# THE MOTION OF ATMOSPHERIC DISTURBANCES

By

B. HAURWITZ†

In a recent paper Rossby has discussed the effect of the latitudinal variation of the Coriolis force on the propagation of oceanic and atmospheric disturbances (*2*). He showed that the velocity $c$ of a disturbance in an incompressible atmosphere on a plane earth when only the effect of the latitudinal variation of the Coriolis force is considered is given with sufficient accuracy by the formula

$$c = U - \frac{\beta L^2}{4\pi^2} \tag{1}$$

Here $U$ denotes the undisturbed velocity of the air, that is the geostrophic wind, $L$ the wave length of the disturbance. $\beta$ is the meridional variation of the Coriolis parameter,

$$\beta = \frac{\partial f}{\partial y} = \frac{2\omega \cos \varphi}{a} \tag{2}$$

Here $\omega$ is the angular velocity of the earth's rotation, $a$ the earth's radius, $\varphi$ the geographic latitude. The direction of the wave propagation is assumed to coincide with the W–E direction which serves as $x$-axis. Rossby assumes further that the disturbance is independent of the $y$-coordinate, that is of the direction normal to the direction of propagation. His investigation has been extended in three particulars. These extensions may prove helpful in future practical work.

1) It will not be assumed that the disturbance is independent of the $y$-coordinate, for that assumption implies infinite lateral extent of the disturbance. Such is not the case in the earth's atmosphere. Since the dimensions of the disturbance may be rather large fractions of the earth's circumference as will be seen later (p. 41) it would appear desirable to take into account the spherical shape of the earth. This extension will be given in a later paper. The mathematical treatment of the spherical case leads to rather complicated formulae. It seems therefore worth while to extend the discussion as far as possible by elementary mathematical methods.

2) The effect of friction will be considered. It will be assumed that the effect of friction on the vorticity is proportional to the vorticity itself.

† Published by permission of the Controller of the Meteorological Service of Canada.

This assumption is admittedly rather crude, but it will at least show qualitatively how the motion is influenced by friction.
3) Finally the action of a force on the vorticity will be considered. This force may, for instance, be due to the concentration of solenoids along the coast lines as pointed out by Rossby (*2*). He has already indicated how the action of such a force may be taken into account by imposing a boundary condition on the solution so that velocity and vorticity have prescribed values in the region where the force is located. To obtain a clearer picture of the dynamical process, however, it is necessary to have the expression for the force appear in the dynamic equations. The force starts to act only at the moment when the air passes over the region where it is concentrated. Therefore the forced oscillation will not be completely developed at the moment when the region of the force is passed but somewhat later.

Reference is here made only to atmospheric disturbances but the application to oceanic motions is obvious.

### *A) Disturbance of finite lateral extent*

A simple expression for the pressure distribution of a disturbance of finite lateral extent is given by

$$p = \text{const.} - A \cdot y + B \cos \frac{2\pi}{D} y \cos \frac{2\pi}{L} x \tag{3}$$

where $D$ represents the width in the $y$-direction of a High and a successive Low. The pressure field defined by (3) gives a succession of Highs and Lows in the $y$-direction as well as in the $x$-direction. The smaller the undisturbed meridional pressure gradient $A$ is compared to the intensity of the superimposed disturbance $B$ the greater is the number of closed isobars. When the ratio $A/B$ increases fewer isobars are closed and the pressure field resembles more closely a succession of wedges and troughs. A small value of the ratio $A/B$ corresponds to the actual atmospheric conditions near the earth's surface where closed Highs and Lows are observed. The intensity $B$ of these disturbances decreases upwards while the undisturbed meridional pressure gradient $A$ remains of the same magnitude. Consequently most of the isobars at upper levels resemble cosine curves with few if any isobars closed.

If only one High or Low is to be considered, the right hand side of (3) may be represented by a different function which can be expressed by a Fourier double integral.

When the waves are of finite lateral extent the velocity of propagation is found to be considerably different from that obtained when the waves are of infinite lateral extent. The motion considered may be purely horizontal, without friction, and the atmosphere may be incompressible and homogeneous as in the case treated by Rossby. It follows from the equations of motion that the absolute vorticity which consists of the sum of the vorticity $\zeta$ of the motion relative to the earth and of the vorticity $f$ due to the rotation of the earth must be constant,

$$\zeta + f = \text{const.} \tag{4}$$

Obviously, only the vertical component of the vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

has to be considered in this case of horizontal motion. The equation of continuity for horizontal incompressible motion states that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5}$$

Differentiation of (4) shows, since $f$ depends only on the latitude, that

$$\frac{d\zeta}{dt} = -\beta v \tag{6}$$

where $\beta$ is defined by (2). Owing to the meridional pressure gradient a geostrophic zonal current of the velocity $U$ must exist. On this zonal current a disturbance with the velocity components $u'$ and $v'$ is superimposed. $u'$ and $v'$ may be considered so small that terms of higher order in the perturbation velocities or their derivatives can be neglected. It follows then from (6) and (5) that

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) = -\beta v' \tag{7}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \tag{8}$$

The quantity $\beta$ depends on the latitude. However, as long as the earth is regarded as a plane this variability of $\beta$ may be neglected since the error involved although considerable is of the same order as the one caused by considering the earth as flat.

The form of eq. (8) suggests that the streamfunction $\psi$ may advantageously be introduced

$$u' = -\frac{\partial \psi}{\partial y} \qquad v' = \frac{\partial \psi}{\partial x} \tag{9}$$

The equation $\psi = \text{const.}$ represents the stream lines.

The equation of continuity is satisfied for any value of $\psi$. The vertical component of the curl is given by

$$\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \tag{10}$$

Using eq. (9) eq. (7) is transformed into

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = -\beta\frac{\partial \psi}{\partial x}. \tag{11}$$

A solution may be assumed in the form

$$\psi = C \cos\frac{2\pi}{D} y \sin\frac{2\pi}{L}(x - ct) \tag{12}$$

where $C$ is an arbitrary constant. $D$ represents the width of the disturbance, $L$ its wave length and $c$ the velocity of the disturbance. Substituting (12) in (11) it follows that

$$c = U - \frac{\beta L^2}{4\pi^2}\frac{D^2}{D^2 + L^2}. \tag{13}$$

If the motion is not a function of the $y$-coordinate, $D = \infty$, and (13) becomes identical with (1). If $D$ is finite, $\dfrac{1}{1 + L^2/D^2}$ is smaller than unity. Thus the smaller the lateral extent of the disturbance, the smaller is the influence of the term which is due to the latitudinal variation of the Coriolis force. The value of this factor for different ratios $L/D$ is shown in Table I. When the lateral extent of the disturbance is equal to the wave length, for instance, the difference between the geostrophic velocity and the velocity of the perturbation is reduced to half the value which is found for an infinite disturbance.

TABLE I

Reduction of the Velocity Deficit $U$-$c$ when It Is Assumed that the Disturbance Is of Finite Lateral Extent

| L/D | 1/5 | 1/3 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $\frac{1}{1+L^2/D^2}$ | .96 | .90 | .50 | .20 | .10 |

The disturbance remains stationary, $c = 0$, when

$$U = \frac{\beta}{4\pi^2} L^2 \frac{1}{1 + L^2/D^2} \tag{14}$$

The length of the stationary wave $L_s$, for a certain wind velocity $U$ is therefore given by

$$L_s = 2\pi \sqrt{\frac{U}{\beta(1 - \frac{4\pi^2}{\beta} U/D^2)}} \tag{15}$$

or

$$L_s = 2\pi \sqrt{\frac{U + (1 + L_s^2/D^2)}{\beta}} \tag{16}$$

if the ratio $L_s/D$ is given, The length of the stationary wave increases as $L_s/D$ increases. If, for instance, $L_s/D = 1$, $L_s$ increases 41% above the value when $D = \infty$. If $L_s/D = \frac{1}{2}$ the increase is only 11%. The effect of the finite lateral extent of the disturbances is therefore to diminish the number of possible stationary disturbances around the circumference of the earth at a given latitude while increasing the size of the perturbation.

Equation (15) shows that for perturbations of finite lateral extent a stationary wave length exists only if

$$D > 2\pi\sqrt{U/\beta}$$

that is when the lateral extent of the perturbation is greater than the stationary wave length for a perturbation whose lateral extent is infinite.

When $D = \infty$ (15) and (16) reduce to the expression

$$L_s = 2\pi\sqrt{U/\beta}$$

which was given by Rossby. When the value of $\beta$ from (2) and the angular velocity of the air motion relative to the earth, given by

$$\dot{\lambda} = \frac{U}{a \cos \varphi},$$

are introduced here it follows that

$$L_s = \pi a \sqrt{\frac{2\dot{\lambda}}{\omega}} \tag{17}$$

The only variable in this equation is $\dot{\lambda}$. The table published by Rossby (*2*) shows that the maximum angular velocity during February is approximately constant at least from 70° to 30° northern latitude, its value being $5.02 \times 10^{-6}$ sec$^{-1}$. This would give a stationary wave length of 7550 km. To show this graphically lines of equal wind velocity have been plotted in Fig. 1 with the geographical latitude as abscissa and the stationary wave length of the disturbances of finite lateral extent as ordinate. The circles indicate the position of the wind velocities given in Rossby's table. They lie fairly closely on a straight line corresponding to a stationary wave length of about 7500 km. According to (16) these considerations concerning

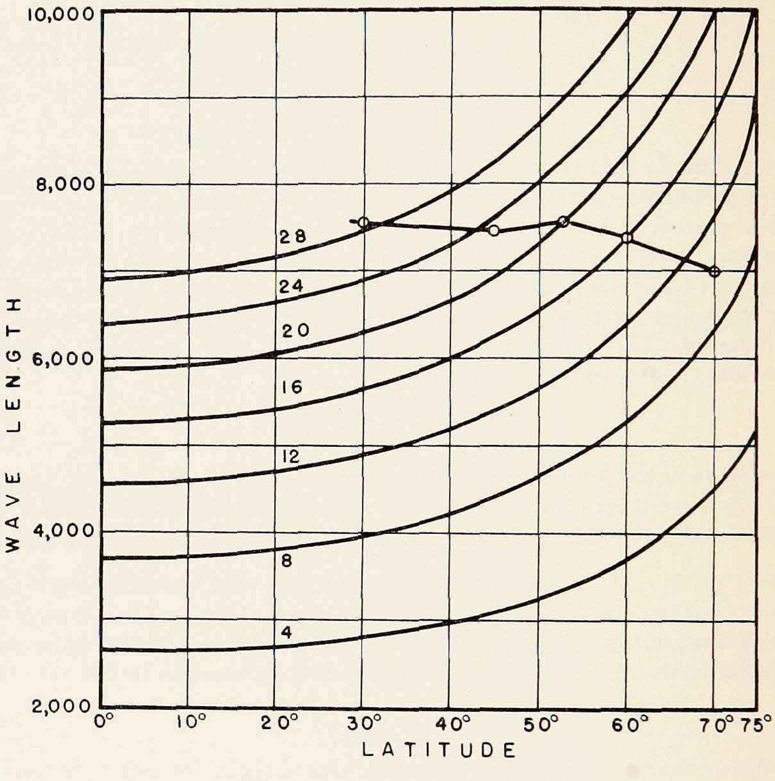


Figure 1. Stationary wave lengths of disturbances of infinite lateral extent. The curves are lines of equal wind velocity in m/sec. The circles indicate the zonal wind velocities at different latitudes on the northern hemisphere.

the constancy of $L_s$ when $\dot{\lambda}$ is constant also hold when the disturbances are of finite lateral extent, provided that the ratio $L_s/D$ is the same everywhere.

Table II and Fig. 2 show the stationary wave lengths for different wind velocities and latitudes when the perturbation extends from the pole to the equator. This value assumed for the magnitude of the lateral extent of the perturbation may be rather an extreme one but for a smaller lateral extent the effect is still greater. For long waves the deviations from the infinite case treated by Rossby become quite large. At latitude 60°, for instance, the stationary wave length is almost 15000 km. as compared to 8300 km. in the infinite case.*

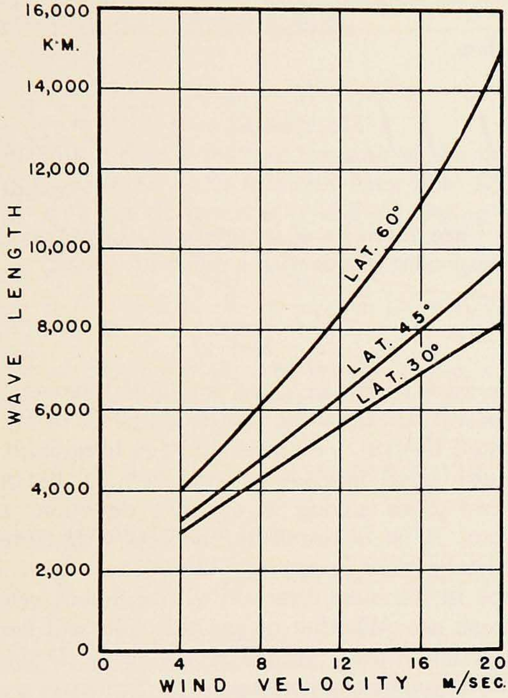


Figure 2. Stationary wave lengths when the perturbation extends from the pole to the equator.

TABLE II

Stationary Wave Length when the Perturbation Extends from Pole to Equator

| $\varphi$ \ $U$ | *4 m/sec* | *8 m/sec* | *12 m/sec* | *16 m/sec* | *20 m/sec* |
|---|---|---|---|---|---|
| 30° | 2940 km. | 4350 km. | 5600 km. | 6850 km. | 8120 km. |
| 45° | 3280 | 4820 | 6440 | 8000 | 9750 |
| 60° | 4000 | 6160 | 8410 | 11100 | 14980 |

* It should however, be understood that the approximation involved in the derivation of these equations is not very good in the case of such long waves. To get more accurate information it will be necessary to take into account the spherical form of the earth.

A more general solution of (11) than the one represented by (13) can be obtained by Cauchy's method. It can be made to satisfy the initial condition that at the time $t = 0$ the stream function is equal to a given function

$$\psi_{t=0} = F(x, y)$$

The solution is then

$$(18)\quad \psi = \frac{1}{\pi^2}\int_0^{\infty}\!\!\int \int_{-\infty}^{+\infty}\!\!\int F(\sigma, \tau)\cos\lambda\left[x-\sigma-\left(U-\frac{\beta}{\lambda^2+\mu^2}\right)t\right]$$
$$\cos\mu(y-\tau)\,d\sigma d\tau d\lambda d\mu.$$

Here $\lambda$, $\mu$, $\sigma$ and $\tau$ are variables of integration. Equation (18) shows that each harmonic component moves with a different velocity

$$U - \frac{\beta}{\lambda^2 + \mu^2}$$

depending on the wave length $2\pi/\lambda$ and the lateral extent $2\pi/\mu$ of the harmonic. The observations show the integrated effect of all the harmonics which is represented by (18). Sufficiently large harmonics move towards the west, sufficiently small ones towards the east, so that the disturbance may split into two parts moving in opposite directions as Rossby has already pointed out. It is, of course, not necessary that the motion of the harmonics should take place in opposite directions. Even if the different harmonics proceed in the same direction with different velocities the disturbance may break up. Whether or not this split will occur depends on the initial distribution $F(x, y)$ of the disturbance. If this is a function such that only the harmonic components in the vicinity of a certain velocity have appreciable amplitudes while the amplitudes of the others are small, no separation will be observed on the map. In this case the disturbance will spread over a larger area and its intensity will weaken.

### *B. The equations when friction and external forces are acting*

The perturbations which have been discussed so far are free perturbations which weaken gradually owing to the influence of friction. The effect of friction will be assumed directly proportional to the intensity of the vorticity. Thus

$$\text{frictional force} = -k\zeta$$

where $k$ is the coefficient of friction. New perturbations may be generated by external forces, for instance by the action of solenoids distributed along the boundary of land and ocean. For the sake of simplicity it may be assumed that the force depends on the $x$-coordinate only, that is on the longitude, and is independent of the latitude. These assumptions about the friction and the external force will at least give an approximate idea as to the effect of these two factors.

Furthermore it will be assumed now that an external force $K(x)$ is acting which generates solenoids and which depends only on the $x$-coordinate. The equation for the vorticity becomes

$$\frac{d\zeta}{dt} + k\zeta + \beta v = K(x). \tag{19}$$

As previously the total velocity may be the sum of the undisturbed (geostrophic) velocity $U$ along the $x$-axis (directed from W to E) and of the perturbation velocity with the components $u'$ and $v'$. Owing to the smallness of $u'$ and $v'$ (19) may be simplified so that

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + k\right)\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) + \beta v' = K(x). \tag{20}$$

Since the equation of continuity (8) remains unchanged a stream function $\psi$ can be introduced again by (9). Equation (20) is then transformed into

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + k\right)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \beta\frac{\partial\psi}{\partial x} = K(x). \tag{21}$$

The solution will be assumed in the form

$$\psi = Ae^{i(\mu x - \nu t)}\, e^{i\lambda y} + g(x). \tag{22}$$

$A$ is an arbitrary constant, representing the amplitude of the disturbance, the wave length $L = 2\pi/\mu$, and the width of the disturbance $D = 2\pi/\lambda$. The time factor $\nu$ has to be determined by substituting (22) in (21). The calculations are simpler when the solution is assumed in complex rather than in real form. To obtain a physical interpretation either the real or the imaginary part of the complex solution or a linear combination of both may be chosen since each part separately represents a solution of the differential equation (21). The function $g(x)$ depends on the $x$-coordinate only. It represents the influence of the external force $K(x)$ on the perturbation. Substituting (22) in (21) the following two relations are obtained

$$\nu = \mu\left(U - \frac{\beta}{\mu^2 + \lambda^2}\right) - ik \tag{23}$$

and

$$\frac{d^3g}{dx^3} + \frac{k}{U}\frac{d^2g}{dx^2} + \frac{\beta}{U}\frac{dg}{dx} = \frac{K(x)}{U}. \tag{24}$$

Equation (23) gives the relation between the time factor $\nu$ and the dimensions of the disturbance which are denoted by $\mu$ and $\lambda$. When the friction is not taken into consideration $k = 0$ and (23) is identical with (13) since the wave velocity $c = \nu/\mu$. Substituting (23) in (22) and retaining only the real part of the solution* $\psi$ may be written

$$(25)\quad \psi = A \cos\frac{2\pi}{L}\left[x - \left(U - \frac{\beta L^2}{4\pi^2}\frac{D^2}{D^2 + L^2}\right)t\right]e^{-kt}\cos\frac{2\pi}{D}y + g(x)$$

## C. Friction

When $K(x) = 0$ the function $g(x)$ may be omitted and the first term of (25) represents the complete solution. This expression shows that the velocity of propagation of the free perturbation remains the same as in the simpler case (eq. 13) when the effect of friction was not considered. Friction causes the amplitude of the disturbance to decrease exponentially with time.

The order of magnitude of the coefficient of friction $k$ has been determined as $10^{-5} - 10^{-4}$ sec$^{-1}$ from surface observations (*1*). This value is probably much too high if the whole atmosphere is considered instead of the surface layers only. Here the values $10^{-6}$ sec$^{-1}$ and $5 \cdot 10^{-6}$ sec$^{-1}$ have been chosen for $k$ in order to obtain a first approximation to the rate of dissipation of the disturbances. The corresponding values of $e^{-kt}$, i. e. the fraction by which the amplitude is diminished after a certain number of days, are given in Table III. With the smaller value of $k$ the amplitude would be reduced to 50% of its original value after 8 days, with the larger one in less than two days.

TABLE III

RATE OF DISSIPATION OF ATMOSPHERIC DISTURBANCES WITH TIME; $k = 10^{-6}$ SEC$^{-1}$ AND $k = 5 \cdot 10^{-6}$ SEC$^{-1}$

| *Time in days* | *2* | *4* | *6* | *8* | *10* |
|---|---|---|---|---|---|
| $k = 10^{-6}$ sec$^{-1}$ | .84 | .71 | .60 | .50 | .42 |
| $k = 5 \times 10^{-6}$ sec$^{-1}$ | .42 | .18 | .075 | .031 | .013 |

For stationary waves the time factor $\nu$ in (22) is zero. Neglecting for the present the influence of the external force $K(x)$ the solution of (21) is given by

$$(26)\qquad \psi = e^{-\frac{k}{2U}x}\,(A_1 \cos \sigma x + A_2 \sin \sigma x)$$

or

* More accurately $A = a + ib$ should be regarded as a complex constant giving a solution of the form $(a + ib)e^{i(\mu x - \nu t)} = (a + ib)\,[\cos(\mu x - \nu t) + i \sin(\mu x - \nu t)]$. The real part of this expression is $a \cos(\mu x - \nu t) - b \sin(\mu x - \nu t) = A' \cos(\mu x - \nu t - \alpha)$ where $A'$ and $\alpha$ are given by $a$ and $b$. The difference between this expression and the one given in the text is, however, obviously not significant.

$$\psi = Ae^{-\frac{k}{2U}x} \cos(\sigma x - \gamma). \tag{27}$$

Here $A_1$ and $A_2$ or $A$ and $\gamma$ are arbitrary constants of integration. The wave length $L_s = 2\pi/\sigma$ is given by the following expression

$$\sigma = \sqrt{\frac{\beta}{U} - \frac{k^2}{4U^2}}.$$

Therefore

$$L_s = \frac{2\pi}{\sqrt{\beta/U - k^2/4U^2}}. \tag{28}$$

Equation (28) shows that the wave length of the stationary wave increases with increasing friction. For the zonal wind velocity the inequality

$$U > \frac{k^2}{4\beta} \tag{29}$$

must hold. Otherwise $\sigma$ becomes imaginary and $\psi$ loses its wave character. Even with the high value $k = 5 \cdot 10^{-6}$ and with $\beta = 1.619 \cdot 10^{-11}\ \mathrm{m}^{-1}\ \mathrm{sec}^{-1}$ (at 45° lat.), for instance, the lower limit for $U$ becomes very small, about 0.4 m/sec so that the presence of friction in general will not affect the possibility of the formation of stationary waves. Similarly equation (28) shows that the effect of friction on the wave length is small.

The values of the stationary wave length for different latitudes and wind velocities are given in Table IV where it is assumed that $k = 10^{-6}$. These figures should be compared with those given by Rossby (*2*). It will be seen that the difference in the wave lengths is negligible.

TABLE IV

STATIONARY WAVE LENGTHS IN KM. WHEN THE COEFFICIENT OF FRICTION $k = 10^{-6}$

| $\varphi$ \ $U$ | *4* | *8* | *12* | *16* | *20 m/sec.* |
|---|---|---|---|---|---|
| 30° | 2830 km. | 4000 km. | 4890 km. | 5650 km. | 6320 km. |
| 45° | 3130 | 4420 | 5410 | 6250 | 6990 |
| 60° | 3720 | 5260 | 6440 | 7430 | 8320 |

The main effect of friction is to produce a decrease in the amplitude of the stationary disturbance with increasing distance from the origin. Table V shows the decrease of the amplitude for the same latitudes and wind velocities as in Table IV and at a distance from the origin which is equal to twice the stationary wave length. The decrease in the amplitude per wave length is greater for smaller than for larger wind velocities even though for

TABLE V

THE DECREASE OF THE AMPLITUDE OF THE PERTURBATION OVER A DISTANCE EQUAL TO TWICE THE STATIONARY WAVE LENGTH. ORIGINAL AMPLITUDE = 1. $k = 10^{-6}$

| $\varphi$ \ $U$ | *4* | *8* | *12* | *16* | *20 m/sec.* |
|---|---|---|---|---|---|
| 30° | .494 | .607 | .665 | .695 | .729 |
| 45° | .457 | .578 | .636 | .669 | .705 |
| 60° | .394 | .519 | .584 | .620 | .660 |

the latter the wave lengths increase. Consequently it may be expected that the larger stationary perturbation patterns repeat themselves while the smaller ones will have a tendency to appear but once or with only a few repetitions. They will not tend to spread around the whole circumference.

According to (10) and (27) the vorticity is given by

$$\zeta = \frac{\partial^2 \psi}{\partial x^2} = -\frac{A}{U}\sqrt{k^2\sigma^2 + \beta^2}\, e^{-\frac{k}{2U}x} \cos(\sigma x - \gamma + \delta) \tag{30}$$

where

$$\tan \delta = \frac{k\sigma}{\beta}.$$

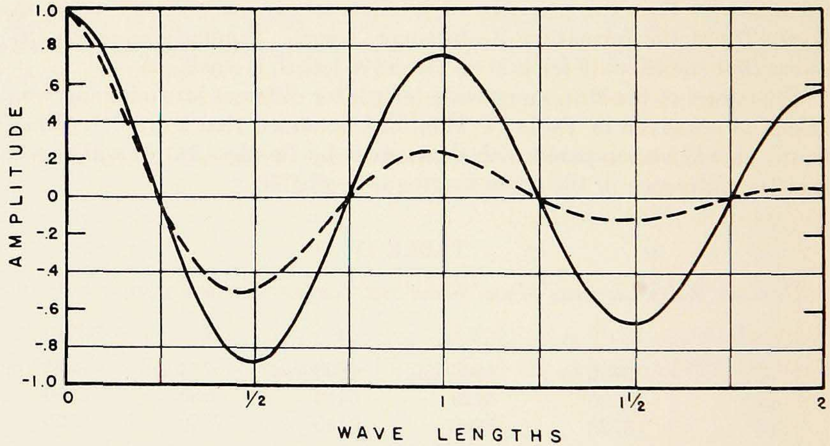


Figure 3. The vorticity distribution in a damped stationary perturbation. Lat. 45°, $U = 8$ m/sec. $k = 10^{-6}$ sec$^{-1}$ (full curve) and $5 \cdot 10^{-6}$ sec$^{-1}$ (broken curve).

The distribution of $\zeta$ along the $x$-axis in a damped stationary disturbance is represented in Fig. 3. The integration constants $A$ and $\gamma$ have been chosen so that

$$-\frac{A}{U}\sqrt{k^2\sigma^2 + \beta^2} = 1$$

and

$$\gamma = \delta$$

It has furthermore been assumed that $\varphi = 45°$, $U = 8$ m/sec, $k = 10^{-6}$ sec$^{-1}$ in one case and $k = 5 \cdot 10^{-6}$ sec$^{-1}$ in the other. When $k = 10^{-6}$ sec$^{-1}$ the length of the stationary wave is 4420 km. When $k = 5 \cdot 10^{-6}$ sec$^{-1}$ it is 4530 km. so that the effect of $k$ on the wave length is small. The effect of friction on the amplitude is small when $k = 10^{-6}$ sec$^{-1}$. But when $k = 5 \cdot 10^{-6}$ sec$^{-1}$ the damping already becomes very noticeable at a distance of half a wave length from the origin (broken curve).

### D. *External Force*

The action of the external force $K(x)$ is represented by the function $g(x)$ in (25) which has to be determined from (24). This equation is similar to the equation for forced oscillations subjected to damping. The main difference is that (24) is a third order equation. But since

$$v' = \frac{\partial \psi}{\partial x} = \frac{dg}{dx} \tag{31}$$

the equation (24) may be written as a second order differential equation in $v'$. The appearance of the third derivative of $g$ is explained physically by the fact that the external force $K(x)$ acts on the changes of the vorticity, not on the velocity directly. Another difference is the appearance of the length coordinate as the independent variable instead of the time. This occurs because the external force depends on the $x$-coordinate. The case of an external force varying with time could, of course, be treated in a similar manner.

The solution of (24) is given by

$$v' = \frac{e^{-\frac{k}{2U}x}}{\sigma U}\left[I_1(x)\sin\sigma x - I_2(x)\cos\sigma x\right] \tag{32}$$

where

$$I_1(x) = \int_0^x K(x)\, e^{-\frac{k}{2U}x}\cos\sigma x\, dx \tag{33a}$$

$$I_2(x) = \int_0^x K(x)\, e^{-\frac{k}{2U}x}\sin\sigma x\, dx \tag{33b}$$

The lower limits of the integrals $I_1(x)$ and $I_2(x)$ have been chosen so that $v' = 0$ when $x = 0$. The forced perturbation starts at the longitude defined by $x = 0$.

The solenoids which generate vorticity are frequently concentrated in a narrow strip near the coast and do not extend very far across the continent and the ocean. It may therefore be assumed that $K(x) \neq 0$ only in the interval $0 < x < \varepsilon$ and that $K(x) = 0$ everywhere else. It follows then from (33a) and (33b) that

$$\left.\begin{aligned} I_1(x) &= I_1(\varepsilon) \\ I_2(x) &= I_2(\varepsilon) \end{aligned}\right\} \text{for } x \geqq \varepsilon$$

Eq. (32) may be written in the form

$$(34) \qquad v' = \frac{1}{\sigma U}\sqrt{I_1^2(\varepsilon) + I_2^2(\varepsilon)}\, e^{-\frac{k}{2U}x} \sin(\sigma x - \delta)$$

where

$$\tan\delta = \frac{I_2(\varepsilon)}{I_1(\varepsilon)}.$$

Thus the distribution of the meridional velocity component outside the region where the external force is acting is given by a damped harmonic curve whose wave length and damping coefficient are equal to the wave length and damping coefficient of the free stationary wave.

To show this by a concrete example it will be assumed that the external force has a constant finite value $K$ in a narrow strip of width $\varepsilon$ and that it vanishes outside this strip, that is

$$(35) \qquad \begin{aligned} K(x) &= K \text{ for } 0 \leqq x \leqq \varepsilon \\ K(x) &= 0 \ \text{ for } x < 0 \text{ and for } x > \varepsilon \end{aligned}$$

Other more elaborate functions might represent the distribution of the solenoids better. Such refinements would give more complicated expressions for $v'$. But they do not appear necessary at present since detailed observations of these disturbances are not yet available.

With the expression (35) for the external force (32) becomes

$$(36a) \quad v' = \frac{K}{\beta}\left[1 - \frac{1}{\sigma}\sqrt{\frac{\beta}{U}}e^{-\frac{k}{2U}x}\cos(\sigma x - \gamma)\right] \qquad \text{if } 0 \leqq x \leqq \varepsilon$$

$$(36b) \quad v' = \frac{K}{\beta}\frac{1}{\sigma}\sqrt{\frac{\beta}{U}}e^{-\frac{k}{2U}x}\left[e^{\frac{k\epsilon}{2U}}\cos[\sigma(x-\varepsilon)-\gamma] - \cos(\sigma x - \gamma)\right] \text{if } x \geqq \varepsilon$$

where

$$\tan\gamma = \frac{k}{2\sigma U}.$$

The perturbation velocity $v'$ and the perturbation vorticity $\frac{dv'}{dx}$ are continuous for $x = \varepsilon$ and vanish when $x = 0$. In the limiting case when the width

$\varepsilon$ of the strip in which the external force acts tends to zero it follows from (36b) that

$$v' = \frac{\varepsilon K}{\sigma U} e^{-\frac{k}{2U}x} \sin \sigma x. \tag{37}$$

Thus, as $\varepsilon$ tends to zero $K$ must tend to infinity in order that $\varepsilon \cdot K$ may remain finite and so give a finite value for $v'$. As long as $\varepsilon$ is small the forced perturbation is represented by a damped sine wave.

When $\varepsilon$ becomes comparable to the stationary wave length $2\pi/\sigma$ the meridional velocity component $v'$ may be computed from (36a) and (36b). To give an example it has been assumed that $U = 8$ m/sec, $\varepsilon = 1000$ km, $\varphi = 45°$, $k = 5 \cdot 10^{-6}$ sec$^{-1}$ and $K = 10\beta$. The velocity has been calculated for two cases:

a) disregarding the influence of friction,
b) taking the influence of friction into account.

The results are represented in Fig. 4. When the effect of friction is neg-

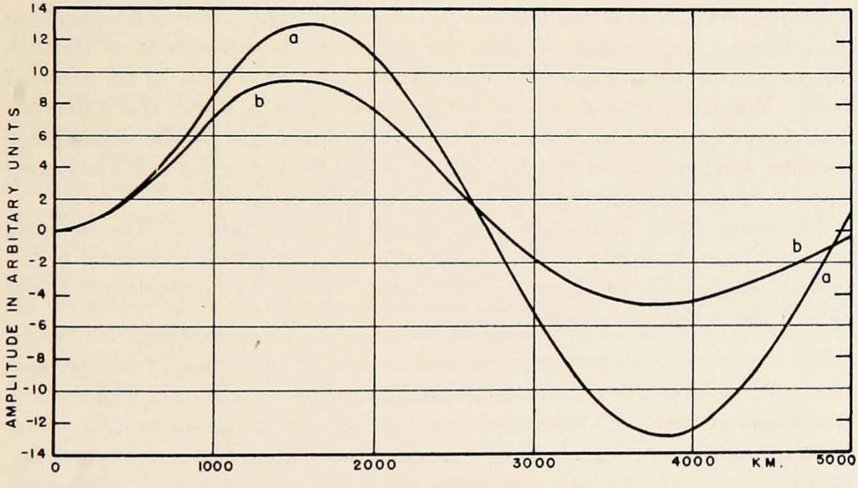


Figure 4. Effect of a concentration of solenoids on the intensity distribution of a stationary perturbation; (a) without friction (b) under the influence of friction.

lected (36a) and (36b) may be changed by a simple trigonometric transformation to

$$v' = 2\frac{K}{\beta}\sin^2\left(\sqrt{\frac{\beta}{U}}\,x\right) \qquad \text{for } 0 \leqq x \leqq \varepsilon \tag{38a}$$

$$v' = 2\frac{K}{\beta}\sin\left(\sqrt{\frac{\beta}{U}}\,\frac{\varepsilon}{2}\right)\sin\left[\sqrt{\frac{\beta}{U}}\left(x - \frac{\varepsilon}{2}\right)\right] \qquad \text{for } x \geqq \varepsilon \tag{38b}$$

These equations show that when no friction is acting $v'$ is proportional to the square of a sine function in the region where the external force acts and proportional to a sine function outside this region. When the frictional forces are effective the conditions are similar as a comparison of the curves (a) and (b) of Fig. 4 will show. In particular, the disturbance has the form of a damped harmonic curve outside the region where the external force is acting.

Over these stationary forced disturbances free migrating disturbances may be superimposed. The appearance of the resultant disturbances may therefore become rather complicated. When five-day mean charts are plotted (*3*), however, the effect of the non-stationary components is partly smoothed out so that the pattern of the charts becomes simpler and resembles more closely the one produced by the external solenoidal forces

## SUMMARY

In this paper an investigation by Rossby concerning the effect of the latitudinal variation of the Coriolis force on the propagation of atmospheric disturbances is extended to allow for the finite lateral extent of the disturbances, for the influence of friction and for the action of an external force. When the lateral extent of the disturbance is finite, the difference between geostrophic wind and velocity of propagation of the disturbance becomes smaller than in Rossby's case. When the disturbance is stationary its wave length is larger than in the case of infinite extent.

The main effect of friction is to damp the disturbance. The external force which is assumed to be a function of the longitude generates a stationary disturbance of the same wave length as the free disturbance would have if there were no external force.

## REFERENCES

(*1*) Exner, F. M.

1925. Dynamische Meteorologie, 2nd ed., p. 110. Wien, 1925.

(*2*) Rossby, C.-G. and collaborators

1939. Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action. Journ. of Mar. Res., vol. II, 1939, pp. 38–55.

(*3*) Willet, H. C.

1939. Certain statistical relationships bearing on the preparation of five-day weather forecasts in the United States. Bull. Am. Met. Soc., vol. 20, 1939, pp. 329–331.