THE THEORY OF THE ELECTRIC FIELD INDUCED IN DEEP OCEAN CURRENTS[1]

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The Problem. Due to the fact that the water in the ocean is a conductor, and that it is everywhere under the influence of the earth's magnetic field, we should expect, by the law of electric induction, that wherever the water is in motion electric potentials and currents will be established. It is the purpose of this paper to discuss the theory of these phenomena in the deep ocean and to indicate certain analytical solutions of the problem which demonstrate the important physical aspects of electric fields associated with ocean currents.

The Fundamental Equation of the Electric Field.[2] Let $\mathbf{H}$ be the vector magnetic field intensity, ρ the scalar resistivity and $\mathbf{i}$ the electric current vector. Consider a closed curve fixed in the fluid. Let $\mathbf{v}$ be the fluid velocity vector. The length of arc along the closed curve is given by the vector $\mathbf{s}$. The element of area of the surface enclosed by the curve is $\mathbf{dS}$. Faraday's law of induction may then be expressed in the following general form,

$$\frac{d}{dt}\int\int \mathbf{H} \cdot \mathbf{dS} + \oint \mathbf{i} \cdot \mathbf{ds} = 0 \ , \tag{1}$$

where the d/dt is to be understood in the manner of the substantial derivative of hydrodynamics. The first equation (1) may be transformed (Abraham and Becker, 1932: 39–40) as follows:

$$\frac{d}{dt}\int\int \mathbf{H} \cdot \mathbf{dS} = \int\int \frac{\partial \mathbf{H}}{\partial t} \cdot \mathbf{dS} + \int\int (\text{div}\ \mathbf{H})\mathbf{v} \cdot \mathbf{dS} - \int\int \Big[\text{curl}\ (\mathbf{v} \times \mathbf{H}) \Big] \cdot \mathbf{dS} \ . \tag{2}$$

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In the ocean a considerable simplification of (2) is possible, because over moderately large areas **H** is constant and uniform. Therefore, the first two terms of the second member vanish, and by Stoke's Theorem the third term may be transformed to a line integral,

$$-\iint \left[\operatorname{curl} (\mathbf{v} \times \mathbf{H}) \right] \cdot d\mathbf{S} = -\oint (\mathbf{v} \times \mathbf{H}) \cdot d\mathbf{s}. \qquad (3)$$

Equation (1) may now be written in the following form:

$$\oint (\mathbf{v} \times \mathbf{H} - \rho \mathbf{i}) \cdot d\mathbf{s} = 0. \qquad (4)$$

The vanishing of this line integral signifies the existence of the electric scalar potential function ϕ, defined in the following way:

$$\nabla \phi = \mathbf{v} \times \mathbf{H} - \rho \mathbf{i}. \qquad (5)$$

The term $\rho\mathbf{i}$ involving the unknown current vector **i** may be eliminated if ρ is assumed to be uniform, for then the div $\rho\mathbf{i}$ vanishes, and upon taking the divergence of (5) one obtains

$$\boxed{\nabla^2 \phi = \mathbf{H} \cdot \operatorname{curl} \mathbf{v}.} \qquad (6)$$

If **H** and **v** are both regarded as known, which is physically the case, then we simply have here Poisson's equation to solve for the electric potential ϕ. It is desirable to emphasize the complete generality of (6). This equation defines the electric field resulting from *any arbitrary velocity field* in the ocean.

The boundary conditions upon this equation are obtained from (5). At a nonconducting boundary the normal component of the vector **i** vanishes. If **n** is the unit vector normal to the boundary,

$$\mathbf{n} \cdot \mathbf{i} = 0,$$

so that

$$\frac{\partial \phi}{\partial \mathbf{n}} = \mathbf{n} \cdot \mathbf{v} \times \mathbf{H}.$$

At a conducting boundary, such as occurs between two media of different specific resistivity, the normal component of **i** is equal in each medium, and the potential ϕ is equal in each medium. The first condition may be expressed in terms of $\partial\phi/\partial\mathbf{n}$ for each medium by using (5).

An Idealized Particular Ocean Current System and its Associated Electric Field. The velocity and potential fields of a natural ocean current system are very complicated functions of position. They may be treated numerically as indicated in the section, Numerical Method of Solution. An analytically simple model is advantageous, inasmuch as it shows the essential features without many complications of a purely formal but not conceptual nature. In order to exhibit some of the features of the problem, an idealized ocean current system is treated here. Rectangular co-ordinates are taken with the x,y-plane in the surface of the ocean, with the z-axis pointing vertically upwards. This ocean is supposed to be divided into two layers. The top layer, no. 1, extends from $z = 0$ to $z = -h_1$. The bottom layer, no. 2, extends from $z = -h_1$ to $z = -h_2$. The bottom of the ocean, $z = -h_2$, is taken in this discussion as nonconducting.

The velocity in layer no. 1 is supposed to be given as

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ v_o \cos \pi/b\, x \\ 0 \end{Bmatrix}. \tag{7}$$

Layer no. 2 is supposed to be at rest. This current system consists of alternate bands of water of breadth b moving in the positive and negative y-direction over a lower layer which does not participate in the motion. This picture is artificial, but may be generalized by implementing the Fourier Integral Theorem.

Let us suppose that the magnetic field of the earth is entirely vertical,

$$\mathbf{H} = \begin{Bmatrix} 0 \\ 0 \\ H_z \end{Bmatrix}. \tag{8}$$

Substitution of (7) into (6) gives the equations

$$\nabla^2 \phi_1 = \gamma \sin \beta\, x, \tag{9}$$

$$\nabla^2 \phi_2 = 0, \tag{9'}$$

where the subscript refers to the layer number, and the following abbreviated forms have been introduced:

$$\gamma = -H_z v_o\, \pi/b\,, \qquad \beta = \pi/b\,. \tag{10}$$

In order to obtain a solution of (9) and (9′), one supposes that the potential is expressible in separable form, which indeed later appears to be the case.

$$\phi_1 = \sin \beta x \cdot Z_1\,, \tag{11}$$

$$\phi_2 = \sin \beta x \cdot Z_2\,. \tag{11'}$$

Substitution of (11) and (11′) into (9) and (9′) yields, after cancelling the common factor,

$$\frac{\partial^2 Z_1}{\partial z^2} - \beta^2 Z_1 = \gamma \, , \tag{12}$$

$$\frac{\partial Z_2}{\partial z^2} - \beta^2 Z_2 = 0 \, . \tag{12'}$$

The solutions of these two equations are simply,

$$Z_1 = A_1 \cosh \beta z + B_1 \sinh \beta z - \delta \, , \tag{13}$$

$$Z_2 = A_2 \cosh \beta z + B_2 \sinh \beta z \, , \tag{13'}$$

where A_1, A_2, B_1, and B_2 are constants of integration and

$$\delta = \beta^{-2} \gamma \, .$$

These constants must now be determined in terms of the boundary conditions. Take $\mathbf{n}$ as the unit vertical vector. At $z = 0$ there must be no current across the water surface:

$$\mathbf{i} \cdot \mathbf{n} = 0 \, . \tag{14}$$

From (5) we have the electric current in terms of the potential

$$\mathbf{i} = \frac{1}{\rho} \left[\mathbf{v} \times \mathbf{H} - \nabla \phi \right] . \tag{15}$$

Substituting into (14) we obtain

$$\mathbf{n} \cdot \mathbf{v} \times \mathbf{H} - \frac{\partial \phi}{\partial z} = 0 \, . \tag{16}$$

In the particular case we are discussing $\mathbf{n} \cdot \mathbf{v} \times \mathbf{H}$ vanishes, so that the boundary condition at the surface $z = 0$ is

$$\frac{\partial \phi}{\partial z} = 0 \, . \tag{17}$$

The boundary condition at the bottom $z = -h_2$ is also one of no current, so it is the same as (17).

At the interface between layers 1 and 2, $z = -h_1$, two conditions prevail, for clearly the potential and current must be continuous. Therefore at $z = -h$,

$$\phi_1 = \phi_2 \, , \tag{18}$$

$$\mathbf{i}_1 = \mathbf{i}_2 \, . \tag{18'}$$

The second of these equations may be expressed in terms of the potential function

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z}. \tag{18''}$$

The conditions (17) at $z = 0$ and $z = -h_2$, and the conditions (18) and (18″) at $z = -h_1$, suffice to determine the constants of integration.

$$\begin{aligned} A_1 &= \delta \frac{\sinh \beta (h_2 - h_1)}{\sinh \beta h_2}, \\ B_1 &= 0 \\ A_2 &= -\delta \frac{\sinh \beta h_1}{\tanh \beta h_2}, \\ B_2 &= -\delta \sinh \beta h_1. \end{aligned} \tag{19}$$

Therefore, the final solution is

$$\phi_1 = \sin \beta x \left[1 - \frac{\sinh \beta (h_2 - h_1)}{\sinh \beta h_2} \cosh \beta z \right] \frac{v_o H_z}{\beta}, \tag{20}$$

$$\phi_2 = \sin \beta x \left[\frac{\sinh \beta h_1}{\tanh \beta h_2} \cosh \beta z + \sinh \beta h_1 \sinh \beta z \right] \frac{v_o H_z}{\beta}. \tag{20'}$$

The exact expressions (20) and (20′) may be replaced by simpler expressions to a first approximation in the case where $\beta h_2 << 1$, that is, where the breadth of the current is much greater than the depth of the ocean, as most frequently happens in nature.

$$\phi_1 \cong \phi_2 \cong \sin \beta x \left[\frac{h_1}{h_2} \right] \frac{v_o H_z}{\beta}. \tag{21}$$

According to this approximate formula the lines of equipotential are vertical straight lines, and the amplitude of the potential developed is directly proportional to the ratio of current depth to ocean depth.

For shallow surface currents the return circuit beneath, in the resting water, is comparatively so large that it effectively shorts the potentials developed.

Numerical Method of Solution. A convenient numerical method of solution may be based upon the relaxation methods developed by Southwell (1946). Any vertical velocity profile can be investigated and the electric potential field computed.

Equation (6) is expressed in finite difference form. The boundary condition at the bottom and at the free surface is expressed by (16). The procedure of the solution is then by liquidation of residuals.

Publication of any of these relaxation solutions is postponed until some *simultaneous* electric potential measurements and sufficient hydrographic stations for computation of the velocity field are available.

The Effect of the Resistance of the Bottom. Longuet-Higgins and Barber (1946) have shown that the potential developed in a tidal current through a shallow channel (such as the English Channel) is greatly controlled by the resistivity of the bottom. It is important, therefore, to discover what the effect of the electric conduction through the bottom will be on the preceding analysis, which was made upon the assumption of a nonconducting bottom.

For this purpose we may introduce a layer 3 extending from $z = -h_2$ to $z = -\infty$ which will represent the bottom. The resistivity of layers 1 and 2 is ρ_1 and ρ_2, and since they are bottom sea water layers $\rho_1 = \rho_2$. The resistivity ρ_3 of the bottom is considerably greater than ρ_2. For convenience we introduce the ratio

$$\bar{\omega} = \frac{\rho_2}{\rho_3}; \qquad 1 >> \bar{\omega} \geq 0 .$$

The boundary conditions at $z = 0$ and $z = -h_1$ remain the same as before. For $z = -h_2$ however we now have

$$\phi_2 = \phi_3 , \tag{22}$$

$$\frac{\partial \phi_2}{\partial z} = \bar{\omega} \frac{\partial \phi_3}{\partial z} . \tag{22'}$$

At $z = -\infty$ the potential vanishes

$$\phi_3 = 0 \text{ at } z = -\infty . \tag{23}$$

The governing equations (13) and (13′) still hold, with the addition of

$$Z_3 = A_3 \cosh \beta z + B_3 \sinh \beta z . \tag{23''}$$

The condition (23) immediately determines B_3, so that we have

$$Z_3 = A_3 (\cosh \beta z - \sinh \beta z) .$$

Again, at $z = 0$ the condition (17) shows that $B_1 = 0$.

There remain four independent constants A_1, A_2, B_2, A_3, to determine and the four boundary conditions (18), (18″), (22) and (22′). These give respectively

$$A_1 \cosh \beta h_1 - A_2 \cosh \beta h_1 + B_2 \sinh \beta h_1 = \delta , \tag{24}$$

$$- A_1 \sinh \beta h_1 + A_2 \sinh \beta h_1 - B_2 \cosh \beta h_1 = 0 ,$$

$$A_2 \cosh \beta h_2 - B_2 \sinh \beta h_2 - A_3 (\cosh \beta h_2 + \sinh \beta h_2) = 0 ,$$

$$- A_2 \sinh \beta h_2 + B_2 \cosh \beta h_2 + A_3 \bar{\omega} (\sinh \beta h_2 + \cosh \beta h_2) = 0 .$$

For the purpose of determining these constants it will be convenient to restrict our thoughts to the case where $\beta h_2 << 1$, that is, broad currents. The above equations are then, to a first approximation, as follows:

$$
\begin{aligned}
A_1 - A_2 + B_2\,\beta h_1 + 0 &= \delta\,, \\
-\,A_1\,\beta h_1 + A_2\,\beta h_1 - B_2 + 0 &= 0\,, \\
0 + A_2 - B_2\,\beta h_2 - A_3\,(1 + \beta h_2) &= 0\,, \\
0 - A_2\,\beta h_2 + B_2 + A_3\,\tilde{\omega}\,(1 + \beta h_2) &= 0\,.
\end{aligned}
\tag{25}
$$

The values of the constants are determined to be (neglecting second and higher powers of βh_1 and βh_2)

$$
\begin{aligned}
A_1 &= \delta\,\frac{\tilde{\omega} + \beta\,(h_1 - h_2)}{\tilde{\omega} - \beta\,h_2} = \frac{\beta\,(h_2 - h_1) - \tilde{\omega}}{\beta\,h_2 - \tilde{\omega}} \\
A_2 &= -\,\delta\,\frac{\beta h_1}{\beta h_2 - \tilde{\omega}}\,, \\
B_2 &= -\,\delta\beta h_1, \\
A_3 &= \frac{\delta\,\beta h_1}{(1 + \beta h_2)\,(\tilde{\omega} - \beta h_2)}\,.
\end{aligned}
\tag{26}
$$

Comparison of these values of A_1, A_2 and B_2 with those obtained for the nonconducting bottom ($\tilde{\omega} = 0$) shows that for small $\tilde{\omega}$ ($\tilde{\omega} << \frac{1}{10}$) the potential developed is affected by the presence of the bottom only negligibly. These considerations make it seem justifiable to us to ignore the resistivity of the bottom in discussion of the potential fields generated by shallow currents in the deep ocean. In the case of currents in shallow channels these conclusions do not apply.

Summary. The physical principles which describe the association of the water velocity and induced electric potential field in the deep ocean are formulated, and with the aid of a simple idealized model the chief features are discussed.

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