NOTE ON THE DYNAMICS OF THE GULF STREAM[1]

By

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ABSTRACT

The nonlinear inertial terms have been neglected in Stommel's and in Munk's theory for the wind-driven ocean circulation. Using a method of successive approximations, the effect of these terms on the mass transport in the Gulf Stream region has been computed under greatly simplifying assumptions. These assumptions involve Reid's model of the vertical density structure, which consists of an exponential decrease in the density upward to the thermocline and a homogeneous upper layer. Assuming first a constant depth of thermocline and a north-south boundary, the principal modification is a displacement downstream by about 7° latitude of the region of maximum currents. The displacement is smaller in the case of a coastline whose orientation deviates from a north-south direction. Assuming next a variation in the depth of thermocline across the stream, this asymmetry of the mass transport circulation relative to the wind circulation is reduced. It is also shown that the density model specifies a relation between the vertically integrated transport and the surface transport, according to which the width of the "cold wall" at the surface is only 40% of that of the integrated transport. Two observed features which could not be accounted for by the linear theory, the countercurrent inshore and the continued sharpness of the Gulf Stream's western boundary long after it has left the American east coast, are apparently not contained in the higher order solutions.

INTRODUCTION

The narrow fast currents that flow along the western boundaries of the oceans are among the most striking oceanographic phenomena. The outstanding examples are the Kuroshio current off the coast of Japan and the Gulf Stream system off the American east coast. During the last few years Iselin and Fuglister (1948) and Von Arx (1950) have made detailed surveys of the Gulf Stream with new instruments and techniques, among them continuous temperature and current recorders and radio aids to navigation. These measurements have only served to emphasize the great speed of the Gulf Stream and the sharpness of its western boundary. This sharp definition of the current, which prompted Benjamin Franklin to call it a great river in

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the sea, is perhaps its most striking characteristic. Any adequate theory, and there have been many, must account for it.

Unfortunately there are two theories, both of which explain, though for different reasons, the existence of such intense and well defined currents. The earlier one is Rossby's (1936) "wake stream" or "jet stream" theory, which represents one of his several remarkable attempts to adapt results in fluid mechanics to meteorology and physical oceanography. The Gulf Stream is interpreted as "a current moving under its own momentum and produced by discharging water into the basin through a 'jet'" (the Straits of Florida). The other theory is that of Stommel (1948), which explains the westward intensification of ocean currents as the result of the earth's rotation on the wind-driven ocean circulation. The theory has been extended by Munk (1950), and for reasons discussed in Munk's paper it will be referred to as the "planetary vorticity" theory.

Both theories have attractive features. The wake stream theory accounts for the increase downstream in width and mass transport of the Gulf Stream as well as a countercurrent on the inshore side of the Gulf Stream.[5] The planetary vorticity theory leads to the right order of magnitude for the mass transport of the Gulf Stream, and it accounts for a countercurrent on the offshore side. One is therefore tempted to combine the two theories into a unified dynamic picture of the Gulf Stream.

The fundamental difference appears to be that in Stommel's and in Munk's linear theory the inertial terms in the equations of motion have been neglected whereas in Rossby's theory these terms play a predominant role. Munk estimates the magnitude of these terms *a posteriori* and finds them to be small "except along the inshore edge of the western current," where he expects his conclusions to be modified in the sense prescribed by Rossby. The procedure here will be to consider Munk's solution to the linear theory as a zero approximation and to introduce the inertial terms into the first and second order approximations.

The mathematics soon becomes so involved that we have found it essential to check each step by independent calculations. Only the outline of the procedure is sketched here. We suggest that the reader confine his attention to the discussion of the somewhat meager results from our tedious calculations.

[5] The first mention of a countercurrent in the slope water was made in 1590 by John White, who remarked that in order to stay within the Gulf Stream one had to stand far out to sea, because along the coast there were "eddy currents setting to the south and southwest."

THE EQUATIONS OF MOTION

The notation follows Munk (1950); equations from this paper will be referred to by the letter A. The inertial terms appear on the left sides of equations A1, A4, A6 in the following forms:[6]

$$\rho\mathbf{v}\cdot\nabla\mathbf{v}\,, \quad \int_{-\infty}^{h}\rho\mathbf{v}\cdot\nabla\mathbf{v}dz\,, \quad -\nabla\times\int_{-\infty}^{h}\rho\mathbf{v}\cdot\nabla\mathbf{v}dz\,.$$

In the linear theory it was found possible to write all equations in terms of vertically integrated functions without having to specify the manner in which density varies with depth. This is no longer possible if the quadratic terms are to be considered.

Reid (1948) has introduced a simple model for the density distribution. Let the z-axis be directed upwards from a level surface just beneath the sea surface, and let h designate the elevation of the sea surface and H the distance beneath the level reference surface to which a homogeneous upper layer of density ρ_0 extends. Beneath the homogeneous layer, Reid assumes an exponential distribution

$$\rho = \rho_\infty - \Delta\rho e^{1+z/H},$$

where $\Delta\rho = \rho_\infty - \rho_0$, and ρ_∞ is the density at great depth ($z = -\infty$).

The horizontal pressure gradient vanishes at great depth, i. e. $\int_{-\infty}^{h}\rho\,dz$ = constant, which yields

$$\frac{h}{H} = \frac{2\Delta\rho}{\rho_0}, \tag{1}$$

provided the reference level is properly selected.

The pressure gradient in the x-direction is then given for the upper layer by

$$\frac{\partial p_0}{\partial x} = g\rho_0\frac{\partial h}{\partial x} \tag{2}$$

and for the lower layer by

$$\frac{\partial p}{\partial x} = \frac{1}{2}\,g\rho_0 e^{1+z/H}\,(1 - z/H)\,\frac{\partial h}{\partial x}, \tag{3}$$

where use has been made of (1). Similar expressions are valid for $\partial p/\partial y$. According to the geostrophic law, the pressure gradients are proportional to the horizontal velocity components. If $\mathbf{v}_0$ denotes the velocity vector in the upper layer, then

[6] Here h and $-\infty$ replace z_0 and $-h$, respectively, in the original notation, without any essential change in meaning.

$$\frac{1}{2}\,\mathbf{v}_0\, e^{1+z/H}(1 - z/H)$$

is the corresponding velocity at depths beneath the homogeneous layer. It follows that

$$\int_{-\infty}^{h} \bar{\rho}\mathbf{v} \cdot \nabla\mathbf{v}dz = \frac{29}{16}\,\bar{\rho}H\,(\mathbf{v}_0 \cdot \nabla)\,\mathbf{v}_0 + \frac{29}{32}\,\bar{\rho}\mathbf{v}_0\,(\mathbf{v}_0 \cdot \nabla)\,H\,.$$

The second term is negligible since H is nearly constant along a stream line. The first term can be combined with

$$\boldsymbol{M} \equiv \int_{-\infty}^{h} \rho\mathbf{v}dz = \frac{5}{2}\,\bar{\rho}H\mathbf{v}_0 \qquad (4)$$

to yield

$$\int_{-\infty}^{h} \bar{\rho}\mathbf{v}\cdot\nabla\mathbf{v}dz = c\boldsymbol{M}\cdot\nabla\boldsymbol{M}\,, \qquad (5)$$

where

$$c = 29/100\bar{\rho}H\,, \qquad (6)$$

provided variations in density are small compared to the mean density $\bar{\rho}$.

The equation of mass transport A6 then becomes

$$\left(A\nabla^4 - \beta\frac{\partial}{\partial x}\right)\psi + \mathrm{curl}_z\,\boldsymbol{\tau} = c\,\mathrm{curl}_z\,(\boldsymbol{M} \cdot \nabla\boldsymbol{M}) = c\,(\boldsymbol{M} \cdot \nabla)\,\nabla^2\psi\,, \qquad (7)$$

where

$$\boldsymbol{M} = \boldsymbol{k} \times \nabla\psi$$

and curl_z designates the vertical component of the vector operator. The term on the right side of equation (7) is essentially the advection of vorticity and represents the inertial effect.

For a wind system we set

$$\tau_x = -\,\Gamma\cos ny\,, \quad \tau_y = 0\,,$$

giving maximum easterlies at $y = 0$ and maximum westerlies at $y = \pi/n$.

THE CASE OF NORTH-SOUTH BOUNDARIES AND CONSTANT DEPTH OF THERMOCLINE

For this case c is a constant. It is convenient to adopt the following nondimensional co-ordinate system:

$$x' = kx \qquad \psi' = \frac{\beta}{\Gamma n r}\psi$$
$$y' = ky \qquad M' = \frac{\beta}{k\Gamma n r} M \tag{8}$$
$$r' = kr$$
$$y'' = ny \qquad \lambda = \frac{n^4 r \Gamma c}{4\beta^2\gamma^7},$$

where

$$k^3 = \beta/A \,, \quad \gamma = n/k \,. \tag{9}$$

It follows that

$$\frac{\partial}{\partial x} = k\frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial y} = k\frac{\partial}{\partial y'}, \qquad \nabla^4 = k^4\nabla'^4, \tag{10}$$

so that the differential equation becomes

$$\left(\nabla'^4 - \frac{\partial}{\partial x'}\right)\psi' = \frac{1}{r'}\sin y'' + 4\gamma^{-1}\lambda\,\mathrm{curl}_z\,(\boldsymbol{M}'\cdot\nabla'\boldsymbol{M}')\,, \tag{11}$$
$$\boldsymbol{M}' = \boldsymbol{k}\times\nabla'\psi'\,. \tag{12}$$

The boundary conditions are

$$\psi' = 0\,, \qquad \frac{\partial\psi'}{\partial x'} = 0\,, \tag{13}$$

at the western boundary ($x' = 0$) and eastern boundary ($x' = r'$); the latter can be replaced by $x' = \infty$ as long as we confine our attention to conditions near the western boundary.

Successive Approximations. The first term on the right-hand side of equation (11) represents the driving force of the wind, the second term incorporates the effect of the neglected inertial terms. The relative importance of these inertial terms depends on the value of λ, and it seems reasonable to expand the solution in the following power series:

$$\psi' = \psi_0' + \lambda\psi_1' + \lambda^2\psi_2' + \ldots \tag{14}$$

We shall show later that the value of λ is about 0.3 so that the series converges fairly rapidly. Substituting (14) into equations (11) and (12) and equating equal powers of λ gives

$$\left(\nabla'^4 - \frac{\partial}{\partial x'}\right)\psi_0' = \frac{1}{r'}\sin y''\,, \tag{11.0}$$

$$\left(\nabla'^4 - \frac{\partial}{\partial x'}\right)\psi_1' = 4\gamma^{-1}\,\mathrm{curl}_z\,(\boldsymbol{M}_0'\cdot\nabla'\boldsymbol{M}_0')\,, \tag{11.1}$$

$$\left(\nabla'^4 - \frac{\partial}{\partial x'}\right)\psi_2' = 4\gamma^{-1}\,\text{curl}_z\,(\boldsymbol{M}_0' \cdot \nabla'\boldsymbol{M}_1' + \boldsymbol{M}_1' \cdot \nabla'\boldsymbol{M}_0')\,, \tag{11.2}$$

and so forth. Equation (12) holds for each order, that is,

$$\boldsymbol{M}_i' = \boldsymbol{k} \times \nabla'\psi_i' \qquad i = 0, 1, 2, \ldots \tag{12i}$$

The Zero Order. To the zero approximation the inertial terms are neglected and the solution to (11.0) becomes (Munk and Carrier, 1950)

$$\psi_0' = X_0 \sin y'', \tag{15.0}$$

where

$$X_0 = 1 + \frac{i\sqrt{3}\omega}{3}\,e^{\omega^2 x'} - \frac{i\sqrt{3}\omega^2}{3}\,e^{\omega x'} - \frac{x'}{r'}\,, \tag{16.0}$$

and

$$\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}\,, \qquad \omega^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}\,, \qquad \omega^3 = 1$$

are the cube roots of one. The last term in (16.0) plays a negligible role in the western boundary zone of a wide ocean.

The First Order. The computation is very cumbersome and hence only the major steps will be outlined here. The right side of equation (11.1) is now a known function of x', since it can be found from equations (15.0) and (16.0). It reduces to the form

$$2F_1 \sin 2y'', \tag{17.1}$$

where

$$F_1 = \frac{\partial X_0}{\partial x'}\,\frac{\partial^2 X_0}{\partial x'^2} - X_0\,\frac{\partial^3 X_0}{\partial x'^3} \tag{18.1}$$

$$= -\,e^{-x'} - \frac{i\sqrt{3}\omega}{3}\,e^{\omega^2 x'} + \frac{i\sqrt{3}\omega^2}{3}\,e^{\omega x'}\,. \tag{19.1}$$

The particular and homogeneous integrals of (11.1) are respectively

$$\psi_1'^{p} = \left[-e^{-x'} - \frac{2i\sqrt{3}\omega}{9}\,x'e^{\omega^2 x'} + \frac{2i\sqrt{3}\omega^2}{9}\,x'e^{\omega x'}\right]\sin 2y''\,, \tag{20.1}$$

$$\psi_1'^{h} = [a_1 e^{\omega^2 x'} + \bar{a}_1 e^{\omega x'}]\sin 2y''. \tag{21.1}$$

The constants a_1 and $\bar{a}_1$ are determined by subjecting the complete solution $\psi_1' = \psi_1'^{p} + \psi_1'^{h}$ to the boundary conditions (13):

$$a_1 = \frac{1}{2} - \frac{7i\sqrt{3}}{18}\,, \qquad \bar{a}_1 = \frac{1}{2} + \frac{7i\sqrt{3}}{18}\,. \tag{22.1}$$

The solution is

$$\psi_1' = X_1 \sin 2y'', \tag{23.1}$$

$$X_1 = -e^{-x'} + (m_1 + in_1)e^{\omega^2 x'} + (m_1 - in_1)e^{\omega x'}, \tag{24.1}$$

where

$$m_1 = \frac{1}{2} + \frac{1}{3}x', \qquad n_1 = -\frac{7\sqrt{3}}{18} + \frac{\sqrt{3}}{9}x'.$$

The Second Order. Substituting the first order solution (23.1) into the right side of the second order equation (11.2) gives

$$(2F_2 - 4G_2)\sin y'' + (2F_2 + 4G_2)\sin 3y'', \tag{17.2}$$

where

$$F_2 = \frac{\partial^2 X_0}{\partial x'^2}\frac{\partial X_1}{\partial x'} - X_0\frac{\partial^3 X_1}{\partial x'^3} \tag{18.2a}$$

$$\left.\begin{aligned} &= \left(\frac{2}{3}x' - \frac{19}{9}\right)e^{-x'} - (m_2 - in_2)e^{\omega^2 x'} - (m_2 + in_2)e^{\omega x'} \\ &\quad - \frac{4}{9}e^{2\omega^2 x'} - \frac{4}{9}e^{2\omega x'} + e^{(\omega^2-1)x'} + e^{(\omega-1)x'}; \end{aligned}\right\} \tag{19.2a}$$

$$G_2 = \frac{\partial X_0}{\partial x'}\frac{\partial^2 X_1}{\partial x'^2} - \frac{\partial^3 X_0}{\partial x'^3}X_1 \tag{18.2b}$$

$$\left.\begin{aligned} &= \left(\frac{2}{3}x' + \frac{1}{9}\right)e^{-x'} + \frac{4}{9}e^{2\omega^2 x'} + \frac{4}{9}e^{2\omega x'} \\ &\qquad + \omega^2 e^{(\omega 2-1)x'} + \omega e^{(\omega-1)x'}; \end{aligned}\right\} \tag{19.2b}$$

and

$$m_2 = -\frac{1}{2} + \frac{x'}{3}, \qquad n_2 = \frac{\sqrt{3}}{18} - \frac{\sqrt{3}x'}{9}.$$

The particular integral equals

$$\left.\begin{aligned} \psi_2'^p = &\left\{\left(-\frac{2}{3}x' - 4\right)e^{-x'} + (m' + in')\,e^{\omega^2 x'} + (m' - in')\,e^{\omega x'} \right. \\ &- \frac{4\omega}{21}e^{2\omega^2 x'} - \frac{4\omega^2}{21}e^{2\omega x'} + \frac{1}{21}(13\omega^2 + 11)\,e^{(\omega 2-1)x'} \\ &\left. + \frac{1}{21}(13\omega + 11)\,e^{(\omega-1)x'}\right\}\sin y'' + \left\{\left(2x' + \frac{28}{9}\right)e^{-x'}\right. \\ &+ (m' + in')\,e^{\omega^2 x'} + (m' - in')\,e^{\omega x'} + \frac{4\omega}{63}e^{2\omega^2 x'} + \frac{4\omega^2}{63}e^{2\omega x'} \\ &\left. + \frac{1}{7}(\omega - 2)\,e^{(\omega 2-1)x'} + \frac{1}{7}(\omega^2 - 2)\,e^{(\omega-1)x'}\right\}\sin 3y'', \end{aligned}\right\} \tag{20.2}$$

where

$$m' = -\frac{1}{9}x'^2 - \frac{1}{9}x', \qquad n' = -\frac{\sqrt{3}}{27}x'^2 + \frac{5\sqrt{3}}{27}x'.$$

The homogeneous solution is

$$\psi_2'^h = [a_2 e^{\omega 2x'} + \bar{a}_2 e^{\omega x'}] \sin y'' + [b_2 e^{\omega 2x'} + \bar{b}_2 e^{\omega x'}] \sin 3y'', \quad (21.2)$$

where, according to the boundary conditions,

$$a_2 = \frac{71}{42} + \frac{115i\sqrt{3}}{378}, \qquad b_2 = -\frac{7}{6} - \frac{173i\sqrt{3}}{378},$$

and $\bar{a}_2$, $\bar{b}_2$ are the corresponding conjugates. The complete second order solution is found by adding $\psi_2'^p$ to $\psi_2'^h$.

THE CASE OF THE INCLINED BOUNDARY

Munk and Carrier (1950) have treated the case of a triangular ocean, with a western boundary inclined by an angle θ from a north-south direction. For an application of the foregoing results to the Gulf Stream, whose over-all direction along the western boundary of the Atlantic is more nearly eastward than northward, it is necessary to see what modifications are introduced by such an inclination.

It will be convenient to introduce the parameter

$$p^3 = \cos^4\theta \quad (25)$$

and to transform to the co-ordinate system

$$\xi = px' - py' \tan\theta, \qquad \zeta = y'. \quad (26)$$

Here ξ represents the (nondimensional) distance from the inclined western boundary. Then

$$\frac{\partial}{\partial x'} = p\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial y'} = -p\tan\theta\frac{\partial}{\partial \xi},$$
$$\nabla' = \boldsymbol{q}\frac{\partial}{\partial s} + \boldsymbol{j}\frac{\partial}{\partial \zeta}, \qquad \nabla'^4 = p\frac{\partial^4}{\partial \xi^4}, \quad (27)$$

where $\boldsymbol{q} = p(\boldsymbol{i} - \boldsymbol{j}\tan\theta)$. The approximations involved are discussed in Munk and Carrier (1950). It follows that

$$\text{curl}_z\,(\boldsymbol{M}' \cdot \nabla' \boldsymbol{M}') = p^{3/2}\,(\psi_\xi' \psi_{\xi\xi\zeta}' - \psi_\zeta' \psi_{\xi\xi\xi}'),$$

whereas previously

$$\text{curl}_z\,(\boldsymbol{M}' \cdot \nabla' \boldsymbol{M}') = \psi_{x'}' \psi_{x'x'y'}' - \psi_{y'}' \psi_{x'x'x'}'.$$

Making the appropriate substitutions, the differential equation (11) becomes

$$\left(\nabla'^4 - \frac{\partial}{\partial\xi}\right)\psi' = \frac{1}{pr'}\sin y'' + 4\gamma^{-1}\sqrt{p}\,\lambda\,\mathrm{curl}_z\,(\boldsymbol{M}'\cdot\nabla'\boldsymbol{M}')\,, \quad (28)$$

with the understanding that all operations now involve ordinary Cartesian operators with respect to ξ and ζ, and not those defined in (27). Thus in equation (28)

$$\nabla'^4 = \frac{\partial^4}{\partial\xi^4} + 2\frac{\partial^4}{\partial\xi^2\partial\zeta^2} + \frac{\partial^4}{\partial\zeta^4}\,, \qquad \boldsymbol{M}' = \boldsymbol{k}\times\left(\boldsymbol{i}\frac{\partial\psi'}{\partial\xi} + \boldsymbol{j}\frac{\partial\psi'}{\partial\zeta}\right),$$

etc. Equations (28) and (11) are equivalent and the previously derived solutions apply, provided we write

$$\sqrt{p}\,\lambda\,, \qquad pr'\,, \qquad \xi$$

for (29)

$$\lambda\,, \qquad r'\,, \qquad x'\,.$$

For $\theta = 0$, $p = 1$, the solution reduces again to the case of north-south boundary.

DISCUSSION OF SOLUTION

The solution in real form for the north-south boundaries is then

$$\psi' = X_0\sin y'' + \lambda X_1\sin 2y'' + \lambda^2(X_2^a\sin y'' + X_2^b\sin 3y'') + \ldots \quad (22)$$

where

$$X_0 = 1 - \left[\cos\frac{\sqrt{3}}{2}x' + \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}x'\right]e^{-\frac{1}{2}x'}\,, \quad (23.0)$$

$$X_1 = \left[\left(1 + \frac{2}{3}x'\right)\cos\frac{\sqrt{3}}{2}x' + \left(-\frac{7\sqrt{3}}{9} + \frac{2\sqrt{3}}{9}x'\right)\sin\frac{\sqrt{3}}{2}x'\right]e^{-\frac{1}{2}x'} - e^{-x'}\,, \quad (23.1)$$

$$X_2^a = \left[\left(-\frac{2}{9}x'^2 - \frac{2}{9}x' + \frac{71}{21}\right)\cos\frac{\sqrt{3}}{2}x' + \left(-\frac{2\sqrt{3}}{27}x'^2 + \frac{10\sqrt{3}}{27} + \frac{115\sqrt{3}}{189}\right)\sin\frac{\sqrt{3}}{2}x'\right]e^{-\frac{1}{2}x'} + \left[\frac{4}{21}\cos\sqrt{3}x' - \frac{4\sqrt{3}}{21}\sin\sqrt{3}x' - \frac{2}{3}x' - 4\right]e^{-x'} + \left[\frac{3}{7}\cos\frac{\sqrt{3}}{2}x' - \frac{13\sqrt{3}}{21}\sin\frac{\sqrt{3}}{2}x'\right]e^{-\frac{3}{2}x'} \quad (23.2a)$$

$$X_2^b = \left[\left(-\frac{2}{9}x'^2 - \frac{2}{9}x' - \frac{7}{3}\right)\cos\frac{\sqrt{3}}{2}x' + \left(-\frac{2\sqrt{3}}{27}x'^2 + \frac{10\sqrt{3}}{27}x' - \frac{173\sqrt{3}}{189}\right)\sin\frac{\sqrt{3}}{2}x'\right]e^{-\frac{1}{2}x'} + \left[-\frac{4}{63}\cos\sqrt{3}\,x' + \frac{4\sqrt{3}}{63}\sin\sqrt{3}x' + 2x' + \frac{28}{9}\right]e^{-x'} + \left[-\frac{5}{7}\cos\frac{\sqrt{3}}{2}x' + \frac{\sqrt{3}}{7}\sin\frac{\sqrt{3}}{2}x'\right]e^{-\frac{3}{2}x'} \qquad (23.2b)$$

We shall first discuss this solution and then point out the modifications introduced by an inclination of the boundary relative to the north-south direction.

The four functions X of the zero, first and second order approximations are plotted in Fig. 1. The streamlines of the zero, first and second order solution are plotted separately in Fig. 2. The three parts of the figure must be combined, each part being weighted according to the value of λ. We shall leave aside for the present the question of the value of λ and consider separately the patterns corresponding to the various orders. Allowance must be made for the exaggeration of the west-east scale relative to the north-south scale by a factor of $10\pi/96\gamma$, or about 8. Thus the north-flowing currents are much more intense than they appear on the figures.

The left part of Fig. 2 gives essentially the solution discussed in Munk (1950). The first order solution (Fig. 2, center) reveals four vortices with a saddle point at their center. The eastern vortex pair is of relatively small interest, as the neglect of the last term in equation (16.0) introduces errors in the Sargasso Sea area. Of the western vortex pair, the southern vortex weakens the Gulf Stream near shore, the northern vortex strengthens it. The second order solution (Fig. 2, right) reveals three vortices.

The net modification introduced by the first and second order terms can be seen qualitatively to consist of strengthening the current in northern latitudes where all of the terms add and of weakening the current in the central portion. As a result the circulation pattern is displaced northward. This result could have been anticipated, as one would naturally expect the effect of inertia to reduce the current strength in a region of acceleration and to increase it in a region of deceleration.[7]

[7] More precisely, to diminish vorticity in regions where the vorticity increases downstream, and vice versa.

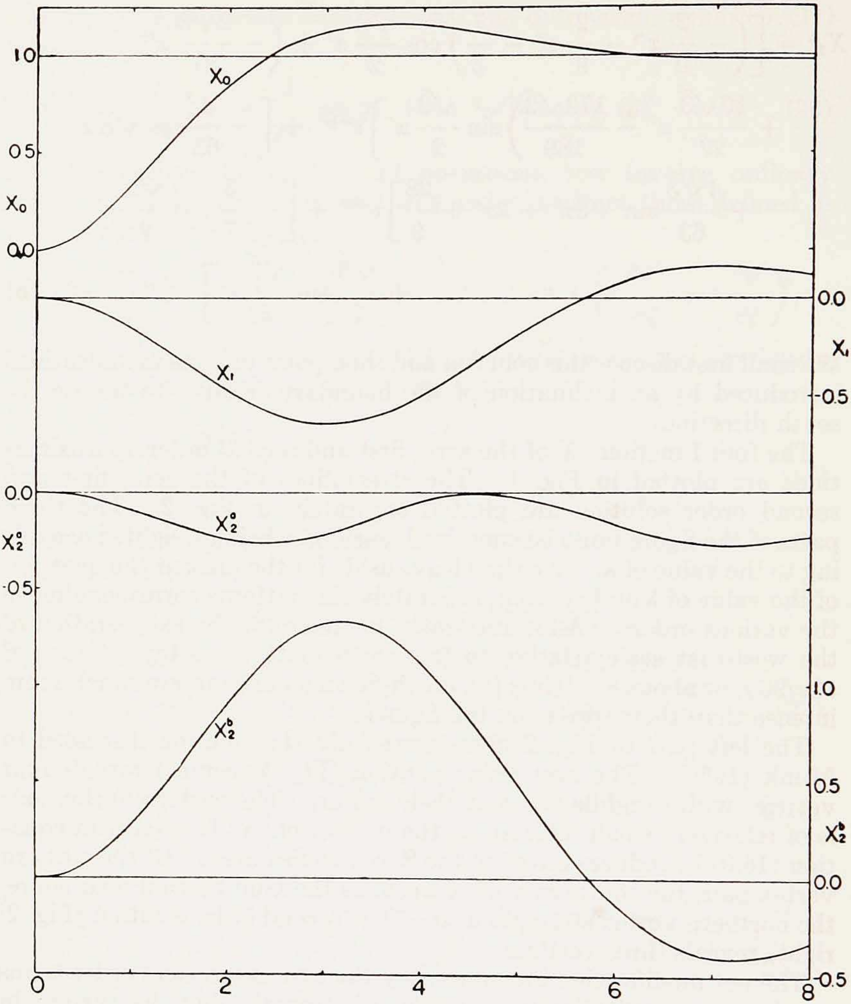

Figure 1. The finctions X of the zero, first and second orders, plotted against the non-dimensional distance x' from the western boundary.

The quantitative extent of these modifications depends upon the numerical value of λ. Setting $\pi/n = 35°$ latitude, $r'' = 6$, $\Gamma = 0.65$ dynes cm^{-2}, $\beta = 2 \times 10^{-13}$ cm^{-1} sec^{-1}, $\gamma = 0.04$, $\bar{\rho} = 1.025$ g cm^{-3} gives [equations (6) and (8)]

$$\lambda = 2784/H \, , \tag{30}$$

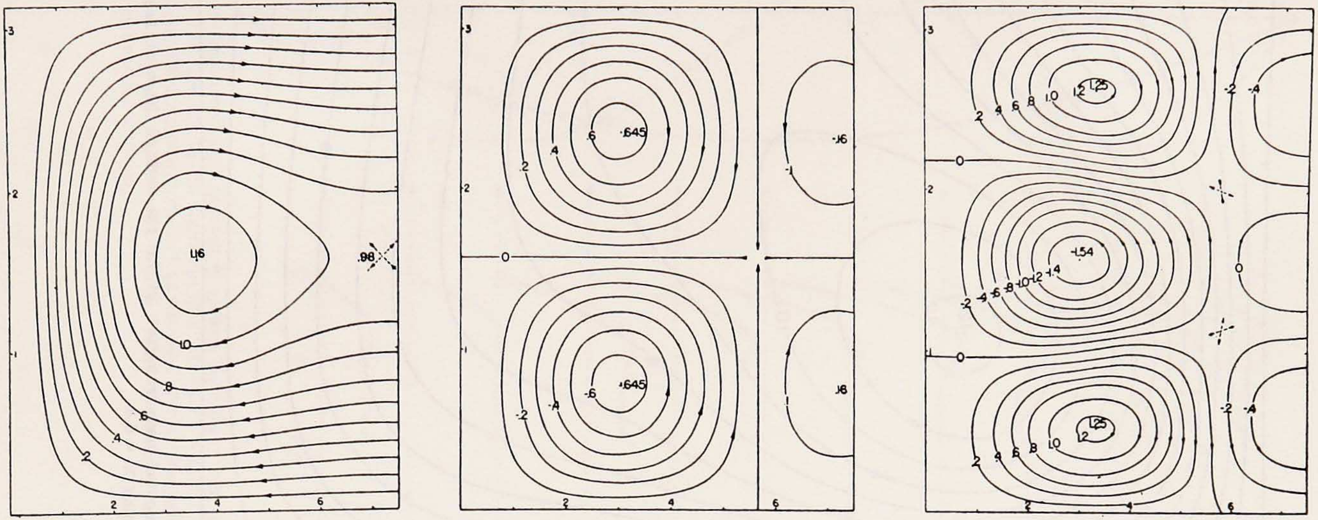

Figure 2. Streamlines of mass transport. The nondimensional west-east scale is exaggerated relative to the north-south scale by a factor of about 8, so that the currents are much more intense than they appear in the figure. Left: zero order solution ψ_0'; center: first order solution ψ_1'; right: second order solution ψ_2'.

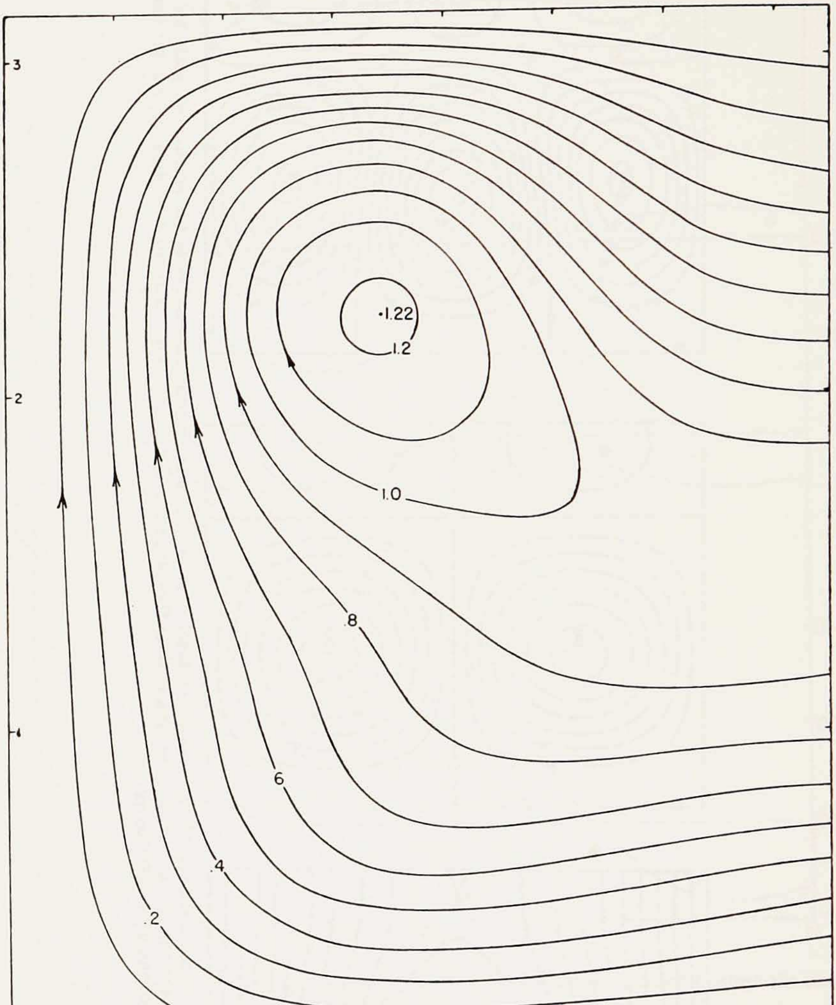

Figure 3. Streamlines of mass transport, ψ', for the sum of the zero, first, and second order solutions, assuming $\lambda = 0.4$.

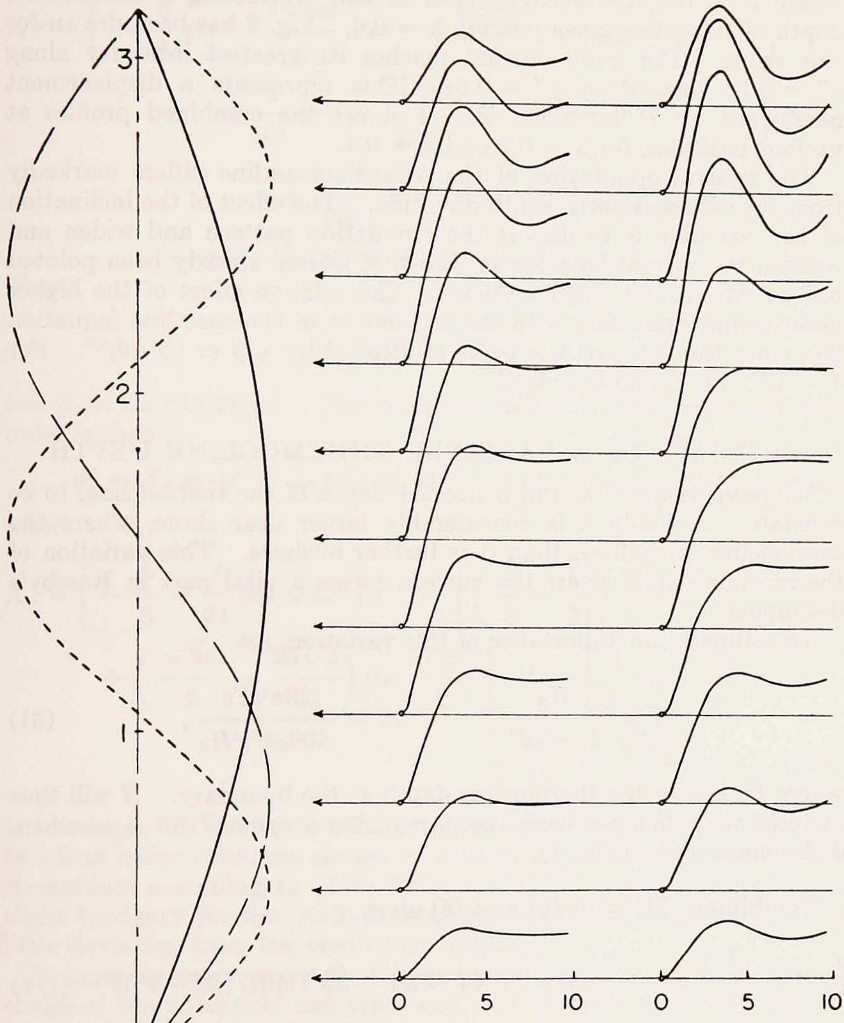

Figure 4. Profiles across Gulf Stream at various latitudes. The curves to the left represent $\sin y''$, $\sin 2y''$, and $\sin 3y''$ as functions of y'', from $y'' = 0$ (15° N) to $y'' = \pi$ (50° N). The two columns to the right represent ψ' as function of x' at the latitudes indicated by the arrows, for $\lambda = 0.2$ (left) and $\lambda = 0.4$ (right). Unit values of ψ' correspond to twice the vertical distance between arrows.

where H is the thermocline depth in cm. Assuming a thermocline depth of 70 meters gives roughly $\lambda = 0.4$. Fig. 3 has been drawn for this value. The main current reaches its greatest intensity along $y'' = 0.7\pi$ instead of $y'' = 0.5\pi$. This represents a displacement northward by 7° latitude. Fig. 4 shows the combined profiles at various latitudes, for $\lambda = 0.2$ and $\lambda = 0.4$.

The general orientation of the Atlantic coastline differs markedly from the assumed north-south direction. The effect of the inclination of the coastline is to distort the circulation pattern and widen and weaken the current by a factor $(\sec\theta)^{1/3}$, as had already been pointed out by Munk and Carrier (1950). The relative effect of the higher order terms diminishes with the inclination of the coastline (equation 29), since the value of λ is to be multiplied by $\sqrt{\bar{p}}$ or $(\cos\theta)^{2/3}$. For $\theta = 50°$, this amounts to 0.74.

THE CASE OF A VARIABLE THERMOCLINE DEPTH

We have assumed λ, and hence the depth of the thermocline, to be constant. Actually λ is considerably larger near shore, where the thermocline is shallow, than it is further offshore. This variation of thermocline depth *across* the current forms a vital part in Rossby's discussion.

To estimate the importance of this variation, set

$$H = \frac{H_0}{1 - \alpha\psi'}, \qquad \lambda_0 = \frac{29n^4 r \Gamma c}{400\bar{\rho}\beta^2\gamma^3 H_0}, \tag{31}$$

where H_0 is now the thermocline depth at the boundary. H will thus increase with distance from shore, reaching a value $H_0/(1 - \alpha)$ where ψ' becomes unity (Fig. 7).

Combining (31) with (6) and (8) gives

$$\left(\nabla'^4 - \frac{\partial}{\partial x'}\right)\psi' = \frac{1}{r'}\sin y'' + 4\gamma^{-1}\lambda_0(1 - \alpha\psi')\,\mathrm{curl}_z\,(\boldsymbol{M}' \cdot \nabla'\boldsymbol{M}') \tag{11v}$$

in place of equation (11). Carrying out solutions of this equation to only zero and first orders leads to equation (11.0) as before, and adds a new term

$$-\,2\alpha X_0 F_1 \sin y'' \sin 2y''$$

to equation (11.1). The particular and homogeneous integrals corresponding to $X_0F_1 \sin y'' \sin 2y''$ are

$$\left.\begin{aligned}\psi_v'^p = \Big[&-\frac{5}{6}e^{-x'} - \frac{i\sqrt{3}\omega}{9}x'e^{\omega^2 x'} + \frac{i\sqrt{3}\omega^2}{9}x'e^{\omega x'} + \frac{1}{42}e^{2\omega^2 x'} \\ &+ \frac{1}{42}e^{2\omega x'} - \frac{i\sqrt{3}}{126}(5+\omega)\,e^{(\omega^2-1)x'} \\ &+ \frac{i\sqrt{3}}{126}(5+\omega^2)\,e^{(\omega-1)x'}\Big] \sin y'' \sin 2y'' ,\end{aligned}\right\} \quad (20\text{v})$$

$$\psi_v'^h = [a_v e^{\omega 2x'} + \bar{a}_v e^{\omega x'}] \sin y'' \sin 2y'' , \quad (21\text{v})$$

where

$$a_v = \frac{8}{21} - \frac{25i\sqrt{3}}{126} ,$$

and $\bar{a}_v$ is its conjugate. The complete solution of the zero and first order is then

$$\psi' = X_0 \sin y'' + \lambda_0 (X_1 \sin 2y'' - 2\alpha X_v \sin y'' \sin 2y'') , \quad (22\text{v})$$

where

$$\begin{aligned}X_v = &\left[-\frac{5}{6} + \frac{1}{21}\cos\sqrt{3}x'\right]e^{-x'} + \left[\left(\frac{x'}{3} + \frac{16}{21}\right)\cos\frac{\sqrt{3}x'}{2}\right. \\ &\left. + \left(\frac{\sqrt{3}x'}{9} - \frac{25\sqrt{3}}{63}\right)\sin\frac{\sqrt{3}x'}{2}\right]e^{-\frac{x'}{2}} + \left[\frac{1}{42}\cos\frac{\sqrt{3}x'}{2}\right. \\ &\left. - \frac{\sqrt{3}}{14}\sin\frac{\sqrt{3}x'}{2}\right]e^{-\frac{3}{2}x'} \quad (23\text{v})\end{aligned}$$

has been plotted in Fig. 5. The function is very similar to X_1, and the two first order solutions cancel to a large degree. Fig. 6 shows the streamlines according to (22v) for $\alpha = 0.75$, $\lambda_0 = 0.6$. Except for a slight tendency for the north-flowing current to "overshoot," there is little deviation from the zero order solution.

It must be emphasized that the results depend somewhat on our choice of the numerical constants and on the rather arbitrary form of equation (31). This description of the variation in thermocline depth was chosen for its suitability in the computation of higher order terms. Actually our model for the density structure, when combined with the geostrophic equations for the currents, completely specifies the law for the variation in thermocline depth, but the resulting expression is not suitable for the computation of higher order terms (see Fig. 7).

Let us define a stream function ϕ, related to the surface mass trans-

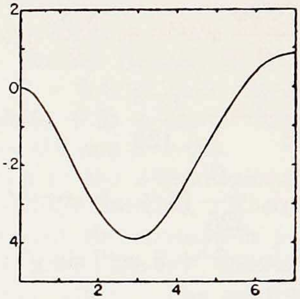

Figure 5. The function X_v (Equation 23v) associated with a variable depth in thermocline.

port $\rho_0 \mathbf{v}_0$ in the same manner as ψ is related to the integrated transport. Then the geostrophic equations, with the aid of (2) and (3), can be written

$$f\nabla\phi = \nabla p_0 = \rho_0 g \nabla h ,$$

$$f\nabla\psi = \int_{-\infty}^{h} \nabla p dz = \frac{5\rho_0^2 g}{8\Delta\rho} \nabla h^2 .$$

Solving for h, and neglecting the variation with latitude of the Coriolis' parameter f, leads to

$$h = \frac{f}{\rho_0 g}\phi + h_0 = \sqrt{\frac{8f\Delta\rho}{5\rho_0^2 g}\psi + h_0^2} ,$$

where h_0 is the value of h for $\theta = \psi = 0$. It follows from (1) that $H_0 = h_0\rho_0/2\Delta\rho$ is the depth of the thermocline at the boundary. The above equations can be solved to yield

$$\phi = \frac{\sqrt{1 + a\psi} - 1}{\frac{5}{4}aH_0}, \qquad H = H_0\sqrt{1 + a\psi} , \qquad (32a, b)$$

where

$$a = \frac{2f}{5g\Delta\rho H_0^2} .$$

In order for (31) to be a suitable approximation to (32b), it would be necessary that $a\psi' << 1$ and $a\psi << 1$. Actually $a\psi$ varies from zero to fourteen.

For a numerical example we choose conditions at $y'' = \pi/2$, the approximate latitude of Cape Hatteras. Setting $f = 10^{-4}$ sec^{-1}, $g =$

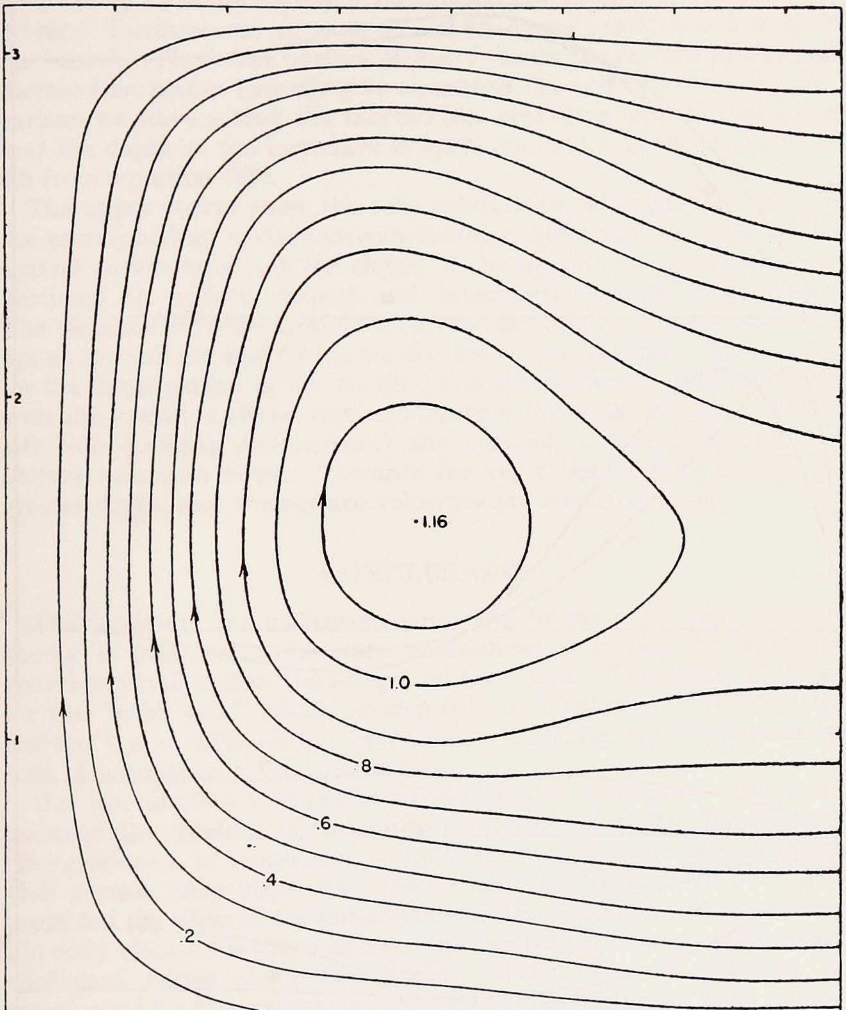

Figure 6. Streamlines of mass transport, ψ', for the combined zero and first order solutions pertaining to a variable depth in thermocline (Equation 22v), assuming $\alpha = 0.75$. The corresponding profile of the thermocline is presented by the dashed curve in the lower portion of Fig. 7.

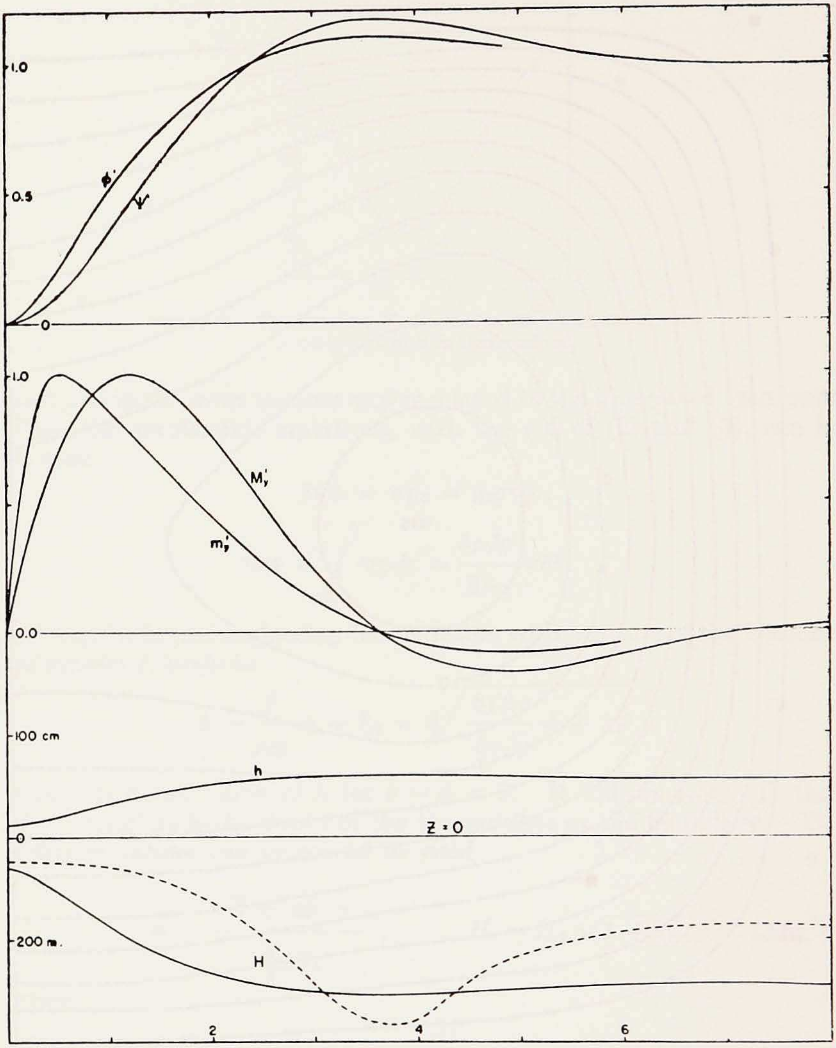

Figure 7. Variation of parameters across the Gulf Stream according to the zero order solution. A unit distance x' equals approximately 50 km. The upper curves denote the surface stream function ϕ' and integrated stream function ψ'. For $\phi' = 1$, $\phi = 4.51 \times 10^8$ g sec^{-1} cm^{-1}; $\psi' = 1$, $\psi = 1.95 \times 10^{13}$ g sec^{-1}. The central curves show the corresponding surface transports and integrated transports. For $m_y' = 1$, $m_y = 55.3$ g sec^{-1} cm^{-2}; $M_y' = 1$; $M_y = 2.13 \times 10^6$ g sec^{-1} cm^{-1}. The lower solid curves represent the surface and thermocline profile according to Equation (32v). The dashed curve is the assumed thermocline according to (31).

10^3 g sec^{-2}, $\Delta\rho = 10^{-3}$ g cm^{-3}, $H_0 = 6000$ cm, gives $a = 1.11 \times 10^{-12}$ g^{-1} sec. Furthermore nr = 6, $\Gamma = 0.65$ dynes cm^{-2}, $\beta = 2 \times 10^{-13}$ cm^{-1} sec^{-1}. The lower portion of Fig. 7 shows the surface profile and thermocline profile according to equations (1) and (32b). For comparison we have added the thermocline according to (31), assuming that the depth at the boundary is 4,640 cm, which gives the value of 0.6 from equation (30).

The upper curves show the zero solution for ψ' (equation 15) and the corresponding surface stream function according to (32a). The central curves represent the slopes of the upper curves and are proportional to surface current and integrated current, respectively. The distance between maximum current and the left edge is about 25 km at the surface and 60 km for the integrated current. The reason for the displacement of the maximum in the surface current has to do with the variation of the vertical structure across the stream. At the left side (looking downstream) the current is superficial and the surface velocities large. Towards the right the current extends to greater depth, and the surface velocities are relatively small.

CONCLUSIONS

The westward intensification according to the planetary vorticity theory is therefore even more pronounced if surface currents are considered rather than the integrated transport. The width of 25 km for the "cold wall," which corresponds to a value of $A = 2.5 \times 10^7$ cm^2 sec^{-1}, $p = 0.445$ (Munk and Carrier, 1950: fig. 4), is in agreement with observations in the Hatteras area.

Our examination is unrealistic for a number of reasons. We have assumed the western boundary to be a vertical wall. The bottom rises gradually, of course, and many of the modifications inshore of the Gulf Stream take place over the Continental Shelf. We have also neglected the effect of the indentations in the coastline. Furthermore, the eddy viscosity is taken as constant, whereas in Rossby's theory this coefficient varies along the Stream. Our zero order solution, as represented by the central curves in Fig. 7, resembles the wake stream profile (Rossby, 1936: fig. 4). However, this resemblance is not noticeably improved by the inclusion of higher order terms.

At the outset of this investigation it was hoped that the higher order terms might contain a clue as to the continued concentration of the Gulf Stream long after it has left the American east coast. Another observed feature not contained in the linear solution is the countercurrent inshore (Bumpus and Wehe, 1949). Even though the validity of the higher order terms is in doubt, largely because of their surprisingly strong dependence on the assumed vertical distribution of density, it does not seem likely that these observed features are contained in the higher order terms. There is some essential feature in the dynamics (or thermodynamics) of the Gulf Stream which has not yet been recognized.

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