOCITY  $V_0$  AND ISOTHERMAL COMPRESSIBILITY  $\beta_0$   
IN DISTILLED WATER AT 0°C, ATMOSPHERIC PRESSURE

<i>Observer</i>	<i>Year</i>	$V_0$ ( <i>m/sec</i> )	$\beta_0$ ( <i>cm<sup>2</sup>/dynes</i> $\times 10^{12}$ )
Hubbard and Loomis	1928	1407	50.50
Randall	1932	1403.5	50.74
Lagemann, et al.	1953	1408	50.41

From the amount of variation in the values of  $\beta_0$  obtained by the different observers, it appears to be almost fortuitous that the Amagat-Ekman-Kuwahara velocity values agree with experimental results as well as they do. At the same time, the results of the more recent experimental measurements, both direct and indirect, indicate that the isothermal compressibility of pure water is somewhat less than the value given by Amagat. If one assumes perfect accuracy for all terms in the Ekman equation other than those dependent on  $\beta_0$ , the Del Grosso velocity value at 0°C can be used to compute the magnitude of  $\beta_0$ . A value of  $50.92 \times 10^{-12}$  cm<sup>2</sup> dyne is obtained. This number lies midway between Amagat's value and that of Randall.

### SALINITY DEPENDENCE

From the foregoing discussion it is evident that the gross difference between the Del Grosso data and the computed values of Kuwahara is due to a variation in the measured value of  $\beta_0$ . There remains, however, a slight difference in the salinity dependence for the two cases. The sound velocity  $V$  for sea water at 0 C, zero depth, is given (in m/sec) as a function of the salinity  $S$  (in ‰) by the two expressions:

$$V = 1445.5 + 1.307(S-35) - 0.00015(S-35)^2 \text{ (Kuwahara),} \quad (9)$$

$$V = 1448.6 + 1.25(S-35) - 0.000002(S-35)^4 \text{ (Del Grosso).}$$

For most salinity values encountered in the ocean, the third term in both of these expressions can be neglected. Since the differences in the first term have already been discussed, this leaves the second term for consideration. Kuwahara obtained this salinity dependence in the form

$$\frac{\partial V}{\partial S} = \frac{1}{2H^{\frac{1}{2}}} \left( 2 \frac{\partial v}{\partial S} - \frac{v}{H} \frac{\partial H}{\partial S} \right), \quad (10)$$

where

$$H = - \frac{\partial v}{\partial p} - \frac{T}{Jc_p} \left( \frac{\partial v}{\partial T} \right)^2 \quad (11)$$

and

$$\frac{\partial H}{\partial S} - \frac{\partial^2 v}{\partial S \partial p} + \frac{T}{Jc_p} \left\{ \frac{1}{c_p} \frac{\partial c_p}{\partial S} \left( \frac{\partial v}{\partial T} \right)^2 - 2 \frac{\partial v}{\partial T} \frac{\partial^2 v}{\partial S \partial T} \right\}. \quad (12)$$

The values of  $c_p$  used by Kuwahara were taken from an expression deduced by Krümmel (1907) from data on the salinity dependence of  $c_p$  at 17.5°C and on the temperature dependence of  $c_p$  for distilled water. At first glance this would appear to be a possible source of significant error. However, substitution of numerical values into (9) indicates that the contribution of the second term (which contains  $\frac{\partial c_p}{\partial S}$ ) is almost completely negligible. Since  $c_p$  is close to unity in all cases, any error in its absolute magnitude could have only a minute effect on those terms that contain  $1/c_p$ .

We must still examine the possible effect of the use of a smaller value for  $\beta_0$ . In this case  $\frac{\partial v}{\partial p}$  would be correspondingly decreased.

Such a drop would result in a decrease in the value of  $H$ , thereby increasing  $\frac{\partial V}{\partial S}$ . Such a consequence would be just opposite to Del Grosso's findings.

Del Grosso, in his report, suggests that the difference in the salinity dependence in the two cases can be attributed to variations in the determination of salinity. Such variations arise from the fact that the concept of salinity is based on the assumption that the relative proportions of different salts in sea water are constant over the entire ocean. Actually, this constancy fails in waters of low salinity, hence the waters used by Knudsen and others to dilute ocean water samples could have actually altered the salt content in a way not observable from chlorinity-salinity measurements. The fact that other sources of error ( $c_p$ ,  $\beta_0$ ) do not contribute to the difference in salinity dependence supports this line of argument.

## CONCLUSIONS

There appears to be little doubt that the velocity values of Kuwahara are too low by about 3m/sec. With almost equal certainty, the discrepancy can be attributed to a measurement of the isothermal compressibility of pure water that is slightly too high.



Because of the lack of independent data, one cannot make any definite comment on the slight differences in the salinity dependence in the two cases. Further direct measurements of sea water compressibility would be useful here.

Finally, as a practical step, it is strongly urged that the formula of Del Grosso be employed in all future calculations of sound velocity from BT data.

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