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NOTES ON THE DISTRIBUTION OF ENERGY AND FREQUENCY IN SURFACE WAVES

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The behavior of irrotational surface waves of small amplitude in deep water has been thoroughly investigated through theoretical studies extending over a period of more than one hundred years. Waves of this type are therefore particularly well suited for demonstration of the characteristics of energy and frequency distribution and propagation in dispersive media. In the present note a few comments will be made on the distribution of energy with respect to frequency in such waves.

A. *On the energy spectrum of waves emanating from a point source.*

In a recently published paper (Rossby, 1945) the author has derived a general equation relating frequency (ν) and wave number (k) in progressive plane waves. If the x -axis represents the direction of propagation of the waves, one finds

$$\frac{\partial \nu}{\partial x} + \frac{\partial k}{\partial t} = 0. \quad (1)$$

The derivation of this equation is based on the assumption that wave crests are conserved and that ν and k are slowly varying functions of x and t .

If the physical theory for the particular wave motion under investigation leads to a relation between ν and k so that

$$\phi(\nu, k) = 0, \quad (2)$$

or

$$c = F(L), \quad (3)$$

where c is the phase velocity and L is the wave length, it follows that (1) may be written in the form

$$\frac{\partial \nu}{\partial t} + c_g \frac{\partial \nu}{\partial x} = 0, \quad (4a)$$

or

$$\frac{\partial c_g}{\partial t} + c_g \frac{\partial c_g}{\partial x} = 0, \quad (4b)$$

where

$$c_g = c - L \frac{dc}{dL} = \frac{dv}{dk} \quad (5)$$

is the group velocity.

If the energy per unit area (or unit length along the axis of wave propagation) is designated by E , it is normally assumed that the energy propagation obeys the law

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x} c_g E = 0. \quad (6)$$

This equation is usually interpreted to mean that energy is propagated with the group velocity. A less restrictive interpretation, and formally equally correct, is to state that the flux of energy across consecutive planes moving with the group velocity does not vary from plane to plane. Such an interpretation would obviously imply that energy may be transferred from one end of the wave train to the other, or that, if the distribution of frequency with respect to x at all times is monotone, energy may be transferred from one end of the energy spectrum to the other.

If ν is a monotone function of x , but varying with time, it is possible to introduce a specific energy per frequency interval, so that

$$E_\nu \delta \nu = \pm E \delta x. \quad (7)$$

In that case (6) changes into

$$\frac{\partial}{\partial t} \left(E_\nu \frac{\partial \nu}{\partial x} \right) + \frac{\partial}{\partial x} \left(E_\nu c_g \frac{\partial \nu}{\partial x} \right) = 0, \quad (8)$$

or, because of (4a),

$$\frac{\partial E_\nu}{\partial t} + c_g \frac{\partial E_\nu}{\partial x} = 0. \quad (9)$$

It follows through comparison of (4a) and (9), that the Jacobian

$\frac{\partial(E_\nu, \nu)}{\partial(x, t)}$ vanishes and hence,

$$E_\nu = f(\nu). \quad (10)$$

Thus the spectral distribution of E_ν is independent of time. It is obvious that instead of (10) one may substitute an equation of the form

$$E_\lambda = \phi(\lambda), \quad (11)$$

where $\phi(\lambda)$ likewise is independent of time.

Cauchy and Poisson¹ have investigated the surface wave trains generated when an elevated strip along the line $x = 0$ is permitted to seek equilibrium. The elevation of the free surface η is originally concentrated near the point $x = 0$ and

$$\int_{-\infty}^{+\infty} \eta \delta x = 1. \quad (12)$$

Two wave trains are generated, moving along the positive and negative x -axes away from the point source. On the positive x -axis the resulting wave train is given by the asymptotic solution

$$\eta = \sqrt{\frac{g}{8\pi x\sqrt{x}}} \frac{t}{x} \left(\cos \frac{gt^2}{4x} + \sin \frac{gt^2}{4x} \right). \quad (13)$$

The frequency of these waves is given by

$$\nu = \frac{gt}{2x}, \quad (14a)$$

the wave length λ by

$$\lambda = \frac{8\pi x^2}{gt^2}, \quad (14b)$$

and the group velocity by

$$c_g = \frac{x}{t}. \quad (15)$$

The amplitude is given by

$$\eta_0 = \sqrt{\frac{g}{8\pi x\sqrt{x}}} \frac{t}{x}, \quad (16)$$

and since

$$E = \frac{g\rho\eta_0^2}{2}, \quad (17)$$

it follows that

$$E = \frac{\rho g^2 t^2}{16\pi x^3}. \quad (18)$$

From the relation

$$E_\lambda \delta \lambda = E \delta x, \quad (19)$$

one finds

$$E_\lambda = \frac{\rho g^3 t^4}{256\pi^2 x^4} = \frac{\rho g}{4\lambda^2}, \quad (20a)$$

and the energy spectrum is therefore independent of time, as indicated in (11). A dimensional constant, measuring the original cross-section

¹ For a summary of the investigations of Cauchy and Poisson, see Lamb (1932).

tional area of the elevated portion of the sea surface has been suppressed through the introduction of condition (12), and thus equation (20a) is dimensionally incorrect, but this obviously does not effect the main result, the invariance of $E_\lambda(\lambda)$ with time. It is perhaps worth noting that by means of the transformation

$$E_\nu \delta \nu = -E_\lambda \delta \lambda, \quad (20b)$$

one obtains

$$E_\nu = \text{constant} \cdot \nu. \quad (20c)$$

The results obtained above suggest that energy cannot be transferred from one frequency to another. To explore this point, it is desirable to investigate the time rate of the establishment of the energy spectrum given in (20a).

It is obvious from the asymptotic solution given above, or from the exact solution,

$$\eta = \frac{\sqrt{g} t}{\pi x \sqrt{x}} \int_0^\omega \cos(\zeta^2 - \omega^2) d\zeta, \quad \left[\omega = \sqrt{\frac{gt^2}{4x}} \right], \quad (21)$$

that the longest, fully developed wave observed at any one time corresponds to a fixed, low value (ω_0) of ω . Thus

$$\frac{gt^2}{4x} = \omega_0^2, \quad (22)$$

and, from (14b)

$$\lambda_{\max} = \frac{\pi gt^2}{2\omega_0^2}. \quad (23)$$

Thus, while the asymptotic energy spectrum given in (20a) is independent of time, it follows from (23) that this spectrum is cut off on the low-frequency end; this portion of the spectrum is gradually extended, at an accelerated rate, to lower and lower frequencies (longer and longer wave lengths). A graphical representation of this process is given in Fig. 24.

The energy for this portion of the spectrum must be obtained, either through propagation of energy across planes moving with the group velocity (transfer of energy from higher to lower frequencies), or through the transformation of an initial energy supply into wave energy. In either case, in view of the absence of the low-frequency portion of the spectrum in the early part of the process, it is necessary to assume that some form of energy is transferred, at some time or another during the process, from left to right (toward increasing values of λ) across the group velocity planes.

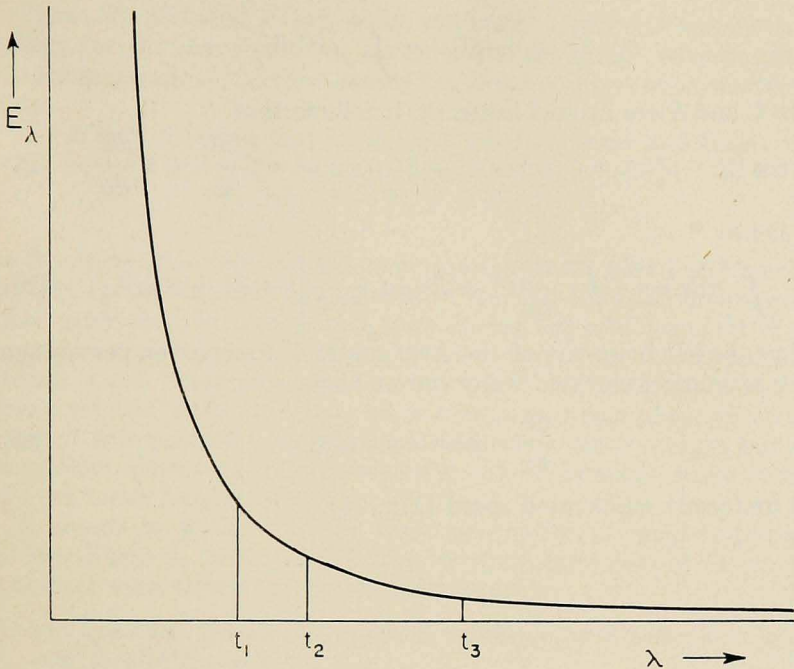


Figure 24. Graphical representation of forward portion of energy spectrum emanating from a point source, and of three successive positions of maximum wave length for fully established wave motion.

To investigate this point further it is helpful to determine the disposition of the original supply of elevated water. We shall compute the total amount of excess water contained at any given time in a unit strip extending from x to infinity.

One finds, from (21)

$$\int_x^\infty \eta dx = \frac{\sqrt{g}}{\pi} \int_0^\infty \frac{tdx}{x\sqrt{x}} \int_0^\infty \cos(\zeta^2 - \omega^2) d\zeta, \quad (24)$$

and, from the definition of ω in (21),

$$d\omega = -\omega \frac{dx}{2x}. \quad (25)$$

Hence

$$\int_x^\infty \eta dx = \frac{4}{\pi} \int_0^\omega d\omega \int_0^\omega \cos(\zeta^2 - \omega^2) d\zeta. \quad (26)$$

If one designates, in a manner convenient for our purposes, but departing from standard usage,

$$C(\omega) + iS(\omega) = \int_0^{\omega} e^{i\xi^2} d\xi, \quad (27)$$

where C and S are Fresnel integrals, it follows that

$$\int_0^{\omega} \cos(\xi^2 - \omega^2) d\xi = C(\omega) \cos \omega^2 + S(\omega) \sin \omega^2 = C \frac{dC}{d\omega} + S \frac{dS}{d\omega}, \quad (28)$$

and hence

$$\int_x^{\infty} \eta dx = \frac{2}{\pi} \int_0^{\omega} \frac{d}{d\omega} (C^2 + S^2) d\omega = \frac{2}{\pi} [C(\omega)^2 + S(\omega)^2]. \quad (29)$$

Thus the left boundary of the strip containing any given percentage of the amount of elevated water moves along a curve,

$$\frac{gt^2}{4x} = \omega^2 = \text{constant}, \quad (30)$$

with uniformly accelerated speed (Fig. 25).

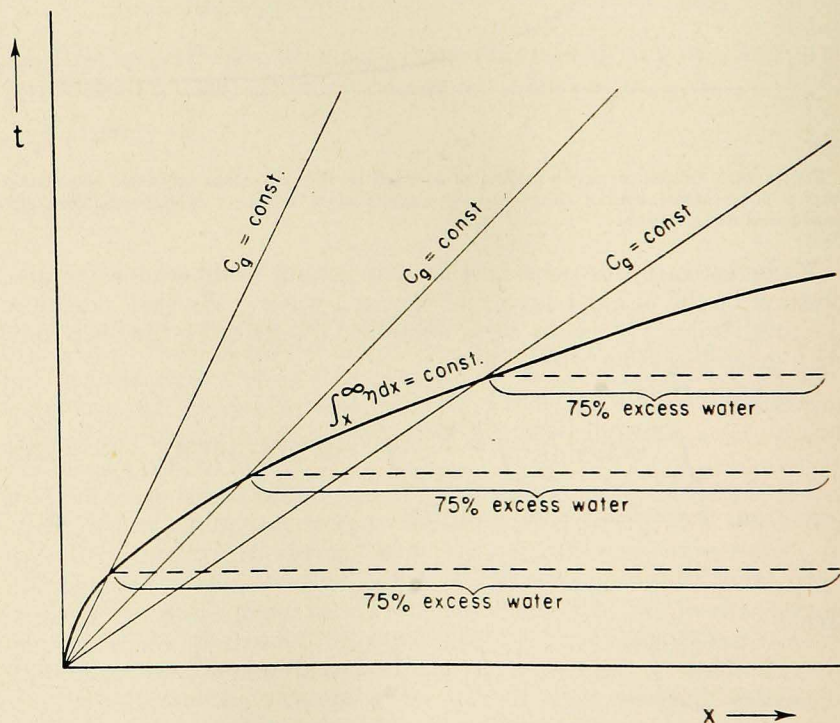


Figure 25. Graphical representation of the sinking and spreading out of excess water mass responsible for the wave motion.

Thus, the elevated water, which obviously represents the source of energy for the wave motion, moves across the group velocity planes at increasing speed. At the same time it spreads out over ever larger areas.

For $x = 0$ one finds that ω becomes infinitely large and thus

$$\int_0^{\infty} \eta dx = \frac{2}{\pi} \left[C_{\infty}^2 + S_{\infty}^2 \right] = \frac{1}{2}, \quad (31)$$

as should be expected, but this asymptotic value is practically reached already for ω_0 values corresponding to the first fully developed wave. One may conclude that at any time during the establishment of the energy spectrum given in (20a) there is a temporary flux of potential energy which has not yet broken down into wave energy, across the group velocity planes and into the low frequency part of the spectrum. Part of this potential energy is left behind as wave energy, but the remainder travels on, at accelerated speed, to establish wave motion of even lower frequency.

It should be added that the basic assumption of small amplitudes is most likely to be satisfied in the low-frequency part of the energy spectrum with which the above analysis deals.

B. *On apparent increases in period resulting from convergent group velocity distributions.*

The general integral of equation (4b) in the preceding section is

$$c_g = f(x - c_g t), \quad (32)$$

or

$$x = c_g t - \phi(c_g), \quad (33)$$

where f and ϕ are arbitrary. In the case of point sources of energy, where no impressed frequencies need to be considered,

$$x = c_g t, \quad (34)$$

represents a typical solution. One may now attempt to generalize (34) by assuming a solution of the form

$$c_g = \frac{x}{t} F\left(\frac{gt^2}{4x}\right), \quad (35)$$

the parameter $\frac{gt^2}{4x}$ being nondimensional. The function F may be

determined through substitution in (4b). Another technique, which leads to similar results, is to deal with (33). From dimensional considerations one should expect that

$$\phi(c_g) = \pm s_o \frac{c_g^2}{g}, \quad (36)$$

g being the only prescribed dimensional constant. Here s_o is a nondimensional, positive constant. Substitution in (33) gives

$$x = c_g t \pm s_o \frac{c_g^2}{g}. \quad (37)$$

For $t = 0$ one obtains

$$x = \pm s_o \frac{c_g^2}{g}. \quad (38)$$

The upper sign pertains to an initial group velocity distribution, beginning with

$$c_g = 0, \quad \text{for } x = 0, \quad (39)$$

corresponding to short wave lengths, and proceeding to very long waves for large positive x -values. This form of (37) represents a divergent group velocity distribution and is of no interest to us in the present analysis. The lower sign gives, for $t = 0$,

$$x = -s_o \frac{c_g^2}{g} \quad (40)$$

or an initial distribution of progressive waves, of vanishing wave length near $x = 0$ and becoming increasingly large for large negative x -values. Since the longer waves travel faster than the shorter, the corresponding group velocity planes must converge and thus give rise to an increase in energy per unit area.

The intersection of two adjacent group velocity lines is obviously obtained, in form (33), as the intersection of

$$x = c_g t - \phi(c_g), \quad (41a)$$

and

$$x = (c_g + \delta c_g) t - \left(\phi + \frac{d\phi}{dc_g} \delta c_g \right). \quad (41b)$$

Thus, in our special case the parametric equation for the curve of intersection is given by (Fig. 26)

$$x = c_g t - s_o \frac{c_g^2}{g}, \quad (42)$$

and

$$0 = t - \frac{2s_o c_g}{g}. \quad (43)$$

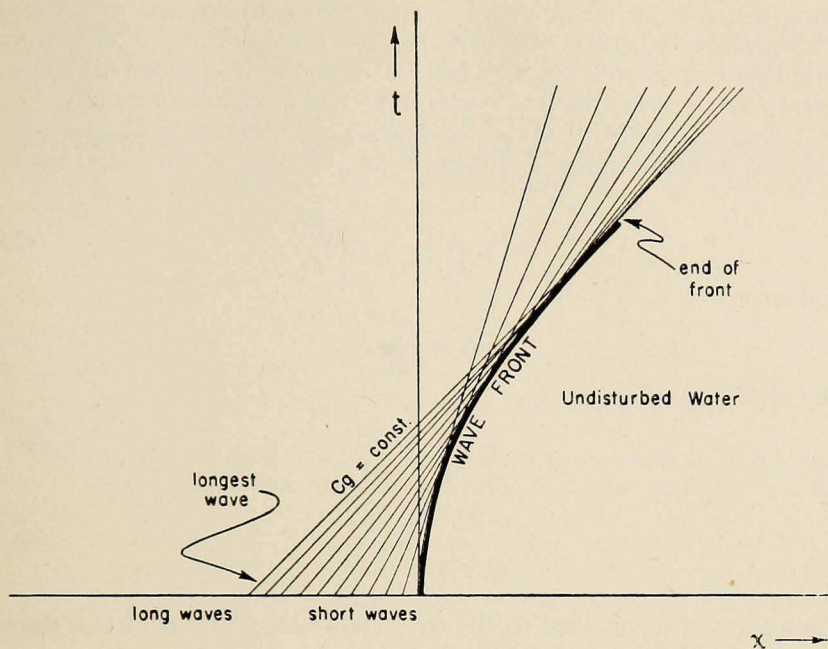


Figure 26. Formation of wave front as a result of a convergent group velocity distribution. Period of wave motion in front increases with time.

Elimination of c_g gives

$$x = \frac{gt^2}{4s_0}. \quad (44)$$

In the region

$$x \geq \frac{gt^2}{4s_0}, \quad (45)$$

the water will be undisturbed. On the line given by (44) a strong concentration of wave energy is suddenly encountered. The larger the value of x (or t), the further from the rear of the original wave train will be the source of the energy, and hence, the longer will be the period.

The variation of period with time is obtained from (43), which gives

$$c_g = \frac{g}{2\nu} = \frac{gT}{4\pi} = \frac{gt}{2s_0}, \quad (46)$$

and thus the period appears to increase with time. *It should be emphasized that the curve given in (44) cannot be extended beyond the maxi-*

imum period T for which energy is available in the original wave train. The process indicated above represents a concentration of energy available in a given frequency belt but does not lead to generation of energy in frequency regions where none was available originally.

It should be added that the nondimensional constant s_0 has a simple physical interpretation. It follows from (40) that in the initial state,

$$c_0 = \sqrt{\frac{-gx}{s_0}} = \frac{gT}{4\pi}, \quad (47)$$

and since

$$L = \frac{gT^2}{2\pi}, \quad (48)$$

one finds

$$-\frac{x}{s_0} = \frac{L}{8\pi}, \quad (49)$$

or

$$\frac{dL}{d(-x)} = \frac{8\pi}{s_0}. \quad (50)$$

Thus $1/s_0$ is proportional to the constant space gradient of the wave length in the initial wave train.

The group velocity distribution given in (37) corresponds to one obtained independently by W. Munk using a different method and recently communicated to the author by letter.

SUMMARY

In the first section of this paper an attempt is made to determine the source of the wave energy in the long-wave forerunner of a train of plane surface gravity waves emanating from a point source. It is shown that the amount of energy $E_\lambda d\lambda$ contained between the wave lengths λ and $\lambda + d\lambda$ is independent of time. Thus there is no transfer of wave energy from one portion of the spectrum to another. It may be shown, however, that the elevated water mass which originally was concentrated at the point source of the wave motion rapidly spreads out and sinks, its potential energy being converted into wave energy. Thus the successive establishment of wave energy E_λ for larger and larger values of λ may be interpreted as the result of a rapid and transient transport of potential energy outward, a small portion of this potential energy supply being converted into wave energy in each successive spectral region.

In the second part of the paper it is shown that a train of progressive surface waves, in which the group velocity of the forward portion is less than that of the rear portion, may give rise to the formation of a sharp progressive wave front, in which the wave period increases with time. This wave front may be given in parametric form as a special solution of the equation for the propagation of group velocity, c_g , namely,

$$\frac{\partial c_g}{\partial t} + c_g \frac{\partial c_g}{\partial x} = 0.$$

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