Journal of Marine Research, 71, 253–288, 2013

Linear dynamics of a stably-neutrally stratified ocean

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ABSTRACT

Linear dynamics of stably-neutrally stratified fluid consisting of the stably stratified upper layer and the homogeneous lower layer is studied with and without rotation. The density and other fields are continuous at the interface between the layers. A special feature of this configuration is existence of the wave mode related to the homogeneous layer. In non-rotating fluid this is the homogeneous layer vortex mode characterized by a stationary three-dimensional velocity field confined to the lower layer. In the presence of rotation, the mode turns into the gyroscopic waves. Besides the mode, the wave spectrum contains internal waves and the zero frequency horizontal vortex mode with zero vertical velocity. In non-rotating fluid, the vertical velocity consists of the dispersive internal waves and of a steady component in the homogeneous layer. With increasing time the internal waves decay at a fixed point because of dispersion, and the vertical velocity decays in the upper layer and becomes stationary in the lower layer. A non-stationary boundary layer develops near the interface in the stratified layer at large times.

In rotating fluid we examined the wave spectrum not using the traditional and hydrostatic approximations, and found the spectrum consists of the super-inertial internal waves, the sub-inertial gyroscopic waves and the sub- and super-inertial internal inertio-gravity waves. In the case of strong stratification $f/N \ll 1$(f is the inertial frequency and N is the stratified layer buoyancy frequency) and for the long wave scales $f^2/N^2 \ll H/L \ll 1$(H and L are the fluid depth and the horizontal scale), the internal and the super-inertial inertio-gravity waves freely penetrate into the lower layer, and the gyroscopic waves are localized in the lower layer and are close to the inertial oscillations. Any long-wave field of the vertical velocity is split into the internal waves, and the inertial oscillations (long gyroscopic waves) confined to the lower layer. With time, the internal waves decay because of dispersion, and the vertical velocity goes to zero in the upper layer and in the lower layer only the inertial oscillations remain.

1. Introduction

In this paper we consider a stably-neutrally stratified (SNS) fluid consisting of two layers. The upper layer is continuously and stably stratified in density i.e. the buoyancy frequency $N(z) > 0$, whereas the lower layer is homogeneous so that $N(z) = 0$ in this layer (z is the vertical coordinate). Contrary to frequently used, two-layer models the density and

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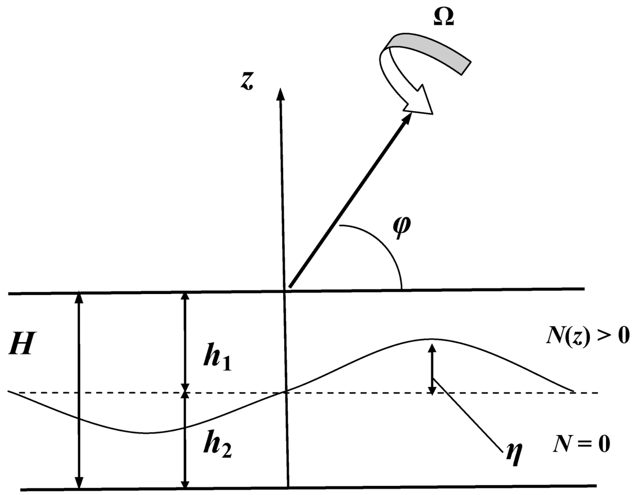

Figure 1. Schematic representation of rotating stably-neutrally stratified fluid.

other fields are assumed to be continuous at the interface between the layers. The buoyancy frequency $N(z)$ is either continuous or it has a finite discontinuity at the interface. This model is of interest for geophysical applications because it was recently shown that at least in some places of the world ocean, the stratification is very weak in the near-bottom layer several hundreds meters thick (e.g. van Haren and Millot 2005; Timmermans et al 2007).

The prime interest here was to study the internal waves (e.g. Miropol'sky 2001; Gerkema and Exarchou 2008). But, as we will see, besides the internal waves, other important wave modes exist in the SNS fluid: the homogeneous layer vortex mode in the case without rotation and the gyroscopic waves (GW) in the presence of rotation. To study evolution of arbitrary initial perturbation one has to take into account all the modes, not only the internal ones. In the first part of the paper we discuss the non-rotating SNS fluid without using the hydrostatic approximation.

The gyroscopic waves exist due to background rotation; in "pure" form they occur (e.g. LeBlond and Mysak 1978) in rotating barotropic fluid layer of constant depth, bounded by two rigid lids. The horizontal component of the Earth's rotation plays an important role in dynamics of the GWs. Under the traditional approximation (TA) when the horizontal component of the Earth's rotation is neglected, the GWs in the barotropic layer are sub-inertial: their frequencies σ do not exceed the vertical component of the double angular speed of rotation $f = 2\Omega \sin \varphi$ (see Figure 1) i.e. $\sigma \leq f$; without TA both sub-inertial and super-inertial waves with $\sigma \geq f$ are possible (e.g. Brekhovskikh and Goncharov 1994; Kasahara 2003).

In stratified fluid under TA the sub-inertial GWs exist together with the super-inertial internal waves only if the minimal buoyancy frequency $N_{\min}$ is smaller than f, $N_{\min} < f$

(e.g. Kamenkovich 1977). In the strongly stratified fluid, i.e. for $N_{\min} > f$, only the super-inertial internal waves are possible. However, without TA the sub-inertial internal inertio-gravity waves (IIGW) occur even in the strongly stratified fluid (Kasahara 2003; Gerkema and Shrira 2005; Gerkema et al 2008). Like the GWs, these waves cannot exist without rotation but the buoyancy effects play an important role in their dynamics.

One aim of this work is to study linear dynamics of a system in which the internal waves co-exist with the gyroscopic ones. The SNS fluid is the simplest system of this kind. We will see that without traditional and hydrostatic approximations, the wave spectrum of the SNS fluid includes all three types of the waves: the internal, the GWs and the super- and sub-inertial IIGWs. It will be shown also that the "non-traditional" terms due to the horizontal component of the Earth's rotation are of importance in weakly stratified domains of the fluid.

The paper is organized as follows: in Section 2 the governing equations with boundary and initial conditions for the SNS fluid are presented; a semi-qualitative analysis of the non-traditional terms depending on the stratification and the horizontal scale of motion is given. In Section 3 dynamics of the SNS fluid without rotation is discussed: the wave modes and evolution of arbitrary initial perturbation are analyzed. The wave modes in rotating fluid are examined in Section 4. Evolution of arbitrary initial perturbation in the rotating fluid in the long-wave approximation is examined in Section 5. Section 6 contains discussion and conclusions. Some details of calculations are given in Appendices A, B and C.

2. Governing equations and some approximations

In this Section we derive the equations governing dynamics of the SNS fluid and discuss some approximations to clarify the role of non-traditional terms for different stratifications and the motion scales. It is shown that the non-traditional terms cannot be neglected in domains with weak stratification, especially if the horizontal scale of motion L does not exceed the vertical scale H, i.e. $L \leq H$. If $L \gg H$ then the non-traditional terms cause a dispersion of near-inertial oscillations.

2.1. Governing equations

We consider a fluid layer of constant depth H, bounded by two rigid lids and rotating as a whole at a constant angular speed $\mathbf{\Omega}$ which can be non-parallel to the gravity (z-axis, see Figure 1). The fluid density ρ, being continuous, depends on the depth z in the upper layer of depth $h_1 - \eta$ and is constant in the lower layer of depth $h_2 + \eta$ where $h_1, h_2 = H - h_1$ are constant mean depths of the layers and $\eta = \eta(x, y, t)$ is the perturbation of interface between the layers. In the linear approximation the governing equations can be written in the form:

$$u_t - fv + f_s w = -p_x/\rho_0, \quad v_t + fu = -p_y/\rho_0, \qquad (2.1\text{a,b})$$

$$w_t - f_s u + g\rho/\rho_0 = -p_z/\rho_0, \quad \rho_t - \rho_0 N^2 w/g = 0, \quad u_x + v_y + w_z = 0 \quad (2.1\text{c,d,e})$$

for $0 \geq z \geq -h_1$, and

$$u_t - fv + f_s w = -p_x/\rho_0, \quad v_t + fu = -p_y/\rho_0, \tag{2.2a,b}$$

$$w_t - f_s u = -p_z/\rho_0, \quad u_x + v_y + w_z = 0 \tag{2.2c,d}$$

for $-h_1 \geq z \geq -H$. Here x- and y-axes are directed along the meridian and the parallel, u, v, w are the velocity components along the x, y, z-axes, ρ and p are the deviations of density and pressure from their hydrostatic values, $N = N(z)$ is the buoyancy frequency, $f = 2\Omega \sin \varphi$ is the Coriolis parameter, $f_s = 2\Omega \cos \varphi$ is the double horizontal component of the angular speed $\mathbf{\Omega}$ and ρ_0 is the constant density of the lower layer.

The velocity, pressure and density fields satisfy the initial conditions:

$$(u, v, w, \rho)_{t=0} = (u_I, v_I, w_I, \rho_I)(x, y, z); \quad w_I = -\int_{-H}^{z} (\partial_x u_I + \partial_y v_I) dz; \tag{2.3a,b}$$

the no-flux conditions at the rigid surface and the bottom:

$$w|_{z=0,-H} = 0, \tag{2.3c}$$

and the continuity conditions at the interface:

$$[u, v, w, p]_{z=-h_1} = 0, \quad \rho|_{z=-h_1} = \frac{\rho_0}{g} N^2(-h_1)\eta, \tag{2.3d,e}$$

where $[a]_{z=-h_1} = a|_{z=-h_1+0} - a|_{z=-h_1-0}$. The interface $z = -h_1 + \eta$ is a material surface therefore in the linear approximation the vertical velocity and the perturbation η are related as follows:

$$w|_{z=-h_1} = \eta_t. \tag{2.4}$$

In (2.3a,b) and below the subscript I denotes initial value of corresponding quantity.

It is of convenience to reduce (2.1) and (2.2) to the equations for vertical velocity w (e.g. Miropol'sky, 2001):

$$(\partial_{tt} + f^2) w_{zz} + \nabla_h^2 w_{tt} + 2 f f_s w_{yz} + f_s^2 w_{yy} + N^2 \nabla_h^2 w = 0, \quad 0 \geq z \geq -h_1 \tag{2.5a}$$

$$(\partial_{tt} + f^2) w_{zz} + \nabla_h^2 w_{tt} + 2 f f_s w_{yz} + f_s^2 w_{yy} = 0, \qquad -h_1 \geq z \geq -H, \tag{2.5b}$$

where $\nabla_h^2 = \partial_{xx} + \partial_{yy}$. The boundary and initial conditions for (2.5) simply follow from (2.3):

$$w|_{z=0,-H} = 0, [w]_{z=-h_1} = [w_z]_{z=-h_1} = 0, w|_{t=0} = w_I, w_t|_{t=0} = \dot{w}_I. \tag{2.6a,b,c,d}$$

The field $\dot{w}_I$ can be expressed in terms of the initial fields u_I, v_I, ρ_I (see Appendices A and C).

2.2. Some approximations

Before proceeding to detail consideration of wave solutions, we examine different approximations to the system (2.5) depending of the motion scales and relationship between N and f. Special attention is paid to the role of the non-traditional terms in (2.5) which are proportional to f_s. The results are summarized in Table 1.

Let the stratification be weak i.e.

$$N \sim f, \tag{2.7}$$

and the horizontal scale L does not exceed the vertical scale H:

$$L \leq H. \tag{2.8}$$

It is readily seen that in this case all terms in (2.5) are of the same order and the non-traditional terms cannot be neglected.

In the long wave approximation

$$L \gg H \tag{2.9}$$

both the equations (2.5) are reduced to approximate equations

$$(\partial_{tt} + f^2) w_{zz}^{\pm} = 0, \tag{2.10}$$

which describe non-dispersive inertial oscillations $\propto \sin ft, \cos ft$. Here and below the superscripts $+$ and $-$ denote the quantities related to the upper and lower layer. The neglected terms induce a slow dispersion of the oscillations on typical times $T_D \gg T_w$ where T_w is a characteristic wave time equal here f^{-1}. The approximate solution to (2.5) can be written in the form:

$$w = w(x, y, z, t^*, t_D^*), \quad t^* = t/T_w = ft, \quad t_D^* = t/T_D. \tag{2.11}$$

The representation (2.11) means that to within small values

$$\partial_{tt} w_{zz} = \frac{1}{T_w^2} \partial_{t^* t^*} w_{zz} + \frac{2}{T_w T_D} \partial_{t^* t_D^*} w_{zz}. \tag{2.12}$$

The dispersion time T_D is determined from the condition that the second term in the r.h.s. part of (2.12) is of the order of maximal neglected term. In our case this term is $2ff_s w_{yz}$ therefore the dispersion is induced by the non-traditional terms and characterized by the dispersion time

$$T_D = \frac{L}{H} f^{-1} \gg f^{-1}. \tag{2.13}$$

Now we consider the case of strong stratification when

$$N \gg f, \tag{2.14}$$

Table 1. Significance of the non-traditional terms depending on stratification and horizontal scale of motion. T_w and T_D are the typical wave and dispersive time scales, W and D mean wave and dispersion, the signs + and – mean significance and insignificance of the non-traditional terms, respectively. For example, the notation W(–), D(+) means that the non-traditional terms are unimportant in dynamics of primary wave (i.e. they can be neglected in the lowest order dispersion relation) but are responsible for dispersion of the wave.

	$L \leq H$	$L_R \gg L \gg H$	$L_R \sim L \gg H$	$L_R \ll L \ll \frac{N}{f} L_R$	$L \geq \frac{N}{f} L_R$
$N \sim f$	Full equations (2.5) $T_w \sim T_D \sim f^{-1}$ $W(+), D(+)$	$(\partial_{tt} + f^2) w_{zz}^{\pm} = 0$ $T_w \sim f^{-1}, T_D \sim \frac{L}{H} f^{-1}$ $W(-), D(+)$	–	–	–
$N \gg f$	Internal waves $\partial_{tt} \nabla^2 w^+ + N^2 \nabla_h^2 w^+ = 0$ $\nabla^2 w^- = 0$ $T_w \sim T_D \sim N^{-1}$ $W(-), D(-)$	Internal waves $\partial_{tt} w_{zz}^+ + N^2 \nabla_h^2 w^+ = 0$ $w_{zz}^- = 0$ $T_w \sim T_D \sim \frac{L}{H} N^{-1}$ $W(-), D(-)$	Internal waves $(\partial_{tt} + f^2) w_{zz}^+ + N^2 \nabla_h^2 w^+ = 0$ $w_{zz}^- = 0$ $T_w \sim T_D \sim f^{-1}$ $W(-), D(-)$	Internal waves $(\partial_{tt} + f^2) w_{zz}^{\pm} = 0$ $T_w \sim f^{-1}$, $T_D \sim \frac{L^2}{L_R^2} f^{-1}$ $W(-), D(-)$	Internal waves $(\partial_{tt} + f^2) w_{zz}^{\pm} = 0$ $T_w \sim f^{-1}$, $T_D \sim \frac{L}{H} f^{-1}$ $W(-), D(+)$
	Gyroscopic waves $w^+ = 0$, w^- -full Eq. (2.5b) $T_w \sim T_D \sim f^{-1}$ $W(+), D(+)$	Gyroscopic waves $w^+ = 0$ $(\partial_{tt} + f^2) w_{zz}^- = 0$ $T_w \sim f^{-1}, T_D \sim \frac{L}{H} f^{-1}$ $W(-), D(+)$	Gyroscopic waves –	Gyroscopic waves $(\partial_{tt} + f^2) w_{zz}^+ = 0$ $T_w \sim f^{-1}$, $T_D \sim \frac{L}{H} f^{-1}$ $W(-), D(+)$	Gyroscopic waves –

and the scales satisfy (2.8). Neglect of small terms in (2.5) does not change (2.5b), whereas (2.5a) takes the form:

$$\partial_{tt} w_{zz}^{+} + N^2 \nabla_h^2 w^{+} + \partial_{tt} \nabla_h^2 w^{+} = 0. \tag{2.15}$$

It is readily seen that two types of the wave motions can exist in the system (2.15) and (2.5b): the internal waves with the timescale $T_w \sim N^{-1}$ and the gyroscopic waves with the timescale $T_w \sim f^{-1}$. For internal wave two last terms and the term $f^2 w_{zz}$ in (2.5b) are small and the equation can be approximately written as

$$w_{zz}^{-} + \nabla_h^2 w^{-} = 0. \tag{2.16}$$

Thus for the strong stratification and moderate horizontal scales (2.8), the rotation is negligible for the internal waves and their dynamics is approximately described by equations for non-rotating SNS fluid (see Section 3).

In the case of gyroscopic waves with $T_w \sim f^{-1}$ the equation (2.5a) degenerates into the Laplace equation $\nabla_h^2 w^{+} = 0$ hence we have

$$w^{+} = 0. \tag{2.17}$$

Thus under the conditions (2.14) and (2.8), the gyroscopic waves are confined to the lower homogeneous layer and are approximately described by the equation (2.5b) with no-flux conditions at the bottom $z = -H$ and the interface $z = -h_1$. Obviously, the non-traditional terms cannot be neglected in the lower layer.

The regimes of motion in the long wave range (2.9) depend on relationships between L and the Rossby scale $L_R = HN/f \gg H$. In the range

$$H \ll L \ll L_R \tag{2.18}$$

the internal waves have timescale

$$T_w \sim \frac{L}{H} N^{-1} = \frac{L}{L_R} f^{-1} \ll f^{-1}, \tag{2.19}$$

approximate equations for these waves are (2.15) and (2.16), in which the last terms are omitted. Rotation does not affect the internal waves in the range (2.18). The gyroscopic waves with the scale $T_w \sim f^{-1}$ are approximately described by the equation of inertial oscillations (2.10) in the lower layer; the oscillations do not penetrate into the upper layer where (2.17) is valid. Therefore, in the range (2.18), the non-traditional terms are negligible in the upper layer and in the lower layer they produce a dispersion of the inertial oscillations on times (2.13).

For practically interesting horizontal scales

$$H \ll L \sim L_R \tag{2.20}$$

the timescale of internal waves is $T_w \sim f^{-1}$, i.e. the rotation becomes of importance. Approximate equations for these waves are as follows:

$$(\partial_{tt} + f^2)w^+_{zz} + N^2\nabla_h^2 w^+ = 0, \quad w^-_{zz} = 0. \tag{2.21}$$

Dynamics of the gyroscopic waves is the same as in the preceding case.

In the range of ultra long scales

$$L \gg L_R \tag{2.22}$$

both the internal and gyroscopic waves degenerates into the inertial oscillations with the approximate equations (2.10). As this takes place, the typical dispersion time for the gyroscopic waves coincides with (2.13), and for the internal waves it depends on relationship between the scales L and $(N/f)L_R \gg L_R$. If

$$L_R \ll L \ll \frac{N}{f}L_R \tag{2.23}$$

then dispersion spreading of the long internal waves with characteristic time

$$T_D = \frac{L^2}{L_R^2}f^{-1} \tag{2.24}$$

dominates. In the case

$$L \gg \frac{N}{f}L_R \tag{2.25}$$

the dispersion is determined by the non-traditional terms and the typical time T_D is given by (2.13). Finally, if

$$L \sim \frac{N}{f}L_R \tag{2.26}$$

then the times (2.13) and (2.24) coincide. The same estimates are also valid for the internal waves in stably stratified fluid, therefore the theory of inertial oscillations developed by Young, Ben Jelloul (1997) under TA, is valid in the scale range (2.23). The results of the Subsection are summarized in Table 1.

3. Waves in the absence of rotation

In this Section we consider the case when the background rotation is absent i.e. $f = f_s = 0$. Subsection 3.1 is devoted to analysis of wave modes in the non-rotating SNS fluid. We show that the wave spectrum consists of the internal waves with non-zero frequencies and two zero frequency vortex modes—the horizontal one and the homogeneous layer one.

The vortex modes are due to conservation of vorticity in the lower layer and the vertical component of vorticity in the upper one.

In Subsection 3.2 we discuss the initial value problem (2.5) and (2.6) in the non-rotating SNS fluid. For simplicity we restrict ourselves to the consideration of the vertical velocity in which the horizontal vortex mode does not manifest itself. An important new element, in comparison with stably stratified fluid, is a three-dimensional stationary vortical circulation confined to the homogeneous lower layer. The resulting motion is a sum of the circulation and internal waves. To prevent penetration of the stationary signal from the homogeneous layer into the stratified one, a non-stationary boundary layer develops near the interface in the upper layer at large times. The boundary layer is analyzed in Subsection 3.3.

3.1. Wave solutions

Without rotation the equations (2.5) are simplified:

$$\partial_{tt}\nabla^2 w^+ + N^2\nabla_h^2 w^+ = 0, \quad \nabla^2 w^- = \nabla^2 w_I^-, \tag{3.1a,b}$$

where $\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz}$. The boundary and initial conditions (2.6) remain unchanged. Substituting the plane wave solution

$$w = W(z)\exp[i(kx + ly - \sigma t)] + c.c., \tag{3.2}$$

into the homogeneous version of (3.1) and assuming the frequency $\sigma \neq 0$ one obtains the following equation for the amplitude W:

$$W_{zz} - s^2(\kappa, z, \sigma)W = 0, \quad s^2 = \begin{cases} \kappa^2(\sigma^2 - N^2)/\sigma^2, & z \geq -h_1, \\ \kappa^2, & z \leq -h_1 \end{cases} \tag{3.3a,b}$$

which should be solved under the conditions

$$W|_{z=0,-H} = 0, [W]_{z=-h_1} = [W_z]_{z=-h_1} = 0. \tag{3.3c,d}$$

Here $c.c.$ denotes the complex conjugate value and $\kappa = \sqrt{k^2 + l^2}$.

Multiplying (3.3a) by the complex conjugate W^* and integrating the resulting equation over z from – H to 0 taking (3.3c,d) into account, one finds that non-trivial solution to (3.3) exists only if

$$0 < \sigma < N_{\max}. \tag{3.4}$$

Obviously the modes are the internal waves existing due to stratification.

The lower layer solution W to within arbitrary constant amplitude is given by the formula:

$$W = W^- = \sinh[\kappa(z + H)]. \tag{3.5}$$

By virtue of (3.5) we have:

$$(W_z^- - \alpha(\kappa)W^-)_{z=-h_1} = 0, \quad \alpha = \kappa \coth \kappa h_2, \tag{3.6}$$

therefore the upper layer problem can be written as

$$W_{zz}^+ - \kappa^2 W^+ = -\lambda\kappa^2 N^2 W^+, \tag{3.7a}$$

$$\left(W_z^+ - \alpha(\kappa)W^+\right)_{z=-h_1} = 0,\ W^+\big|_{z=0} = 0, \tag{3.7b.c}$$

where $\lambda = \sigma^{-2}$. The problem (3.7) is a classical Sturm-Liouville eigenvalue problem (e.g. Korn & Korn, 1968). The spectrum of the eigenvalues $\lambda = \lambda_\mu, \mu = 1, 2, \ldots$ is discrete and positive i.e. $\lambda_\mu > 0$, and the corresponding eigenfunctions W_μ^+ comprise a complete orthogonal basis. The amplitude function W_μ in (3.2) corresponding to the μ-th vertical mode of internal waves can be represented as:

$$W_\mu = \begin{cases} W_\mu^+(z,\kappa), & z \geq -h_1 \\ W_\mu^- = \dfrac{\sinh \kappa(z+H)}{\sinh \kappa h_2}, & z \leq -h_1 \end{cases}, \tag{3.8}$$

where W_μ^+ is properly normalized.

It is seen from (3.8) that the long internal waves with $\kappa h_2 \ll 1$ freely penetrate into the homogeneous layer, the corresponding vertical (horizontal) velocities being approximately linear in z (independent of z). In the same time, the short internal waves with $\kappa h_2 \gg 1$ are confined to the stratified layer since in this case $W_\mu^- \sim e^{\kappa(z+h_1)}$. The lower layer field induced by the internal wave does not depend on the mode number μ and is determined only by the horizontal wavenumber κ. Therefore, the total field induced by the internal waves in the homogeneous layer can be represented in the form:

$$w_u^- = \frac{1}{2\pi}\int \tilde{w}_u^+(k,l,-h_1,t)\frac{\sinh \kappa(z+H)}{\sinh \kappa h_2} e^{i(kx+ly)}dkdl, \tag{3.9}$$

where $\tilde{w}_u^+ = \tilde{w}_u^+(k,l,z,t)$ is the Fourier-amplitude of the upper layer internal waves field $w_u^+(x,y,z,t)$.

In the simple case $N = const$ (see also Miropol'sky 2001) the dispersion relation and the amplitude functions W_μ can be written as:

$$\sigma_\mu = \sigma_\mu(\kappa) = \frac{N}{\sqrt{1 + s_\mu^2(\kappa)/\kappa^2 h_1^2}}, \mu = 1, 2, \ldots \tag{3.10a}$$

$$W_\mu = \begin{cases} -\dfrac{\sin q_\mu z}{\sin q_\mu h_1}, & z \geq -h_1 \\ \dfrac{\sinh \kappa(z+H)}{\sinh \kappa h_2}, & z \leq -h_1 \end{cases}. \tag{3.10b}$$

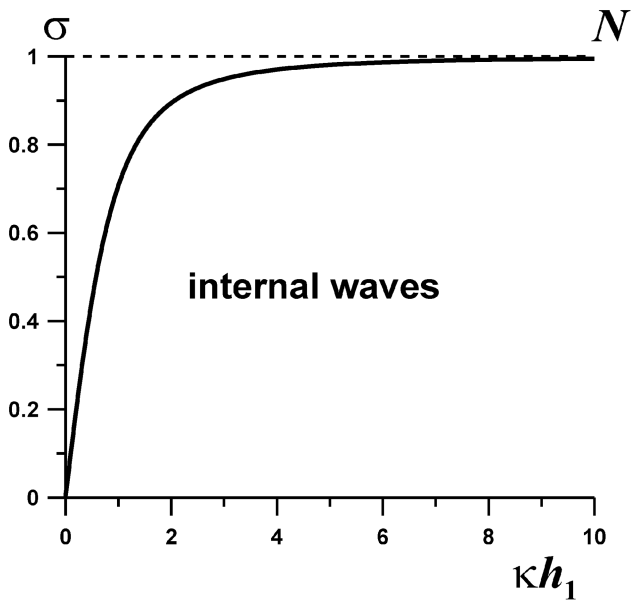

Figure 2. Dispersion curves in non-rotating fluid. The dispersion curve (3.10a) is shown for $\mu = 1$; the curves with $\mu > 1$ have a similar form.

Here $q_\mu = s_\mu(\kappa)/h_1$, where s_μ is the μ-th root of the equation:

$$s \cot s = -\kappa h_1 \coth \kappa h_2. \tag{3.10c}$$

Dispersion curve $\sigma_1(\kappa)$ is shown in Figure 2, the curves with $\mu > 1$ have a similar form.

One can readily check that without rotation the equations (2.1) and (2.2) conserve vorticity: the vertical component of vorticity in the upper layer,

$$\Omega_z^+ = v_x^+ - u_y^+ = \Omega_{zI}^+, \tag{3.11a}$$

and all three components in the lower one:

$$\Omega_z^- = v_x^- - u_y^- = \Omega_{zI}^-, \quad \Omega_x^- = w_y^- - v_z^- = \Omega_{xI}^-, \quad \Omega_y^- = u_z^- - w_x^- = \Omega_{yI}^-. \tag{3.11b,c,d}$$

The internal waves considered above are characterized by the zero vorticity in the homogeneous layer and zero vertical component of vorticity in the stratified one i.e. in these waves

$$\Omega_z^\pm = v_x^\pm - u_y^\pm = 0, \quad \Omega_x^- = w_y^- - v_z^- = 0, \quad \Omega_y^- = u_z^- - w_x^- = 0$$

Two zero-frequency vortex modes correspond to the non-zero vorticity components $\Omega_z^\pm, \Omega_{x,y}^-$. One of these modes will be referred to as the *horizontal* vortex mode and is due to the conservation of the vertical component of vorticity $\Omega_z^{\pm}$. The mode is characterized by zero vertical velocity, density, pressure and an arbitrary non-divergent horizontal velocity, i.e. in this mode

$$w = p = \rho = 0, u_x + v_y = 0. \tag{3.12a}$$

Given Ω_{zI}^{+} the velocities u, v of the horizontal vortex mode are determined from the continuity equation in (3.12a) and the vertical vorticity equation

$$v_x - u_y = \begin{cases} \Omega_{zI}^{+}, & z \geq -h_1 \\ \Omega_{zI}^{+}\big|_{z=-h_1}, & z \leq -h_1 \end{cases}. \tag{3.12b}$$

Another mode will be referred to as the *homogeneous layer* vortex mode. In this mode the upper layer fields and the lower layer pressure are zero:

$$u^{+} = v^{+} = w^{+} = p^{+} = \rho = p^{-} = 0. \tag{3.13a}$$

The lower layer vertical velocity $w^{-}(x, y, z)$ is a solution to equation following from (3.11c,d):

$$\nabla^2 w^{-} = \partial_y \Omega_{xI}^{-} - \partial_x \Omega_{yI}^{-} = \nabla^2 w_I^{-}, \tag{3.13b}$$

satisfying the boundary conditions

$$w^{-}\big|_{z=-h_1,-H} = 0. \tag{3.13c}$$

The lower layer horizontal velocities are related to the vertical one via the continuity equation (2.2d). We assume that all fields decay at infinity as $\sqrt{x^2 + y^2} \to \infty$, in this case the problem (3.13b,c) is well-defined and $w^{-}(x, y, z)$ can be readily calculated. Obviously, the condition (3.13c) ensures continuity of the vertical velocity at the interface $z = -h_1$, but the *derivative* w_z and, therefore, the horizontal velocity can be discontinuous at $z = -h_1$.

Thus, without rotation, the wave spectrum of the stably-neutrally stratified fluid consists of the internal waves with $\sigma > 0$ and two steady vortex modes. An important difference from the case of stable stratification (when $h_1 = H$), is that in the stably stratified fluid only the horizontal vortex mode with $w = 0$ exists.

3.2. Initial value problem

Now we consider the initial value problem (3.1) and (2.6). The solution is sought as a sum

$$w = w_a + w_I, \tag{3.14}$$

where the auxiliary function w_a obeys the following equations:

$$\partial_{tt}\nabla^2 w_a^+ + N^2\nabla_h^2 w_a^+ = -N^2\nabla_h^2 w_I^+, \quad \nabla^2 w_a^- = 0, \tag{3.15a,b}$$

$$w_a^+\big|_{z=0} = 0, \quad w_a^-\big|_{z=-H} = 0, \tag{3.16a,b}$$

$$w_a^+\big|_{z=-h_1} = w_a^-\big|_{z=-h_1}, \quad w_{az}^+\big|_{z=-h_1} = w_{az}^-\big|_{z=-h_1}, \tag{3.16c,d}$$

$$w_a^\pm|_{t=0} = 0, \quad w_{at}^+|_{t=0} = \dot{w}_I. \tag{3.16e,f}$$

All functions in (3.15), (3.16) are represented in the form of Fourier integrals:

$$w = \frac{1}{2\pi}\int \tilde{w}(k,l,z,t)e^{i(kx+ly)}dkdl, \tag{3.17}$$

here and below the tilde denotes the Fourier amplitude of the corresponding function. From (3.15a,b) we have

$$\left(\tilde{w}_{azz}^+ - \kappa^2\tilde{w}_a^+\right)_{tt} - \kappa^2 N^2\tilde{w}_a^+ = \kappa^2 N^2\tilde{w}_I^+, \tag{3.18a}$$

$$\tilde{w}_{azz}^- - \kappa^2\tilde{w}_a^- = 0. \tag{3.18b}$$

The boundary and initial conditions for $\tilde{w}_a^\pm$ coincide with (3.16) in which the functions are replaced by their Fourier amplitudes. Using (3.18b) and (3.16b) one obtains:

$$\tilde{w}_a^- = A(k,l,t)\sinh\kappa(z+H), \tag{3.19}$$

where $A(k,l,t)$ is as yet an unknown function. The solution (3.19) and the continuity conditions (3.16c,d) give the following relationship at the interface $z = -h_1$ (cf. (3.7b)):

$$\left(\tilde{w}_{az}^\pm - \alpha(\kappa)\tilde{w}_a^\pm\right)_{z=-h_1} = 0. \tag{3.20}$$

The solution $\tilde{w}_a^+$ is determined from the equation (3.18a), boundary conditions (3.16a), (3.20) and initial conditions (3.16e,f). Since the boundary condition (3.20) coincides with (3.7b) and the basis W_μ^+ is complete and orthogonal, the solution $\tilde{w}_a^+$ is sought in the form of the following expansion:

$$\tilde{w}_a^+ = \sum_{\mu=1}^{\infty}\tilde{w}_{a\mu}^+(k,l,t)W_\mu^+(k,l,z). \tag{3.21}$$

Here and below the subscript μ denotes the μ-th coefficient of corresponding expansion. The field $\tilde{w}_a$ is readily obtained and has the form:

$$\tilde{w}_a^+ = -\tilde{w}_I^+ + \tilde{w}_u^+, \tag{3.22a}$$

$$\tilde{w}_u^+ = \sum_{\mu=1}^{\infty}[\tilde{w}_{I\mu}^+\cos\sigma_\mu t + (\tilde{\dot{w}}_{I\mu}^+/\sigma_\mu)\sin\sigma_\mu t]W_\mu^+. \tag{3.22b}$$

$$\tilde{w}_a^- = -\tilde{w}_I(k,l,-h_1)\frac{\sinh\kappa(z+H)}{\sinh\kappa h_2} + \tilde{w}_u^+(k,l,-h_1,t)\frac{\sinh\kappa(z+H)}{\sinh\kappa h_2}. \tag{3.22c}$$

The resulting solution to the initial value problem (3.1) and (2.6) can be written in the form of Fourier integral (3.17) where the amplitude $\tilde{w}$ is a sum of the unsteady $\tilde{w}_u$ and steady $\tilde{w}_{st}$ components:

$$\tilde{w} = \tilde{w}_u(k, l, z, t) + \tilde{w}_{st}(k, l, z). \tag{3.23a}$$

Here

$$\tilde{w}_u = \begin{cases} \tilde{w}_u^+, & z \geq -h_1 \\ \tilde{w}_u^-, & z < -h_1 \end{cases}, \quad \tilde{w}_{st} = \begin{cases} 0, & z \geq -h_1 \\ \tilde{w}_{st}^-, & z < -h_1 \end{cases}; \tag{3.23b,c}$$

and the functions $\tilde{w}_u^-$, $\tilde{w}_{st}^-$ are given by the formulae:

$$\tilde{w}_u^- = \tilde{w}_u^+(k, l, -h_1, t)\frac{\sinh \kappa(z + H)}{\sinh \kappa h_2}, \tag{3.24}$$

$$\tilde{w}_{st}^- = \tilde{w}_I^- - \tilde{w}_I(k, l, -h_1)\frac{\sinh \kappa(z + H)}{\sinh \kappa h_2}. \tag{3.25}$$

Thus the solution consists of the stationary field w_{st} and the internal waves w_u engendered by the initial fields w_I, $\dot{w}_I$. One can readily see that w_{st} coincides with the homogeneous layer vortex mode previously considered. The existence in the linear problem of three-dimensional stationary circulation related to the vertical velocity w_{st} is possible only due to the presence of the homogeneous fluid layer. In stably stratified fluid, the time-independent vertical velocity would result in unbounded growth of density by virtue of equation (2.1d). Averaging (3.18) with respect to time and using (3.14) and (3.16b,c) one finds that the component w_{st} is the averaged in time vertical velocity w i.e.

$$w_{st} = \langle w \rangle_t = \lim_{T\to\infty} \frac{1}{T}\int_0^T w dt. \tag{3.26}$$

It follows from (3.25) that

$$\tilde{w}_{st}^-\big|_{z=-h_1} = 0,\ \partial_z \tilde{w}_{st}^-\big|_{z=-h_1} = (\partial_z \tilde{w}_I - \alpha(\kappa)\tilde{w}_I)_{z=-h_1}. \tag{3.27a,b}$$

In the special case when the initial field w_I satisfies the condition

$$(\partial_z \tilde{w}_I - \alpha(\kappa)\tilde{w}_I)_{z=-h_1} = 0, \tag{3.28}$$

we have

$$\partial_z w_{st}^-\big|_{z=-h_1} = 0, \tag{3.29}$$

i.e. the horizontal velocity $\mathbf{u}_{st}$ corresponding to the stationary component w_{st} is continuous at $z = -h_1$. If in addition the initial field is sufficiently smooth, so that the contribution of high harmonics with large numbers μ is small, then the solution remains smooth with time. If the initial field is localized in the horizontal plane, then the horizontal dispersion of internal waves results in decay of the field w_u and the solution w with increasing t tends to the stationary field w_{st}.

If (3.28) is not valid then the situation becomes more complicated since the vertical derivative $\partial_z w_{st}$ is discontinuous at the interface $z = -h_1$ together with $\mathbf{u}_{st}$. This means that even for smooth initial conditions, the internal wave field should contain a sufficient number of low-frequency harmonics with large numbers μ to provide continuity of the derivative w_z, i.e. of the horizontal velocities, at the interface $z = -h_1$. At large times the joint impact of these high harmonics results in developing a non-stationary boundary layer with large vertical gradients of the horizontal velocities in the stratified layer near $z = -h_1$; a similar process takes place for the forced Rossby waves in a bounded basin (Kamenkovich and Kamenkovich 1993).

3.3. Near-interface boundary layer

To study the boundary layer we (following Kamenkovich and Kamenkovich 1993) introduce a time averaged velocity $\bar{w}$ instead of w:

$$\bar{w} = \frac{1}{t}\int_0^t w dt. \tag{3.30}$$

Knowing $\bar{w}$ one can readily calculate w:

$$w = (t\bar{w})_t. \tag{3.31}$$

An important property of the new variable $\bar{w}$ is that on large times contribution of the rapidly oscillating part of the field w into $\bar{w}$ tends to zero and only the slowly oscillating boundary layer remains in $\bar{w}$.

In terms of $\bar{w}$ the problem (3.1) and (2.6) takes the form:

$$\left(t\nabla^2\bar{w}^+\right)_{tt} + t\nabla_h^2\bar{w}^+ = \nabla^2\dot{w}_I^+, \tag{3.32a}$$

$$\nabla^2\bar{w}^- = \nabla^2 w_I^-, \tag{3.32b}$$

$$\bar{w}^+\big|_{z=0} = 0,\ \bar{w}^-\big|_{z=-1} = 0, \tag{3.33a,b}$$

$$\bar{w}^+\big|_{z=-h_1} = \bar{w}^-\big|_{z=-h_1},\ \bar{w}_z^+\big|_{z=-h_1} = \bar{w}_z^-\big|_{z=-h_1}. \tag{3.33c,d}$$

Equations (3.32) and (3.33) are written in non-dimensional form using the length scale H and the time scale $1/N$; for simplicity N is assumed to be constant and the notation for h_1 remains unchanged.

In view of (3.22) and (3.26) at large times the solution $\bar{w}^-$ is close to w_{st}^-:

$$\bar{w}^- = w_{st}^- + O(1/t). \tag{3.34}$$

In the boundary layer near $z = -h_1$ in the domain $z \geq -h_1$ the function $\bar{w}^+$ should be $O(1/t)$ because of (3.27a), (3.34) but the derivative $\bar{w}_z^+ = O(1)$ to provide the continuity of the horizontal velocity at the interface. The leading order solution in the boundary layer is sought in the form (cf. Kamenkovich and Kamenkovich 1993):

$$\bar{w}^+ = \frac{1}{t}\hat{w}(x, y, \xi), \xi = (z + h_1)t, \tag{3.35}$$

where ξ is the boundary layer stretched coordinate. Substituting (3.35) into (3.32a) and neglecting small terms one obtains for $t \gg 1$:

$$\xi^2 \hat{w}_{\xi\xi\xi\xi} + 4\xi \hat{w}_{\xi\xi\xi} + 2\hat{w}_{\xi\xi} + \nabla_h^2 \hat{w} = \nabla^2 \dot{w}_I^+(x, y, -h_1). \tag{3.36}$$

Representing $\hat{w}$, $\dot{w}_I^+$ in the form of Fourier integral (3.17) we find from (3.36) the equation for the Fourier amplitude $\tilde{\hat{w}}$:

$$\xi^2 \tilde{\hat{w}}_{\xi\xi\xi\xi} + 4\xi \tilde{\hat{w}}_{\xi\xi\xi} + 2\tilde{\hat{w}}_{\xi\xi} - \kappa^2 \tilde{\hat{w}} = \tilde{R} = \left((\partial_{zz} - \kappa^2)\tilde{\dot{w}}_I^+\right)_{z=-h_1}. \tag{3.37}$$

A non-singular solution of (3.37) (Kamke 1976) is given by the formula $\tilde{\hat{w}} = C J_0(2\sqrt{\kappa\xi}) - \tilde{R}/\kappa^2$ hence we have

$$\tilde{\bar{w}}^+ = \frac{C}{t} J_0(2\sqrt{\kappa\xi}) - \frac{\tilde{R}}{\kappa^2 t}. \tag{3.38}$$

Here J_0 is the Bessel function of zero order and the coefficient $C = C(k, l)$ is determined from the continuity of $\bar{w}_z$ at $z = -h_1$:

$$C = -\frac{1}{\kappa}\left(\partial_z \tilde{w}_I - \alpha(\kappa)\tilde{w}_I\right)_{z=-h_1}. \tag{3.39}$$

As seen from (3.35) and (3.38) the boundary layer vertical velocity is small, but the corresponding horizontal velocity is of the order of unity and rapidly oscillates in z, its vertical gradients growing proportionally to t. Thickness of the boundary layer decreases with increasing t proportionally to $1/t$. If a small viscosity is included into the consideration, (not given here) then the non-stationary boundary layer becomes a narrow stationary one. In the stationary boundary layer the time averaged horizontal velocity changes from some non-zero value in the homogeneous layer to zero in the stratified one. Thus in either case, with or without viscosity, the nearest vicinity of the interface is characterized by strong vertical gradients of the horizontal velocity, which can result in strong mixing and instability in this domain.

4. Waves in rotating fluid

In this Section the waves in rotating SNS fluid are considered without using the traditional and hydrostatic approximations. The buoyancy frequency N in stratified layer is assumed to be constant. In Subsection 4.1 we obtain the vertical eigenfunctions of the wave modes with non-zero frequencies and demonstrate existence of three different frequency ranges depending on the parameters f, f_s, N, and the wave vector (k, l).

In Subsection 4.2 the waves in barotropic and stably stratified fluids are discussed. In the barotropic fluid the wave spectrum consists of super- and sub-inertial gyroscopic waves, in the stably stratified one—of super-inertial internal waves and sub-inertial IIGWs.

Dispersion curves and structure of the vertical modes in the SNS fluid are examined in Subsection 4.3. The wave spectrum consists of the sub-inertial gyroscopic waves oscillating (in the vertical) in the homogeneous layer and having no zeros in the stratified one, super-inertial internal waves oscillating in the stratified layer and having no zeros in the homogeneous one, and sub- and super-inertial IIGWs oscillating in both layers.

In Subsection 4.4 a case of long waves in the SNS fluid with strongly stratified ($N \gg f$) upper layer is considered. All types of the waves are close to the inertial oscillations if their horizontal scale L is sufficiently large in comparison with the fluid depth H. An important point is that the gyroscopic waves become near-inertial if $L \gg H$, whereas the internal waves and IIGWs are near-inertial only in the ultra long-wave limit $L \gg L_R = HN/f \gg H$.

Besides the modes with non-zero frequency, a zero frequency vortex mode exists in the rotating SNS fluid. The mode is discussed in Subsection 4.5. It obeys geostrophic equations and has a zero vertical velocity.

4.1. Dispersion relations

With rotation the amplitude W in the wave solution (3.2) obeys the following equations:

$$(f^2 - \sigma^2)W_{zz} + 2iff_s lW_z + [\kappa^2(\sigma^2 - N^2) - f_s^2 l^2]W = 0, \quad 0 \ge z \ge -h_1 \qquad (4.1a)$$

$$(f^2 - \sigma^2)W_{zz} + 2iff_s lW_z + (\kappa^2\sigma^2 - f_s^2 l^2)W = 0, \quad -h_1 \ge z \ge -H. \qquad (4.1b)$$

and the boundary conditions (3.3c,d). One can readily show that for non-zero κ non-trivial solutions W exist only if $\sigma \neq f$. For simplicity, we assume the buoyancy frequency N to be constant. In this case the amplitude W is given by the following formula:

$$W = \begin{cases} A^+(e^{\lambda_1^+ z} - e^{\lambda_2^+ z}), & 0 \ge z \ge -h_1 \\ A^-(e^{\lambda_1^-(z+H)} - e^{\lambda_2^-(z+H)}), & -h_1 \ge z \ge -H \end{cases}, \qquad (4.2)$$

where $A^\pm$ are constant amplitudes and

$$\lambda_1^+ = a + ib^+, \lambda_2^+ = a - ib^+, \lambda_1^- = a + ib^-, \lambda_2^- = a - ib^-, \qquad (4.3a,b,c,d)$$

$$a = -\frac{iff_s l}{f^2 - \sigma^2}, b^- = \frac{\sigma\kappa}{|f^2 - \sigma^2|}\sqrt{f^2 - \sigma^2 + \bar{f}_s^2}, \bar{f}_s = f_s\frac{|l|}{\kappa}, \qquad (4.4a,b,c)$$

$$b^+ = \frac{\kappa}{\left|f^2 - \sigma^2\right|}\sqrt{(f^2 - \sigma^2)(\sigma^2 - N^2) + \bar{f}_s^2\sigma^2} = \frac{\kappa}{\left|f^2 - \sigma^2\right|}\sqrt{(\bar{\sigma}_1^2 - \sigma^2)(\sigma^2 - \bar{\sigma}_2^2)}, \tag{4.5a}$$

$$\bar{\sigma}_{1,2}^2 = \frac{1}{2}(f^2 + \bar{f}_s^2 + N^2) \pm \sqrt{\frac{1}{4}(f^2 + \bar{f}_s^2 + N^2)^2 - f^2N^2}; \tag{4.5b}$$

in (4.5b) the subscript 1 (2) corresponds to + (−).

The parameters $b^{\pm}$ can be imaginary or real. Without loss of generality we set them to be positive if they are real or their imaginary parts to be positive if they are imaginary. The solution (4.2) satisfies the boundary conditions (3.3c,d) if:

$$\frac{A^-}{A^+} = -e^{-aH}\frac{e^{ib^+h_1} - e^{-ib^+h_1}}{e^{ib^-h_2} - e^{-ib^-h_2}}, \tag{4.6a}$$

$$b^+\frac{e^{ib^+h_1} + e^{-ib^+h_1}}{e^{ib^+h_1} - e^{-ib^+h_1}} = -b^-\frac{e^{ib^-h_2} + e^{-ib^-h_2}}{e^{ib^-h_2} - e^{-ib^-h_2}}. \tag{4.6b}$$

The equation (4.6b) can be satisfied if at least one of the parameters $b^{\pm}$ is real. Since

$$\bar{\sigma}_1^2 \geq \max(f^2 + \bar{f}_s^2, N^2),\ \bar{\sigma}_2^2 \leq f^2, \tag{4.7a,b}$$

the waves with frequencies $\sigma > \bar{\sigma}_1$ do not exist.

Using (4.6b), (4.4a), (4.5) and (4.7) one obtains the ranges of allowable frequencies and the corresponding dispersion relations:

$$\sigma \leq \bar{\sigma}_2 \Longrightarrow b^+ \text{-imaginary}, b^-\text{-real}, \tag{4.8a}$$

$$\left|b^+\right| \coth\left|b^+\right| h_1 = -b^- \cot b^- h_2; \tag{4.8b}$$

$$\bar{\sigma}_2 \leq \sigma \leq \bar{\sigma}_3 \Rightarrow b^{\pm} \text{-real}, \tag{4.9a}$$

$$b^+ \cot b^+ h_1 = -b^- \cot b^- h_2; \tag{4.9b}$$

$$\bar{\sigma}_3 \leq \sigma \leq \bar{\sigma}_1 \Rightarrow b^+ \text{-real}, b^-\text{-imaginary}, \tag{4.10a}$$

$$b^+ \cot b^+ h_1 = -|b^-| \coth |b^-| h_2. \tag{4.10b}$$

Here

$$\bar{\sigma}_3 = \sqrt{f^2 + \bar{f}_s^2} \tag{4.11}$$

and one assumes that $f^2 + \bar{f}_s^2 \leq N^2$.

4.2. Dispersion curves in barotropic and stably stratified cases

In barotropic fluid $N = 0$ and $\bar{\sigma}_1 = \bar{\sigma}_3, \bar{\sigma}_2 = 0$ i.e. the branches (4.8) and (4.10) disappear, and only the gyroscopic waves branch (4.9) remains. The parameters b^- and b^+ are equal to each other:

$$b^+ = b^- = b, \tag{4.12}$$

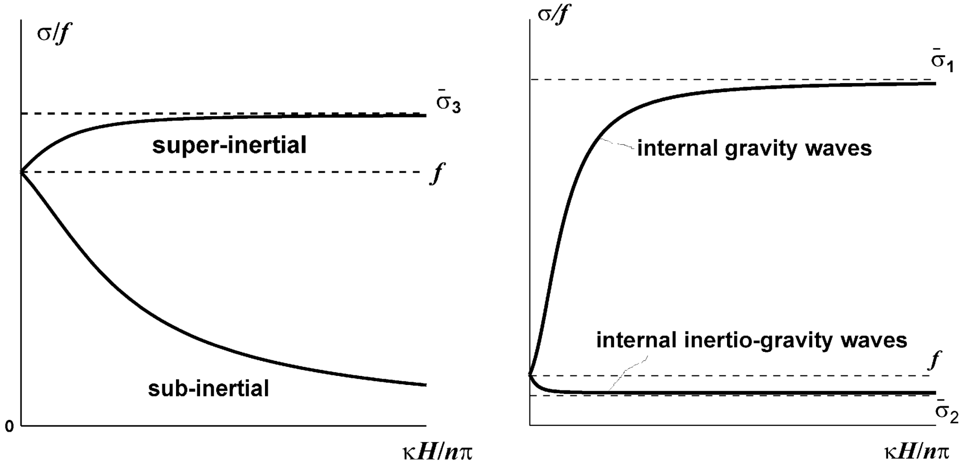

Figure 3. Dispersion curves in rotating fluid. *Left*: gyroscopic waves in the barotropic layer, *right*: waves in the stably stratified layer, $N = \text{const}$; $N/f = 2$, $\bar{f}_s/f = 1/\sqrt{2}$.

and we have from (4.9b) that

$$bH = n\pi, n = 1, 2, \ldots, \tag{4.13}$$

n is the number of vertical mode. The corresponding dispersion relation (e.g. Brekhovskikh and Goncharov 1994; Kasahara 2003) follows from (4.4b) and (4.13):

$$\sigma_n^2 = \frac{2f^2\bar{b}_n^2 + f^2 + \bar{f}_s^2}{2(1+\bar{b}_n^2)} \pm \frac{1}{2(1+\bar{b}_n^2)}\sqrt{(f^2+\bar{f}_s^2)^2 + 4f^2\bar{f}_s^2\bar{b}_n^2}; \tag{4.14}$$

here $\bar{b}_n = n\pi/\kappa H$ and the signs + and – correspond to the super- and the sub-inertial branches. The dispersion curves are represented in Figure 3. It is seen that the super-inertial (sub-inertial) frequencies lie in the range $[f, \bar{\sigma}_3]([0, f])$. In the long-wave limit one obtains from (4.14):

$$\sigma_n = f\left(1 \pm \frac{\bar{f}_s}{2n\pi f}\kappa H + O(\kappa^2H^2)\right), \kappa H \ll 1. \tag{4.15}$$

The stably stratified case when the near bottom homogeneous layer is absent, i.e. $h_1 = H$, was examined in a number of papers (Brekhovskikh and Goncharov 1994; Kasahara 2003; Gerkema and Shrira 2005). The solution is given by the "upper" line in (4.2) and should satisfy only the boundary conditions (3.3c) whence one obtains that

$$b^+ = n\pi/H, \; n = 1, 2, \ldots \tag{4.16}$$

Since b^+ is real we have from (4.5a) that

$$\bar{\sigma}_2 \le \sigma \le \bar{\sigma}_1, \tag{4.17}$$

i.e. the frequencies $\bar{\sigma}_1$ and $\bar{\sigma}_2$ given by (4.5b) are the super- and the sub-inertial boundaries of the frequency range. The dispersion relation follows from (4.5a) and (4.16):

$$\sigma_n^2 = \frac{2f^2\bar{b}_n^2 + f^2 + \bar{f}_s^2 + N^2}{2(1+\bar{b}_n^2)} \pm \frac{1}{2(1+\bar{b}_n^2)}\sqrt{(f^2 + \bar{f}_s^2 + N^2)^2 - 4f^2N^2 + 4f^2\bar{f}_s^2\bar{b}_n^2} \tag{4.18}$$

and is shown in Figure 3. It is seen that the boundary frequencies $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are approached at the short-wave limit $\kappa H \to \infty$. The long-wave asymptotics (4.15) remains valid in the stratified case under the condition

$$\kappa H \ll n\pi f/N, \tag{4.19}$$

i.e. when the horizontal scale greatly exceeds the corresponding Rossby scale $NH/n\pi f$.

The super-inertial branch $f \le \sigma \le \bar{\sigma}_1$ corresponds to the internal-gravity waves. The sub-inertial branch $\bar{\sigma}_2 \le \sigma \le f$ exists only if $N \ne 0$ and both components of the angular speed $\boldsymbol{\Omega}$ are taken into account (i.e. $f \ne 0$, $f_s \ne 0$); the waves on this branch are termed internal inertio-gravity waves (Gerkema et al 2008). In the case $f/N \ll 1$ the frequencies $\bar{\sigma}_{1,2}$ can be written as

$$\bar{\sigma}_1 = N\left(1 + \frac{\bar{f}_s^2}{2N^2} + O\left(\frac{f^4}{N^4}\right)\right), \; \bar{\sigma}_2 = f\left(1 - \frac{\bar{f}_s^2}{2N^2} + O\left(\frac{f^4}{N^4}\right)\right), \tag{4.20a,b}$$

i.e. the sub-inertial branch is very close to the inertial frequency f for all wavenumbers k, l. One can readily see that on the sub-inertial branch $|a|H \sim (N^2/f^2)\kappa H$, therefore vertical scales of the sub-inertial waves are very small for $\kappa H \gg f^2/N^2$, i.e. almost everywhere excluding a narrow vicinity near $\kappa H = 0$ by width $O(f^2/N^2)$. If initial fields are smooth in depth, then only a small part of energy transfers to the sub-inertial waves while the main part of the energy comes to the super-inertial waves in the frequency range $f \le \sigma \le \bar{\sigma}_1$, which are free of the limitations on vertical scale. We note that the super-inertial waves are close to the inertial oscillations for $\kappa H \ll f/N$, in the case $\kappa H \sim f/N$ the frequency of super-inertial wave differs from f by the value of the order of f.

4.3. General case

Dispersion curves for the general case are represented in Figure 4. Like the preceding particular cases, each curve consists of sub-inertial and super-inertial branches given number n of the vertical mode. The sub-inertial frequencies lie in the range $[0, f]$, and as seen from Figures 3 and 4 the sub-inertial branch is similar to that in the barotropic case. In

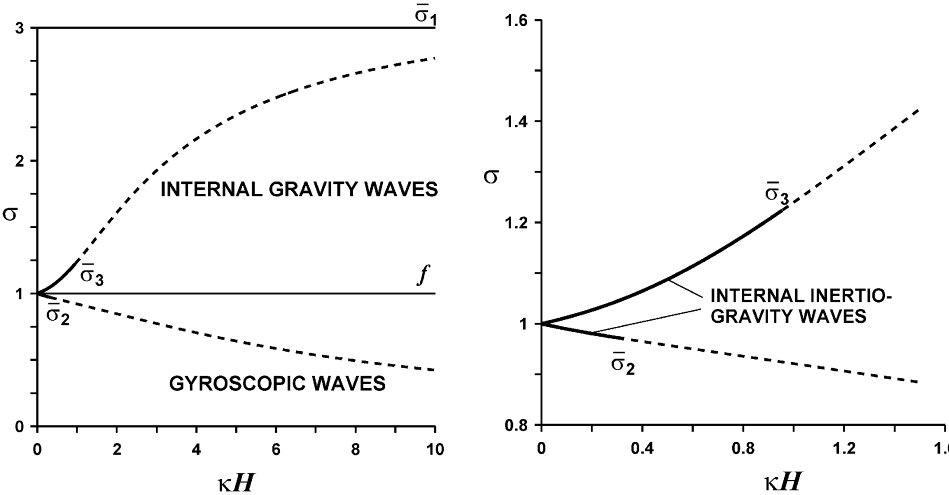

Figure 4. Dispersion curves in stably-neutrally stratified fluid. *Left*: general view, *right*: long waves domain; $N/f = 3$, $f_s/f = 1/\sqrt{2}$, $h_1 = 0.5$.

turn, the super-inertial branch is similar to that in the stably stratified case, and the super-inertial frequencies lie in the range $[f, \bar{\sigma}_1]$. The frequencies $\bar{\sigma}_2$ and $\bar{\sigma}_3$ here are "separating" frequencies: the frequency $\bar{\sigma}_2$ divides the sub-inertial waves into the sub-inertial IIGWs with frequencies from the sub-range $[f, \bar{\sigma}_2]$ and the gyroscopic waves from the sub-range $[0, \bar{\sigma}_2]$. Analogously, the frequency ranges $[f, \bar{\sigma}_3]$ and $[\bar{\sigma}_3, \bar{\sigma}_1]$ at the super-inertial branch correspond to the super-inertial IIGWs and the internal waves.

The above classification is based on behavior of the eigenfunction (4.2). Using (4.3) and (4.6a) one can represent (4.2) as:

$$W = e^{az} \begin{cases} \dfrac{e^{ib^+z} - e^{-ib^+z}}{e^{ib^+h_1} - e^{-ib^+h_1}}, & 0 \geq z \geq -h_1 \\ -\dfrac{e^{ib^-(z+H)} - e^{-ib^-(z+H)}}{e^{ib^-h_2} - e^{-ib^-h_2}}, & -h_1 \geq z \geq -H, \end{cases} \tag{4.21}$$

In the range of gyroscopic waves $\sigma \leq \bar{\sigma}_2$ the parameter b^+ is imaginary and b^- is real. Therefore here

$$W = e^{az} \begin{cases} \sinh \left|b^+\right| z \big/ \sinh \left|b^+\right| h_1, & 0 \geq z \geq -h_1 \\ -\sin b^-(z+H) \big/ \sin b^- h_2, & -h_1 \geq z \geq -H \end{cases}. \tag{4.22}$$

In the range of sub- and super-inertial IIGWs $\bar{\sigma}_2 \leq \sigma \leq \bar{\sigma}_3$ both the parameters $b^{\pm}$ are real and the eigenfunction (4.21) oscillates in both layers:

$$W = e^{az} \begin{cases} \sin b^+ z \big/ \sin b^+ h_1, & 0 \geq z \geq -h_1 \\ -\sin b^-(z+H) \big/ \sin b^- h_2, & -h_1 \geq z \geq -H \end{cases}. \tag{4.23}$$

Finally, in the range of internal waves $\bar{\sigma}_3 \leq \sigma \leq \bar{\sigma}_1$ the parameter b^+ is real and b^- is imaginary i.e.

$$W = e^{az} \begin{cases} \sin b^+ z \big/ \sin b^+ h_1, & 0 \geq z \geq -h_1 \\ -\sinh \left|b^-\right| (z+H) \big/ \sinh \left|b^-\right| h_2, & -h_1 \geq z \geq -H \end{cases}. \tag{4.24}$$

When $\sigma \to 0$ the wavenumber $\kappa H \to \infty$ and the parameter $|b^+| \to \infty$, therefore it follows from (4.22) that for sufficiently short gyroscopic waves:

$$W \cong e^{az} \begin{cases} e^{-|b^+|(z+h_1)}, & 0 \geq z \geq -h_1 \\ -\sin b^-(z+H) \big/ \sin b^- h_2, & -h_1 \geq z \geq -H \end{cases}, \tag{4.25}$$

i.e. the waves are confined to the near-bottom homogeneous layer. Analysis of the branch (4.8) gives that the dispersion relation for the short gyroscopic waves can be approximately written as

$$\sigma = \frac{f^2}{\sqrt{f^2 + \bar{f}_s^2}} \frac{n\pi}{\kappa h_2}, \ \kappa H \gg 1, \tag{4.26}$$

that, as follows from (4.14), coincides with the corresponding asymptotics for the sub-inertial gyroscopic waves in barotropic layer of depth h_2.

As $\sigma \to \bar{\sigma}_2$ the parameter $b^+ \to 0$ (see (4.5a)) and we have:

$$W = e^{az} \begin{cases} z/h_1, & 0 \geq z \geq -h_1 \\ -\sin b^-(z+H)/\sin b^- h_2, & -h_1 \geq z \geq -H \end{cases}, \tag{4.27}$$

i.e. the eigenfunction (if not to take into account the oscillating term e^{az}) linearly depends on z in the upper layer and oscillates in the lower one.

Similarly, as $\sigma \to \bar{\sigma}_3$ the parameter $b^- \to 0$ and using (4.23) and (4.24) one obtains that

$$W = e^{az} \begin{cases} \sin b^+ z \big/ \sin b^+ h_1, & 0 \geq z \geq -h_1 \\ -(z+H) \big/ h_2, & -h_1 \geq z \geq -H \end{cases}, \tag{4.28}$$

i.e. the eigenfunction is linear in z in the lower layer and oscillates in the upper one.

In the range of short internal waves the frequency σ is close to $\bar{\sigma}_1$, the non-dimensional wavenumber $\kappa H \gg 1$ and $|b^-| h_2 \gg 1$, therefore

$$W = e^{az} \begin{cases} \sin b^+ z \big/ \sin b^+ h_1, & 0 \geq z \geq -h_1 \\ -e^{|b^-|(z+h_1)}, & -h_1 \geq z \geq -H \end{cases}; \tag{4.29}$$

we see that the short internal wave is confined to the upper stratified layer.

Finally, as $\sigma \to f$ we have $\kappa H \to 0$, $b^- \cong b^+ \cong n\pi/H$ and in view of (4.23):

$$W = e^{az} \frac{\sin n\pi z/H}{\sin n\pi h_1/H}, \tag{4.30}$$

i.e. sufficiently long waves in the close vicinity of inertial frequency do not feel stratification. We note, however, that this is valid only if $\bar{f}_s \neq 0$.

4.4. *The case $f/N \ll 1$ and the long-wave approximation*

In the strongly stratified case $f/N \ll 1$ the frequencies $\bar{\sigma}_{1,2}$ take the form (4.20) i.e. the range $\bar{\sigma}_2 \leq \sigma \leq f$ corresponding to the sub-inertial IIGWs strongly contracts, while the range of super-inertial IIGWs remains wide (Figure 4). In Appendix B we examine the long waves with

$$\kappa H \ll 1 \tag{4.31}$$

and show that frequencies of these waves are given by the formulae

$$\sigma = f(1 + O(\kappa H)) \tag{4.32}$$

in the range $\sigma \leq f$ of sub-inertial IIGWs and gyroscopic waves, and

$$\sigma = f(1 + O(\kappa^2 H^2 N^2/f^2)) \tag{4.33}$$

in the range $\sigma \geq f$ of super-inertial IIGWs and internal waves.

Moreover, in the range

$$f^2/N^2 \ll \kappa H \ll 1 \tag{4.34}$$

the following approximate dispersion relations are valid:

$$\sigma \cong f - \frac{\bar{f}_s}{2n\pi}\kappa h_2, \quad n = 1, 2, \ldots, \ \sigma \leq f; \tag{4.35a}$$

$$\sigma \cong f\sqrt{1 + \kappa^2 h_1^2 N^2/s_n^2 f^2}, \quad n = 1, 2, \ldots, \ \sigma > f. \tag{4.35b}$$

Here s_n is the n-th root of the equation

$$s \cot s = -h_1/h_2. \tag{4.36}$$

As for the eigenfunctions (4.21), in the range (4.34) the sub-inertial modes are confined to the lower homogeneous layer and exponentially decay in the upper one, and the super-inertial modes oscillate in the stratified layer and depend linearly on z in the lower one. One can say that the long waves in the range (4.34) behave like the waves without rotation discussed in Section 3, only the zero frequency homogeneous layer vortex mode is replaced by the sub-inertial inertial oscillations. An important point is that in accordance with (4.33) and (4.34) the sub-inertial gyroscopic waves (the super-inertial modes) become close to the inertial oscillations if $L \gg H(L \gg L_R = HN/f)$ where L is typical horizontal scale of wave and L_R is the Rossby scale. In the case $f/N \ll 1$ the Rossby scale greatly exceeds the depth H i.e. the sub-inertial gyroscopic waves make possible inertial oscillations, which are shorter than the "traditional" inertial oscillations related to the long super-inertial internal modes. One can assume that namely this fact explains the large values of vertical velocities observed in the inertial oscillations in the nearly barotropic deep Western Mediterranean (van Haren and Millot 2005).

4.5. Vortex mode

In addition to the previously mentioned wave modes with nonzero σ, there exists a steady vortex mode which obeys the following equations:

$$-fv^+ = -p_x^+/\rho_0, \quad fu^+ = -p_y^+/\rho_0, \quad w^+ = 0, \tag{4.37a,b,c}$$

$$-f_s u^+ + g\rho/\rho_0 = -p_z^+/\rho_0, \quad u_x^+ + v_y^+ = 0; \tag{4.37d,e}$$

$$-fv^- + f_s w^- = -p_x^-/\rho_0, \quad fu^- = -p_y^-/\rho_0, \tag{4.38a,b}$$

$$f_s u^- = p_z^-/\rho_0, \quad u_x^- + v_y^- + w_z^- = 0. \tag{4.38c,d}$$

It readily follows from (4.38b,c) that the velocity u^- can be written in the form:

$$u^- = u^-(x, fy - f_s z); \tag{4.39}$$

using (4.39) and (4.38a,d) one can show that the same is valid for other fields. Because of the no-flux condition on the bottom the vertical velocity w^- is zero, and the lower layer field is geostrophic like the upper layer one. An important property is that in the homogeneous layer given x all variables are constant at the planes $fy - f_s z = const$, which are parallel to the rotation speed $\mathbf{\Omega}$. The effect takes place only in the barotropic flow, the structure of the baroclinic steady flow depends on the density distribution (compare (4.37d) with (4.38c)). In the case $f = f_s = 0$ the geostrophic mode (4.37) and (4.38) transforms into the horizontal vortex mode considered in the Section 3. The gyroscopic waves are analog of the homogeneous layer vortex mode in non-rotating fluid.

5. Long-wave dynamics

In this Section we consider the initial value problem (2.5) and (2.6) in the case when the upper layer is strongly stratified, i.e. $f/N \ll 1$, and the horizontal scale of motion L is much greater than the fluid depth H, but does not exceed the Rossby scale $L_R = HN/f$ in the order of magnitude: $H \ll L \leq L_R$. The zero frequency geostrophic mode does not manifest itself in the vertical velocity, therefore the solution is a sum of gyroscopic waves, which are close to inertial oscillations in this approximation, and of dispersive internal waves (see above Subsection 4.4).

Neglecting small terms in (2.5a,b) one obtains:

$$(\partial_{tt} + f^2)w_{zz} + N^2\nabla_h^2 w = 0, \quad 0 \geq z \geq -h_1 \tag{5.1a}$$

$$(\partial_{tt} + f^2)w_{zz} = 0, \quad -h_1 \geq z \geq -H. \tag{5.1b}$$

The equations (5.1a,b) should be solved under conditions (2.6); the initial condition for w_t is derived in Appendix C.

The solution is represented in the form (cf. (3.14)):

$$w = w_a + w_I \cos ft + \frac{1}{f}\dot{w}_I \sin ft, \tag{5.2}$$

for w_a we have a forced problem with homogeneous boundary and initial conditions:

$$(\partial_{tt} + f^2)w^+_{azz} + N^2\nabla_h^2 w^+_a = -N^2\nabla_h^2[w^+_I \cos ft + (1/f)\dot{w}^+_I \sin ft], \tag{5.3a}$$

$$(\partial_{tt} + f^2)w^-_{azz} = 0; \tag{5.3b}$$

$$w_a|_{z=0,-H} = 0, [w_a]_{z=-h_1} = [w_{az}]_{z=-h_1} = 0, \tag{5.4a,b}$$

$$w_a|_{t=0} = 0, \;\; w_{at}|_{t=0} = 0. \tag{5.4c,d}$$

The solution to (5.3b) obeying (5.4a,c,d) is readily obtained:

$$w^-_a = (z + H)[A(x, y, t) - (1/f)A_t(x, y, 0)\sin ft - A(x, y, 0)\cos ft], \tag{5.5}$$

here $A(x, y, t)$ is an arbitrary differentiable function. It follows from (5.5) that at the interface $z = -h_1$:

$$\left(w^-_{az} - \frac{1}{h_2}w^-_a\right)_{z=-h_1} = 0, \tag{5.6}$$

therefore, in view of (5.4b), the equation (5.3a) should be solved under the condition

$$\left(w^+_{az} - \frac{1}{h_2}w^+_a\right)_{z=-h_1} = 0, \tag{5.7}$$

and the conditions (5.4a,c,d).

Similarly to Section 3, the solution is sought as a superposition of the wave modes

$$w^+_\mu = W^+_\mu(z)\exp[i(kx + ly - \sigma_\mu t)], \quad \mu = 1, 2, \ldots \tag{5.8}$$

where each the mode (5.8) is a solution of the homogeneous problem (5.3a), (5.4a), (5.7). The amplitude $W^+_\mu(z)$ is eigenfunction of the eigenvalue problem (cf. (3.7)):

$$W^+_{zz} = -\lambda N^2 W^+, \tag{5.9a}$$

$$[W^+_z - (1/h_2)W^+]_{z=-h_1} = 0, \; W^+\big|_{z=0} = 0, \tag{5.9b,c}$$

where λ is the eigenvalue. Like (3.7) the spectrum of the eigenvalues $\lambda = \lambda_\mu$, $\mu = 1, 2, \ldots$ is discrete and positive and the corresponding eigenfunctions W_μ^+ comprise a complete orthogonal basis. The frequencies σ_μ are related to λ_μ in the following way:

$$\sigma_\mu = \sqrt{f^2 + \kappa^2/\lambda_\mu}. \tag{5.10}$$

Obviously, these waves are super-inertial and lie on the branch (4.10). In the case $N = const$ we have (cf. (3.10)):

$$W_\mu^+ = \sin q_\mu z, \quad q_\mu = s_\mu/h_1 = N\sqrt{\lambda_\mu}, \quad \sigma_\mu = \sqrt{f^2 + \kappa^2 N^2/q_\mu^2} \tag{5.11a,b,c}$$

where s_μ is the μ-th root of equation

$$s \cot s = -h_1/h_2. \tag{5.11d}$$

We now represent all functions in (5.3a) in the form of Fourier integrals (3.17), write the Fourier amplitudes as expansions in the form (3.21) and calculate the coefficients $\tilde{w}_{a\mu}^+(k, l, t)$. As a result we have

$$\tilde{w}_a^+ = \sum_{\mu=1}^{\infty} [\tilde{w}_{I\mu}^+ \cos\sigma_\mu t + (\tilde{\dot{w}}_{I\mu}^+/\sigma_\mu) \sin\sigma_\mu t] W_\mu^+ - \tilde{w}_I^+ \cos ft - \frac{1}{f}\tilde{\dot{w}}_I^+ \sin ft, \tag{5.12}$$

and using (5.2)

$$\tilde{w}^+ = \sum_{\mu=1}^{\infty} [\tilde{w}_{I\mu}^+ \cos\sigma_\mu t + (\tilde{\dot{w}}_{I\mu}^+/\sigma_\mu) \sin\sigma_\mu t] W_\mu^+. \tag{5.13}$$

Knowing w^+ one can determine the function $A(x, y, t)$ in (5.5). In view of (5.2) the solution in the lower layer can be written as

$$\begin{aligned} w^- = {} & A(x, y, t)(z + H) + [w_I^- - A(x, y, 0)(z + H)] \cos ft \\ & + \frac{1}{f}[\dot{w}_I^- - A_t(x, y, 0)(z + H)] \sin ft. \end{aligned} \tag{5.14}$$

The solution w^+ is orthogonal to the inertial harmonics $\sin ft$, $\cos ft$, as it is seen from (5.13) and (5.10) and can be obtained directly from (5.1a). Thus the continuity conditions (5.4b) require that

$$A(x, y, t) = \frac{1}{h_2} w^+(x, y, -h_1, t) \tag{5.15}$$

and the resulting solution in the lower layer can be represented as a sum (cf. (3.23)):

$$w^- = w_w^- + w_{osc}^-, \tag{5.16}$$

where

$$w_w^- = w^+(x, y, -h_1, t)\frac{z + H}{h_2}, \tag{5.17}$$

$$w_{osc}^- = \left[w_I^- - w_I|_{z=-h_1}\frac{z+H}{h_2}\right]\cos ft + \frac{1}{f}\left[\dot{w}_I^- - \dot{w}_I|_{z=-h_1}\frac{z+H}{h_2}\right]\sin ft. \tag{5.18}$$

The structure of the solution is similar to that obtained in Section 3. As seen from (5.13), the upper layer field consists of the super-inertial internal waves. In the lower layer the term w_w^- describes the field induced by the internal waves, and it does not contain the inertial oscillations $\sin ft$, $\cos ft$. On the contrary, the term w_{osc}^- consists of the inertial oscillations which represent the long gyroscopic waves. At the interface $z = -h_1$ the term w_{osc}^- is zero and $\partial_z w_{osc}^-$ is equal to

$$\partial_z w_{osc}^-\big|_{z=-h_1} = \left[\left(w_{Iz} - \frac{1}{h_2}w_I\right)\cos ft + \frac{1}{f}\left(\dot{w}_{Iz} - \frac{1}{h_2}\dot{w}_I\right)\sin ft\right]_{z=-h_1}. \tag{5.19}$$

The equation (5.19) means that the behavior of the solution qualitatively depends on the initial fields. By virtue of (5.2), (5.7) and (5.19) we have

$$\left(w_z^+ - \frac{1}{h_2}w^+\right)_{z=-h_1} = \partial_z w_{osc}^-\big|_{z=-h_1}. \tag{5.20}$$

If the initial fields w_I, $\dot{w}_I$ satisfy the conditions (cf. (3.28)):

$$\left(w_{Iz} - \frac{1}{h_2}w_I\right)_{z=-h_1} = \left(\dot{w}_{Iz} - \frac{1}{h_2}\dot{w}_I\right)_{z=-h_1} = 0, \tag{5.21}$$

then any partial sum of the series (5.13) satisfies (5.20) and a smooth initial field (characterized by a small contribution of high harmonics with large numbers μ) results in a smooth solution. But if (5.21) is not valid, then the inertial signal in w_z^- (i.e. in the lower layer horizontal velocity) proportional to $\sin ft$, $\cos ft$ is not zero at the interface. To prevent its penetration into the stratified layer a non-stationary boundary layer should arise in the interface vicinity $z \geq -h_1$ at large times as it was discussed at the end of Section 3. The boundary layer is a result of joint impact of high harmonics (5.8), which tend to the inertial oscillations as $\mu \to \infty$.

6. Summary and conclusions

A special feature of the stably-neutrally stratified fluid is the wave mode related to the homogeneous layer. In the non-rotating fluid this is the zero frequency homogeneous layer vortex mode in which a three-dimensional velocity field is confined to the homogeneous layer, the vertical velocity is zero at the interface and the horizontal velocity can be discontinuous at $z = -h_1$. Besides the mode, the wave spectrum contains internal waves and the zero frequency horizontal vortex modes with zero vertical velocity. Short internal waves with large horizontal wavenumbers κ, $\kappa h_2 \gg 1$, are confined to the upper stratified layer and decay exponentially with depth in the lower one. At the same time, the internal modes with large or moderate length, $\kappa h_2 \leq 1$, penetrates down to the bottom, long waves with $\kappa h_2 \ll 1$ inducing in the bottom layer the velocity field in which the vertical velocity is approximately linear in depth and the horizontal one does not depend on the depth.

Arbitrary vertical velocity field consists of the dispersive internal waves and a stationary part confined to the lower layer—homogeneous layer vortex mode. If the horizontal velocity in this mode is discontinuous at $z = -h_1$, then a non-stationary boundary layer develops at large times near the interface between layers to prevent penetration of stationary signal into the upper stratified layer. If the initial state is horizontally localized then at a fixed point the internal waves decay with increasing time because of dispersion and the vertical velocity tends to the lower layer stationary component, i.e. it decays in the upper layer and becomes stationary in the lower one.

This process can be called the *wave adjustment* by analogy with the well-known geostrophic adjustment (e.g. Reznik et al. 2001; Zeitlin et al. 2003). Two key elements are needed for the wave adjustment to exist in a linear system: *linear* invariants and linear waves harmonically depending on time. The invariants are determined by initial conditions and are not influenced by the waves, which are characterized by the *zero* linear invariants. Evolution of such a system can be represented in natural way as a sum of stationary component with non-zero invariants and a non-steady wave part with zero invariants. A nice example of the wave adjustment was presented by Lighthill (1996) for non-rotating uniformly stratified fluid, where the invariant is the vertical component of vorticity. In the non-rotating SNS fluid, the invariants are the vertical component of vorticity in the stratified layer and the vorticity vector in the homogeneous one. The waves in both cases are the internal waves having the zero invariants and propagating away from the initial perturbation domain. The stationary component, determined by the invariants, coincides with the residual motion that is left behind after the wave propagation. We note that the concept of wave adjustment can be applied to any linear system (not necessarily a hydrodynamic one) in which linear conserving quantities and the waves occur. From this point of view, the geostrophic adjustment can be treated as a particular case of the wave adjustment in rotating fluid. A paper with detail presentation of the wave adjustment concept is in preparation.

In rotating fluid, the horizontal and the homogeneous layer vortex modes turn into the geostrophic mode and the gyroscopic waves. We explored the wave spectrum of this system taking into account the horizontal component of the angular speed of the Earth's rotation i.e. without the traditional and hydrostatic approximations. The spectrum combines the sub-inertial gyroscopic waves in homogeneous fluid (e.g. Le Blond and Mysak 1978; Brekhovskikh and Goncharov 1994) and the internal waves in stably stratified fluid (Badulin et al 1991; Kasahara 2003; Gerkema and Shrira 2005; Gerkema et al 2008). Each dispersion curve $\sigma(k, l)$ in the spectrum consists of super-inertial and sub-inertial branches corresponding to $\sigma > f$ and $\sigma < f$. Both the branches start at the inertial frequency f for infinitely long waves with $\kappa H = 0$. Very long waves are near-inertial oscillations with $\sigma \cong f$ and are not affected by stratification. With increasing κH one arrives at the ranges of the super- and the sub-inertial internal inertio-gravity waves oscillating in both layers. Greater κH correspond to the range of internal waves at the super-inertial branch and to the range of gyroscopic waves at the sub-inertial branch. The internal modes oscillate in the vertical in the upper layer and have no zeros in the lower one; the short internal waves with $\kappa H \gg 1$ do not penetrate into the lower layer. By contrast, the gyroscopic waves oscillate in the homogeneous layer and do not oscillate in the stratified one. The short gyroscopic waves with $\kappa H \gg 1$ decay exponentially in the upper layer with increasing distance from the interface.

In important case of the strong stratification when $f/N \ll 1$, the range of sub-inertial IIGWs almost disappears, the range of gyroscopic waves enlarges almost up to the inertial frequency, and the ranges of super-inertial IIGWs and internal waves almost do not change. In interesting range of long wave scales $f^2/N^2 \ll \kappa H \ll 1$ the internal and the super-inertial internal inertio-gravity waves freely penetrate into the lower layer, whereas the gyroscopic waves are localized in the lower layer and are close to the inertial oscillations. An important point is that the gyroscopic waves are close to the inertial oscillations if $L \gg H$, whereas the super-inertial modes – if $L \gg L_R = HN/f$, where L is the typical horizontal scale of wave. Thus the gyroscopic waves provide existence of inertial oscillations, which are shorter than the "traditional" inertial oscillations related to the long internal modes. This fact can explain the large values of vertical velocities observed in inertial oscillations in the nearly barotropic deep Western Mediterranean (van Haren and Millot 2005).

There are some important differences between the sub-inertial IIGWs in stably stratified fluid (Badulin et al 1991; Kasahara 2003; Gerkema and Shrira 2005; Gerkema et al 2008) and the GWs in the SNS fluid, especially in the case of strong stratification $f/N \ll 1$. Firstly, the IIGWs exist only if the horizontal component of the Earth's rotation is non-zero while the GWs occur under TA. Secondly, the sub-inertial IIGW frequencies lie in the range $[\bar{\sigma}_2, f]$, which is very narrow if $f/N \ll 1$; the frequencies of the GWs are in a much wider range $[0, \bar{\sigma}_2]$. Thirdly, the IIGWs are characterized by a very small aspect ratio $H/L \sim f^2/N^2$ while the aspect ratio of the GW can be an arbitrary one.

Analysis of an arbitrary long wave initial perturbation demonstrates that in the course of time the motion is split into the internal waves "filling" the upper layer and penetrating into the lower one, and the inertial oscillations (long gyroscopic waves) confined to the lower layer. If the initial fields are localized in the horizontal plane, then the internal waves decay because of dispersion and the system tends to the state when the upper layer is motionless and in the lower layer only the inertial oscillations remain. Note that we restrict ourselves to the analysis of the vertical velocity field and do not consider the geostrophic component, which in the linear approximation does not affect the vertical velocity. A complete analysis will be given in a future work on the nonlinear geostrophic adjustment in the stably-neutrally stratified fluid.

In addition to the study of waves, we also performed a semi-qualitative analysis of governing equations for rotating SNS fluid to examine the role of non-traditional terms related to the horizontal component of the Earth's rotation. In weakly stratified domains (i.e. in the homogeneous layer if $N \gg f$ or in both layers if $N \sim f$) the terms are of importance, especially for moderate horizontal scales $L \leq H$. In the long wave approximation $L \gg H$ the non-traditional terms determine a dispersion of near-inertial oscillations. Analogous role the terms play in strongly stratified domains for ultra-long horizontal scales $L \geq \frac{N}{f} L_R$.

Acknowledgments. The work was supported by the RFBR grants 11-05-00099 and 12-05-00889. The author is grateful to Sergey Badulin, Eugene Benilov and Victor Shrira for helpful comments and discussions.

APPENDIX A

Calculation of the initial field $\dot{w}_I$ in the case of non-rotating fluid

In the case $f = f_s = 0$ the field $\dot{w}_I = w_t|_{t=0}$ can be determined from (2.1c) knowing initial density and pressure fields ρ_I and p_I. To calculate p_I we use the equations which readily follow from (2.1a,b,c,e) and (2.2):

$$\nabla^2 p^+ = -g\rho_z, \nabla^2 p^- = 0. \tag{A1a,b}$$

Boundary conditions for p at $z = 0, -H$ are obtained from (2.3a,c), (2.1c,d) and (2.2c):

$$p_z^+\big|_{z=0} = -\, g\rho_I|_{z=0}\, ,\; p_z^-\big|_{z=-H} = 0. \tag{A2a,b}$$

At the interface $z = -h_1$ the pressure is continuous:

$$p^+\big|_{z=-h_1} = p^-\big|_{z=-h_1} \tag{A3a}$$

and by virtue of (2.1c), (2.2c) and the continuity of w_t, the vertical gradients of pressure are related as follows:

$$(p_z^+ - p_z^-)_{z=-h_1} = -\, g\rho|_{z=-h_1}\, . \tag{A3b}$$

Knowing density ρ one can obtain the pressures $p^{\pm}$ from (A.1) – (A.3). We represent p^+ as

$$p^+ = p_a^+ - g\int_{-h_1}^{z} \rho dz; \tag{A4}$$

in this case the system (A.1) – (A.3) takes the form:

$$\nabla^2 p_a^+ = g\nabla_h^2 \int_{-h_1}^{z} \rho dz, \quad \nabla^2 p^- = 0 \tag{A5a,b}$$

$$p_{az}^+\big|_{z=0} = 0, \; p_z^-\big|_{z=-H} = 0. \tag{A6a,b}$$

$$p_a^+\big|_{z=-h_1} = p^-\big|_{z=-h_1}, \tag{A7a}$$

$$(p_{az}^+ - p_z^-)_{z=-h_1} = 0. \tag{A7b}$$

The solution to the problem (A5)–(A7) is sought in the form of the Fourier integral (3.17); the Fourier amplitudes obey the equations:

$$\partial_{zz}\tilde{p}_a^+ - \kappa^2\tilde{p}_a^+ = -g\kappa^2 \int_{-h_1}^{z} \tilde{\rho} dz, \quad \partial_{zz}\tilde{p}^- - \kappa^2\tilde{p}^- = 0. \tag{A8a,b}$$

$$\tilde{p}_{az}^+\big|_{z=0} = 0, \; \tilde{p}_z^-\big|_{z=-H} = 0. \tag{A9a,b}$$

$$\tilde{p}_a^+\big|_{z=-h_1} = \tilde{p}^-\big|_{z=-h_1}, \tag{A10a}$$

$$\left(\tilde{p}_{az}^+ - \tilde{p}_z^-\right)_{z=-h_1} = 0. \tag{A10b}$$

The function $\tilde{p}^-$ is found from (A8b) and (A9b):

$$\tilde{p}^- = A^- \cosh \kappa(z + H), \tag{A11}$$

where A^- is a constant amplitude. It follows from (A11) that

$$\left(\tilde{p}_z^- - \beta(\kappa)\tilde{p}^-\right)_{z=-h_1} = 0, \quad \beta(\kappa) = \kappa \tanh \kappa h_2, \tag{A12}$$

therefore in view of (A10)

$$\left(\tilde{p}_{az}^+ - \beta(\kappa)\tilde{p}_a^+\right)_{z=-h_1} = 0. \tag{A13}$$

Solving (A8a) with boundary conditions (A9a) and (A13) one expresses the function p_a^+ in terms of the density ρ. Setting $\rho = \rho_I$ in the obtained solution gives the initial field p_{aI}^+ and, therefore, the function p_I^+ by (A4). The resulting field $\dot{w}_I = w_t|_{t=0}$ is found from (2.1c):

$$\dot{w} = -\frac{1}{\rho_0}\partial_z p_{aI}^+. \tag{A14}$$

APPENDIX B

Analysis of long waves in rotating fluid

First we consider the range of gyroscopic waves (4.8a) $\sigma \le \bar{\sigma}_2$. The lowest root of the dispersion relation (4.8b) satisfies the inequality

$$\frac{\pi}{2} \le b^- h_2 \le \pi. \tag{B1}$$

By virtue of (4.4b) and (4.31) in the range $\sigma \leq \bar{\sigma}_2 \leq f$ the inequalities (B1) are possible only for $\sigma \cong f$; in this case we have

$$b^- \cong \frac{\bar{f}_s \kappa}{2(f-\sigma)}. \tag{B2}$$

It follows from (B1) and (B2) that

$$f - \frac{\bar{f}_s}{\pi}\kappa h_2 \leq \sigma \leq f - \frac{\bar{f}_s}{2\pi}\kappa h_2, \tag{B3}$$

therefore (4.32) is valid here. If we impose an additional restriction on the wave vector:

$$f^2/N^2 \ll \kappa H \ll 1, \tag{B4}$$

then one obtains from (B3), (4.5a) and (B2) that

$$\frac{\bar{\sigma}_2 - \sigma}{f} \gg \frac{f^2}{N^2} \Rightarrow \frac{|b^+|}{b^-} \cong \sqrt{2}\frac{N}{\bar{f}_s}\sqrt{\frac{\bar{\sigma}_2 - \sigma}{f}} \gg 1. \tag{B5}$$

Using (B5) and (4.8b) we find that in the range (B4) $b^- h_2 \cong \pi$ i.e. in view of (B2) we arrive at (4.35a) for $n = 1$:

$$\sigma \cong f - \frac{\bar{f}_s}{2\pi}\kappa h_2. \tag{B6}$$

The dispersion relation (B6) is valid for

$$\kappa H \sim f/N \ll 1. \tag{B7}$$

The range (B1) corresponds to the lowest root $b^- h_2$ of the equation (4.8b); other roots are analyzed in the same manner if we replace in (B1) the limits $\pi/2, \pi$ by the limits $n\pi - \pi/2, n\pi, n = 1, 2 \ldots$. The dispersion relation (4.35a) is fulfilled in this case.

IIGWs from the range $\bar{\sigma}_2 \leq \sigma \leq f$ obey the dispersion relation (4.9b). In view of (4.20b) $(\sigma - \bar{\sigma}_2)/f \leq f^2/N^2$, therefore (B2) remains valid and by virtue of (4.5a) we have:

$$b^+ = g(\sigma)b^-, \quad g(\sigma) \cong \frac{\sqrt{2}N}{\bar{f}_s\sqrt{f}}\sqrt{\sigma - \bar{\sigma}_2} \sim 1. \tag{B8}$$

We now rewrite (4.9b) in the form

$$g \cot \alpha g x = -\cot x, \tag{B9}$$

where $\alpha = h_1/h_2, x = b^- h_2$. A simple graphic analysis of (B9) gives the lowest root x lying in the range

$$0 \leq x = b^- h_2 \leq \pi. \tag{B10}$$

It follows from (B2), (B10) and $(\sigma - \bar{\sigma}_2)/f \le f^2/N^2$ that

$$\kappa H \le f^2/N^2, \tag{B11}$$

i.e. (4.32) is valid for the sub-inertial inertio-gravity waves too.

Now we proceed to the super-inertial range $\sigma \ge f$. In the sub-range of the IIGWs $f \le \sigma \le \bar{\sigma}_3$ (4.9b) is valid and here

$$b^- = \frac{\sigma\kappa}{\sigma^2 - f^2}\sqrt{f^2 - \sigma^2 + \bar{f}_s^2}, \quad b^+ \cong \frac{\kappa N\sqrt{\sigma + \bar{\sigma}_2}}{\sigma^2 - f^2}\sqrt{\sigma - \bar{\sigma}_2}. \tag{B12a,b}$$

It follows from (B12) that:

$$\frac{b^+}{b^-} \cong \frac{N\sqrt{\sigma + \bar{\sigma}_2}\sqrt{\sigma - \bar{\sigma}_2}}{\sigma\sqrt{f^2 + \bar{f}_s^2 - \sigma^2}} \ge \frac{N\sqrt{\sigma + \bar{\sigma}_2}\sqrt{\sigma - \bar{\sigma}_2}}{\sqrt{f^2 + \bar{f}_s^2}\sqrt{f^2 + \bar{f}_s^2 - \sigma^2}} > \frac{N\sqrt{\bar{\sigma}_2}\sqrt{\sigma - \bar{\sigma}_2}}{f^2 + \bar{f}_s^2}$$
$$\cong \frac{f^2}{f^2 + \bar{f}_s^2}\frac{N}{f}\frac{\sqrt{\sigma - \bar{\sigma}_2}}{\sqrt{f}}. \tag{B13}$$

For $(\sigma - \bar{\sigma}_2)/f \le f^2/N^2$ the problem coincides with that considered in the previous paragraph, i.e. (B11) and (4.32) are valid. In the case $(\sigma - \bar{\sigma}_2)/f \gg f^2/N^2$ we have from (B13):

$$b^- = g_1(\sigma)b^+, \; g_1(\sigma) \ll 1. \tag{B14}$$

Graphic analysis of the equation (4.9b) under the conditions (B14) gives that

$$0 \le b^+h_1 \le \pi \tag{B15}$$

i.e.

$$b^-h_2 \ll 1, \tag{B16}$$

and (4.9b) takes the following approximate form:

$$b^+h_1 \cot b^+h_1 \cong -h_1/h_2. \tag{B17}$$

From (B12b) and (B17) we find the approximate dispersion relation (4.35b) for the lowest mode:

$$\sigma \cong f\sqrt{1 + \kappa^2 h_1^2 N^2/s^2 f^2}, \tag{B18a}$$

where s is the root of the equation (cf. (4.36))

$$s \cot s = -h_1/h_2, \; \pi/2 \le s \le \pi. \tag{B18b}$$

The validity of (4.35b) and (4.36) for higher modes (when $n\pi \le b^+h_1 \le (n+1)\pi, n = 1, 2, \ldots$ is fulfilled instead of (B15)) is demonstrated in the same manner. The dispersion relation (4.35b) remains valid under the condition (B7). In the range of internal wave $\bar{\sigma}_3 \le \sigma \le \bar{\sigma}_1$ (4.10b) is fulfilled and the waves with frequencies $\sigma \sim f$ are the long waves from the range (B4); the approximate dispersion relation (4.35b) remains valid in this case too.

APPENDIX C

Calculation of the initial field $\dot{w}_I$ in the case of rotating fluid

The equations (5.1) are derived under the long-wave approximation when the system (2.1) and (2.2) takes the form:

$$u_t - fv = -p_x/\rho_0, \ \ v_t + fu = -p_y/\rho_0, \ \ g\rho = -p_z, \tag{C1a,b,c}$$

$$\rho_t - \rho_0 N^2 w/g = 0, \ \ u_x + v_y + w_z = 0 \tag{C1d,e}$$

for $0 \ge z \ge -h_1$, and

$$u_t - fv = -p_x/\rho_0, \ \ v_t + fu = -p_y/\rho_0, \ \ p_z = 0, \ \ u_x + v_y + w_z = 0 \tag{C2a,b,c,d}$$

for $-h_1 \ge z \ge -H$.

From (C1a,b,e) and (C2a,b,d) we have:

$$w_{zt}^{\pm} = -f\zeta^{\pm} + \frac{1}{\rho_0}\nabla_h^2 p^{\pm}, \ \ \varsigma = v_x - u_y, \tag{C3}$$

hence using (C1c) and (C2c) one obtains that:

$$w_{zzt}^{+} = -f\zeta_z^{+} - \frac{g}{\rho_0}\nabla_h^2\rho, w_{zzt}^{-} = -f\zeta_z^{-}. \tag{C4a,b}$$

Integrating (C4a) twice over z from $z = -h_1$ to z and (C4b)—from z to $z = -h_1$, we arrive at the following equations:

$$w_t^{+} = K + (z+h_1)M - f\int_{-h_1}^{z} \zeta^{+} dz + f(z+h_1)\,\zeta^{+}\big|_{z=-h_1} - \frac{g}{\rho_0}\nabla_h^2 \int_{-h_1}^{z} dz' \int_{-h_1}^{z'} \rho dz'', \tag{C5a}$$

$$w_t^{-} = K + (z+h_1)M + f\left[\int_{z}^{-h_1} \zeta^{-} dz + (z+h_1)\,\zeta^{+}\big|_{z=-h_1}\right], \tag{C5b}$$

where

$$K = w_t^{\pm}\big|_{z=-h_1}, M = w_{zt}^{\pm}\big|_{z=-h_1}. \tag{C6a,b}$$

Using the boundary conditions (2.3c) one obtains two equations for K and M:

$$K + h_1 M = f \int_{-h_1}^{0} \zeta^+ dz - f h_1 \, \zeta^+ \big|_{z=-h_1} + \frac{g}{\rho_0} \nabla_h^2 \int_{-h_1}^{0} dz \int_{-h_1}^{z} \rho dz', \tag{C7a}$$

$$K - h_2 M = -f \int_{-H}^{-h_1} \zeta^- dz + f h_2 \, \zeta^- \big|_{z=-h_1}. \tag{C7b}$$

Solving (C7) with respect to K and M and substituting the resulting expressions to (C5) we find the initial field $\dot{w}_I$:

$$\dot{w}_I^+ = f \int_{z}^{0} \zeta_I dz + \frac{g}{\rho_0} \nabla_h^2 \int_{z}^{0} dz' \int_{-h_1}^{z'} \rho_I dz'' + \frac{z}{H} \left[f \int_{-H}^{0} \zeta_I dz + \frac{g}{\rho_0} \nabla_h^2 \int_{-h_1}^{0} dz \int_{-h_1}^{z} \rho_I dz' \right], \tag{C8a}$$

$$\dot{w}_I^- = f \int_{z}^{0} \zeta_I dz + \frac{g}{\rho_0} \nabla_h^2 \int_{-h_1}^{0} dz \int_{-h_1}^{z} \rho_I dz' + \frac{z}{H} \left[f \int_{-H}^{0} \zeta_I dz + \frac{g}{\rho_0} \nabla_h^2 \int_{-h_1}^{0} dz \int_{-h_1}^{z} \rho_I dz' \right]. \tag{C8b}$$

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Received: *Feb. 2, 2013*; revised: *July 2, 2013*.