ON DISPLACEMENTS AND INTENSITY CHANGES OF ATMOSPHERIC VORTICES

By
C.-G. ROSSBY
Institutet för Meteorologi
Stockholm, Sweden

Aerological investigations conducted during the last few years have revealed that the long, meandering disturbances characteristic of the middle latitude belt of westerlies in the upper troposphere frequently increase in amplitude until troughs and ridges finally are cut off from the main current, thus forming cold cyclonic vortices on the equatorial side of the west-wind belt and warm anticyclonic vortices on the north side thereof. True vortices, with nearly vertical axes, are thus characteristic phenomena not only of equatorial regions where they appear as tropical cyclones but also of middle latitudes. The middle latitude vortices seem to be phenomena of great stability which have sometimes been observed to persist for several weeks.

In view of these observations it may be of interest to call attention to a dynamic factor which appears to be of some significance to the intensity fluctuations and displacements of simple vortices with vertical axes. A few elementary statements with regard to this factor will be made below and an attempt will be made to suggest a few practical consequences of the results obtained.

We shall consider first a simple vortex, in which the air *at a given instant* spins around the vertical axis with a relative angular velocity ω, which may be a function of r, the horizontal distance from the axis, and of z, the height above sea level, but *not* of the azimuth angle φ. Outside a certain distance R, the outer boundary of the vortex, the air does not participate in the vortex motion.

In such a vortex the Coriolis' force acting on an air parcel to the north of the center will be somewhat greater than the Coriolis' force acting on a correspondingly located parcel south of the center, the difference depending upon the rate of variation of the Coriolis' parameter with latitude. Thus the vortex will be acted upon by a resultant force F, the magnitude of which may be computed quite readily. If one introduces a coordinate system with its center in the axis of the vortex, with the x-axis pointing towards the east and the y-axis towards the north, it follows that the distribution of the Coriolis' parameter (f) may be given by

(1) $$f = f_0 + \beta y = f_0 + \beta r \sin \varphi ,$$

where f_0 is the value of this parameter in the latitude of the center of the vortex and β its rate of change northward. Variations in β will be neglected. The resultant F of the Coriolis' forces acting on a horizontal unit slice is directed northward and has the value

(2) $$F = \rho \int_0^R rdr \int_0^{2\pi} fr\omega \sin \varphi \, d\varphi .$$

The relative angular velocity ω is positive for cyclonic motion. Substitution of (1) in (2) gives

(3) $$F = \beta \rho \pi \int_0^R r^3 \omega dr .$$

If $\bar{\omega}$ represents a mean value of the angular velocity within the vortex, one finds

(4) $$F = \beta M \frac{\bar{\omega} R^2}{4} ,$$

where

(5) $$M = \rho \pi R^2$$

represents the mass of a horizontal unit slice of the vortex. Since there are no pressure gradients in the environment to balance the force F, it is evident that the vortex will receive an acceleration, northward for $\bar{\omega} > 0$ (cyclonic vortices) and southward for $\bar{\omega} < 0$ (anticyclonic vortices).[1]

Before computing the resulting displacement we shall make a comparison of the kinetic energy of the vortex in two different latitude positions. In this comparison we shall assume the vortex to be completely barotropic, so that the velocity distribution is independent of height. Under those circumstances the distribution of mass will remain practically constant and a fluid ring in a given distance r from the axis will therefore remain in approximately the same distance throughout the life history of the vortex. The absolute circulation C of such a fluid ring remains constant, in spite of the displacement of the vortex. If one indicates the mean value of the Coriolis' parameter in the first position as f_1 and in the second position as f_2, it follows that

[1] The acceleration discussed above should not be confused with the so-called rotor effect, which depends upon the presence of a circulation around the vortex in the outer fluid and upon a translation of the vortex relative to the surrounding medium.

$$2\pi r^2\left(\omega_1 + \frac{f_1}{2}\right) = 2\pi r^2\left(\omega_2 + \frac{f_2}{2}\right), \tag{6}$$

or, in differential form,

$$\delta\omega = -\tfrac{1}{2}\,\delta f\,. \tag{7}$$

The kinetic energy K of a horizontal slice of unit thickness is given by

$$K = \pi\rho\int_0^R r^3\omega^2 dr\,. \tag{8}$$

For the case of solid rotation one finds

$$K = \pi\rho\omega^2\,\frac{R^4}{4}\,. \tag{9}$$

Since in this case the vertical component of the absolute vorticity Z is given by

$$Z = 2\omega + f\,, \tag{10}$$

and since Z, according to (6), must remain constant during the displacement of the vortex, it follows that the kinetic energy conveniently may be written in the form

$$K = \frac{\pi}{16}\,\rho\,(Z - f)^2R^4 = \frac{M}{16}\,(Z - f)^2R^2\,, \tag{11}$$

showing that solid cyclonic barotropic vortices ($Z > f$) suffer a decrease in kinetic energy with northward displacements but gain kinetic energy if forced to move southward.

If the vortex possesses a translational velocity v northward through the surrounding medium, and if one neglects the kinetic energy of the environment, its energy equation must be of the form

$$K + M\,\frac{v^2}{2} = \text{constant}\,. \tag{12}$$

Individual differentiation with time gives

$$\frac{\partial K}{\partial f}\,\frac{df}{dt} + Mv\,\frac{dv}{dt} = 0\,, \tag{13}$$

or, since

$$\frac{df}{dt} = \beta v\,, \tag{14}$$

$$M\,\frac{dv}{dt} = -\,\beta\,\frac{\partial K}{\partial f}\,. \tag{15}$$

It follows from (11) that

$$-\beta \frac{\partial K}{\partial f} = \frac{\pi}{8} \rho (Z - f) \beta R^4 = M \frac{\beta \omega R^2}{4}. \tag{16}$$

Thus

$$M \frac{dv}{dt} = M \frac{\beta \omega R^2}{4}, \tag{17}$$

in complete agreement with the expression for the resultant Coriolis' force given in (4).

To compute the magnitude of the translational velocity produced by this force, we might make use of (12), assuming that $v = 0$ in the first position ($f = f_1$). It follows from (11) and (12) that

$$\frac{v_2^2}{2} = \frac{R^2}{16} [(Z - f_1)^2 - (Z - f_2)^2], \tag{18}$$

or

$$v_2^2 = \frac{R^2}{4} \left[\left(Z - \frac{f_1 + f_2}{2} \right) (f_2 - f_1) \right]. \tag{19}$$

To arrive at the significance of this result we shall apply it to a small, intense vortex of hurricane type. It is then permissible to neglect f in comparison with Z, and (19) reduces to

$$v_2 = \frac{R}{2} \sqrt{Z(f_2 - f_1)} = \sqrt{\frac{R^2 \omega (f_2 - f_1)}{2}}. \tag{20}$$

Assuming $R\omega = 25$ mps, $R = 200$ km, $f_2 - f_1 = 10^{-5}$ sec^{-1}, one finds

$$v_2 = 5 \text{ mps}$$

for the northward drift. This value would be reduced, perhaps by fifty per cent or more, if one were to include in (12) also the kinetic energy of the environment. The order of magnitude of the result appears to be in good agreement with the tendency of tropical hurricanes to drift somewhat to the north and right of the trade wind current in which they are embedded.

We shall next attempt to determine the resulting Coriolis' force on a particularly simple barocline and anticyclonic vortex. For this purpose we select a cold core anticyclonic vortex with the air density ρ, surrounded by a lighter air mass of density ρ', at rest relative to the surface of the earth. To simplify the numerical work as much as possible, we shall select a vortex in which the absolute vorticity vanishes, so that

(21)
$$f + \frac{1}{r}\frac{\partial ur}{\partial r} = 0\,,$$

u being the tangential linear velocity, counted positive for cyclonic motion, and *f* the mean (or center) value of the Coriolis' parameter. It follows that

(22)
$$u = -\frac{fr}{2}\,.$$

This represents an anticyclonic vortex with a cold dome of maximum steepness. To determine the shape of the dome one may make use of the gradient wind equation, which in this case takes the form

(23)
$$\frac{u^2}{r} + fu - \gamma\frac{\partial D}{\partial r} = 0\,,$$

γ being an abbreviation for $\frac{\rho-\rho'}{\rho}\,g$ and D being the depth of a cold air column at a given distance r from the center. From a combination of (22) and (23) one finds

(24)
$$\frac{\partial D}{\partial r} = -\frac{f^2 r}{4\gamma}\,,$$

and, upon integration,

(25)
$$D = \frac{f^2}{8\gamma}\,(R^2 - r^2)\,,$$

R being the distance in which the cold dome boundary reaches the ground. If V is the total volume of the cold air, one finds

(26)
$$fR^2 = 4\sqrt{\frac{\gamma V}{\pi}}\,,$$

and hence

(27)
$$D = \frac{2V}{\pi R^4}\,(R^2 - r^2)\,.$$

The kinetic energy KE of the vortex is readily computed. One finds

(28)
$$KE = \frac{\rho V}{24}\,f^2R^2 = \frac{\rho V}{6}\sqrt{\frac{\gamma V}{\pi}}\,f\,.$$

The variable part of the potential energy of the system (PE) is given by

(29)
$$PE = \int_0^R \rho\gamma\,\frac{D^2}{2}\,2\pi r dr = \frac{\rho V}{6}\sqrt{\frac{\gamma V}{\pi}}\,f$$

and thus potential and kinetic energy are equal. During any displacement of this vortex the total volume V or mass M must remain constant. The total energy E is given by

$$E = \frac{\rho V}{3} \sqrt{\frac{\gamma V}{\pi}} f, \tag{30}$$

and this quantity is thus proportional to the Coriolis' parameter f. The individual variation of E is given by

$$\frac{dE}{dt} = \frac{\rho V}{3} \sqrt{\frac{\gamma V}{\pi}} \beta v = \frac{M}{3} \sqrt{\frac{\gamma V}{\pi}} \beta v, \tag{31}$$

v being the rate of translation northward.

We shall next compute directly the force F acting upon this vortex as a result of the nonbalance of the Coriolis' forces. One finds, as before, if F is counted positive northward,

$$F = \rho \int_0^R rDdr \int_0^{2\pi} (f_0 + \beta r \sin \varphi) u \sin \varphi d\varphi, \tag{32}$$

and, after proper substitutions,

$$F = - M \frac{\beta}{3} \sqrt{\frac{\gamma V}{\pi}}. \tag{33}$$

If one neglects the kinetic and potential energy of the environment, the energy equation for this system must be of the form

$$M \frac{v^2}{2} + E = \text{constant}, \tag{34}$$

or

$$Mv \frac{dv}{dt} = - \frac{dE}{dt} = - \frac{M}{3} \sqrt{\frac{\gamma V}{\pi}} \beta v. \tag{35}$$

It follows that

$$M \frac{dv}{dt} = - M \frac{\beta}{3} \sqrt{\frac{\gamma V}{\pi}}, \tag{36}$$

and this expression is identical with the resultant force F given in (33).

From the two simple examples treated so far we may infer that, if the sense of rotation remains the same at all levels:

A. Axially symmetric cyclonic vortices are driven northward and the energy associated with the rotational motion is decreased and changed into a corresponding increase of the translational kinetic energy;

B. Axially symmetric anticyclonic vortices are driven southward and *the potential and kinetic energy associated with the rotational motion is decreased and changed into a corresponding increase of the translational kinetic energy.*

One should expect that the translation of vortices northward or southward would give rise to Coriolis' forces of the form Mfv, and that these forces would deflect northward moving (cyclonic) vortices towards the east and southward moving vortices towards the west. However, in an earlier study [1] the author has shown that in a nondivergent horizontal fluid sheet of density ρ' and subjected to a constant Coriolis' parameter f, a vertical cylinder of radius a, moving through the fluid at a constant speed v, will be acted upon by a resultant pressure gradient force of the magnitude $\rho'\pi a^2 fv$ which is directed 90° to the left of the translation.[2] This pressure gradient balances the Coriolis' force of the translation completely, or very nearly so, and will thus permit the vortex to move practically straight northward or southward in response to the resultant force F. In the case of a cyclonic vortex of hurricane type which is carried in an easterly trade wind current, the effect of the force F would be to deflect the direction of motion of the storm somewhat poleward of the direction of the steering current.

The resultant force F stems from the local lack of balance between the horizontal pressure gradient on the one hand and the sum of Coriolis' and centrifugal forces on the other hand. One may ask if this local unbalance might not produce form changes whereby the vortex would lose its initial symmetry and ultimately acquire a shape characterized by dynamic balance in all points. If such equilibrium shapes exist, as suggested by Bjerknes and Holmboe [2], they must be characterized by the condition that F vanishes.

It has not yet been possible to undertake an exhaustive study of such possible equilibrium shapes, but a preliminaty analysis indicates that the simple concentrated vortices considered in the present paper are far removed from internal equilibrium. In the case of purely

[2] This result is readily obtained if one makes use of the Bernoulli equation. If the observed motion is steady relative to a coördinate system travelling with the speed c in the positive x-direction (which may have an arbitrary direction as long as f is considered constant), this equation takes the form

$$\frac{d}{dt}\left[p + \rho' \frac{(U - c)^2 + V^2}{2} + \rho' f c y\right] = 0,$$

U and V being the x- and y-components of motion. The contributions to the pressure resultant from the quadratic kinetic energy term cancel out as long as the motion in the environment remains symmetric with respect to the x-axis, and this condition is satisfied if the motion is nondivergent and if f is considered as a constant.

horizontal, nondivergent motion, the problem may be approached as follows:

The horizontal velocity components may be expressed through a stream function ψ, and are given by

$$V = \frac{\partial \psi}{\partial x}, \quad U = -\frac{\partial \psi}{\partial y}. \tag{37}$$

The equations of motion may then be written:

$$\rho \frac{dU}{dt} = \rho f \frac{\partial \psi}{\partial x} - \frac{\partial p}{\partial x}, \tag{38}$$

$$\rho \frac{dV}{dt} = \rho f \frac{\partial \psi}{\partial y} - \frac{\partial p}{\partial y}, \tag{39}$$

or, if the y-axis is directed northward

$$\rho \frac{dU}{dt} = \frac{\partial \Phi}{\partial x}, \tag{40}$$

$$\rho \frac{dV}{dt} = \frac{\partial \Phi}{\partial y} - \rho \beta \psi, \tag{41}$$

where

$$\Phi = \rho f \psi - p. \tag{42}$$

If the atmosphere surrounding the vortex is to be at rest, it follows that ψ must approach zero in great distances from the center of the vortex and p must approach a constant value. If the circulation along the various stream lines surrounding the vortex center is always of the same sense, it follows that ψ must vary in a monotone fashion from the center and outward. Thus, in case of a cyclonic vortex, ψ must grow from a negative minimum value to zero, while in the case of an anticyclonic vortex, ψ must fall from a positive maximum value to zero.

If one integrates (41) over the entire area of such a monotone vortex, it follows that

$$F = \iint \frac{dV}{dt} \rho \, dxdy = -\rho \beta \iint \psi \, dxdy, \tag{43}$$

and this expression is positive for arbitrary cyclonic monotone vortices and negative for arbitrary anticyclonic monotone vortices. It is easily shown that (43), for the case of axially symmetric vortices, leads to the same value for F as was given in (3). Within the limitations set by the simplified theory, one must conclude that there are no monotone vortices of equilibrium type; cyclonic vortices are always subjected to a resultant force directed northward, while in the case of anticyclonic vortices the force is directed southward.

The results of this analysis remain unchanged if one superimposes upon the entire system a zonal current U_0, which may depend upon y. To the previous value of the stream function ψ one must then add a basic function

$$\psi_0 = -\int U_0 dy ,$$

and to the previous pressure p a basic pressure field

$$p_0 = -\rho\int\int U_0 dy ,$$

corresponding to the geostrophic requirement.

The effect of axially symmetric flow out from or in towards the center of the vortex is easily evaluated by an analogous method. Such motions may be described with the aid of a velocity potential θ, which is a function of r only. The corresponding velocity components would be

$$U = \frac{\partial\theta}{\partial x} = \frac{x}{r}\frac{\partial\theta}{\partial r}, \quad V = \frac{\partial\theta}{\partial y} = \frac{y}{r}\frac{\partial\theta}{\partial r} . \tag{46}$$

The horizontal divergence would be given by

$$\operatorname{div}_2 \mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right), \tag{47}$$

and to insure that the area integral of this quantity vanish, one must assume that $\frac{\partial\theta}{\partial r}$, for large values of r, decreases more rapidly than r^{-1}. Radial outflow would then correspond to a divergence area at the center, surrounded by a ring-shaped convergence area. In the case of radial inflow, the distribution of convergence and divergence areas would be reversed. Radial outflow would correspond to a θ-function increasing from a negative minimum at the center to zero at the periphery. Inflow would correspond to a θ-function decreasing from a positive maximum at the center to zero at the periphery. One then finds that these radial motions give rise to a force H, directed eastward and having the value

$$H = -\rho\beta\int\int\theta \, dx \, dy . \tag{48}$$

Thus vortices with radial outflow are accelerated eastward, vortices with radial inflow are accelerated westward.

Displacements eastward or westward of barocline vortices are not accompanied by immediate and direct changes in the thermal wind balance, as is the case when such vortices are displaced northward or southward. Therefore, we shall devote the remainder of this paper to a discussion of the effects of the force F.

The vortices analyzed above are unstable in the sense that they must undergo a continuous displacement and decrease in energy until their rotational energy has been converted into translational energy. It is clear, however, that the entire process is most appropriately described in terms of a relaxation time, defining the total length of the life cycle of the vortex.

In the analysis of the displacement of a cold core vortex presented above, we neglected to consider the kinetic energy of the environment. Inspection of (26) shows that the sea level radius R of the cold core must vary in such a manner that

$$R \approx \frac{1}{\sqrt{f}},$$

and this result, in combination with (26), shows that

$$D_{\max} \approx f.$$

It follows that a displacement of the vortex from 45° N to 30° N would result in an increase in R of about 20% and a decrease in the maximum height of the dome to about 71% of its initial value. This divergence of the cold air must result in convergence of the lighter air aloft and the generation of cyclonic circulation at these levels. As a consequence, the total force F, in this case directed southward, will be diminished, and less rotational (potential and kinetic) energy becomes available for conversion into translational energy.

In the case of purely barotropic vortices we have shown that the displacement in latitude results in a fairly rapid dispersion of the rotational kinetic energy. The qualitative analysis of the effect of low level divergence presented above shows that this dispersion may be reduced, and to some extent replaced, by a concentration of rotational kinetic energy aloft. To explore how far this concentration may proceed it is useful to inspect anew the structure of the force of displacement F, as given in (3). It is evident from this expression, in which ω is multiplied by r^3, that in a cold core vortex the negative contribution to F from a feeble but widespread anticyclonic circulation in the lower layers may dominate over the positive contribution to F from a more intense but restricted cyclonic circulation aloft[3]. *The*

[3] It follows from simple dimensional considerations that a maximum low level anticyclonic wind of 4 mps at a distance of 600 km from the axis would be sufficient to balance a maximum high level cyclonic wind of 64 mps at 200 km from the center, in so far as the force F is concerned.

complex barocline vortex with a broad anticyclonic base may thus continue its displacement southward in spite of strong cyclonic winds aloft until the resultant value of F *vanishes. The vortex has then reached its equilibrium latitude and cyclonic vorticity has been brought southward at high levels.*

Since the source of the rotational cyclonic energy aloft must come from the energy supply in the lower layers, it is important to show that the intensification of the circulation aloft is compensated for by a decrease in the potential energy of the system, *i. e.*, through a sinking of the cold air. Such a transfer of energy upward is generally not possible as long as the vortex remains in a fixed latitude, because of the normally prevailing inertia stability. It would appear from the preceding analysis, however, that a cold core vortex, anticyclonic in the lower layers but cyclonic aloft and subjected to a slight decrease in f, must experience an initial centripetal acceleration in the upper layers and an initial centrifugal acceleration below. Through the resulting direct circulation around the solenoids, with descending motion near the center and ascending motion in the periphery, part of the potential energy will be converted into cyclonic kinetic energy aloft. In this connection it is perhaps of interest to mention that the cold upper level cyclones which form as a result of excessive meandering normally seem to develop over broad low-level anticyclones [3]. It is not suggested that the initial cutting-off process should be attributed to subsidence, since synoptic experience clearly indicates that the mechanism of cutting-off is associated with changes in the field of motion upstream, but it is suggested that the subsequent intensification aloft may be associated with subsidence in the underlying cold anticyclone as it is displaced southward.

In connection with some recent aerological investigations of flow patterns over the western Pacific (region of the Marianas), Riehl [4] finds indications that typhoons have a tendency to develop under anticyclonic cells which have been observed to exist at the 30,000–50,000-foot levels over this part of the Pacific. Riehl attributes the formation of typhoon circulations in the lower and middle troposphere to divergence associated with these upper level anticyclones. It is not improbable that centrifugal accelerations associated with the southward displacement of such upper anticyclones may account for the dynamic unbalance required to initiate the divergence postulated by Riehl. The subsequent, continued removal of air would presumably require a different and more effective mechanism. This problem, recently investigated by Sawyer [5], will not be considered here.

SUMMARY

It is shown that, in any horizontal plane in the atmosphere, isolated closed circulation cells of arbitrary shape are subjected to a resultant force F which, in the case of cyclonic circulations, is directed northward, and in the case of anticyclonic circulations southward. It follows that simple atmospheric vortices with vertical axes have a tendency to move across the latitude circles in such a direction that the absolute value of their circulation relative to the surface of the earth decreases. Furthermore, it is shown, with the aid of two simple examples, that the decrease in potential and rotational kinetic energy of a vortex so displaced is sufficient to account for the increase in translational kinetic energy produced by the force F. It is well known that under normal conditions a symmetric, anticyclonically rotating dome of cold air is prevented by inertia stability from spreading out while remaining in a fixed location. This stability is overcome by the force F, which brings about a displacement of the cold dome equatorwards into regions where the thermal wind balance is such as to permit further sinking and spreading out.

The southward displacement and consequent sinking of cold anticyclonic domes must result in the generation of cyclonic circulations at higher levels. Thus composite, cold-core vortices may arise with a broad anticyclonic base and a concentrated cyclonic circulation aloft, and these vortices must continue to move southward with increasing cyclonic circulation aloft, until the resultant value of F, obtained through integration over the entire vertical extent of the vortex, vanishes. The composite vortex has then reached its equilibrium latitude; in the course of its displacement it has contributed to the southward transport of cyclonic vorticity aloft and to its accumulation near the equilibrium latitude, presumably at the tropopause level.

In a similar manner warm, anticyclonically rotating air masses embedded between the troposphere and the stratosphere, must tend to drift southward and spread out laterally. In so doing they must create a certain amount of convergence and ascending air motion in the underlying parts of the troposphere. If the underlying portion of the atmosphere is convectively unstable, this convergence may facilitate the development of tropical storms.

REFERENCES

(1) Rossby, C.-G.
1936. Dynamics of steady ocean currents in the light of experimental fluid mechanics. Pap. Phys. Oceanogr. Meteorol., *5* (1): 1–43.

(2) Bjerknes, J. and J. Holmboe
1944. On the theory of cyclones. J. Meteorol., *1* (1): 1–22.

(3) Palmén, Erik
1948. Origin of high-level cyclones south of the maximum westerlies. U. G. G. I., Reunion d'Oslo. Progr. Rés. Mémoires, pp. 35–37.

(4) Riehl, Herbert
1948. On the formation of typhoons. Tech. Rep. (mimeo.) Dept. Meteorol., Univ. Chicago.

(5) Sawyer, J. S.
1947. Notes on the theory of tropical cyclones. Quart. J. R. met. Soc., *73* (315–316): 101–126.