STATIC STABILITY PARAMETERS IN OCEANOGRAPHY[1]

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ABSTRACT

The use of static vertical stability criteria in oceanography is discussed and two basic types of such stability parameters are derived. It is shown how these can be determined by using sound velocity as one of the variables. Computation can be simplified by means of a graph.

The concept of static vertical stability in a fluid is a simple one, yet the use of static stability criteria in physical oceanography has been extremely limited. Among the reasons for this situation may be (a) the somewhat haphazard treatment of the subject in the literature and (b) the rather cumbersome computation required for the exact evaluation of the terms involved. This has led to the occasional use of crude approximations to a stability parameter in oceanographic literature, such as the term $10^{-3}\dfrac{d\sigma_t}{dz}$ (Sverdrup, et al., 1942), or the "coefficient of stability," $\dfrac{1}{[\rho]}\dfrac{\partial[\rho]}{dz}$ which appears in Richardson's number (Proudman, 1953).

Since vertical stability is a measure of the restoring force (or the resistance to change) in the vertical direction, it may be expected to have a marked effect on any process involving vertical motion, such as the vertical velocity components of turbulent flow. This fact is generally recognized, and a number of attempts have been made to incorporate a stability term in the vertical eddy coefficients that have been determined theoretically or experimentally for the sea. In view of the continuously increasing attention that oceanographers are devoting to problems of turbulent flow, and because of the evident relationship of vertical stability to nonisotropic turbulence, the present paper is an attempt to remove some of the vagueness that has crept into the concept of stability in physical oceanography.

When a particle of water of unit mass is displaced vertically, it will acquire a vertical acceleration equal to the buoyant force per unit

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mass exerted on it by its environment. This acceleration per unit change of geometrical depth is the definition of vertical stability derived by Hesselberg in the last of three papers on the subject (Hesselberg and Sverdrup, 1915; Hesselberg, 1918, 1929). The physical significance of the term remains unaltered if the acceleration is given per unit change of geopotential or per unit change of pressure. This criterion of stability, an acceleration per unit of displacement, is identical to that used in meteorology (Holmboe, et al., 1945), where it is generally written in terms of temperature gradients.

While the above-mentioned definition, in terms of acceleration, is the one conventionally used, any other continuous function which is a parameter of the resistance to deformation of the fluid column would be equally valid. An example of such a function is frequently encountered in meteorology. When the acceleration form of the stability function is transformed into terms of temperature gradients by means of the perfect gas law and when the constant acceleration of gravity is dropped from the expression, we have a function which no longer has the dimensions of an acceleration but which is, nevertheless, an exact parameter of stability.

An alternative definition of stability, one which also has a clear-cut physical meaning, is the amount of energy required to move a particle of unit mass a unit measure of displacement. In both this definition and the one in terms of acceleration, the stability parameter is a function of the difference between the observed vertical mass distribution and the mass distribution corresponding to neutral equilibrium. Consequently, in both of them, a value of zero corresponds to neutral equilibrium, positive values to stable conditions, and negative values to instability.

Stability could also be defined arbitrarily as the ratio of the observed mass distribution to the mass distribution of neutral equilibrium. While such a function lacks the physical meaning of the more conventional expressions, it possesses the advantage of being nondimensional. In terms of this parameter, neutral stability is represented by a value of one, stable equilibrium by values greater than one, and unstable equilibrium by values less than one.

The following section will give the development of some of the stability parameters mentioned, and it will be shown how the velocity of sound can be used as one of the variables for their evaluation. Conventional symbols are used and their units given: h = depth in meters; G = geopotential in dynamic meters (0.1 m^2/sec^2); p = pressure in decibars (10^5 $dynes/cm^2$); g = acceleration of gravity = 9.81m/sec^2 (assumed constant); T = temperature in °C; S = salinity in ‰;

$\rho = \rho(T,S,p)$ = density in g/cm³; $c = c(T,S,p)$ = velocity of sound in m/sec; subscript $_A$ refers to an adiabatic process.

Let $\frac{d\rho}{dp}, \frac{d\rho}{dG}, \frac{d\rho}{dh}$ be the observed density gradient in a column of water expressed in terms of pressure, geopotential, and geometric depth. Further, let $\left(\frac{d\rho}{dp}\right)_A$ be the density gradient of neutral equilibrium. This is the mass distribution which would exist in an insulated column which has been filled through its open upper end from an infinite reservoir of homogeneous water (T = constant, S = constant, $p = 0$) in a period of time that is sufficiently short to make molecular conductivity negligible. Because water is compressible, $\left(\frac{d\rho}{dp}\right)_A = \frac{\partial\rho}{\partial p} + \frac{\partial\rho}{\partial T}\left(\frac{dT}{dp}\right)_A$. In order to make the adiabatic density gradient comparable to the observed gradient based on other depth measures, the hydrostatic equation can be applied to the above expression. Hence, $\left(\frac{d\rho}{dp}\right)_A = \frac{1}{\rho}\left(\frac{d\rho}{dG}\right)_A = \frac{1}{g\rho}\left(\frac{d\rho}{dh}\right)_A$.

If a unit volume of water is given a vertical displacement corresponding to Δp, the buoyant force on it will be $g\left[\frac{d\rho}{dp} - \left(\frac{d\rho}{dp}\right)_A\right]\Delta p$. Then the corresponding buoyant force on a unit mass will be $\frac{g}{\rho}\left[\frac{d\rho}{dp} - \left(\frac{d\rho}{dp}\right)_A\right]\Delta p$. This latter expression has the dimensions of acceleration irrespective of the measure of vertical displacement used. However, since the force per unit mass is proportional to the increment of displacement, its magnitude is clearly dependent on the units used. For instance, the difference in density between a particle and its environment will not necessarily be the same at a displacement of one meter as it will be at a displacement of one dynamic meter or one decibar. We finally write, force per unit mass per unit pressure equals $\frac{g}{\rho}\left[\frac{d\rho}{dp} - \left(\frac{d\rho}{dp}\right)_A\right]$. Hesselberg's analogous expression in geometric units is his stability term $E'' = \frac{g}{\rho}\frac{\delta\rho}{dh} \equiv \frac{g}{\rho}\left[\frac{d\rho}{dh} - \left(\frac{d\rho}{dh}\right)_A\right]$, in our notation. We thus obtain a set of three stability functions:

$$E_p = \frac{9.81}{\rho}\left[\frac{d\rho}{dp} - \left(\frac{d\rho}{dp}\right)_A\right] \quad \text{in m/sec}^2\text{/dbar,} \tag{1a}$$

(1b) $$E_G = \frac{9.81}{\rho}\left[\frac{d\rho}{dG} - \left(\frac{d\rho}{dG}\right)_A\right] \quad \text{in m/sec}^2\text{/dyn.m},$$

(1c) $$E_h = \frac{9.81}{\rho}\left[\frac{d\rho}{dh} - \left(\frac{d\rho}{dh}\right)_A\right] \quad \text{in m/sec}^2\text{/m},$$

where $E_p = \frac{1}{\rho} E_G = \frac{1.02}{\rho} E_h$.

If we wish to redefine stability in terms of energy per unit mass per unit of displacement, calling this function E^*, it is merely necessary to multiply one-half the E term (in other words, the mean acceleration over the displacement) by the geometric equivalent of the displacement: E_p by $\frac{0.51}{\rho}$ m, E_G by 0.51 m, and E_h by 0.5 m. Thus we have another set of three functions derived from the same basic stability parameter:

(2a) $$E^*_p = \frac{5}{\rho^2}\left[\frac{d\rho}{dp} - \left(\frac{d\rho}{dp}\right)_A\right] \quad \text{in (m/sec)}^2\text{/dbar},$$

(2b) $$E^*_G = \frac{5}{\rho}\left[\frac{d\rho}{dG} - \left(\frac{d\rho}{dG}\right)_A\right] \quad \text{in (m/sec)}^2\text{/dyn.m},$$

(2c) $$E^*_h = \frac{4.9}{\rho}\left[\frac{d\rho}{dh} - \left(\frac{d\rho}{dh}\right)_A\right] \quad \text{in (m/sec)}^2\text{/m}.$$

For purposes of computing the stability E or E^*, the terms in brackets can be written in the following manner:

$$\left[\frac{d\rho}{dp} - \left(\frac{d\rho}{dp}\right)_A\right] = \left[\frac{\partial\rho}{\partial p} + \frac{\partial\rho}{\partial T}\frac{dT}{dp} + \frac{\partial\rho}{\partial S}\frac{dS}{dp}\right] - \left[\frac{\partial\rho}{\partial p} + \frac{\partial\rho}{\partial T}\left(\frac{dT}{dp}\right)_A\right].$$

Combining terms: $$\left[\frac{d\rho}{dp} - \left(\frac{d\rho}{dp}\right)_A\right] = \frac{\partial\rho}{\partial T}\left[\frac{dT}{dp} - \left(\frac{dT}{dp}\right)_A\right] + \frac{\partial\rho}{\partial S}\frac{dS}{dp}.$$

Except for the use of h in place of p, the three terms, $\frac{\partial\rho}{\partial T}$, $\left(\frac{dT}{dp}\right)_A$, and $\frac{\partial\rho}{\partial S}$ have been tabulated by Hesselberg and Sverdrup (1915). These can be combined with the observed temperature and salinity gradients, $\frac{dT}{dh}$ and $\frac{dS}{dh}$, to obtain the stability functions.

A different type of stability parameter, which has already been mentioned and which is represented by the symbol E_R, is the ratio of the observed to the adiabatic density gradient. Then:

(3a) $$E_{R(p)} = \frac{d\rho}{dp} \bigg/ \left(\frac{d\rho}{dp}\right)_A,$$

(3b) $$E_{R(G)} = \frac{d\rho}{dG} \bigg/ \left(\frac{d\rho}{dG}\right)_A,$$

(3c) $$E_{R(h)} = \frac{d\rho}{dh} \bigg/ \left(\frac{d\rho}{dh}\right)_A.$$

It is immediately apparent that $E_{R(p)} = E_{R(G)} = E_{R(h)}$ and that this parameter is therefore independent of the units used in measuring the density gradients.

It has been pointed out by Bjerknes, et al. (1934) that the adiabatic density gradient $\left(\frac{d\rho}{dp}\right)_A$ is identical to the term which appears in the Laplacian expression for the velocity of sound, $c = \sqrt{\left(\frac{dp}{d\rho}\right)_A}$, and can therefore be represented by $1/c^2$. Since we are dealing with only the total change in p, and since, in both cases, $\rho = \rho(p, T)$ with salinity constant, $\left(\frac{d\rho}{dp}\right)_A$ always represents the same characteristic of the fluid.

The stability can now be expressed in the following forms:

(4) $$E_p = \frac{9.81}{\rho}\left[\frac{d\rho}{dp} - \frac{10}{c^2}\right],$$

(5) $$E_p{}^* = \frac{5}{\rho^2}\left[\frac{d\rho}{dp} - \frac{10}{c^2}\right],$$

(6) $$E_R = \frac{c^2}{10}\frac{d\rho}{dp}.$$

The coefficient, 10, of $1/c^2$ is a conversion factor resulting from the use of decibars and m per sec in our stability terms in place of dynes and cm per sec. In the comparable expressions for dynamic meters and geometric meters the term $\frac{10}{c^2}$ is replaced by $\frac{10\rho}{c^2}$ and $\frac{9.81\rho}{c^2}$, respectively.[2]

[2] In 1928 the meteorologist Väisälä apparently derived an expression for stability, designated N^2, which is identical to our E_h when the latter is written in terms of sound velocity. Consequently, N has the dimensions of frequency. Väisälä's expression is mentioned in the unpublished notes of a seminar given by Eckart at Scripps

To test the suitability of equations (4), (5), and (6) for practical applications, it was expedient to use the data from MICHAEL SARS Station No. 44 which had been given as an example of stability computation by Hesselberg and Sverdrup (1915) and which is reproduced in Sverdrup, et al. (1942: table 62). The stability term $10^8 E$, computed by Hesselberg and Sverdrup from their tables, is equivalent to the bracketed term times 10^8 in our expression $E_h = \frac{9.81}{\rho}\left[\frac{d\rho}{dh} - \frac{9.81\,\rho}{c^2}\right]$, or $10^8 E = 10^8 \frac{\rho E_h}{9.81}$.

The stability parameter $\left[\frac{d\rho}{dh} - \frac{9.81\,\rho}{c^2}\right]$ was evaluated as follows. The density, ρ, was determined for atmospheric pressure from Knudsen's (1901) tables for σ_t and was corrected for pressure by means of the tables computed by Ennis (1944). Here it should be pointed out, parenthetically, that the density obtained from Knudsen's σ_t values is actually specific gravity or density in grams per milliliter. Density in grams per cubic centimeter (the units appropriate to dynamic equations) is obtained by dividing $[1 + 10^{-3}\sigma_t]$ by 1.000027 or, approximately, by using the expression $[1 + 10^{-3}(\sigma_t - 0.03)]$. The velocity of sound, c, was determined for atmospheric pressure from Del Grosso's (1952) values and was corrected for depth according to the British Admiralty tables by Matthews (1939). Table I gives the results of this computation, together with the pertinent MICHAEL SARS data.

While the values of $10^8 \frac{\rho E_h}{9.81}$ are not identical with those of $10^8 E$, it is felt that the use of sound velocity in stability computations is validated by these results. The differences between the two tabulated sets of stability parameters can probably be explained on the following grounds.

It will be noted that the largest and least systematic differences occur in the upper 200 m where the depth increments are 50 m or less. With $\frac{\Delta\rho}{\Delta h}$ equal to the mean value of $\frac{d\rho}{dh}$ in the interval Δh, provided ρ is a continuous single-valued function of h, an error of 10^{-5} in $\Delta\rho$

Institution of Oceanography in 1949 and recorded by R. O Reid. Eckart's source is a lecture by Solberg at UCLA in 1946. I have been unable to find the original reference despite a comprehensive search of the meteorological and oceanographic literature.

TABLE I. MICHAEL SARS STATION NO. 44
(28° 37′ N, 19° 08′ W, 28 May 1910)†

Depth (*m*)	*Temp.* (°C)	*Sal.* (°/₀₀)	10^8 *E* (*Hesselberg and Sverdrup*)	$10^8 \frac{\rho E_h}{9.81}$ (*Pollak*)
0	19.2	36.87		
			−440	−434
10	19.31	36.85		
			−150	−167
25	19.34	36.83		
			− 13	− 15
50	19.24	36.79		
			610	604
75	18.65	36.79		
			390	364
100	18.24	36.78		
			38*	64
150	17.50	36.56		
			270	242
200	16.45	36.40		
			160	160
300	14.52	36.02		
			120	127
400	13.08	35.77		
			150	155
500	11.85	35.64		
			130	134
600	10.80	35.54		
			100	107
800	9.09	35.39		
			89	96
1000	8.01	35.37		
			84	86
1200	7.27	35.42		
			48	50
1400	6.40	35.35		
			26*	29
2000	4.52	35.15		
			11	14
3000	2.84	34.92		
			8	11
4000	2.43	34.90		
			1	3
5000	2.49	34.90		

† First four columns from Sverdrup, et. al. (1942: table 62).
* Recomputed by M. J. P. by means of Hesselberg and Sverdrup's tables (1915).

when Δh = 50m will produce an error of 20 in the term $10^8 \frac{\rho E_h}{9.81}$. This is the same order of magnitude as the differences found in Table I. A total error of 10^{-5} in $\Delta\rho$ is easily introduced in the determination of the ρ's from temperature, salinity, and depth. In contrast, when Δh = 200 m, the corresponding error in the stability parameter is only 5. However, the differences are not necessarily due solely to errors in ρ. Values obtained by means of the Hesselberg and Sverdrup tables may also contain small errors as a result of the large intervals in the arguments of those tables and the need for two-way interpolation.

The small systematic differences below 1000 m result from the use of Del Grosso's sound velocity tables in place of Kuwahara's (1939) or Matthews'. Del Grosso's values are based on direct experimental determination and are about 3 m per sec higher than those given in the latter two tables which were derived theoretically from somewhat obsolete and incomplete experimental data on the elastic and thermal properties of sea water (Ekman, 1908; Krümmel, 1907); these latter sets of data are the same ones on which the Hesselberg and Sverdrup stability tables are based. A 3 m per sec increase in sound velocity corresponds to an increase of approximately two units in the term $10^8 \frac{\rho E_h}{9.81}$. However, even with the use of Del Grosso's tables, the present method of stability computation still contains potential sources of error. The pressure corrections for density and sound velocity, taken from Ennis' and Matthews' tables, respectively, are derived from the same data for elastic and thermal properties that are the basis of all the other tables mentioned above. In other words, the only improvement in accuracy which can be claimed for the proposed stability computation stems from the introduction of a more accurate measure of sound velocity under atmospheric pressure. This increased accuracy, however, may be more apparent than real.

Dorsey (1940) raises an interesting point which may have some bearing on the difference between Del Grosso's and Kuwahara's or Matthews' sound velocity tables and therefore on the stability values computed by the above method. In section 34 he states:

> There are no direct determinations of the values of specific heat of water at constant volume, of the ratios of specific heats, or of their differences, but all these can be computed from the observed compressibility, thermal expansion, and specific heat at constant pressure. Values so determined may be called *static* values. They can also be determined from the velocity of sound, in which case they may be described as *dynamic*. Likewise, the increase in temperature

> on adiabatic compression may be determined either statically, from the thermal expansion, or dynamically, from the observed drop in temperature that accompanies a sudden release of pressure.

He then points out that, since water probably contains associated molecules of more than one type and since the internal state of a molecule may be affected by gross dynamic changes in the substance, the static and dynamic values of these thermal properties may be expected to differ under certain conditions.

If Dorsey's assumptions are correct for fresh water, we could expect these effects to be even more pronounced in such a complex ionized solution as sea water. Finally, if we extend his reasoning one step further, we might conclude that sound velocities, which have been derived from the static values of thermal properties, would differ from those obtained by direct measurements. The latter values, as exemplified by Del Grosso's data, correspond to sound velocity as a function of the dynamic values of thermal properties. One implication of these ideas is that the term $\left(\frac{d\rho}{dp}\right)_A$ is not a unique function of adiabatic compressibility but is dependent also on the time element involved in the process.

If the suggested theory is true—and the evidence for this is by no means conclusive—it may be necessary to use computed sound velocity values, rather than measured ones, in stability computations. However, as was pointed out previously, the values of the thermal and elastic properties on which Kuwahara's and Matthews' computations are based are not entirely reliable, and hence there is little basis for assuming that their sound velocity tables would give more accurate stability values than Del Grosso's.

In considering the practicability of the proposed method of stability computations, it should be pointed out that the term $\frac{10}{c^2}$ lends itself to graphical presentation as a function of temperature, salinity, and pressure without first determining c. The range of $\frac{10}{c^2} \times 10^8$ is only 100 units, from about 400 to 500. There is little justification for carrying out these values to one more place, since an error of only 0.00001 g/cm^3 in $\Delta\rho$, when Δp equals 1000 dbar, corresponds to an error of 1 in the term $\frac{d\rho}{dp} \times 10^8$. A simple design for such a graph gives $\frac{10}{c^2}$, i.e., $\left(\frac{d\rho}{dp}\right)_A$ for arguments of temperature and salinity at

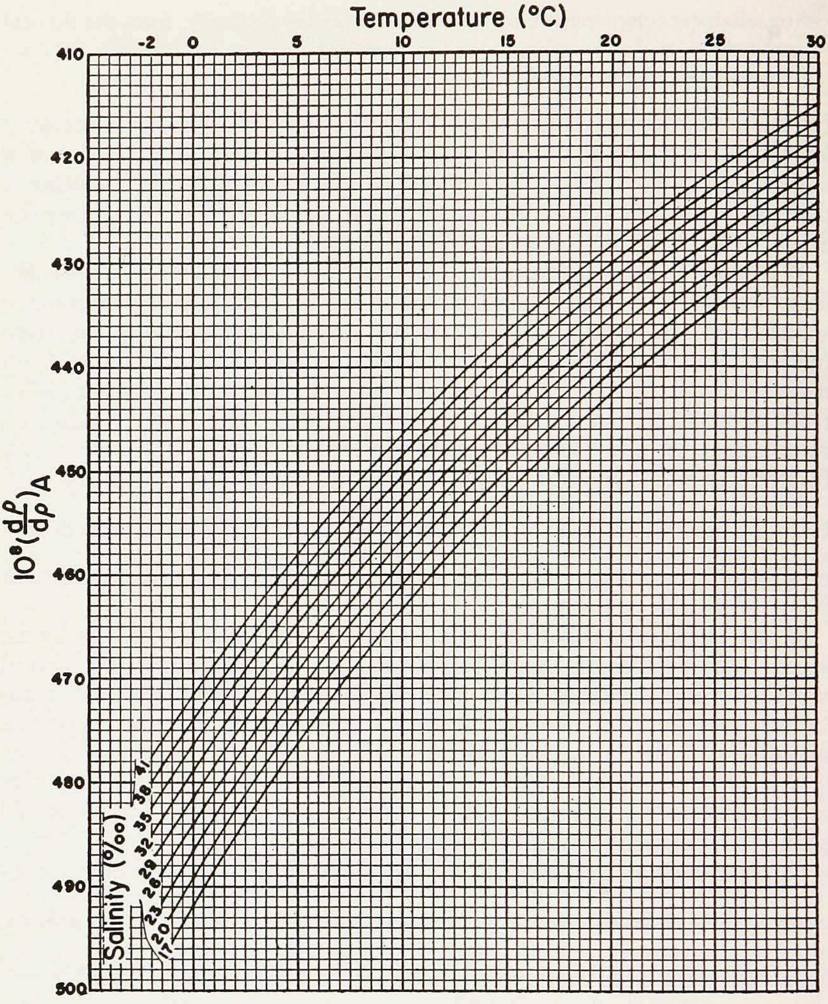

Figure 1. Adiabatic density gradient at atmospheric pressure as a function of temperature and salinity.

atmospheric pressure, with an auxiliary graph for the pressure correction (Figs. 1, 2). Even if the use of sound velocity does not improve the accuracy of stability parameters over that obtainable from Hesselberg's and Sverdrup's tables, the graphical determination of the adiabatic density gradient would probably simplify and speed up the required computations to a considerable extent.

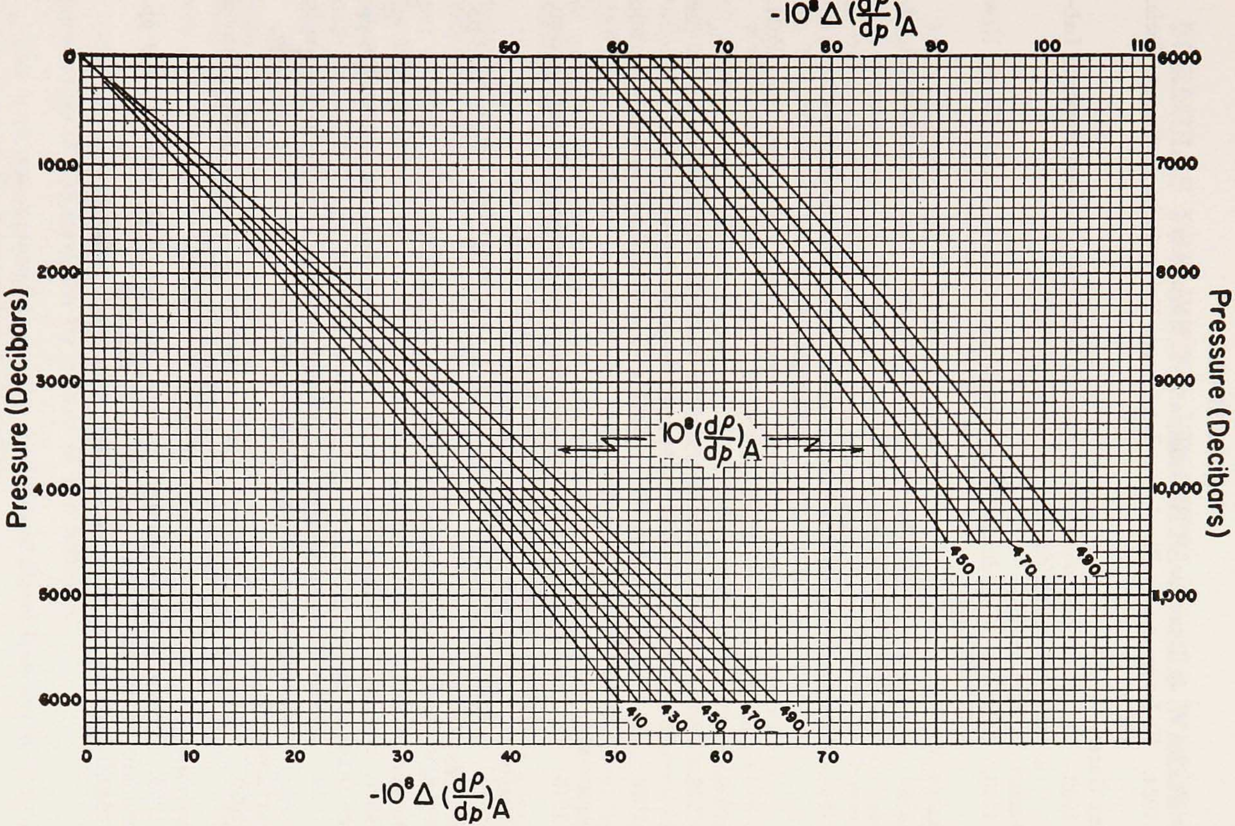

Figure 2. Pressure correction (always subtractive) as a function of pressure and adiabatic density gradient at atmospheric pressure.

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