SEARS FOUNDATION FOR MARINE RESEARCH
BINGHAM OCEANOGRAPHIC LABORATORY, YALE UNIVERSITY

JOURNAL OF MARINE RESEARCH

VOLUME VII 1948 NUMBER 1

ESTIMATES OF WATER TRANSPORT PRODUCED BY WAVE ACTION

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The writer has received requests on several occasions to give estimates in concise form of the intensity of the transport of water caused by surface waves. It is hoped that the following discussion will be helpful in answering such questions, although it should be pointed out from the beginning that the actual conditions in the oceans depart very significantly from the ideal circumstances presupposed in the theory of the subject, so that great caution must be exercised in applying the theoretical results to practical problems in oceanography, pending comparison with observational data.[1]

[1] Under the actual circumstances in the oceans there are present such additional factors as the effect of the earth's rotation, pronounced irregularities in the waves themselves and, generally speaking, a certain horizontal stress at the surface due to the wind. It would therefore be of great interest to secure data on the actual transport produced by the waves in order to see whether, in spite of the added complexities, the simple theory here presented can be used at least as a first approximation.

In order to speculate concerning the possible state of affairs in an actual case, let us suppose that we are dealing with a large deep ocean which (for simplicity only) we assume to be free from permanent currents. Let us suppose further that there is present a steady and uniform wind which, after a suitable interval of time, establishes a stationary wave regime and also a system of steady drift currents near the surface. Since under these conditions it is necessary that the surface wind stress be exactly balanced by the Coriolis force associated with the momentum of the water, it follows that the *total* water transport must be at right angles to the stress. If there is to be

Dealing only with plane and periodic waves on the surface of a basin of very great depth, the instantaneous transport of mass across a fixed vertical plane normal to the direction of wave propagation (per unit distance along the crests) is given by

$$\int_{-\infty}^{z_0} \rho u dz \, . \tag{1}$$

Here z is the vertical co-ordinate counted positive upward, z_0 is the elevation of the free surface, ρ is the (constant and uniform) density and u is the x-component of the particle velocity. The x-axis is taken positive in the direction of wave propagation. If expression (1) is integrated with respect to time t over one wave period T, we obtain an expression for the net mass of water transferred, μ_t. Thus

$$\mu_t = \int_0^T \int_{-\infty}^{z_0} \rho u dz dt \, . \tag{2}$$

Since we assume that the waves move without alteration of their structure, it is possible to replace the time integration in (2) by a space integration over a wave length. Thus we shall employ the transformation

$$dx = cdt \, ; \quad L = cT \, , \tag{3}$$

where c is the wave speed and L is the wave length, so that (2) becomes

$$\boxed{c\mu_t = m} \, . \tag{4}$$

In equation (4) m is the horizontal momentum per wave length. We thus see that *the horizontal momentum per wave length and per unit*

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a component of this total transport in the direction of wave propagation, it would seem that the waves should run somewhat at an angle to the stress, a little to the right in the northern hemisphere and to the left in the southern. As an illustration, a wind of 15 m/sec would eventually establish waves which we might estimate to be 3 m in amplitude and to have a period of 13 sec. The same wind would produce a stress of about 7 dynes/cm² according to customary formulae. Computation shows that at middle latitudes the ratio of the wave transport to the total transport should be roughly about as 2 is to 7, so that the angle between the stress and the direction of wave propagation is 16 degrees. For the purpose of this illustration it has been assumed that the wave transport is given by Table I.

It is not clear, however, what the situation might be when waves are present but when the surface stress vanishes, since under such circumstances the total transport should also vanish.

TABLE I. APPROXIMATE VOLUME TRANSPORT DUE TO SURFACE WAVES IN DEEP WATER

(g = 980 cm per sec per sec)

				a = 0.5 m			a = 1.0 m			a = 2.0 m			a = 3.0 m			a = 4.0 m			a = 5.0 m			a = 6.0 m		
T	L	c	d	V_t	h	u_0	V_t	h	u_0	V_t	h	u_0	V_t	h	u_0	V_t	h	u_0	V_t	h	u_0	V_t	h	u_0
(sec)	(m)	(m per sec)	(m)	(m^3 per m per sec)	(cm)	(cm per sec)	(m^3 per m per sec)	(cm)	(cm per sec)	(m^3 per m per sec)	(cm)	(cm per sec)	(m^3 per m per sec)	(cm)	(cm per sec)	(m^3 per m per sec)	(cm)	(cm per sec)	(m^3 per m per sec)	(cm)	(cm per sec)	(m^3 per m per sec)	(cm)	(cm per sec)
3	14	04.7	01.1	0.26	05.6	23.4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
4	25	06.2	02.0	0.20	03.1	09.9	0.79	12.6	39.5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
5	39	07.8	03.1	0.16	02.0	05.1	0.63	08.1	20.2	2.51	32.2	81.0	—	—	—	—	—	—	—	—	—	—	—	—
6	56	09.4	04.5	0.13	01.4	02.9	0.52	05.6	11.7	2.09	22.4	46.9	4.71	50.4	105.5	—	—	—	—	—	—	—	—	—
7	76	10.9	06.1	0.11	01.0	01.8	0.45	04.1	07.4	1.80	16.4	29.5	4.04	37.0	66.4	7.18	65.8	118.1	—	—	—	—	—	—
8	100	12.5	07.9	0.10	00.8	01.2	0.39	03.1	04.9	1.57	12.6	19.8	3.53	28.3	44.5	6.28	50.4	79.1	9.82	78.7	123.6	—	—	—
9	126	14.0	10.1	0.09	00.6	00.9	0.35	02.5	03.5	1.40	09.9	13.9	3.14	22.4	31.2	5.59	39.8	55.6	8.73	62.2	86.8	12.57	89.5	125.0
10	156	15.6	12.4	0.08	00.5	00.6	0.31	02.0	02.5	1.26	08.1	10.1	2.83	18.1	22.8	5.03	32.2	40.5	7.85	50.4	63.3	11.31	72.5	91.1
11	189	17.2	15.0	0.07	00.4	00.5	0.29	01.7	01.9	1.14	06.7	07.6	2.57	15.0	17.1	4.57	26.6	30.4	7.14	41.6	47.5	10.28	59.9	68.5
12	225	18.7	17.9	0.07	00.3	00.4	0.26	01.4	01.5	1.05	05.6	05.9	2.36	12.6	13.2	4.19	22.4	23.4	6.54	35.0	36.6	9.42	50.4	52.7
13	264	20.3	21.0	0.06	00.3	00.3	0.24	01.2	01.2	0.97	04.8	04.6	2.17	10.7	10.4	3.87	19.1	18.4	6.04	29.8	28.8	9.00	42.9	41.5
14	306	21.8	24.3	0.06	00.3	00.2	0.22	01.0	00.9	0.90	04.1	03.7	2.02	09.2	08.3	3.59	16.4	14.8	5.61	25.7	23.1	8.08	37.0	33.2
15	351	23.4	27.9	0.05	00.2	00.2	0.21	00.9	00.7	0.84	03.6	03.0	1.88	08.1	06.7	3.35	14.3	12.0	5.24	22.4	18.7	7.54	32.2	27.0
16	399	25.0	31.8	0.05	00.2	00.2	0.20	00.8	00.6	0.79	03.1	02.5	1.77	07.1	05.6	3.14	12.6	09.9	4.91	19.7	15.4	7.07	28.3	22.2
17	451	26.5	35.9	0.05	00.2	00.1	0.19	00.7	00.5	0.74	02.8	02.1	1.66	06.3	04.6	2.96	11.2	08.2	4.62	17.4	12.9	6.65	25.1	18.5
18	505	28.1	40.2	0.04	00.2	00.1	0.17	00.6	00.4	0.70	02.5	01.7	1.57	05.6	03.9	2.79	09.9	06.9	4.36	15.5	10.9	6.28	22.4	15.6
19	563	29.6	44.8	0.04	00.1	00.1	0.17	00.6	00.4	0.66	02.2	01.5	1.49	05.0	03.3	2.65	08.9	05.9	4.14	13.9	09.2	5.95	20.1	13.3
20	624	31.2	49.6	0.04	00.1	00.1	0.16	00.5	00.3	0.63	02.0	01.3	1.41	04.5	02.8	2.51	08.1	05.1	3.93	12.6	07.9	5.65	18.1	11.4

distance along the crests is equal to the wave speed multiplied by the mass of water transported during one wave period per unit distance along the crests. The momentum per wave length is thus the same as that of a mass of water μ_t moving with the wave speed.

In papers published previously by the writer (Starr, 1947a, b), attention was called to a relationship originally established by Levi-Civita, namely that for periodic waves in an irrotational medium,

$$cm = 2e \, ; \left(e = \int_0^L \int_{-\infty}^{z_0} \rho \frac{u^2 + w^2}{2} dz dx \right), \tag{5}$$

where w is the vertical component of the particle velocity, so that e is the kinetic energy per wave length and per unit distance along the crests. Elimination of m between (4) and (5) yields the equation

$$\boxed{e = \tfrac{1}{2} \mu_t c^2} \, . \tag{6}$$

Thus the kinetic energy per wave length and per unit distance along the crests is equal to one half the mass transported per wave period and per unit distance along the crests, multiplied by the square of the speed of propagation. The kinetic energy per wave length is thus the same as that of a mass of water μ_t moving with the wave speed.[2]

In order to provide a means readily grasped by the imagination for representing some of the concepts discussed above, let us suppose that a mass of water μ_t per wave length is spread horizontally in the x-direction so as to form a layer of uniform thickness h. It follows that

$$h = \frac{\mu_t}{\rho L} \, . \tag{7}$$

If v is the potential energy of the actual waves per wave length (and per unit distance along the crests), this energy is equivalent to the work done in lifting the layer h through an appropriate vertical distance d. If, in addition, we impart a uniform horizontal translation to the layer h, equal to the wave speed c, the following statements are true:

(1) The average potential energy of the waves per unit area of sea surface is equal to the work done per unit area in lifting the layer h through a vertical distance

[2] In the papers by the writer already alluded to, it was shown that an equation of the same form as (5) is valid not only for the entire depth of the water, but also for any material layer bounded above and below by streamlines for the motion relative to the moving wave. It remains to be pointed out that an equation of the form (4), and consequently also one of from (6), is also valid for such a material layer.

$$d = \frac{v}{g\rho hL} = \frac{v}{g\mu_t}, \tag{8}$$

where g is the acceleration of gravity.

(2) The average volume transport per unit time and per unit distance along the crests is the same as that of the layer h, namely,

$$V_t = ch = \frac{c\mu_t}{\rho L}. \tag{9}$$

(3) The average horizontal wave momentum per unit area of the sea surface is the same as the horizontal momentum of the layer h per unit area, namely,

$$\frac{m}{L} = \rho ch = \frac{c\mu_t}{L}. \tag{10}$$

(4) The average kinetic energy per unit area of the sea surface is the same as the kinetic energy per unit area of the layer h, namely,

$$\frac{e}{L} = \tfrac{1}{2}\rho hc^2 = \frac{c^2\mu_t}{2L}. \tag{11}$$

It is, however, important to observe that the layer h cannot serve as a dynamic substitute for the wave motion in certain other ways. In particular it is especially important to note that in the layer h the transport of kinetic plus potential energy during one wave period is equal to $e + v$, whereas in the actual wave motion it is generally recognized that this transport is about one half as great, at least as a first approximation.

The material presented above is independent of any restriction to small amplitudes. However, in order to present some idea of actual magnitudes in specific cases we shall now use certain approximate expressions derived from the small-amplitude theory. These are that

$$e = v = \tfrac{1}{4} g\rho a^2 L, \tag{12}$$

$$c^2 = \frac{gL}{2\pi}, \tag{13}$$

where a is the amplitude of the waves at the free surface which is assumed to be of sinusoidal form. From (6), (12) and (13) it follows that

$$\mu_t = \pi\rho a^2. \tag{14}$$

From (7) and (14) it follows that

$$h = \frac{\pi a^2}{L}. \tag{15}$$

From (8), (12) and (14) it follows that

$$d = \frac{L}{4\pi}. \tag{16}$$

From (3), (9) and (14) it follows that

$$V_t = \frac{\pi a^2}{T}. \tag{17}$$

Numerical values of c, T, d, h and V_t are given in Table I, together with u_0 the surface particle transport velocity for various values of amplitude and wave length. The surface particle transport velocity is calculated by means of an approximate formula[3] given by Lamb (1932), namely,

$$\frac{4\pi^2 a^2 c}{L^2} \, exp \, \frac{4\pi}{L} z. \tag{18}$$

APPENDIX

Although it may seem somewhat surprising that the surface wave problem is characterized by three exact relationships as simple as (4), (5) and (6), still it might have been anticipated purely on *a priori* grounds that equations of this form should exist, provided that the present subject is brought into proper relation with another topic in classical hydrodynamics. The writer has in mind the theory of the motion of a solid through an infinite liquid, according to which the solid in certain cases reacts to the action of an external force in a manner as though its inertia were artificially increased by a constant amount depending upon the configuration of the solid and the direction of the applied force. For a discussion of this theory the reader is referred to Lamb (1932).

In order to apply the principle to the wave problem, let us consider an admittedly artificial but nevertheless theoretically satisfactory mode of generating surface waves of a given wave length and height. We shall suppose that we have at our disposal a rigid corrugated metal sheet of negligible mass per wave length, having the precise shape of the profile of the waves to be generated and also having a large number of wave lengths. We may suppose that the waves are to be generated

[3] This formula was obtained through the use of the second-order approximation to the solution of the wave problem. It is to be noted that when integrated it gives a result in agreement with (17). Mr. G. W. Platzman has informed the writer that he has carried out a computation of the transport velocity which includes terms of the eighth order. From his results it appears that, for waves of the extreme form, the surface transport velocity is equal to about one fifth of the wave speed.

in a canal of great depth but of unit width. If the sheet is placed on the surface of the water in such a way that no space remains between it and the water, an amount of work equal to v will have been done per wave length in producing the deformations.

Let us suppose next that, with the sheet constrained to move horizontally, a certain constant arbitrary force f per wave length is applied horizontally to the sheet. The effect will be to accelerate the sheet until after some appropriate time t_1 it will be moving with the velocity c, the (predetermined) wave speed corresponding to the chosen wave length and wave height. At this time the force f may be discontinued and the sheet removed.

The process which has taken place may be analyzed as follows. Assuming that, with the sheet in place at time $t = o$, the water is at rest, all the motions subsequently generated are irrotational and hence determined completely by the motion of the boundary. Except for end effects which become of negligible importance if a very large number of wave lengths are used, at time $t = t_1$ the motion of the boundary is the same as the motion of the geometric curve representing the moving profile of the desired waves, hence the particle motions must be those corresponding to the desired waves and the dynamic reactions of these motions will produce a constant pressure at the upper boundary as is needed. If the force f is taken very large, in the limit the motions may be considered as being started instantaneously by impulsive action, although it will be more convenient to suppose that the process takes place during a finite time interval.

At this point an assumption must be made which however seems reasonable. The sheet used, instead of being immersed in fluid, is in contact with water only on the lower side. Since, in the limit when the number of wave lengths used is very large, the two regions above and below are independent of each other, we shall assume that the artificial mass of the sheet is simply some fraction of what it would be in the case of immersion. Actually the presence of the artificial inertia is due to the fact that as the solid object is accelerated certain motions must be generated in the fluid. In our case the motions generated in the water are known to be periodic in the x-direction so that the artificial mass must be regularly distributed along the sheet. Without making any supposition as to magnitude, we thus assume that this virtual mass is an amount μ_t per wave length and proceed to write an equation of motion for it, namely,

$$\mu_t \frac{d^2x}{dt^2} = f\,. \tag{19}$$

Since μ_t is a constant determined by the shape of the sheet, a single integration gives

$$\mu_t \frac{dx}{dt} = ft\,; \quad \left(\frac{dx}{dt} = 0 \text{ for } t = 0\right). \tag{20}$$

For time $t = t_1$, $dx/dt = c$ and $ft = m$, so that $c\mu_t = m$, which is (4). A second integration gives

$$\mu_t(x - x_0) = \tfrac{1}{2} ft^2 \;\; ; \;\; (x = x_0 \text{ for } t = 0)\,. \tag{21}$$

For time $t = t_1$, the quantity $f(x - x_0)$ is the work done, which must be equal to e the added energy, so that, with the aid of (20),

$$e = f\,(x_1 - x_0) = \tfrac{1}{2}\frac{f^2 t_1^2}{\mu_t} = \tfrac{1}{2}\,\mu_t c^2\,, \tag{22}$$

which is (6). Elimination of μ_t between (4) and (6) finally gives (5).

The writer wishes to express his thanks to Mr. G. W. Platzman for reading the manuscript.

SUMMARY

In this paper it is shown that for periodic surface gravity waves in a basin of great depth there exists a theoretical integral relationship according to which the horizontal momentum per wave length and per unit distance along the crests is equal to the wave speed multiplied by the mass of water transported horizontally during one wave period per unit distance along the crests. It is also shown that, in the case of irrotational waves, the kinetic energy per wave length and per unit distance along the crests is equal to one half the mass of water transported per wave period and per unit distance along the crests, multiplied by the square of the speed of propagation. An attempt is made to explain the existence of these relationships, as well as the existence of another similar integral discussed by the writer previously, by bringing the surface-wave problem under the general scope of the classical theory of the motions of solids through liquids.

The various concepts arising in the discussion are illustrated by means of a simple dynamic analogue consisting of a plane horizontal sheet of water moving with the wave speed and having the same potential energy, mass transport, and kinetic energy as the actual wave motion. A table giving the magnitude of the water transport in specific cases is included. This table was prepared with the aid of the small-amplitude theory of gravity waves.

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