THE EFFECT OF OCEAN CURRENTS ON INTERNAL WAVES

By

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In a fluid whose density changes with the height, either abruptly at a sharp surface of discontinuity or gradually, it is possible to have internal wave motions whose amplitude is greatest at the density discontinuity or, in the case of a gradual density change, somewhere in the interior of the fluid and not at the free upper surface where the surface waves have their maximum amplitude. The occurrence of such internal waves has been demonstrated by Ekman (1904), Seiwell (1937, 1942), Ufford (1947), and others. A review of our theoretical and empirical knowledge of internal waves has been given by Sverdrup, *et al.* (1942: 585–602). In the theoretical investigations it has always been assumed that the mean velocity of each layer, *i. e.*, the velocity apart from the wave motion, is zero. In many cases, especially where ocean currents are present, this assumption will not be satisfied. It is the purpose of the present paper to investigate whether or not modifications in the internal wave motion would be brought about by the existence of such currents on which the internal wave motion may be superimposed.

The presence of undisturbed currents on which the wave motions are superimposed complicates considerably the expressions for the wave periods, wave velocities, amplitude ratios at different levels, etc. Therefore, a very simple model will be considered which consists of two fluid layers (Fig. 1). The free upper surface (0) may be at the level $z = 0$, the rigid lower surface (2) at the level $z = -H$. Both fluids are separated by an internal boundary at the level $z = -h^I$. Each of the two fluids is incompressible and homogeneous, the densities being ρ^I and ρ^{II}, respectively, for the upper and lower fluid layers. The velocities in both layers, in the absence of wave motion, are U^I and U^{II}, respectively, both parallel to the x-axis. These current

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velocities may also be assumed to be constant within each layer and to change abruptly at the surface of discontinuity. In the absence of wave motion the pressure distribution is given by the expression

$$P = g\rho z + \text{const.},$$

where ρ is the density and g the acceleration of gravity. The wave motion may be considered as a small perturbation superimposed on the horizontal current U, and the velocity components in this case may be $U + u$, w while the pressure is $P + p$. Under the assumption that the perturbation quantities are small (designated by small letters), the equations of motion and continuity may be written

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0\,,$$

(1)

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0\,,$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\,.$$

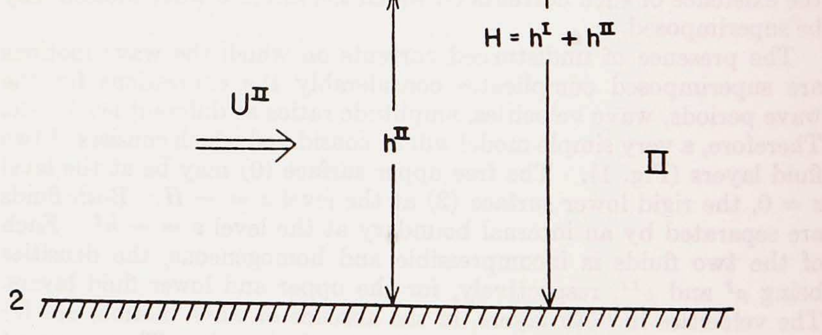

Figure 1. Stratification of fluid.

The equation of continuity is satisfied if a stream function ψ exists, such that

$$u = -\frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \psi}{\partial x}. \tag{2}$$

Since the rigid lower boundary at $z = -H$ must be a stream line, the solution of (1) for the lower layer in the case of waves travelling in the x-direction may be written in the form

$$\psi^{II} = K^{II} \sinh \alpha(z + H)\, e^{i(\alpha x - \beta t)}, \tag{3}$$

$$p^{II} = -\alpha c^{II} \rho^{II} K^{II} \cosh \alpha(z + H)\, e^{i(\alpha x - \beta t)}. \tag{4}$$

Here, K^{II} is an arbitrary constant, $\alpha = 2\pi/L$, where L is the wave length, $\beta = 2\pi/T$, where T is the period, $c = \beta/\alpha$, the wave velocity, and $c^{II} = c - U^{II}$.

For the upper layer, the solution of (1) may be written in the form

$$\psi^{I} = (K_1{}^{I} e^{\alpha z} + K_2{}^{I} e^{-\alpha z})\, e^{i(\alpha x - \beta t)}, \tag{5}$$

$$p^{I} = -\alpha c^{I} \rho^{I} (-K_1{}^{I} e^{\alpha z} + K_2{}^{I} e^{-\alpha z})\, e^{i(\alpha x - \beta t)}, \tag{6}$$

where $K_1{}^I$ and $K_2{}^I$ are arbitrary constants and where $c^I = c - U^I$. Since at the free surface the pressure must remain constant,

$$P^I + p_{z=0}{}^I = -g\rho^I z + \alpha c^I \rho^I (-K_1{}^I + K_2{}^I)\, e^{i(\alpha x - \beta t)} = \text{const.} \tag{7}$$

In the expression for p^I the actual height of the free surface has been replaced by the height of the free surface in the undisturbed position, $z = 0$. This simplification involves only an error of higher order. It follows that the equation of the free surface may be written in the form

$$z - Z_0\, e^{i(\alpha x - \beta t)} = \text{const.}, \tag{8}$$

where the amplitude of the free surface

$$Z_0 = \frac{\alpha}{g} c^I (-K_1{}^I + K_2{}^I).$$

The kinematic boundary condition requires that a particle which was once part of the free surface remains in this surface. If in general the equation of the disturbed free surface is

$$F + f = 0,$$

where $F = 0$ denotes the equation for the undisturbed boundary, the kinematic condition becomes

$$\left(\frac{\partial}{\partial t} + U^I \frac{\partial}{\partial x}\right) f + w_{z=0}{}^I \frac{\partial F}{\partial z} = 0.$$

Here, only an error of higher order is involved if the value of w at the actual disturbed position of the boundary is replaced by the value at the undisturbed position. The last equation leads to the following equation for Z_0,

$$c^I Z_0 = -\,(K_1{}^I + K_2{}^I)\,.$$

By means of the two expressions for Z_0, (5) and (6) may be written as follows:

$$\text{(9)}\qquad \psi^I = -\,Z_0\left[\frac{g}{\alpha c^I}\sinh \alpha z + c^I \cosh \alpha z\right] e^{i(\alpha x - \beta t)}$$

$$\text{(10)}\qquad p^I = \alpha c^I \rho^I Z_0\left[\frac{g}{\alpha c^I}\cosh \alpha z + c^I \sinh \alpha z\right] e^{i(\alpha x - \beta t)}\,.$$

The equation of the internal boundary, which, in the absence of wave motion, is $z = -\,h^I$, is given by the condition that the pressure at this discontinuity must be continuous. There are two kinematic boundary conditions which express the fact that particles in the upper as well as in the lower layer, which at one time were at the boundary, must remain there. Similar to the procedure in the case of the free surface, these conditions permit one to express the arbitrary constant K^{II} in (3) and (4) by the amplitude of the internal boundary Z_1,

$$\text{(11)}\qquad K^{II} = -\,\frac{Z_1\, c^{II}}{\sinh \alpha h^{II}}\,.$$

Furthermore, the ratio of the amplitudes of the free surface Z_0, and of the internal boundary Z_1, is found to be

$$\text{(12)}\qquad \frac{Z_1}{Z_0} = -\,\frac{g}{\alpha c^{I}{}_{2}}\sinh \alpha h^I + \cosh \alpha h^I\,,$$

and the wave velocity c must satisfy the equation

$$\text{(13)}\quad c^{I^4} + \sigma a^I a^{II} c^{I^2} c^{II^2} - \frac{g}{\alpha}\,\sigma(a^{II} c^{II^2} + a^I c^{I^2}) + \frac{g^2}{\alpha^2}(\sigma - 1) = 0.$$

Here the following abbreviations have been used:

$$\frac{\rho^{II}}{\rho^I} = \sigma, \quad \coth \alpha h^I = a^I, \quad \coth \alpha h^{II} = a^{II}\,.$$

The amplitudes Z_0 and Z_1 of the free and of the internal boundary give the amplitudes of the vertical motion at the levels $z = 0$ and $z = -\,h^I$ where the two boundaries are situated in the undisturbed case. In order to find the amplitude z of the vertical motion at other levels, an additional integration has to be performed. If again quantities of higher than the first order in the perturbation terms are neglected,

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + U\frac{\partial z}{\partial x} = w\,.$$

By integration of this equation, for the upper layer it follows that

$$z^I = \left(\frac{g}{\alpha c^{I^2}} \sinh \alpha z + \cosh \alpha z\right) Z_0\, e^{i(\alpha x - \beta t)}\,,$$

and for the lower layer, if the expression (11) is used for K^{II},

$$(14) \qquad z^{II} = \frac{\sinh \alpha\,(z + H)}{\sinh \alpha h^{II}}\, Z_1\, e^{i(\alpha x - \beta t)}\,.$$

At the internal boundary, $z = -\,h^I$, the vertical displacements in both layers must be the same, which leads again to the relation (12). If by means of this relation Z_0 in the foregoing expression for z^I is replaced by Z_1,

$$(15) \qquad z^I = \frac{(g/\alpha c^{I^2}) \sinh \alpha z + \cosh \alpha z}{-\,(g/\alpha c^{I^2}) \sinh \alpha h^I + \cosh \alpha h^I}\, Z_1\, e^{i(\alpha x - \beta t)}\,.$$

As (14) shows, the amplitude of the vertical motion increases from zero at the bottom, $z = -\,H$, to Z_1 at the internal boundary, $z = -\,h^I$. In the upper layer the amplitude of the vertical motion at a given depth depends on the magnitude of c^I in addition to the parameters which determine the vertical amplitude in the lower layer. It is necessary, therefore, first to determine the wave velocity c from (13).

A rigorous solution is quite complicated because equation (13) is of the fourth degree, but approximate solutions can be obtained without much difficulty. For this purpose the conditions in the absence of currents, $U^I = U^{II} = 0$, may first be reviewed briefly. Let $\varepsilon = (\rho^{II} - \rho^I)/\rho^I$, which in all actual cases is small compared to unity. If powers higher than the first of ε are neglected, the following two well known roots of (13) are obtained:

$$(16) \quad c_1^2 = \frac{g}{\alpha}\frac{a^I + a^{II}}{1 + a^I a^{II}}\left[1 + \varepsilon\,\frac{a^{I^2} + a^{II^2} - a^{I^2}a^{II^2} - 1}{(a^I + a^{II})^2\,(1 + a^I a^{II})}\right] \quad \text{(surface wave)}$$

and

$$(17) \qquad c_2^2 = \frac{g}{\alpha}\frac{\varepsilon}{a^I + a^{II}} \qquad \text{(internal wave).}$$

The bracket in (16) is very closely equal to one and may be replaced by this value. The expressions for c_1^2 and c_2^2 show that, of the four roots of (13), the two for the internal waves are considerably smaller than the other two, at least as long as the density discontinuity is small, a condition which is always true for internal boundaries in the ocean.

For this reason it may now be assumed that in the more general case where U^I and U^{II} are not both zero, the values of c^I and c^{II} for the internal waves are also sufficiently small so that the first two terms in (13) may be neglected, these being of the fourth and third degree in these quantities. The solution of the remaining quadratic equation,

$$(18) \qquad \sigma\,(a^{II}c^{II^2} + a^I c^{I^2}) - \frac{g}{\alpha}(\sigma - 1) = 0\,,$$

results in the following expression for c,

$$(19) \quad c = \frac{U^I a^I + U^{II} a^{II}}{a^I + a^{II}} \pm \left[\frac{g}{\alpha}\,\frac{\rho^{II} - \rho^I}{\rho^{II}}\,\frac{1}{a^{II} + a^I} - \frac{(U^I - U^{II})^2}{(a^I + a^{II})^2}\,a^I a^{II}\right]^{\frac{1}{2}}.$$

In order to obtain a correction to this approximation let the rigorous solution of (13) be $c + j$, where j is a small quantity if c is a good approximation. If $c + j$ is substituted for c in (13), if powers higher than the first in j are neglected, and if it is taken into account that the value of c given by (18) makes the combination of the last two terms in (13) equal to zero, the following expression is obtained for the correction j to the approximate wave velocity

$$(20) \quad \left[4c^{I^3} + 2\sigma a^I a^{II} c^I c^{II}(c^I + c^{II}) - 2\frac{g}{\alpha}\sigma\,(a^{II}c^{II} + a^I c^I)\right] j = -\,c^{I^4} - \sigma a^I a^{II} c^{I^2} c^{II^2}\,.$$

It should be noted that the value for c to be inserted in this formula is the one given by (19). For any particular numerical example it is simplest to check the approximation represented by (19) by direct computation of the correction term j from (20).

After the wave velocity c has been determined, the distribution of the vertical amplitude can be found from (14) and (15). As (14) shows, the velocity of the undisturbed current does not affect the distribution of the vertical amplitude in the lower layer if the amplitude of the internal discontinuity Z_1 is taken as the basic constant. But

U^I and U^{II} affect the value of the vertical amplitude in the upper layer, since (15) contains $c^I = c - U^I$, and since c depends on U^I and U^{II}. However, it can be seen that the effect of the current on the distribution of the vertical amplitude as a function of the depth will be small. With the actually occurring current velocities, c^I will be considerably smaller than g/α. Consequently, the first terms in the numerator and denominator of (15) will be considerably larger than the second terms, except very close to the free surface, $z = 0$. Consequently

$$z^I \approx - \frac{\sinh \alpha z}{\sinh \alpha h^I},$$

except in the closest vicinity of the free surface, and the distribution of the vertical amplitude is practically independent of the current velocity.

In order to amplify this statement, consider the smallest possible value of g/α for which the waves do not become exponentially unstable. In this case the square root in (19) is zero, so that

$$\frac{g}{\alpha} = \frac{(U^I - U^{II})^2}{a^I + a^{II}} a^I a^{II} \frac{\rho^{II}}{\rho^{II} - \rho^I},$$

and

$$\frac{g}{\alpha c^{I2}} = \frac{a^I + a^{II}}{a^{II}} a^I \frac{\rho^{II}}{\rho^{II} - \rho^I}.$$

Since the percentual density difference $(\rho^{II} - \rho^I)/\rho^{II}$ is a small quantity, $g/\alpha c^{I2}$ could be comparable to one only if $(a^I + a^{II})a^I/a^{II}$ were small. But since a^I is a hyperbolic cotangent it cannot be less than one, and it follows that a^{II} would have to be negative, which is impossible.

The case of long waves may also be considered separately. When the wave length is large compared to the depth of the fluid layer, the hyperbolic cotangent may be replaced by the reciprocal of its argument. The expression (19) for the velocity of the internal wave then becomes

$$(21)\quad c = \frac{U^I h^{II} + U^{II} h^I}{H} \pm \left[g \frac{h^I h^{II}}{H} \frac{\rho^{II} - \rho^I}{\rho^{II}} - \frac{(U^I - U^{II})^2 h^I h^{II}}{H^2} \right]^{\frac{1}{2}}.$$

Further, since the hyperbolic sine may be replaced by its argument, from (14)

$$z^{II} = \frac{z + H}{h^{II}} Z_1 \, e^{i(ax-\beta t)} \,, \tag{22}$$

and since the hyperbolic cosine may be replaced by one,

$$z^{I} = \frac{\dfrac{gz}{c^{I^2}} + 1}{-\dfrac{gh^I}{c^{I^2}} + 1} Z_1 \, e^{i(ax-\beta t)} \,. \tag{23}$$

Equation (21) shows that in the case of long waves the vertical amplitude increases linearly from the bottom to the internal boundary, as is well known in the absence of a basic current.

It can be shown similarly that in the upper layer the vertical amplitude decreases practically linearly from its maximum value at the internal boundary to the free surface. This statement is certainly true as long as the depth h^I of the upper layer is not very small, because for sufficiently large h^I the first terms in the numerator and denominator of (22) are much larger than one. Only for very small depths z the first term in the numerator of (22) will be of the order one, but at these levels the vertical motion due to the internal waves is very small anyway. If h^I is small compared to the total depth H of the fluid, it follows from (20) that approximately

$$c^{I^2} = gh^I \frac{\rho^{II} - \rho^I}{\rho^{II}} - \frac{(U^I - U^{II})^2 \, h^I}{H} \,.$$

If $h^I = 10$ m, $H = 1000$ m and $(\rho^{II} - \rho^I)/\rho^{II} = 2 \cdot 10^{-3}$, for instance,

$$\frac{gh^I}{c^{I^2}} \sim \frac{\rho^{II}}{\rho^{II} - \rho^I} \,,$$

so that the first terms in the denominator and numerator of (22) are still considerably larger than unity except very close to the surface, where the amplitude is very small, as pointed out before.

In order to show the modifications in the wave velocity produced by a current, the following example may be considered:

$$\rho^I = 1.024 \qquad \rho^{II} = 1.026$$
$$h^I = 100 \text{ m} \qquad h^{II} = 3000 \text{ m}$$
$$L = 1000 \text{ m}.$$

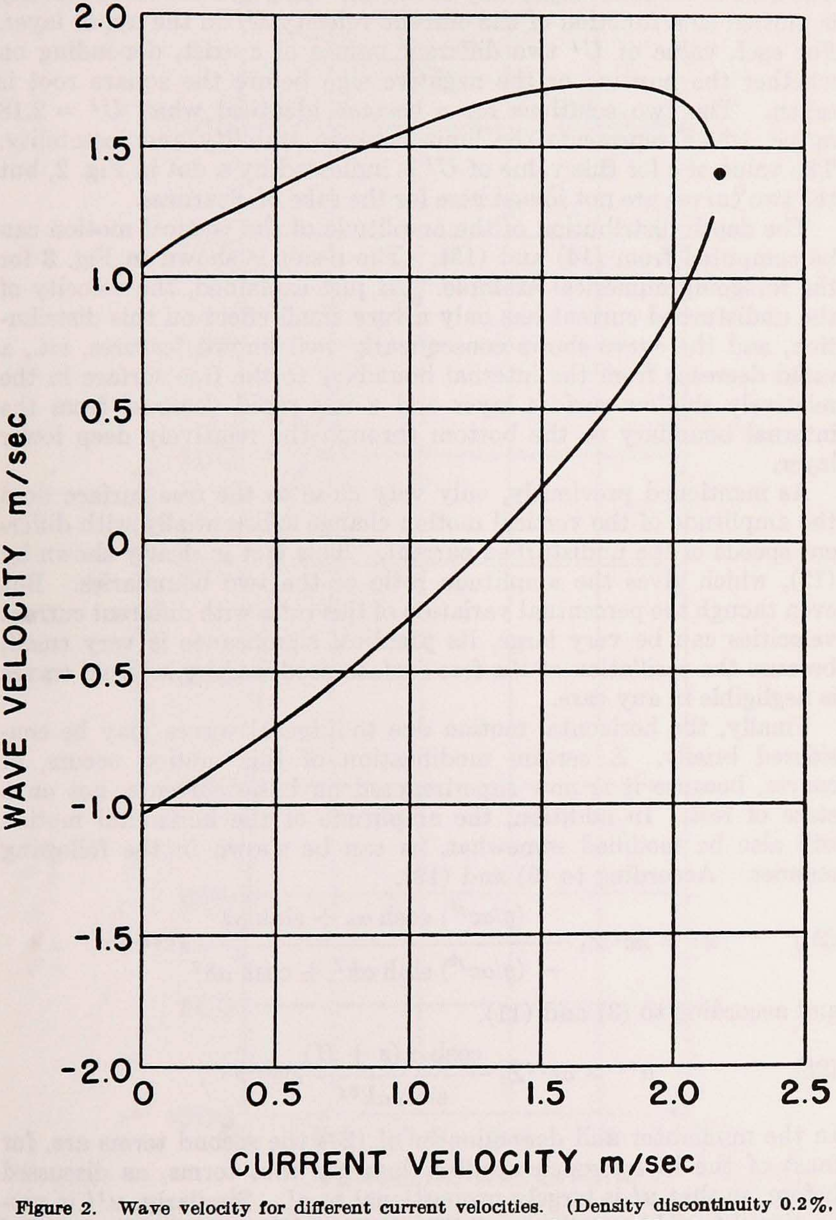

Figure 2. Wave velocity for different current velocities. (Density discontinuity 0.2%, depth of upper layer 100 m and of lower layer 3,000 m, wave length 1,000 m.)

The fluid in the lower layer may be at rest. In Fig. 2 the wave velocity is plotted as a function of the current velocity U^I in the upper layer. For each value of U^I two different values of c exist, depending on whether the positive or the negative sign before the square root is taken. The two solutions for c become identical when $U^I = 2.18$ m/sec, which represents the limit between stability and instability. The value of c for this value of U^I is indicated by a dot in Fig. 2, but the two curves are not joined here for the sake of clearness.

The depth distribution of the amplitude of the vertical motion can be computed from (14) and (15). The result is shown in Fig. 3 for the foregoing numerical example. As just explained, the velocity of the undisturbed current has only a very small effect on this distribution, and the curve shows consequently well known features, *viz.*, a rapid decrease from the internal boundary to the free surface in the relatively shallow surface layer and a less rapid decrease from the internal boundary to the bottom through the relatively deep lower layer.

As mentioned previously, only very close to the free surface does the amplitude of the vertical motion change substantially with different speeds of the undisturbed current. This fact is clearly shown by (12), which gives the amplitude ratio of the two boundaries. But even though the percentual variation of this ratio with different current velocities can be very large, its practical significance is very small, because the oscillation of the free surface produced by internal waves is negligible in any case.

Finally, the horizontal motion due to internal waves may be considered briefly. A certain modification of this motion occurs, of course, because it is now superimposed on basic currents, not on a state of rest. In addition, the amplitude of the horizontal motion will also be modified somewhat, as can be shown in the following manner. According to (9) and (12),

$$(24) \qquad u^I = \alpha c^I Z_1 \frac{(g/\alpha c^{I^2}) \cosh \alpha z + \sinh \alpha z}{-(g/\alpha c^{I^2}) \sinh \alpha h^I + \cosh \alpha h^I} e^{i(\alpha x - \beta t)}$$

and according to (3) and (11),

$$(25) \qquad u^{II} = \alpha c^{II} Z_1 \frac{\cosh \alpha (z + H)}{\sinh \alpha h^{II}} e^{i(\alpha x - \beta t)} .$$

In the numerator and denominator of (23) the second terms are, for most of the layer, much smaller than the first terms, as discussed before, so that u^I is largely proportional to c^I. Similarly, u^{II} is proportional to c^{II} according to (24). Any modifications in the value of

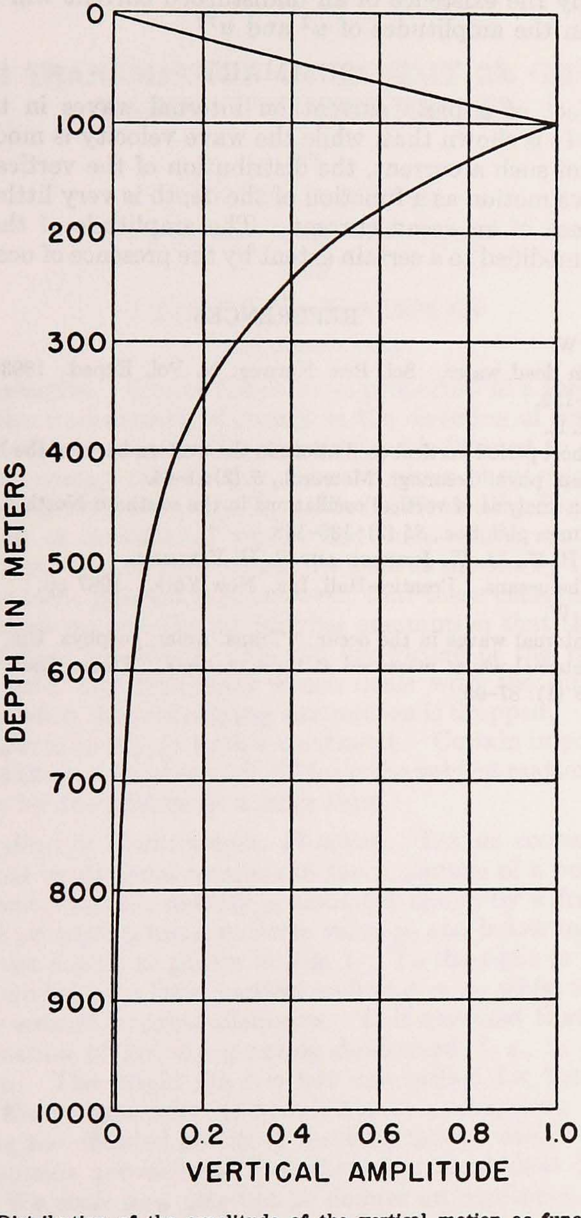

Figure 3. Distribution of the amplitude of the vertical motion as function of depth (amplitude at the internal discontinuity equal to one).

c caused by the existence of an undisturbed current will therefore be reflected in the amplitudes of u^I and u^{II}.

SUMMARY

The effect of a basic current on internal waves in the ocean is studied. It is shown that, while the wave velocity is modified by the existence of such a current, the distribution of the vertical amplitude of the wave motion as a function of the depth is very little affected by the presence of an ocean current. The amplitude of the horizontal motion is modified to a certain extent by the presence of ocean currents.

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