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SEARS FOUNDATION FOR MARINE RESEARCH
BINGHAM OCEANOGRAPHIC LABORATORY, YALE UNIVERSITY

JOURNAL OF MARINE RESEARCH

VOLUME VI

1946

NUMBER 1

A THEORY OF *T-S* CURVES AS A METHOD FOR STUDYING THE MIXING OF WATER MASSES IN THE SEA¹

By

W. B. STOCKMAN

Laboratory of Oceanology, Academy of Science, U. S. S. R.

A sea or an ocean, according to Defant (1929), is composed of water masses which are more or less homogeneous volumes of water characterized by definite relationships of physico-chemical properties such as temperature and salinity. No less interesting than the important problem of the origin of different water masses is the study of their transformation by the processes of mixing. The so-called method of *T-S* diagrams of Helland-Hansen (1918) have proved to be very useful in this respect. According to this method, temperature (ϑ) and salinity (s) of sea water are expressed as rectangular coordinates in relation to two parameters: depth (z) and time (t). Thus, $s = f(\vartheta, z, t)$. The lines, which on the *T-S* diagram correspond to a constant value of the parameter t , will be referred to as *T-S* curves.

Although the method of *T-S* diagrams is widely used in oceanographic practice, there is as yet no satisfactory theoretical treatment of the subject. In this study an attempt is made to set up the foundations of an analytical theory of *T-S* curves which gives a deeper insight into the essentials of this interesting method. A series of theorems is established which forms the basis of a distinctive "geometry of *T-S* curves" and which can be applied practically. A study is made of the well known method of Jacobsen (1927) for determining the coefficient of convectivity by means of *T-S* curves, and it is shown that

¹ Proofs not seen by author.

there is an error of principle in its geometrical part. The error can be avoided, however, by a deduction (from the basic formula of Jacobsen) which is developed and which is based on demonstrated geometrical properties of T - S curves. Finally, a new and simple geometrical method is developed for determining the coefficient of convectivity by means of T - S curves.

I. GEOMETRICAL PROPERTIES OF T - S CURVES IN THE MIXING OF THREE WATER MASSES

Let us consider the vertical mixing of three water masses in which the intermediate one is limited, and let us locate the plane $z = 0$ in the center of this intermediate water mass, whose boundaries are determined by values $z = \pm h$. The other two water masses are unlimited, extending from $z = h$ to $z = \infty$ and from $z = -h$ to $z = -\infty$. Since we consider here only processes of vertical heat conductivity and diffusion, which tend to equalize the temperature and salinity of water masses, we shall describe these processes by differential equations of the Fourier's type,

$$(1) \quad \frac{\partial \vartheta}{\partial t} = K \frac{\partial^2 \vartheta}{\partial z^2}; \quad \frac{\partial s}{\partial t} = K \frac{\partial^2 s}{\partial z^2},$$

assuming that the coefficient of turbulent mixing K in the region of all three water masses remains constant and is similar for the exchange of both salt and thermal properties.

Let us also assume at the initial moment that the temperature and salinity of the water masses are equal to

$$(2) \quad \begin{array}{ll} 1) \text{ for } \infty > z > h & \vartheta = \Theta_1, s = S_1, \\ 2) \text{ for } h > z > -h & \vartheta = \Theta_2, s = S_2, \\ 3) \text{ for } -h > z > -\infty & \vartheta = \Theta_3, s = S_3. \end{array}$$

Evidently the initial condition is represented on the T - S diagram by three points with coordinates Θ_1, S_1 ; Θ_2, S_2 ; and Θ_3, S_3 , forming a "triangle of mixing." This conception is fundamental for the whole subsequent theory of T - S curves.

Apart from the initial conditions (2), the solution of equations (1) must be subordinated to conditions of continuity of heat and salt exchange at the interfaces of the water masses, *i. e.*,

$$(3) \quad \left(K \frac{\partial \vartheta}{\partial z} \right)_{z=h}^I = \left(K \frac{\partial \vartheta}{\partial z} \right)_{z=h}^{II}; \quad \left(K \frac{\partial \vartheta}{\partial z} \right)_{z=-h}^{II} = \left(K \frac{\partial \vartheta}{\partial z} \right)_{z=-h}^{III}$$

$$\vartheta_{z=h}^I = \vartheta_{z=h}^{II}; \quad \vartheta_{z=-h}^{III} = \vartheta_{z=-h}^{II},$$

and similarly for salinity. In the case of temperature the solutions of equations (1) will be:

for the region $\infty > z > h$ (water mass I)

$$(4) \quad \vartheta = \frac{1}{2} \left[\Theta_1 + \Theta_3 + (\Theta_1 - \Theta_2) \Phi \left(\frac{z-h}{2\sqrt{Kt}} \right) + (\Theta_2 - \Theta_3) \Phi \left(\frac{z+h}{2\sqrt{Kt}} \right) \right],$$

for the region $h > z > -h$ (water mass II)

$$(4') \quad \vartheta = \frac{1}{2} \left[\Theta_1 + \Theta_3 - (\Theta_1 - \Theta_2) \Phi \left(-\frac{z-h}{2\sqrt{Kt}} \right) + (\Theta_2 - \Theta_3) \Phi \left(\frac{z+h}{2\sqrt{Kt}} \right) \right],$$

for the region $-h > z > -\infty$ (water mass III)

$$(4'') \quad \vartheta = \frac{1}{2} \left[\Theta_1 + \Theta_3 - (\Theta_1 - \Theta_2) \Phi \left(-\frac{z-h}{2\sqrt{Kt}} \right) - (\Theta_2 - \Theta_3) \Phi \left(-\frac{z+h}{2\sqrt{Kt}} \right) \right].$$

In these formulas (4), $\Phi ()$ stands for Gauss' integral of errors (function of probabilities)

$$\Phi (x) = \frac{2}{\pi} \int_0^x e^{-\eta^2} d\eta.$$

The formulas for salinity are analogous (4). An analysis of the formulas of this type (4) will make it possible to establish a series of important deductions referring to the geometrical properties of *T-S* curves.

First of all it should be noted that $\Phi (\infty) = 1$. Therefore at $t = 0$ the temperature and salinity at the limits of the intermediate water mass II are equal respectively to half the sums of the temperatures and of the salinities adjoining the given limit:

$$(5) \quad \vartheta_{z=h} = \frac{\Theta_1 + \Theta_2}{2}; \quad s_{z=h} = \frac{S_1 + S_2}{2}$$

$$\vartheta_{z=-h} = \frac{\Theta_2 + \Theta_3}{2}; \quad s_{z=-h} = \frac{S_2 + S_3}{2}.$$

The latter values evidently correspond to the initial values of temperature and salinity established at the interface between two limited or unlimited water masses. However, as seen from formulas (4), with an increase of time (t) the intermediate layer degenerates, and temperatures and salinities of the three water masses tend to reach new stationary values,

$$\text{limit } (\vartheta)_{t \rightarrow \infty} = \frac{\Theta_1 + \Theta_3}{2}; \quad \text{limit } (s)_{t \rightarrow \infty} = \frac{S_1 + S_3}{2},$$

corresponding to the final stage of mixing.

Let us now follow the transformation in the system $s = f(\vartheta)$ of the "nucleus" of the intermediate water mass II, defining this term as the central region corresponding to a value of parameter $z = 0$. On the basis of the second formula (4), assuming $z = 0$, we obtain the following expression for the temperature of the "nucleus,"

$$(6) \quad \vartheta = \frac{\Theta_1 + \Theta_3}{2} + \Phi\left(\frac{h}{2\sqrt{Kt}}\right)\left(\Theta_2 - \frac{\Theta_3 + \Theta_1}{2}\right),$$

and we obtain an analogous formula for salinity:

$$(7) \quad s = \frac{S_1 + S_3}{2} + \Phi\left(\frac{h}{2\sqrt{Kt}}\right)\left(S_2 - \frac{S_3 + S_1}{2}\right).$$

Excluding the function $\Phi(\)$ from (6) and (7), we obtain an equation that characterizes the transformation of the "nucleus" of water mass II in the system $s = f(\vartheta)$,

$$(8) \quad s - \frac{S_1 + S_3}{2} = \frac{S_2 - \frac{S_1 + S_3}{2}}{\Theta_2 - \frac{\Theta_1 + \Theta_3}{2}}\left(\vartheta - \frac{\Theta_1 + \Theta_3}{2}\right).$$

This formula (8) is nothing more than an equation for a straight line passing through two points: point II (Fig. 1) with coordinates Θ_2, S_2 and point P with coordinates $\frac{\Theta_1 + \Theta_3}{2}, \frac{S_1 + S_3}{2}$. In Fig. 1 the scheme of transformation of the water masses is indicated by arrows. The straight line II P, which is "the principal median" of the triangle of mixing I II III, divides its plane into a positive and a negative region. The results obtained are summarized by the following theorem:

1. *The geometrical position of points with a value of parameter $z = 0$, characterizing the transformation of the "nucleus" of the intermediate water mass with the course of time, represents the principal median of the triangle of mixing drawn from that apex of the triangle which corresponds to the intermediate water mass.*

Now let us determine the slope of the tangent to the T - S curve. Since the temperature is a function of two variables, s and z , at every point of the T - S curve, the slope of the tangent at any point of the T - S curve may be computed according to formula

$$(9) \quad \frac{d\vartheta}{ds} = \frac{\partial\vartheta}{\partial z} \bigg/ \frac{\partial s}{\partial z}.$$

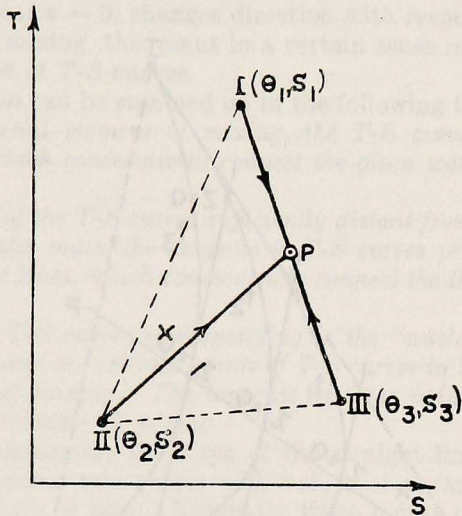


Figure 1.

Constructing $\frac{\partial \vartheta}{\partial z}$ and $\frac{\partial s}{\partial z}$ comparably to expressions of type (4), after some simple transformations we obtain, according to (9),

$$(10) \quad \frac{d\vartheta}{ds} = \frac{\Theta_1 - \Theta_2 + (\Theta_2 - \Theta_3) e^{-\frac{hz}{Kt}}}{S_1 - S_2 + (S_2 - S_3) e^{-\frac{hz}{Kt}}},$$

from which it follows that, at the initial moment of mixing, when t is very small and $\frac{hz}{Kt}$ very high, the slope of the T - S curve for the region $z > 0$ is determined by expression

$$(11) \quad \frac{d\vartheta}{ds} = \frac{\Theta_1 - \Theta_2}{S_1 - S_2} = \text{Constant}$$

and for the region $z < 0$ by expression

$$(12) \quad \frac{d\vartheta}{ds} = \frac{\Theta_2 - \Theta_3}{S_2 - S_3} = \text{Constant}.$$

Thus, at the initial moment of mixing the T - S curve is characterized by straight lines, which on the T - S diagram consecutively connect the water masses I-II, II-III, as shown by the broken line in Fig. 2.

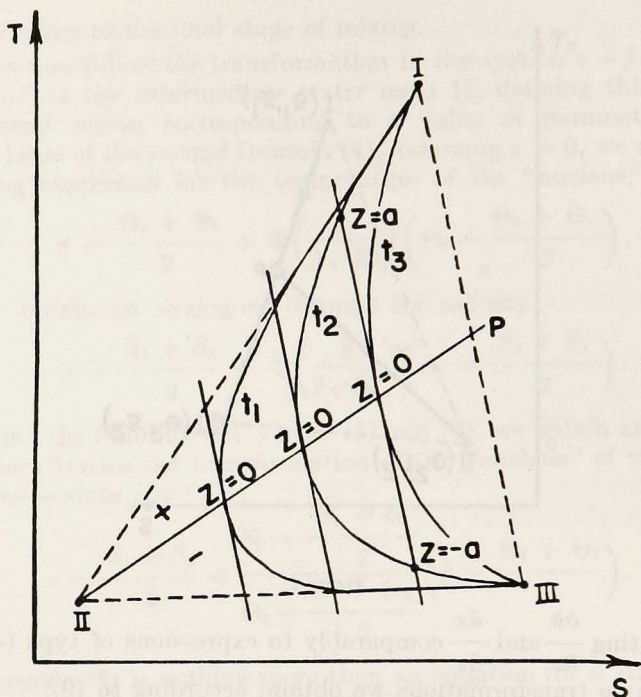


Figure 2.

It is also obvious from formulas (4) and (10) that, at $t = \text{Constant}$, the branches of the T - S curve asymptotically approach these same straight lines I-II, II-III as z increases, *i. e.*, with increasing distance from the central region of the intermediate water mass. This approach is the more rapid the greater the width ($2h$) of the intermediate water mass, or the lower the value of t (for given values of h and z).

And finally, it follows from formula (10) that the slope of the tangent at points of the T - S curve corresponding to the "nucleus" of the intermediate water mass (*i. e.*, at points with a value of parameter $z = 0$) is determined by the expression

$$(13) \quad \left(\frac{d\theta}{ds} \right)_0 = \frac{\Theta_1 - \Theta_3}{S_1 - S_3} = \text{Constant.}$$

This signifies that tangents drawn through the points mentioned are parallel to the side of the triangle of mixing which is opposite to the intermediate water mass (Fig. 2). This side will be named the *base of the triangle of mixing*. Since the slope of the T - S curve, when

crossing the point $z = 0$, changes direction with respect to the base of the triangle of mixing, this point in a certain sense may be called the maximum point of T-S curves.

Our deduction can be summed up in the following theorems:

2. *At the initial moment of mixing, the T-S curve consists of two straight lines which consecutively connect the given water masses on the T-S diagram.*

3. *At points of the T-S curves sufficiently distant from the limits of the intermediate water mass, the tangents to T-S curves practically coincide with the straight lines, which consecutively connect the three water masses considered.*

4. *Points of T-S curves corresponding to the "nucleus" of the intermediate water mass are extreme points of T-S curves in respect to the base of the triangle of mixing. The tangents at these points are parallel to the base of the triangle of mixing.*

Now let us determine the slope of the straight line which, on the T-S curve, connects two points with values of parameter z that are equal but opposite in sign. Evidently these points of the T-S curve are located on different sides of the principal median of the triangle of mixing. If the value of the parameter z at the points considered is equal, $z = a$ and $z = -a$, the slope of the straight line connecting these points will be determined by the formula

$$\operatorname{tg} \alpha = \frac{\vartheta_a - \vartheta_{-a}}{S_a - S_{-a}}.$$

By substituting alternatively values $z = a$ and $z = -a$ in formulas (4) and by simple transformations we obtain

$$\begin{aligned} \vartheta_a - \vartheta_{-a} &= \frac{\Theta_1 - \Theta_3}{2} \left[\Phi \left(\frac{a-h}{2\sqrt{Kt}} \right) + \Phi \left(\frac{a+h}{2\sqrt{Kt}} \right) \right] \\ s_a - s_{-a} &= \frac{S_1 - S_3}{2} \left[\Phi \left(\frac{a-h}{2\sqrt{Kt}} \right) + \Phi \left(\frac{a+h}{2\sqrt{Kt}} \right) \right]; \end{aligned}$$

hence

$$\operatorname{tg} \alpha = \frac{\Theta_1 - \Theta_3}{S_1 - S_3}.$$

Thus we may regard as demonstrated the theorem:

5. *Straight lines, which on a T-S curve connect two points having identical values of parameter z but with opposite signs, are parallel to the base of the triangle of mixing.*

Combining this theorem with theorem 4 we obtain a new deduction:

6. *A tangent, drawn at the point of intersection of the principal median of the triangle of mixing with the T-S curve, locates on the other T-S*

curve, prior in time, two points having equal values of the parameter z but with opposite signs.

From formula (4) issues another no less remarkable property possessed by points of a T - S curve with a parameter value of $z = \pm h$, i. e., points located on the boundaries of the intermediate water mass. To demonstrate this let us consider two points Q and D with an identical value of parameter $z = h$ but with different values of parameter $t = t'$ and $t = t''$ belonging to two successive T - S curves (Fig. 3).

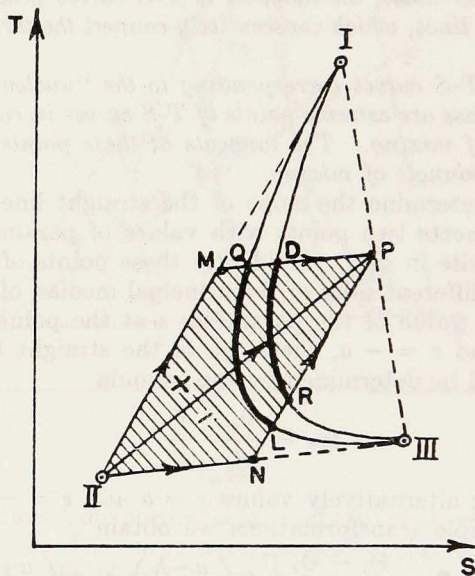


Figure 3.

The equation of a straight line passing through points $Q(h, t')$ and $D(h, t'')$ is written as follows:

$$t - t(h, t') = \frac{t(h, t') - t(h, t'')}{s(h, t') - s(h, t'')} [s - s(h, t')].$$

Substituting values for t and s from formulas of type (4) we obtain:

$$(14) \quad t - \frac{\Theta_1 + \Theta_3}{2} - \frac{\Theta_2 - \Theta_3}{2} \Phi\left(\frac{h}{\sqrt{Kt'}}\right) = \frac{\Theta_2 - \Theta_3}{S_2 - S_3} \left[s - \frac{S_1 + S_3}{2} - \frac{S_2 - S_3}{2} \Phi\left(\frac{h}{\sqrt{Kt'}}\right) \right].$$

It follows from equation (14) that the straight line considered is parallel to side II-III of the triangle of mixing, since the slope m of this straight line is equal to:

$$m = \frac{\Theta_2 - \Theta_3}{S_2 - S_3}.$$

In an analogous way we discover that the straight line passing through points L $(-h, t')$ and R $(-h, t'')$ (Fig. 3) is parallel to side I-II of the triangle of mixing. On the basis of equation (14) it is easy to show that the straight line which passes through points Q and D passes also through point P located in the middle of the base of the triangle of mixing. But if so, the straight line considered also passes through point M lying in the middle of side I-II of the triangle of mixing.

In exactly the same way can we demonstrate that the straight line passing through points L and R also passes through points P and N (Fig. 3). But, as already established in formulas (5), points M $\left(\frac{\Theta_1 + \Theta_2}{2}, \frac{S_1 + S_2}{2}\right)$ and N $\left(\frac{\Theta_2 + \Theta_3}{2}, \frac{S_2 + S_3}{2}\right)$ characterize the properties of water on the boundaries of the intermediate layer at the initial moment of mixing, these properties varying continuously with time.

Now we come to the conclusion that this process is illustrated on the T - S curve by two straight lines MP and NP which are medians of the triangles IPII and IPIII; to distinguish them from the principal median they will be referred to as "secondary" medians of the triangle of mixing.

It is obvious that any point within the parallelogram IINMP, shown by hatched lines in Fig. 3, will characterize the water of the intermediate layer.

Thus this parallelogram may be regarded as a distinctive geometrical interpretation of the transformation of the intermediate water layer, and it should be kept in mind that on the boundaries of this layer the passage from point II to points M and N occurs instantaneously, whereas further transformation along the secondary medians proceeds continuously with time t . The secondary medians of the triangle of mixing cut off arcs on the successive T - S curves; every point of these arcs corresponds to the water of the intermediate layer in the system $s = f(\vartheta)$. These results may be summed up in the form of the following theorem:

7. All points in the system $s = f(\vartheta)$ corresponding to the water on the boundaries of the intermediate layer lie on the secondary medians of the triangle of mixing. The secondary medians cut off, on the T - S curves,

arcs which correspond to the water of the intermediate layer in the system $s = f(\vartheta)$.

The theorems here demonstrated are of interest, not only in respect to an abstract "geometry of T - S curves," but because they are also of practical value. Thus, on the basis of theorem 3 we are often able to reconstruct initial thermohaline indices of the intermediate water mass. For this purpose, according to the above theo-

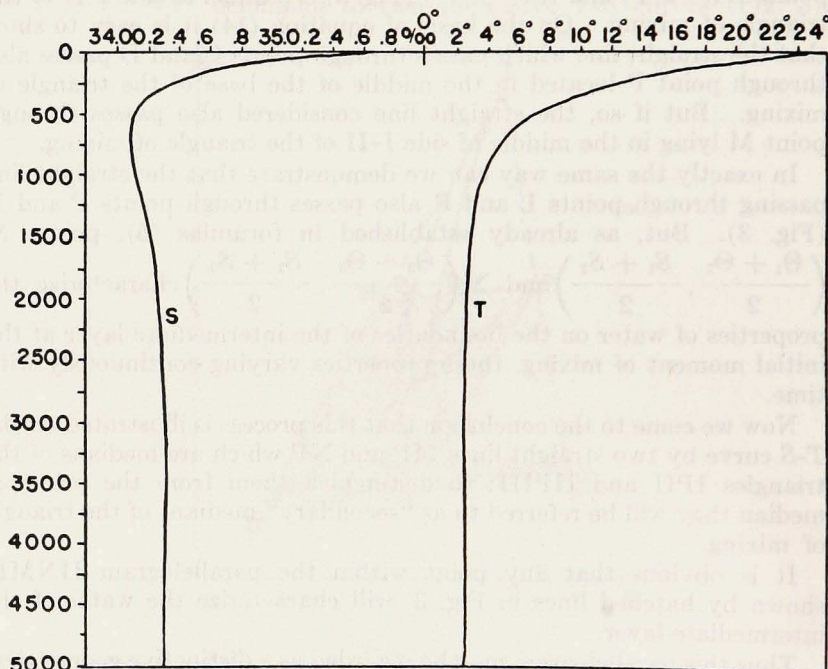


Figure 4.

rem, tangents must be drawn at points of the T - S curve of the type considered sufficiently distant from the region of its rounding, or, in other words, prolongations of the rectilinear branches of the curve. The intersection of such straight lines will determine the coordinates of the point corresponding to the temperature and salinity of the intermediate water mass.

Theorem 7 is of no less practical value. Up to now in oceanographic investigations the question has remained open as to the vertical dimensions of water masses which could be detected in the sea by the method of T - S curves. The conclusions of marine scientists regarding

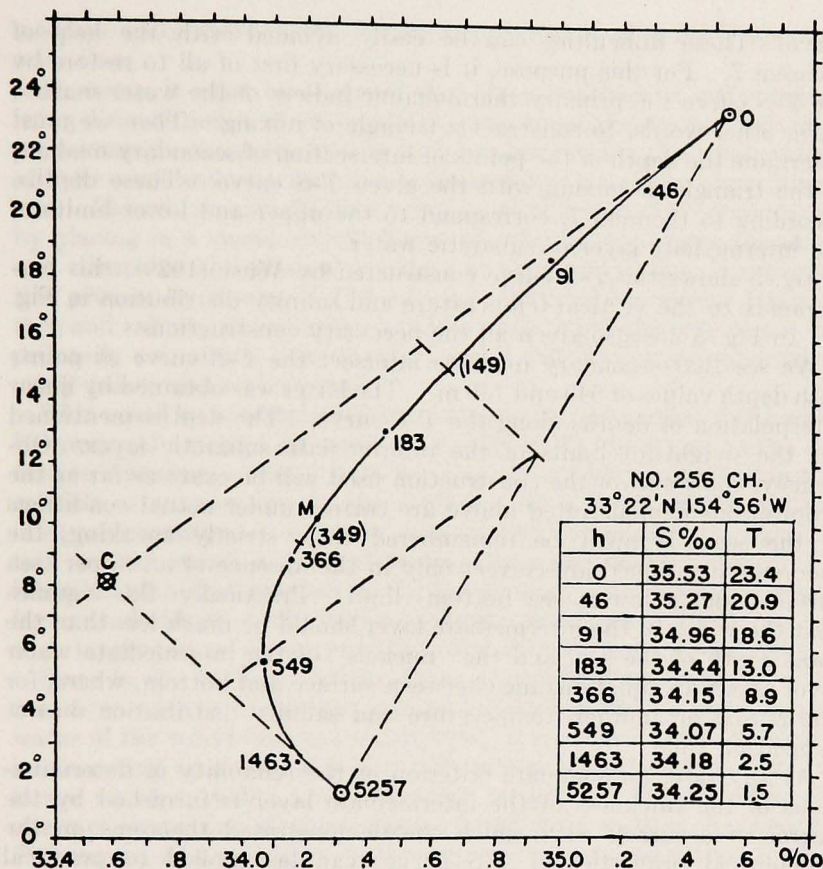


Figure 5.

the vertical limits of some layer or other were rather vague and conventional. This is accounted for by the continuous character of the vertical distribution of different properties of sea water, owing to a levelling effect exercised by processes of mixing.

For example, Fig. 4 shows the curves of vertical temperature and salinity distribution in the central Pacific Ocean, plotted by Wüst (1929) on the basis of data from the "Challenger" expedition, station ch. 256. We see immediately the difficulties that arise from the attempt to determine, by means of these data alone, the vertical limits of the layer of fresh subarctic water, whose presence is clearly shown by the peculiar curvature of the vertical salinity distribution

curve. These difficulties can be easily avoided with the help of theorem 7. For this purpose, it is necessary first of all to restore by the T - S curve the primary thermohaline indices of the water masses, or, in other words, to construct a triangle of mixing. Then we must determine the depth of the points of intersection of secondary medians of the triangle of mixing with the given T - S curve. These depths, according to theorem 7, correspond to the upper and lower limits of the intermediate layer of subarctic water.

Fig. 5 shows the T - S curve constructed by Wüst (1929); this corresponds to the vertical temperature and salinity distribution in Fig. 4. In Fig. 5 are also given all the necessary constructions.

We see that secondary medians intersect the T - S curve at points with depth values of 549 and 149 m. The latter was obtained by linear interpolation of depths along the T - S curve. The depths mentioned are the sought-for limits of the intermediate subarctic layer. Obviously, the result of the construction used will be exact as far as the propositions demonstrated above are correct under actual conditions in the sea. It must be remembered that, strictly speaking, the theorems mentioned are correct only in the absence of an upper (sea surface) and a lower (sea bottom) limit. Practically, this signifies that the width of the intermediate layer should be much less than the total depth of the sea, and the "nucleus" of the intermediate water layer about an equal distance between surface and bottom, where, for one reason or another, temperature and salinity distribution do not depend on time.

At any rate, an adequate criterion of the reliability of determinations of the thickness of the intermediate layer is furnished by the degree of accuracy with which the demonstrated theorems on the geometrical properties of T - S curves can be applied to practical circumstances. Thus, on the basis of these theorems, it should be expected that the point of intersection of the principal median with the empirical T - S curve shown on Fig. 5 will be situated at an equal distance (with respect to depth z) from the points of intersection of the secondary medians with the same T - S curve. The point of intersection of the principal median with the T - S curve on Fig. 5 (point M) corresponds to a depth of 307 m. The latter does not quite correspond to the center of the intermediate layer, whose depth is 349 m. and whose position (with respect to parameter z) is indicated in the figure by a straight line drawn across the curve. However, there is no great discrepancy between the depth of point M (307 m.) and the center of the intermediate layer (349 m.), the relative value of the error in this case not exceeding 12 per cent. Thus the demonstrated propositions used in the determination of the thickness of the inter-

mediate water mass in the Pacific are proved to give fairly accurate estimates.

However, it must be observed that the conception of water masses being discrete bodies of water, which, prior to mixing possess individual properties (a conception that is both fundamental to the whole theory of *T-S* curves and very useful in oceanography) is a rather conventional one. Indeed, we can obtain a precise reproduction of this conception by placing in a vessel several layers of a liquid, one above the other, and observing the gradual levelling of their properties, interpreting this process by means of *T-S* curves. Obviously a similar temperature and salinity distribution could be obtained in the case of a vessel primarily containing non-stratified water with identical properties at every point within the vessel.

This could be accomplished by changing the properties of the water by means of external influences, *i. e.*, subjecting the water on the surface to warming, cooling, or to a change in salinity brought about by rainfall or evaporation at the surface. As a result of the external influences mentioned, together with mixing processes, we might obtain in our vessel a vertical temperature and salinity distribution identical with that reached in the former case (intermixing of several layers, individual at the initial moment), although in the latter case such layers (water masses) at the initial moment were nonexistent.

Since in the author's opinion there are, strictly speaking, no water masses in nature, excepting perhaps the fresh water on land and salt water of the world oceans (merely two), it must be kept in mind that in dissecting oceanic water into different water masses we deal with layers which in reality are nonexistent in the sea. Nevertheless, this abstract conception of water masses, forming the basis of the theory of *T-S* curves, and analogous to the conception of aerial masses, plays a useful part like many other abstract conceptions of precise natural science.

II. DETERMINATION OF THE COEFFICIENT OF MIXING BY MEANS OF *T-S* CURVES (ON THE CORRECTNESS OF JACOBSEN'S METHOD)

T-S curves serve not only to determine the structure of water masses in the sea, but, as shown by Jacobsen (1927), they enable computation of the coefficient of turbulent mixing in the sea. The essentials of the method are as follows: one of two curves is selected; one (later in respect to time), with a tangent drawn at the point of its maximum curvature, locates on the other *T-S* curve (that prior in time) two points with known values of parameters [z' and z''] $z = z'$

and $z = z''$. Then, by computing the difference of parameters we can calculate the coefficient K according to Jacobsen by the formula

$$(15) \quad K = \frac{(z' - z'')^2}{8\Delta t}$$

where Δt is the lapse of time separating the first T - S curve from the second.

Owing to its extreme simplicity, the ingenious method of Jacobsen has been widely applied in oceanography; this, however, cannot prevent us from expressing some doubts as to its correctness which arise as the result of a closer scrutiny of its theoretical foundations.

Indeed, beside the difficulties occasionally experienced in practice in determining the point of maximum curvature of a T - S curve and drawing a tangent, it must be kept in mind that the *curvature of a T - S curve varies considerably in relation to the scale of temperature and salinity on the T - S diagram.*

Obviously *the position of the point of maximum curvature of a T - S curve on such a deformed diagram does not correspond to its position in natural (not distorted) coordinates.* Nevertheless, this important fact is completely overlooked both in the theory of the method itself and in its practical applications. Moreover, even a detailed study of the fundamentals of the theory of Jacobsen's method fails to answer the question of whether drawing a tangent just at the point of maximum curvature is essential to the method itself, or whether the tangent could be as well drawn at any other point of the T - S curve.

It must be noted that the method of Jacobsen is applicable to the usual type of T - S curves in the case of the mixing of three water masses. Moreover, as seen from the theory of the method as stated by Jacobsen, the water medium is assumed to be unlimited (in his computations z is integrated within the limits $z = -\infty$ to $z = \infty$).

Thus the initial statements of Jacobsen's theory of procedure fully comply with the conditions of the problem considered in the foregoing chapter. Hence, we are able to use fully the demonstrated theorems on the geometrical properties of T - S curves for a more exact and very simple basis for a theory of Jacobsen's method.

For this purpose let us consider the two T - S curves illustrated in Fig. 6. The first is assumed to correspond to the moment of time t_0 and the second to $t_0 + \Delta t$. Let us then draw a tangent at point A having parameters $z = 0$, $t_0 + \Delta t$ and which consequently belongs to the second T - S curve; this tangent intersects the branches of the first curve at points B and C. According to theorem I of the first chapter, point A lies on the principal median IIE of the triangle of mixing and

forms at the same time the maximum point of the second T - S curve in relation to the base of the triangle of mixing.

On the other hand, according to theorems 5 and 6 of the same chapter, the tangent mentioned above must be parallel to the base of the triangle of mixing and points B and C must have equal values, but opposite in signs, of parameter z in relation to point D ($z = 0, t_0$). We emphasize that the last circumstance is essential to the following computations which finally lead to the formula of Jacobsen. According to the above let the parameters of points B and C, which belong to the first T - S curve, be characterized by values $B\left(-\frac{\Delta z}{2}, t_0\right)$, $C\left(\frac{\Delta z}{2}, t_0\right)$, where Δz is the difference of parameters z between the points B and C.

Let us assume, together with Jacobsen (1927) and Okada (1938), that temperature and salinity at points A, B, C can be presented precisely enough for practical purposes in the form of the sum of the first terms of Taylor's series,

$$(16) \quad \begin{cases} \vartheta_C = \vartheta_0 + \frac{\Delta z}{2} \left(\frac{\partial \vartheta}{\partial z}\right)_0 + \frac{(\Delta z)^2}{8} \left(\frac{\partial^2 \vartheta}{\partial z^2}\right)_0 \\ s_C = s_0 + \frac{\Delta z}{2} \left(\frac{\partial s}{\partial z}\right)_0 + \frac{(\Delta z)^2}{8} \left(\frac{\partial^2 s}{\partial z^2}\right)_0 \\ \vartheta_B = \vartheta_0 - \frac{\Delta z}{2} \left(\frac{\partial \vartheta}{\partial z}\right)_0 + \frac{(\Delta z)^2}{8} \left(\frac{\partial^2 \vartheta}{\partial z^2}\right)_0 \\ s_B = s_0 - \frac{\Delta z}{2} \left(\frac{\partial s}{\partial z}\right)_0 + \frac{(\Delta z)^2}{8} \left(\frac{\partial^2 s}{\partial z^2}\right)_0 \\ \vartheta_A = \vartheta_0 + \frac{\partial \vartheta}{\partial t} \Delta t; \quad s_A = s_0 + \frac{\partial s}{\partial t} \Delta t, \end{cases}$$

while the equation of tangent BC may be inscribed in the form of

$$(17) \quad \vartheta_A - \vartheta_B = \left(\frac{d\vartheta}{ds}\right)_0 (s_A - s_B)$$

where $\left(\frac{d\vartheta}{ds}\right)_0$ is the slope of the tangent to the T - S curve at point $z = 0$.

By substituting in formula (17) corresponding values from formula (16) and rearranging the terms we obtain

$$(18) \quad \left[\frac{\partial \vartheta}{\partial t} - \left(\frac{d\vartheta}{ds} \right)_0 \left(\frac{\partial s}{\partial t} \right) \right] \Delta t = - \frac{\Delta z}{2} \left[\left(\frac{\partial \vartheta}{\partial z} \right)_0 - \left(\frac{d\vartheta}{ds} \right)_0 \left(\frac{\partial s}{\partial z} \right)_0 \right] \\ + \frac{(\Delta z)^2}{8} \left[\left(\frac{\partial^2 \vartheta}{\partial z^2} \right)_0 - \left(\frac{d\vartheta}{ds} \right)_0 \left(\frac{\partial^2 s}{\partial z^2} \right)_0 \right].$$

Let us now decipher the expressions in brackets [] from formula (18). On the basis of formula (9), the first bracket to the right of (18) is reduced to 0. Differentiating (9) by z as a complex function, we obtain

$$(19) \quad \frac{\partial^2 \vartheta}{\partial z^2} = \frac{d^2 \vartheta}{ds^2} \left(\frac{\partial s}{\partial z} \right)^2 + \frac{d\vartheta}{ds} \left(\frac{\partial^2 s}{\partial z^2} \right).$$

Consequently the remaining item on the right of (18) can be inscribed as:

$$(20) \quad \frac{(\Delta z)^2}{8} \left[\left(\frac{\partial^2 \vartheta}{\partial z^2} \right)_0 - \left(\frac{d\vartheta}{ds} \right)_0 \left(\frac{\partial^2 s}{\partial z^2} \right)_0 \right] = \frac{(\Delta z)^2}{8} \left(\frac{d^2 \vartheta}{ds^2} \right)_0 \left(\frac{\partial s}{\partial z} \right)^2.$$

Combining equation (1) with formulas (9) and (19), it is easy to demonstrate that

$$\frac{\partial \vartheta}{\partial t} - \frac{d\vartheta}{ds} \frac{\partial s}{\partial t} = K \frac{d^2 \vartheta}{ds^2} \left(\frac{\partial s}{\partial z} \right)^2.$$

Substituting the right hand side of this last equation for the brackets on the left of formula (18) and omitting index 0, we obtain

$$K \frac{d^2 \vartheta}{ds^2} \left(\frac{\partial s}{\partial z} \right)^2 \Delta t = \frac{(\Delta z)^2}{8} \left(\frac{d^2 \vartheta}{ds^2} \right) \left(\frac{\partial s}{\partial z} \right)^2$$

or, finally

$$(21) \quad K = \frac{(\Delta z)^2}{8 \Delta t}.$$

As we see, the last formula is identical with Jacobsen's.

An essential feature of our reasoning in deducing Jacobsen's formula is that we draw the tangent BC (Fig. 6), not at the point of maximum curvature, but *at the point of intersection of the principal median of the triangle of mixing with the second T-S curve*. Generally speaking, this point does not coincide with the point of maximum curvature as illustrated by Fig. 7, which shows two T-S curves constructed according to formula (4) on the proposition that: $\Theta_1 = 2^\circ$, $S_1 = 34.00^\circ/\text{‰}$; $\Theta_2 = 1^\circ$, $S_2 = 33.10^\circ/\text{‰}$; $\Theta_3 = 5^\circ$, $S_3 = 33.20^\circ/\text{‰}$; $K = 10 \frac{\text{cm}^2}{\text{cl} K}$; $2h = 120 \text{ m.}$ for two values of $t = 30$ days and $t = 50$ days. As seen

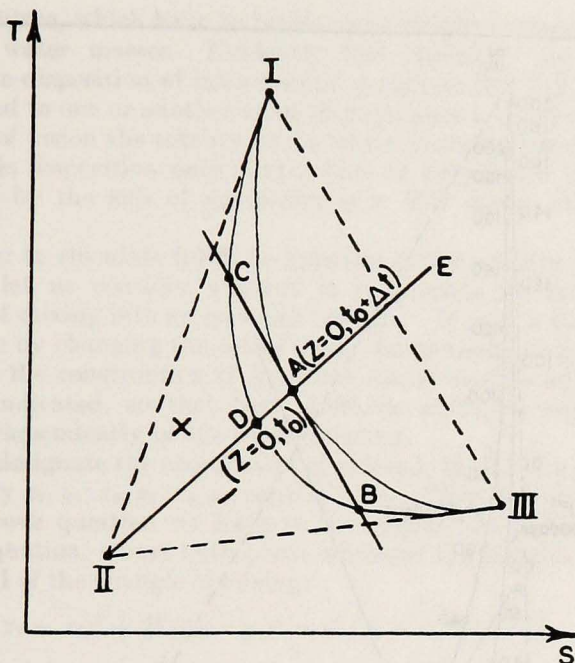


Figure 6.

from Fig. 7, the maximum curvature of the curve for $t = 50$ days lies in the vicinity of the point with parameter $z = 40$ m.

It is easily seen that Jacobsen's construction could be identical with ours only in one particular case; *i. e.*, when the triangle of mixing would be isosceles in relation to sides I-II and II-III (Fig. 6) and when, consequently, its principal median coincides with the axis of symmetry of the T-S curve (bisects the angle I, II, III). Since in this case the point of maximum curvature is actually located on the principal median (bisector) of the triangle of mixing, both constructions will have a completely identical significance. Jacobsen considered just this particular case, though he expected to obtain a method of determining the coefficient of mixing applicable to any case. Thus it is no wonder that the calculations of Jacobsen gave a result homotypical with formula (21). There is also no difficulty in discovering the cause of the error made by Jacobsen. The explanation is that the theory of Jacobsen completely ignores the important element of the triangle of mixing.

Contrary to our principle, Jacobsen sets no limits to the branches of

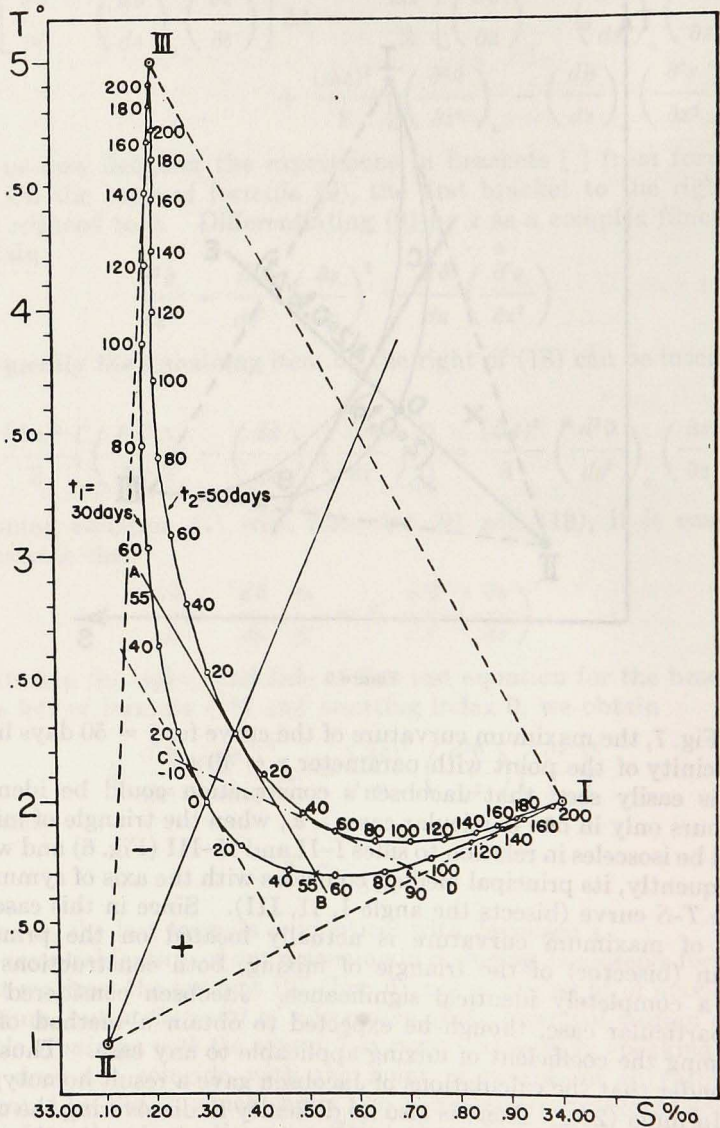


Figure 7.

the T - S curve, which have to break up at points corresponding to the extreme water masses. Evidently this important fact essentially affects the disposition of intermediate points on the T - S curve, which correspond to one or another value of parameter z . In excluding from the field of vision the totality of the water masses, Jacobsen naturally had at his disposition only that means of orientation which can be furnished by the axis of symmetry of a T - S curve with unlimited branches.

In order to elucidate fully the question of the validity of Jacobsen's method, let us consider whether it is possible to transform any triangle of mixing into an isosceles triangle. If such a transformation is possible by changing the scales along the rectangular axes T - S , then evidently the construction of Jacobsen can always be adjusted to the method indicated, so that both methods will give equally correct results independently of the scales selected.

Let us designate the coordinates of apices I, II, III of a given mixing triangle by $x_1, y_1; x_2, y_2; x_3, y_3$, respectively. Then, to obtain an answer to the above question we have to investigate the following indeterminate equation, whose both parts represent the lengths of sides I-II and II-III of the triangle of mixing:

$$(22) \quad \alpha^2 (x_1 - x_2)^2 + \beta^2 (y_1 - y_2)^2 = \alpha^2 (x_3 - x_2)^2 + \beta^2 (y_3 - y_2)^2,$$

where α and β are coefficients of proportionality of scales on the axes x and y (s and ϑ) respectively.

The last equation may easily be reduced to

$$\alpha^2 A + \beta^2 B = 0$$

where

$$A = (x_1 - x_2)^2 - (x_3 - x_2)^2; \quad B = (y_1 - y_2)^2 - (y_3 - y_2)^2.$$

Consequently α is related to β by

$$(23) \quad \alpha = \beta \sqrt{-\frac{B}{A}}.$$

Formula (23) indicates that transformation into an isosceles triangle is possible only in the case where the values A and B have *opposite* signs. On the contrary, in the case of similar signs such a transformation cannot be accomplished. That is why, generally speaking, the construction of Jacobsen is erroneous in its principle, whereas our mode of constructing a tangent, owing to the invariability of our constructions in relation to the scales of the T - S curve, is the only method valid in all cases of transformable and nontransformable (nonisosceles) triangles of mixing.

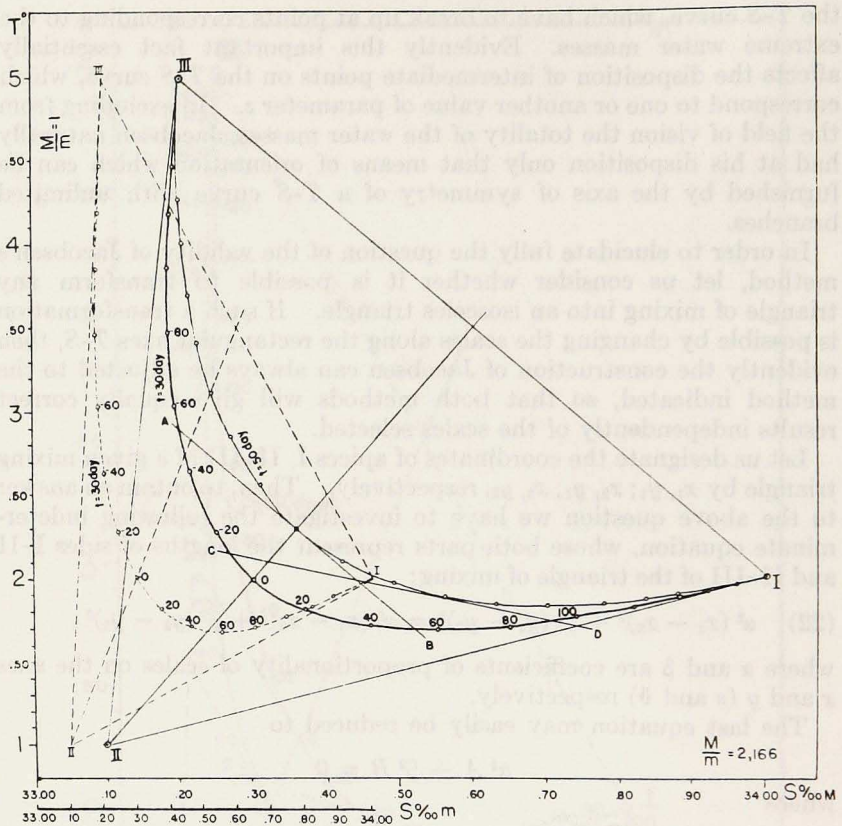


Figure 8.

Fig. 8 gives an example of the transformation of the triangle of mixing (illustrated in Fig. 7) into an isosceles triangle. It may easily be seen from formula (23) that in this case such a transformation is possible and can be accomplished by changing only one scale on the x axis (i. e., s) ($\beta = 1$), increasing the scale of salinity 2,166 times, as compared with the previous one ($\alpha = 2,166$).

Fig. 8 shows also what essential changes are brought about in the curvature of a T - S curve by changes of scales; thus, the point of maximum curvature ($z = 40$) of the dotted T - S curve, corresponding to the "original," with the change of scale is converted almost into the point of minimum curvature (with the same parameter $z = 40$ on the transformed curve). As is to be expected, the T - S curve changed by

the transformation of this triangle into an isosceles one becomes a symmetrical curve in relation to the principal median (bisector) of the triangle of mixing. Naturally, a tangent drawn at the point of maximum curvature of such a symmetrical T-S curve will be parallel to the base of the triangle of mixing, so that in this case the construction of Jacobsen will fully coincide with the construction required by the deduction of formula (21) and the theorems of Chapter I.

In contrast to this example, in Fig. 9 is shown a triangle of mixing with T-S curves constructed for $t = 30$ days and $t = 50$ days, which cannot be transformed into an isosceles triangle, as is easily proved by formula (22).

Finally, we suggest a new and simple graphical method for determining the coefficient of mixing with the help of T-S curves.

Indeed, as follows from formula (10), the slope of the tangent at the point corresponding to the limit of the intermediate layer (assuming $z = -h$) is determined for the period of time t_1 by the expression

$$tg\alpha_1 = \frac{\Theta_1 - \Theta_2 + (\Theta_2 - \Theta_3) e^{\frac{h^2}{Kt_1}}}{S_1 - S_2 + (S_2 - S_3) e^{\frac{h^2}{Kt_1}}}$$

and for the period of time t_2 by the expression,

$$tg\alpha_2 = \frac{\Theta_1 - \Theta_2 + (\Theta_2 - \Theta_3) e^{\frac{h^2}{Kt_2}}}{S_1 - S_2 + (S_2 - S_3) e^{\frac{h^2}{Kt_2}}}$$

It follows from these formulas that

$$(24) \quad e^{\frac{h^2}{Kt_1}} = A ; \quad e^{\frac{h^2}{Kt_2}} = B ,$$

where A and B are

$$(25) \quad \begin{cases} A = \frac{\Theta_1 - \Theta_2 - (S_1 - S_2) tg\alpha_1}{(S_2 - S_3) tg\alpha_1 - \Theta_2 + \Theta_3} ; \\ B = \frac{\Theta_1 - \Theta_2 - (S_1 - S_2) tg\alpha_2}{(S_2 - S_3) tg\alpha_2 - \Theta_2 + \Theta_3} . \end{cases}$$

Taking the logarithm of (24) we obtain

$$t_1 = \frac{h^2}{K \ln A} ; \quad t_2 = \frac{h^2}{K \ln B} .$$

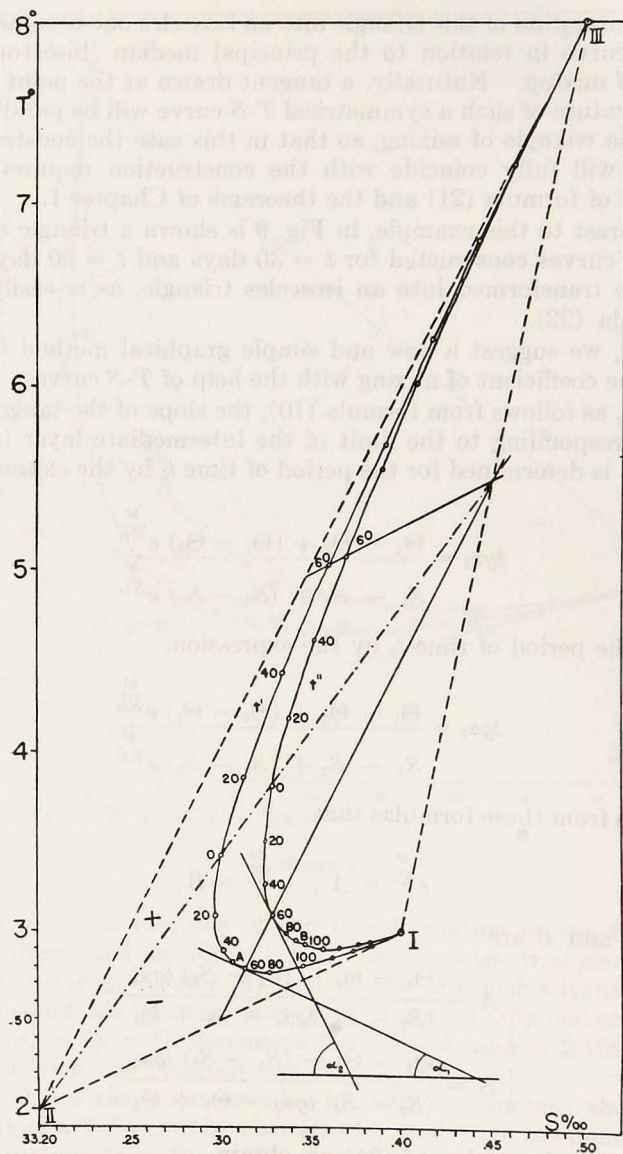


Figure 9.

Forming the difference $t_2 - t_1 = \Delta t$, we obtain

$$\Delta t = \frac{h^2}{K} \left(\frac{\ln B - \ln A}{\ln B \cdot \ln A} \right)$$

hence

$$(26) \quad K = \frac{h^2}{\Delta t} \left(\frac{\ln B - \ln A}{\ln B \cdot \ln A} \right).$$

This is the precise formula for computing the coefficient of mixing, instead of the approximate formula (21). The determination of parameter h on the basis of theorem 7 is quite simple. It must be kept in mind that h equals only one-half the thickness of the intermediate layer:

$$h = \frac{z'' - z'}{2}$$

where $z'' - z'$ is the difference of depths located on the *T-S* curve by the secondary medians of the triangle of mixing. Consequently, formula (26) can be inscribed more conveniently in the following form:

$$(26') \quad K = \frac{(z'' - z')^2 M}{4\Delta t}.$$

It is interesting that formula (26') somewhat resembles the approximate formula (21), though the significance of values $z'' - z'$ in both formulas is quite different. To calculate the multiplier M , we must draw tangents at the points of intersection of one of the secondary medians of the triangle of mixing with two *T-S* curves in the manner indicated in Fig. 9, and determine the angles formed by tangents with axis s . Evidently this method, like that of Jacobsen, is valid only in the case of mixing of three water masses.

The theory of *T-S* curves described above has an interesting hydrodynamic interpretation. Indeed, if we consider a two-dimensional horizontal flow of a viscous incompressible liquid in the absence of external forces, the equations of such motion, as known, will have the following aspect:

$$\frac{\partial u}{\partial t} = v \frac{\partial^2 u}{\partial z^2}; \quad \frac{\partial v}{\partial t} = v \frac{\partial^2 v}{\partial z^2}$$

formally analogous to equations (1).

In this system of equations, u and v are components of the horizontal vector of velocity along the rectangular axes of coordinates x and y , respectively. It is easy to understand that in this case of motion of a liquid the hodograph of velocities presents an analogy to the *T-S*

curve, while the apices of the triangle of mixing are the ends of the three vectors, corresponding to the initial distribution of velocities in the layers of moving liquid.

Thus, in the case considered, the hodographs of velocities possess all the geometrical properties of T - S curves and with the help of formulas (21) and (26) it is possible to determine the value of the kinematic coefficient of viscosity.

A generalization of the theory of T - S curves applicable to any finite number of intermediate layers in the sea is now being prepared.

REFERENCES

DEFANT, ALBERT

1929. Stabile Lagerung ozeanischer Wasserkörper und dazu gehörige Stromsysteme. Veröff. Inst. Meeresk., Univ. Berlin, Neue Folge, A., Heft 19: 5-33.

HELLAND-HANSEN, BJØRN

1918. Nogen hydrografiske metoder. Forh. skand. naturf. Møte, 16: 357-359.

JACOBSEN, J. P.

1927. Eine graphische Methode zur Bestimmung des Vermischungskoeffizienten im Meere. Beitr. Geophys., 16: 404-412.

OKADA, MITUYO

1938. Use of T-S-diagram for determining the rates of mixing and of replacement of sea water in the coastal region I. Bull. Jap. Soc. sci. Fish., 6 (5): 233-236.

WUST, GEORG

1929. Schichtung und Tiefenzirkulation des Pazifischen Ozeans. Veröff. Inst. Meeresk., Univ. Berlin, Neue Folge, A., Heft 20: 5-63.