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Does the nonlinearity of the equation of state impose an upper bound on the buoyancy frequency?

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ABSTRACT

Mixing in the ocean is usually accompanied by a net reduction in volume caused by the nonlinear nature of the equation of state. This contraction-on-mixing at a certain depth implies that the whole water column above this depth slumps a little and so suffers a reduction in gravitational potential energy. Under certain circumstances the gravitational potential energy of the entire water column can decrease as a consequence of mixing activity at a certain depth. We examine Fofonoff's hypothesis that in these circumstances the net reduction of gravitational potential energy of the whole water column causes a local increase in the turbulent mixing activity at the location of the original mixing. Fofonoff proposed that this increased local mixing diffuses the local property gradients until the criterion for positive feedback is no longer satisfied, so providing an upper bound for the vertical stratification in the ocean. Bearing in mind the relatively inefficient nature of turbulent mixing at causing diapycnal fluxes (the majority of the turbulent kinetic energy goes directly into internal energy), we find that the criterion for positive feedback is a factor of approximately seven more difficult to achieve than has been realized to date. An examination of oceanic data shows that while Fofonoff's original criterion for positive feedback is often exceeded, the more appropriate criterion is almost never approached.

The positive feedback hypothesis assumes that the reduction in the gravitational potential energy of the whole water column appears at the location of the original mixing as an increase in the turbulent mixing activity. We show that this very focused oceanic response is extremely difficult to justify. For example there is no such feedback in a strictly one-dimensional water column; rather all of the reduction in gravitational potential energy appears as an increase in internal energy at the depth of the original mixing and there is no possibility of any positive feedback to increase the turbulent mixing. As the positive feedback hypothesis is lacking a convincing theoretical basis and is not supported by oceanic data, we do not believe that it acts as an effective upper bound on oceanic stratification.

1. Introduction

Fofonoff (1961, 1998, 2001) pointed out that the contraction-on-mixing that occurs when mixing occurs at a particular depth causes the entire water column above this depth to sink. That is, mixing at a certain depth causes a reduction of the gravitational potential

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energy of all the fluid above that depth. For a sufficiently large vertical gradient of potential temperature, θ_z, Fofonoff showed that this reduction of gravitational potential energy by the adiabatic slumping of the water column is larger than the local increase in gravitational potential energy caused by diapycnal mixing, DN^2, where D is the diapycnal diffusivity and N is the buoyancy frequency.

Fofonoff's gravitational potential energy feedback hypothesis states that when diapycnal mixing at a particular depth leads to a net decrease of the gravitational potential energy of the whole water column, a feedback process acts to enhance the diapycnal mixing activity at the location of the original mixing. This enhanced mixing is assumed to smooth the vertical stratification until the criterion for the positive feedback is eliminated. This hypothesis implies a local upper bound on the vertical stratification so that the criterion for the existence of the positive feedback should never be exceeded. We show that the feedback hypothesis is best regarded as providing an upper limit to the buoyancy frequency rather than to the vertical temperature gradient. We also extend Fofonoff's discussion of this feedback process to include double-diffusive convection and isopycnal mixing.

Fofonoff's feedback process has 100% of the loss of the water column's gravitational potential energy appearing as an increase in the local kinetic energy, and then being converted into an increase in the local gravitational potential energy with an efficiency of 100%. However, small-scale turbulent mixing is known to be relatively inefficient in causing a local increase in gravitational potential energy and hence we argue that the upper bound on the vertical density gradient is about seven times larger than the one advanced by Fofonoff (2001). An analysis of historical ocean data demonstrates that this larger limit is almost never approached in the ocean whereas Fofonoff's original limit is often exceeded. Furthermore, it seems unlikely that a physical process exists that would localize any increased mixing activity to exactly the depth and horizontal location where the original mixing occurred. In the absence of such a physical process, and without support for the hypothesis from oceanic data, we conclude that this feedback mechanism is unlikely to be real.

2. The impact of mixing on gravitational potential energy

Following Fofonoff (2001), the gravitational potential energy (GPE) of an individual water column with respect to a fixed height, z_0, is given by (using the hydrostatic balance and integrating by parts)

$$\text{GPE} = \int_{z_0}^{z_s} \rho g(z - z_0)dz = \int_0^M p/\rho dm, \tag{1}$$

where z_s is the height of the free surface and $m(z)$ is the mass of fluid per unit horizontal area above the reference height, z_0, as a function of height ($dm = \rho dz = -g^{-1}dp$). This derivation has assumed that the pressure at the sea surface is zero. If the atmospheric pressure is included, an additional term appears which is very small and we ignore it. Here we will compare oceanic states before and after mixing events so any deviation from hydrostatic balance of the mixing processes is unimportant.

For this analysis of the effects of mixing processes on a single water column, we are not interested in any changes in GPE caused by the adiabatic and isohaline rearrangement of water parcels (that is of central concern for the more common calculation of available potential energy). Hence, the rate at which local mixing processes affect the GPE of the water column is the material derivative of the integrand in (1) minus the adiabatic and isohaline contribution to that material derivative, namely

$$\frac{d(p/\rho)}{dt} - \left.\frac{\partial(p/\rho)}{\partial p}\right|_{S,\Theta} \frac{dp}{dt} = \chi, \tag{2}$$

where we use the shorthand label χ for this nonadvective local contribution to the change of GPE. Here density has been regarded as a function of salinity, S, conservative temperature, Θ, and pressure, p, that is as $\rho(S, \Theta, p)$. Conservative temperature is very similar to potential temperature but is more conservative than is potential temperature by more than two orders of magnitude (McDougall, 2003). In this paper, the distinction between θ and Θ is not important and indeed we will sometimes refer to Θ as simply temperature.

Eq. (2) can be written as

$$\chi = \frac{p}{\rho}\left(\alpha \frac{d\Theta}{dt} - \beta \frac{dS}{dt}\right) = \frac{p}{g\rho} eN^2 \tag{3}$$

where $\alpha = -\rho^{-1}\partial\rho/\partial\Theta|_{S,p}$ and $\beta = \rho^{-1}\partial\rho/\partial S|_{\Theta,p}$ are the thermal expansion coefficient and the saline contraction coefficient respectively and e is the dianeutral velocity which is given by (McDougall, 1987)

$$eN^2/g = \alpha(D\Theta_z)_z - \beta(DS_z)_z - K(C_b\nabla_n\Theta \cdot \nabla_n\Theta + T_b\nabla_n\Theta \cdot \nabla_n p) + (\beta F_z^S - \alpha F_z^\Theta), \tag{4}$$

where D is the isotropic diffusivity of small-scale mixing, K is the lateral (i.e. epineutral) eddy diffusivity for tracers, $C_b = \alpha_\Theta + 2\alpha_S(\alpha/\beta) - \beta_S(\alpha/\beta)^2$ is the cabbeling parameter, $T_b = \alpha_p - \beta_p(\alpha/\beta)$ is the thermobaric parameter, F^S and F^Θ are the (vertical) double-diffusive fluxes of salt and conservative temperature, ∇_n denotes the gradient operator along the locally-referenced potential density surface (i.e. along the neutral tangent plane), and the subscript z denotes a vertical derivative. Thus, the rate of change of GPE is

$$\frac{d(\mathrm{GPE})}{dt} = \int_0^M \frac{p}{g\rho} eN^2 dm = \int_0^M \chi dm. \tag{5}$$

The equations above have assumed that the total mass is constant on an individual water column such as would be the case if we were considering the consequences of mixing processes in a tall and rigid glass test tube. We have included lateral mixing in the equations and this is allowable so long as there is no net horizontal convergence of mass into the water column. Any such lateral advection of mass would be considered in the calculation of available potential energy; such a calculation involves the adiabatic advection of all fluid parcels to a state of rest and gravitational stability. The present paper does not consider issues related to the available potential energy, but rather we concentrate on the consequences of mixing processes and of the nonlinear nature of the equation of state on changes in gravitational potential energy as raised by Fofonoff.

The contributions to χ from small-scale turbulence and from double-diffusive convection are now written as the vertical divergence (with respect to pressure) of a flux and remaining terms so that χ becomes (from (3) and (4))

$$\begin{aligned}\chi = \frac{p}{g\rho} eN^2 = -(\alpha p)_z \frac{D\Theta_z}{\rho} + (\beta p)_z \frac{DS_z}{\rho} - \frac{p}{\rho} K(C_b \nabla_n\Theta \cdot \nabla_n\Theta + T_b \nabla_n\Theta \cdot \nabla_n p) \\ + (\alpha p)_z \frac{F^\Theta}{\rho} - (\beta p)_z \frac{F^S}{\rho} - \{pDN^2 + pg(\beta F^S - \alpha F^\Theta)\}_p.\end{aligned} \tag{6}$$

For any amount of diapycnal and double-diffusive mixing that is localized in space, the terms in curly brackets in (6) will not contribute to the vertical integral of χ in (5) and so will not affect the rate of change of GPE of the water column. Hence, following Fofonoff (2001), we ignore these terms. We now express (6) as

$$\chi = \chi_D + \chi_K + \chi_{dd}, \tag{7}$$

where the contributions of isotropic small-scale turbulence, of epineutral mixing, and of double-diffusive fluxes are given respectively by (using the hydrostatic relationship $p_z = -g\rho$)

$$\chi_D = DN^2 - \frac{p}{\rho} D\alpha\Theta_z\left(\frac{\alpha_z}{\alpha} - \frac{1}{R_\rho}\frac{\beta_z}{\beta}\right), \tag{8}$$

$$\chi_K = -\frac{p}{\rho} K(C_b \nabla_n\Theta \cdot \nabla_n\Theta + T_b \nabla_n\Theta \cdot \nabla_n p) \qquad \text{and} \tag{9}$$

$$\chi_{dd} = g(\beta F^S - \alpha F^\Theta) + \frac{p}{\rho}(\alpha_z F^\Theta - \beta_z F^S), \tag{10}$$

where the stability ratio, $R_\rho \equiv \alpha\Theta_z/\beta S_z$, is the ratio of the contributions of the vertical gradients of Θ and S to the relevant vertical density gradient. For a linear equation of state, (8) confirms that small-scale turbulent mixing increases the GPE of the water column at the rate DN^2.

The development above is based on Fofonoff's (2001) work, but because it has been developed in terms of diffusivities, it has circumvented an unknown volume fraction of mixing events that appears as an undetermined variable in Fofonoff's (1998, 2001) approach. Also, the present development is a little more general because the effects of lateral mixing (cabbeling and thermobaricity) are now explicitly included as is double-diffusive convection.

3. Dianeutral mixing

For the small-scale mixing case, (8) can be written as

$$\begin{aligned} D^{-1}\chi_D &= N^2 - \frac{p}{\rho}(\alpha_z\Theta_z - \beta_z S_z) \\ &= N^2 - \frac{p}{\rho}(\alpha_\Theta \Theta_z^2 + 2\alpha_S\Theta_z S_z - \beta_S S_z^2) + gp(\alpha_p\Theta_z - \beta_p S_z), \end{aligned} \tag{11}$$

which separates the influence of the cabbeling-type nonlinearities of the equation of state (the middle terms) from the thermobaric terms (the ones in α_p and β_p). Fofonoff's (2001) expression for χ_D (his Eq. (15)) is the same as (11) if his dilution factor is assumed to be infinite.

We now choose to write (8) or (11) as

$$D^{-1}\chi_D = N^2(1 - F) \tag{12}$$

where the factor F is

$$\begin{aligned} F &= \frac{p}{g\rho}\frac{R_\rho}{(R_\rho - 1)}\left(\frac{\alpha_z}{\alpha} - \frac{1}{R_\rho}\frac{\beta_z}{\beta}\right) \\ &= p(\alpha_\Theta\Theta_z^2 + 2\alpha_S\Theta_z S_z - \beta_S S_z^2)/(\rho N^2) - gp(\alpha_p\Theta_z - \beta_p S_z)/N^2. \end{aligned} \tag{13}$$

If we ignore the small variations of the saline contraction coefficient compared with the vertical variation of the thermal expansion coefficient, it is apparent that when

$$\frac{p\alpha_z}{g\rho}\frac{R_\rho}{(R_\rho - 1)} > \alpha, \tag{14}$$

then $F > 1$ or equivalently χ_D is negative so that the reduction of gravitational potential energy by the slumping of the water column (caused by the contraction on mixing at this depth) will be larger than the local increase in GPE produced by the vertical mixing, DN^2. The dominant contributions to α_z are $\alpha_\Theta\Theta_z + \alpha_p p_z$ and given the typical vertical profiles of conservative temperature in the ocean, this combination is usually zero at a depth of about 1500 m where Θ_z is about $2.3 \times 10^{-3}\mathrm{K\ m}^{-1}$ (McDougall, 1988), with α_z being positive above this depth and negative in the deep ocean where the second term, $\alpha_p p_z$, dominates. Substituting $\alpha_\Theta\Theta_z + \alpha_p p_z$ for α_z in (14), we find that the vertical gradient of conservative temperature must satisfy

$$\left(\Theta_z - g\rho\frac{\alpha_p}{\alpha_\Theta}\right)\frac{R_\rho}{(R_\rho - 1)} > \frac{g\rho}{p}\frac{\alpha}{\alpha_\Theta} \tag{15}$$

if the net effect of diapycnal turbulent mixing is to reduce the GPE. Using typical values of α, α_Θ and α_p, (15) becomes approximately

$$(\Theta_z - 0.0023\text{K m}^{-1}) \frac{R_\rho}{(R_\rho - 1)} > \frac{9}{|z|}$$

where $|z|$ is the depth (in m). If, for example, the stability ratio is about 2, then Θ_z must exceed $4.5/|z| + 2.3 \times 10^{-3}\text{K m}^{-1}$ in order for χ_D to be negative.

Interestingly, it is mostly the magnitude of Θ_z that needs to be large in order for χ_D to be negative; the sign of Θ_z is less important. If $\Theta_z < 0$, vertical stability ($N^2 > 0$) dictates that $0 < R_\rho < 1$ which is the sense of vertical stratification conducive to the "diffusive" form of double-diffusive convection. Then, ignoring the generally small term in α_p, (15) can be written as $\Theta_z^2 > \rho N^2/(p\alpha_\Theta)$. Hence, a very negative value of Θ_z is as effective as a large positive value in assisting to make χ_D negative.

Fofonoff (1998, 2001) considered not only turbulent mixing events but also the molecular diffusion of heat as a trigger for the positive GPE feedback process. However, the positive GPE feedback involves an assumed increase in the dissipation of mechanical energy and an increase in oceanic microstructure which leads to *turbulent* mixing that diffuses both temperature and salinity. Hence, we believe it is inconsistent to imagine a positive feedback that diffuses only temperature. That is, if one is to hypothesize that the reduction in depth-integrated GPE causes increased vertical *turbulent* mixing in a positive feedback sequence of events, then the criterion for the existence of this positive feedback must be cast in terms of turbulent mixing, not molecular diffusion.

Thus, the enhanced vertical turbulent mixing of the Fofonoff feedback hypothesis diffuses not only temperature but also salinity, while not changing the value of the stability ratio, $R_\rho = \alpha\Theta_z/\beta S_z$. It follows that the GPE feedback hypothesis is not so much a mechanism for limiting $|\Theta_z|$ but rather is a process for limiting the buoyancy frequency. Since N^2 is equal to $g\alpha\Theta_z(1 - R_\rho^{-1})$, we find that the criterion for χ_D to be negative can be written as (from (15))

$$N^2 > N_{\text{crit}}^2 \tag{16}$$

where

$$N_{\text{crit}}^2 = \frac{g^2\rho\alpha^2}{p\alpha_\Theta}\left(\frac{R_\rho - 1}{R_\rho}\right)^2 + \frac{g^2\rho\alpha\alpha_p}{\alpha_\Theta}\left(\frac{R_\rho - 1}{R_\rho}\right) = A + B. \tag{17}$$

We have labeled the first and second parts of this expression for N_{crit}^2 as A and B and we note that this expression, like (14) and (15), is approximate in the sense that the variations of β have been ignored in comparison with those of α (including ignoring two terms in $\beta_\theta = -\alpha_S$).

It may appear from a cursory inspection of (8) or (11) that the GPE feedback process would be assisted by a low value of N^2 but the opposite is the case since (for a given stability ratio R_ρ) a larger value of N^2 also implies a larger value of the magnitude of the vertical temperature gradient. For a given stability ratio, the Θ_z^2 term in (11) scales as N^4 and this explains why a larger value of N^2 tends to reduce χ_D; just the opposite of what one might imagine at first sight.

4. Energy considerations and the GPE feedback hypothesis

Fofonoff (1998, 2001) argued that oceanic data suggest that there is a tendency for the vertical gradient of potential temperature to rarely exceed the value that would make χ_D negative. To a good approximation, this suggestion implies that Θ_z should be rarely greater than that given by equality in (15). Fofonoff provided the following tentative explanation for this observation. He hypothesized that if χ_D were to become negative (essentially due to a large value of $p\alpha_z$, see (14)), then the excess GPE released "will enhance local mixing" so increasing the turbulent diffusivity D and further decreasing χ_D (see (8)). This amounts to a positive feedback process exhibiting explosive growth of local mixing activity. This escalation of local vertical mixing then vertically diffuses temperature, so reducing $|\Theta_z|$ and thereby increasing the value of χ_D toward zero.

This hypothesis describes a feedback mechanism whereby the GPE of the whole water column that is released due to the local contraction-on-mixing somehow causes the ocean circulation to alter in just such a way as to provide the prevailing conditions necessary for an increase in the small-scale turbulent mixing at just the depth where it can be most effective at further reducing the GPE. This feedback process is then thought to continue until $|\Theta_z|$ is reduced by the enhanced vertical diffusion and χ_D becomes non-negative. Fofonoff (2001) recognized that this feedback mechanism is an unproven hypothesis that is lacking a dynamical basis to date.

Is it correct to expect that a positive feedback to vertical mixing activity occurs when $\chi_D < 0$, that is when $F > 1$? To answer this question, we need to consider the total energy budget. The local rate of increase in gravitational potential energy, DN^2, is related to the amount of dissipated kinetic energy through the mixing efficiency, Γ, so that the rate of dissipation of kinetic energy per unit mass is $\Gamma^{-1}DN^2$. The central tenant of Fofonoff's GPE feedback hypothesis is that the reduction of gravitational potential energy associated with the vertical slumping of the water column ". . . is converted to the microstructure necessary to maintain the mixing . . ." (quote from Fofonoff (2001)). If we accept this hypothesis then the amount of extra dissipation of kinetic energy that appears at the site of the original mixing activity is FDN^2 (see (8) and (12)). This extra dissipation will cause extra diapycnal mixing only in proportion to the relatively small mixing efficiency Γ. In order for a positive feedback to exist, the extra dissipation from the slumping water column, FDN^2, must be greater than the initial amount of dissipation, namely $\Gamma^{-1}DN^2$. Equivalently, the extra amount of diapycnal mixing, ΓFDN^2, must be greater than the initial amount, DN^2.

Hence the corrected criterion for the GPE positive feedback process is that

$$F > \Gamma^{-1} \approx 7 \pm 2 \tag{18}$$

where the range of values reflect the range of mixing efficiencies from the value of 0.2 recommended by Osborn (1980) to the value of 0.11 of Arneborg (2002) which includes the slumping of partially mixed fluid after an intermittent mixing event. If F is merely greater than 1 (but not as large as Γ^{-1}) the extra energy arising from the slumping water column is less that the dissipation that is needed to start the chain of events and so there is no positive feedback. This criterion (18) on F is equivalent to realizing that the GPE positive feedback process can only operate when χ_D is more negative than $DN^2(1 - \Gamma^{-1}) \approx -6DN^2$. Note that Fofonoff's choice of the critical value of F of unity implies a mixing efficiency of 100%, contrary to our current understanding of turbulent diffusion in the ocean. The criterion for the positive GPE feedback based on N^2/N^2_{crit} also has to be changed in the light of our energetic argument. The realization that the energy source from the non-linear nature of the equation of state can only increase the vertical mixing activity through the relatively small multiplying factor of the mixing efficiency, Γ, implies that while the approximate criterion for $\chi_D < 0$ is $N^2/N^2_{\text{crit}} > 1$ as given by (16), the criterion for the existence of a positive feedback process is

$$N^2/N^2_{\text{crit}} > \Gamma^{-1} \approx 7 \pm 2. \tag{19}$$

5. Tests of the GPE feedback hypothesis with ocean observations

We now examine the hydrographic atlas of Koltermann *et al.* (2004) to see if the ocean appears to respect the upper bound on N^2 that is implied by Fofonoff's GPE positive feedback hypothesis. The data set used here was sub-sampled to a horizontal resolution of 1 degree of both latitude and longitude with 45 bottles in the vertical. In Figure 1, the two competing contributions to $D^{-1}\chi_D$ are illustrated for four different locations in the world ocean. The left-hand panels of Figure 1 show the $S - \Theta$ curves for the four vertical casts and the right-hand panels show vertical profiles of N^2 and of $FN^2 \equiv p\rho^{-1}\alpha\Theta_z(\alpha_z/\alpha - R_\rho^{-1}\beta_z/\beta)$ (see (8), (12) and (13)). When FN^2 exceeds N^2 Fofonoff (2001) hypothesizes that the GPE positive feedback process sets in and enhances the vertical mixing activity in an exponential positive feedback loop. This is thought to continue until the vertical stratification is reduced so that F is reduced to the limiting value of unity. If this mechanism was operating in the ocean, an observable fraction of the ocean's volume would be expected to have $F \approx 1$ and there should be very few, if any, observations of $F > 1$. Figure 1(b) shows that χ_D is negative between approximately 600 db and 800 db in the vicinity of the Gulf Stream. In Figure 1(d), we see that F exceeds 2 at about 1500 db and is greater than 1 between 1300 db and 1750 db in the depth range just below the intrusion of Mediterranean Water in the eastern North Atlantic. Panel 1(f) shows the same kind of behavior at 20S in the central Pacific Ocean where the conditions are such that F exceeds 1 in the Central Water over a pressure range of about 150 db centered at 400 db. The last example in Figure 1 is from the Weddell Sea (panels (g) and (h)). In contrast to the other examples in Figure 1 that all have $R_\rho > 1$ at the depth range where $F > 1$, the Weddell Sea

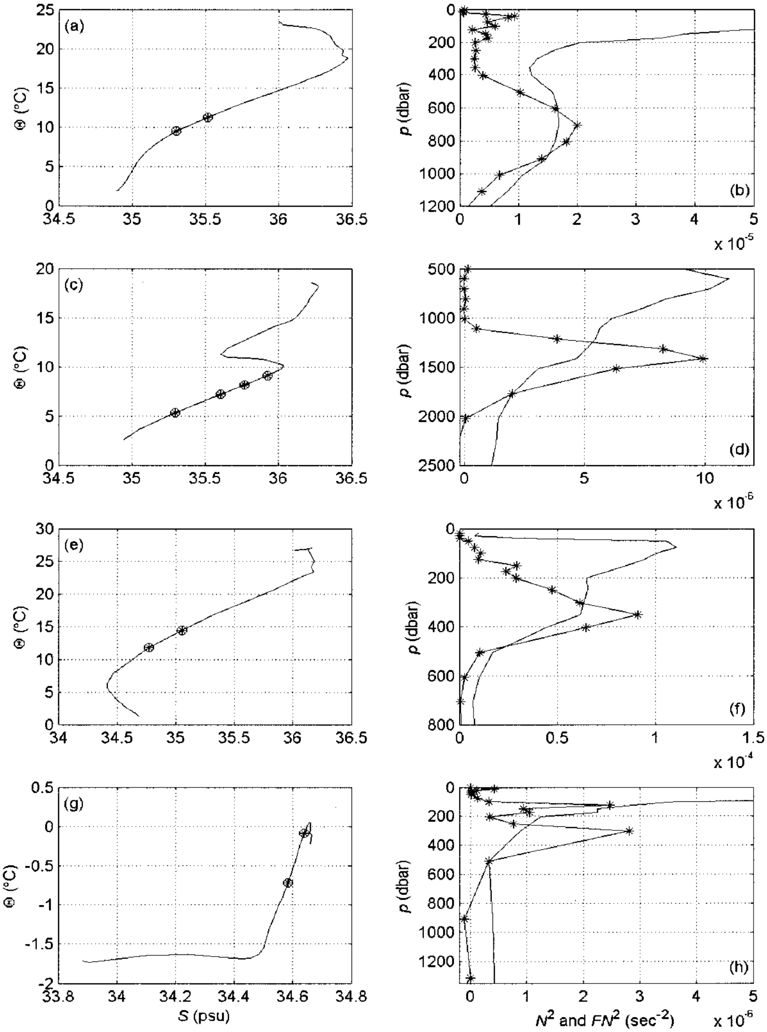

Figure 1. The left-hand panels show the $S - \Theta$ curves for four vertical casts and the right-hand panels shows their vertical profiles of N^2 and of $FN^2 \equiv p\rho^{-1}\alpha\Theta_z(\alpha_z/\alpha - R_\rho^{-1}\beta_z/\beta)$ (see (12) and (13)). The GPE feedback hypothesis states that when FN^2 exceeds N^2 enhanced vertical turbulent mixing should result so as to limit F to be no larger than 1. In this way, one should not be able to observe F greater than 1 in the ocean. The data of panels (a) and (b) are from the Gulf Stream region at 287E, 35N, the data of panels (c) and (d) from the Mediterranean Outflow region of the North Atlantic at 345E, 37N, panels (e) and (f) are from the Central Pacific at 220E, 20S, and panels (g) and (h) are from the Weddell Sea at 314E, 72S. On the right-hand panels N^2 is shown by the full line, FN^2 is shown by the full line with asterisks at the central pressure between the "bottles" of the data of the Koltermann *et al.* (2004) atlas. The $S - \Theta$ values at the mid-point between "bottles" where $F > 1$ are circled on the left-hand panels.

example has the temperature and salinity increasing with depth ($\Theta_z < 0$) and so has $0 < R_\rho < 1$.

The GPE feedback hypothesis, at least as described by Fofonoff (2001), suggests that the vertical profiles of FN^2 and N^2 should coincide over these depth intervals since χ_D would be clamped at zero there. The examples illustrated in Figure 1 show that F does not appear to be limited to 1. However, when there is a subsurface peak in F, it often significantly exceeds unity over a depth range of several hundred meters but does not appear to approach the critical value of seven.

We have also searched every cast of data in the Koltermann *et al.* (2004) hydrographic atlas and isolated the depth of the maximum value of F. This maximum value of F is shown on the color map of Figure 2(a). Values of F near 1 are colored green while those near 2 are red. It is apparent that it is quite common for the maximum value of F on an oceanic cast to exceed 1. In fact 22.5% of the casts in the atlas have a maximum value of F larger than 1. When weighted by the cosine of latitude it is found that 16% of the ocean area has a vertical stratification that exhibits $F > 1$ at some depth. Substantial areas of the South Pacific and the North and South Atlantic have values of F greater than 1.25 at some depth. The only locations where F exceeds 2 are near the Mediterranean and in the southernmost parts of the Southern Ocean while values of F as large as 7 are almost never observed. The pressures at which the maximum values of F occur are shown in Figure 2(b).

The contribution of the thermobaric term to the critical value of N^2 is illustrated in Figure 2(c). Specifically, the ratio B/A (from Eq. (17)) at the depth of maximum F is contoured in Figure 2(c). This ratio is positive when $R_\rho > 1$ and is negative when $0 < R_\rho < 1$ so that Figure 2(c) shows that it is mainly in the Southern Ocean where $0 < R_\rho < 1$ at the depth of maximum F. The other ocean basins mostly have $R_\rho > 1$ at the depth of maximum F. Also, it is apparent that the ratio B/A is rarely less than -0.5 or greater than 0.5, confirming that cabbeling is generally more effective than thermobaricity at causing changes in GPE.

Figure 3(a) shows the histogram of all the values of F for the whole world ocean atlas of Koltermann *et al.* (2004). The histogram is for every pair of bottles down every cast except that we have not plotted any data with $N^2 < 3 \times 10^{-7}\ \mathrm{s}^{-2}$. It is apparent that there is a substantial fraction of the ocean volume where F is negative. This mainly occurs at depth where, as pointed out by McDougall (1988), $(\alpha_z/\alpha - R_\rho^{-1}\beta_z/\beta)$ is negative. The solid line in Figure 3(c) is an expanded view of this histogram in the range $0.5 < F < 2$. The percentage of bottle-pairs from the ocean atlas that have $F > 1$ is 1.8%. The histogram of F in Figure 3(c) does not show any evidence of F being limited at a value of 1. Rather in the region of $F \approx 1$, the histogram varies smoothly through this value and it exhibits the curvature typical of a near-Gaussian probability distribution. In contrast, if the GPE feedback hypothesis of Fofonoff (2001) were real, we would expect to often observe F to be close to 1 and so there should be a peak in the histogram of F near $F = 1$ and a sharp drop off with very few observations having $F > 1$, as indicated by the dashed line in

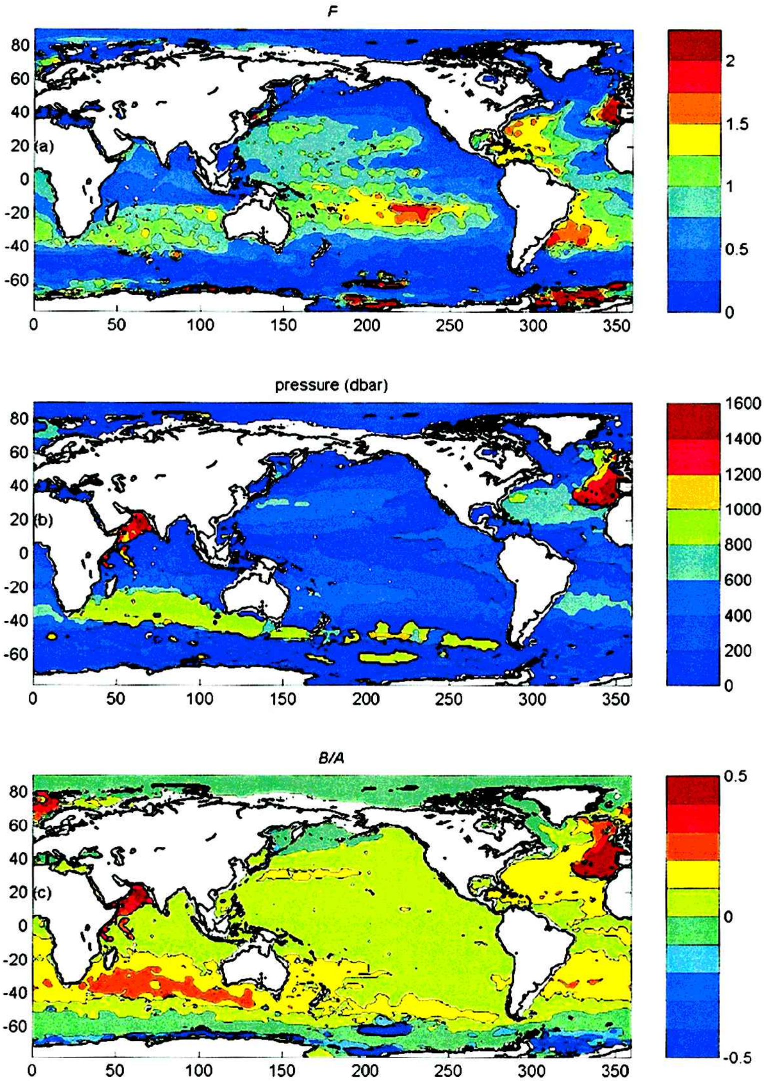

Figure 2. (a) The maximum value of F down each vertical cast is displayed for the world ocean, with the pressure at which this maximum value of F occurs illustrated in panel (b). (c) The ratio B/A (see Eq. (17)) of the two contributions to the critical value of the square of the buoyancy frequency, N^2_{crit}.

Figure 3(c). Instead, the oceanic data show that there appears to be nothing special happening in the vicinity of $F = 1$.

In panels (b) and (d) of Figure 3 we show the histogram of N^2/N^2_{crit}. The interpretation of these figures is as follows. For an observed value of N^2/N^2_{crit} of say 0.5, if the vertical stratification were increased by a factor of two (by say internal gravity wave stretching)

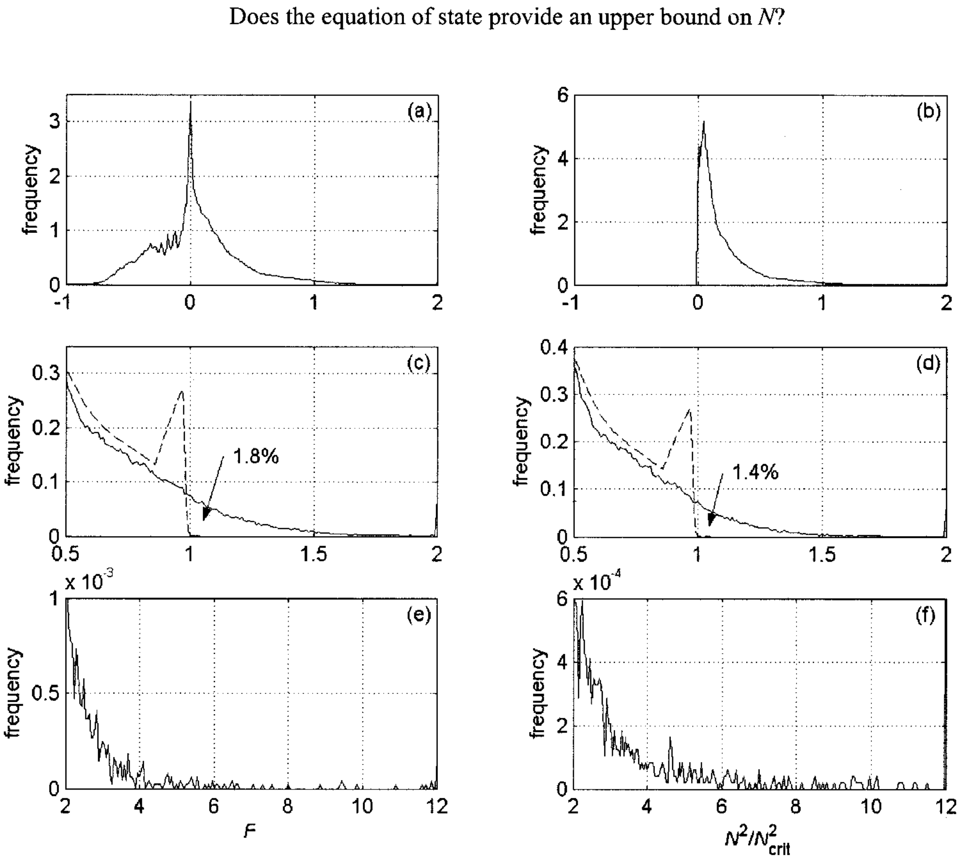

Figure 3. (a) Histogram of values of F for the whole volume of the world ocean. Panels (c) and (e) are expanded views of panel (a). (b) Histogram of N^2/N^2_{crit} (see Eq. (17)) with panels (d) and (f) being expanded views of panel (b). The dashed lines in panels (c) and (d) indicate the histogram that one would expect if the GPE positive feedback process were operative in the manner described by Fofonoff (2001).

then the water column would be critical to the GPE positive feedback process. The figure indicates that 1.4% of the data has $N^2/N^2_{\text{crit}} > 1$ and this percentage is different to the value of 1.8% for F because of the approximations that have been made in arriving at (17) (namely the neglect of α_S and β_S in comparison with α_Θ, and the neglect of β_p in comparison with α_p). The shape of the histogram in Figure 3(d) gives the same conclusion that we have drawn from Figure 3(c), namely that there is no oceanic evidence for any special behavior at $F = 1$ or at $N^2/N^2_{\text{crit}} = 1$. The analysis of Figure 3 has been repeated (not shown here) using other oceanographic atlases with essentially identical results.

Figure 3(e) shows an expanded view of the probability distribution of F in the range $2 < F < 12$ and again there is no hint of a peak in the distribution at some value of F near 7.

Indeed there are only a handful of "bottle pairs" in the entire three-dimensional global ocean that have F approaching or exceeding the critical value of seven. There is simply no observational support for a limiting value of F in Figures 3(c) or (e) and there is no indication of such a limiting value of N^2/N^2_{crit} in Figures 3(d) or (f).

6. Epineutral mixing

McDougall (1987), McDougall and You (1990) and Davis (1994) have examined the dianeutral consequences of the strong lateral mixing of salinity and temperature achieved by mesoscale eddies mixing properties along locally-referenced potential density surfaces. The nonlinear nature of the equation of state leads to dianeutral advection given by

$$e^{\mathrm{cabb}} + e^{\mathrm{tb}} = -gN^{-2}K(C_b\nabla_n\Theta \cdot \nabla_n\Theta + T_b\nabla_n\Theta \cdot \nabla_n p). \tag{20}$$

The dianeutral cabbeling velocity e^{cabb} is always negative while the dianeutral thermobaric velocity e^{tb} can take either sign although its magnitude is generally less than e^{cabb}. McDougall and You (1990) have plotted $e^{\mathrm{cabb}} + e^{\mathrm{tb}}$ on a number of density surfaces in the world ocean and find that this combination is generally negative and that in the southern ocean the dianeutral sinking achieved by these mechanisms is several times larger than the canonical upwelling velocity metric of 10^{-7} m s^{-1}.

What then is the criterion for a positive feedback process to exist in the presence of both lateral cabbeling and diapycnal mixing? We believe that the necessary criterion is not simply that $\chi_K + \chi_D$ is negative. Again, the criterion needs to take into account the relatively small proportion of any dissipated kinetic energy that appears as a local increase in gravitational potential energy. For a given diapycnal diffusivity, D, kinetic energy needs to be locally dissipated at the rate $\Gamma^{-1}DN^2$. If there is to be a positive feedback process then the energy available from the slumping of the water column, $FDN^2 - \chi_K - \chi_{ddc}$ (see (7) and (12)) must exceed $\Gamma^{-1}DN^2$. Here we also include double-diffusive convection, and in the next section we show that χ_{ddc} is expected to be negative. That is, a positive feedback and hence growth of D requires that

$$FDN^2 - \chi_K - \chi_{ddc} > \Gamma^{-1}DN^2. \tag{21}$$

Since we expect $(-\chi_K - \chi_{ddc})$ to be positive, it is clear from (21) that if the criterion for positive feedback in the absence of cabbeling, thermobaricity and double-diffusive convection is satisfied, namely if $F > \Gamma^{-1}$, then the criterion (21) remains satisfied and the growth of the diapycnal diffusivity would continue. Having said that, the difficulty in imagining a series of physical processes that would focus the ocean's response back to the original location of the cabbeling remains. We believe this is a very serious caveat to the whole feedback concept, but for the present discussion, we put this to one side and assume that the ocean is capable of arranging such a focused response.

Concentrating now on the situation where the vertical stratification is such that $F < \Gamma^{-1}$ it is instructive to write (21) in the form

$$D < \frac{\Gamma(-\chi_K - \chi_{ddc})}{N^2}\frac{1}{(1 - F\Gamma)} = D^{K+ddc}\frac{1}{(1 - F\Gamma)} \tag{22}$$

where D^{K+ddc} is the vertical diffusivity that one would calculate from the Osborn (1980) method using the dissipation of kinetic energy $-(\chi_K + \chi_{ddc})$. If the diapycnal diffusivity, D, is small enough to obey (22) then the existence of the positive dissipation $-(\chi_K + \chi_{ddc})$ means that the assumed GPE feedback process could cause the diffusivity, D, to grow, following Fofonoff's assumption that the energy is focused back to the site of the original mixing. This growth does not, however, continue indefinitely because when equality is reached in (22) the feedback process is no longer a positive feedback. Rather, when $F < \Gamma^{-1}$ the GPE feedback hypothesis implies that D grows until it reaches the finite value $(1 - F\Gamma)^{-1}D^{K+ddc}$ when the actual dissipation of kinetic energy, ϵ, will be $(1 - F\Gamma)^{-1}(-\chi_K - \chi_{ddc})$. If the existing diapycnal diffusivity is larger than the limit in (22) then there is no positive feedback and there is no growth in the diffusivity.

Continuing to accept the focused oceanic GPE feedback, we conclude that F needs to satisfy $F > \Gamma^{-1} \approx 7$ in order for a positive feedback to be expected. If $F < \Gamma^{-1}$ the GPE feedback process does not allow growth of diapycnal mixing without bound but instead any such growth is limited by the amount of lateral cabbeling, lateral thermobaricity and double-diffusive convection. For example, for $F \approx 1$ the GPE feedback mechanism would ensure that the diapycnal diffusivity would never be less than $(1 - F\Gamma)^{-1}D^{K+ddc} \approx 1.17D^{K+ddc}$ but the feedback process does not allow growth of the diffusivity larger than this value.

7. Double-diffusive mixing

Consider now the influence of double-diffusive convection on the evolution of GPE. From (10) we see that if the equation of state is linear, then $\chi_{dd} = g(\beta F^S - \alpha F^\Theta)$ and it is a characteristic of double-diffusive convection that this density flux is negative, that is, it is directed vertically downward (and incidentally, in the *up-gradient* direction with respect to the vertical density gradient). That is, even for a linear equation of state, double-diffusive convection acts to reduce the GPE of a water column.

If one applies the GPE contraction-on-mixing feedback hypothesis to this situation, it would seem that double-diffusive convection may often carry the seeds of its own destruction. That is, a small amount of double diffusive convection with its negative χ_{dd} would lead to a turbulent diapycnal diffusivity of magnitude corresponding to equality in (22) at just the location of the initial double-diffusive convection. This diapycnal diffusivity $-(\Gamma^{-1} - F)^{-1}\chi_{ddc}/N^2$ would then tend to disrupt the organized structures that are typical of double-diffusive convection. However, since we do not believe in the positive feedback GPE hypothesis, we do not intend for this suggestion to be taken seriously.

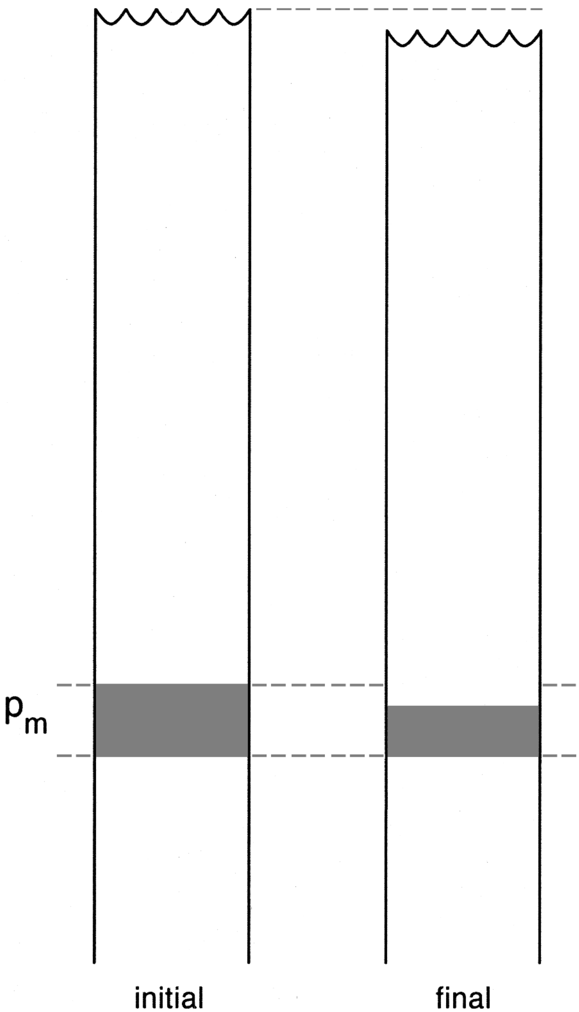

Figure 4. Sketch of the slumping of a strictly one-dimensional water column (like a very tall test tube) due to turbulent mixing in the vicinity of pressure p_m. Each parcel in the water column above p_m retains its enthalpy, its internal energy and its pressure. The reduction in gravitational potential energy of the entire water column appears as an increase in the internal energy of the fluid that undergoes the mixing.

8. The one-dimensional counter-example

Consider the strictly one-dimensional water column sketched in Figure 4. Mixing occurs over a small pressure interval centered at p_m and as a consequence each parcel in the water column above this pressure slumps as shown in the figure. The first law of thermodynamics can be written as (McDougall, 2003)

$$\rho\left(\frac{dI}{dt}+\frac{d(p/\rho)}{dt}\right)-\frac{dp}{dt}=-\nabla\cdot\mathbf{F}_Q+\rho\varepsilon_M, \tag{23}$$

where I is the internal energy, $\mathbf{F}_Q$ is the flux of heat by all manner of molecular fluxes and by radiation, and $\rho\varepsilon_M$ is the rate of dissipation of kinetic energy (in units of W m^{-3}) into thermal energy. Also, specific enthalpy, h, is defined by $h \equiv I + p/\rho$. During the mixing and slumping process, each fluid parcel in the upper un-mixed part of the water column retains its salinity, its conservative temperature and its pressure, hence both the internal energy $I = I(S, \Theta, p)$ and the enthalpy $h = h(S, \Theta, p)$ of these fluid parcels remain constant as they all lose gravitational potential energy. When vertically integrated over the small mixing region, the heat flux divergence $-\nabla\cdot\mathbf{F}_Q$ integrates to zero so that the change in the enthalpy (per unit mass) of the fluid in this region, δh, is given by $\varepsilon_M\delta t$ where δt is the time interval. Note that this increase in enthalpy is independent of the nonlinearity in the equation of state. By contrast, in the mixing region where $d(p/\rho)/dt = \chi$ (see (2)) the change in the internal energy per unit mass of fluid, δI, is $(\varepsilon_M - \chi)\delta t$. Hence we see that in this one-dimensional example, the reduction in the gravitational potential energy of the entire water column appears as an extra increase in the internal energy, $-\chi\delta t$, of the mixed fluid. Because the water column is strictly one-dimensional, we have ruled out the possibility of any physical process (such as Rossby waves or internal gravity waves) taking some part of the reduction of GPE and partitioning it into kinetic energy.

If, on the other hand, one allows horizontal nonhomogeneity in the mixing activity, so that for example, one water column is mixed like in Figure 4 but now there is a surrounding ocean that is unmixed, then one sees that the immediate slumping on one water column produces a horizontal pressure gradient over the full depth of the slumped fluid. This opens up the possibility of internal and Rossby wave activity, but since the horizontal pressure gradient exists over the full water column, would it not be reasonable to assume that any resulting increase in kinetic energy and mixing activity would also be spread out over the full water column? There seems to be no reason to assume that the oceanic response would be to concentrate this energy at the original depth of the mixing. This issue of the assumed focusing of the oceanic response lies at the heart of the GPE feedback hypothesis and it seems this assumption is unsustainable.

9. Discussion

We have explored the implications of the "gravitational potential energy local feedback hypothesis" that has been advanced by Fofonoff (1961, 1995, 1998, 2001). This is a rather intriguing hypothesis and it can be summarized as follows. When mixing processes occur at any depth in the ocean, the nonlinear nature of the equation of state (and also double-diffusive convection) usually cause a net contraction in the vicinity of the mixing and consequently the gravitational potential energy of the fluid above this depth decreases. The argument then notes that in some circumstances the loss of depth-integrated gravitational potential energy by the vertical slumping of this water column can exceed the local increase in gravitational potential energy that is associated with the mixing. So far so good. Then the hypothesis states that when there is a net reduction in the gravitational potential energy of the whole water column, 100% of this net energy feeds back and causes extra diapycnal mixing activity at exactly the location of the original mixing activity. This argument describes a positive feedback process that is assumed to greatly increase the vertical turbulent mixing, thereby diffusing the vertical property gradients and providing an upper limit to the vertical stratification.

We have shown that because only the fraction Γ (between 0.11 and 0.2) of kinetic energy that is dissipated appears as a local increase in gravitational potential energy, the relevant criterion for a positive feedback to exist is a factor of Γ^{-1} harder to achieve than that described by Fofonoff. This factor is approximately 7 and there are very few places in the world ocean where the corrected criterion is exceeded. The revised criterion for positive GPE feedback ((18) with F given by (13)) is equivalent to realizing that the process can only operate when χ_D is more negative than $DN^2(1 - \Gamma^{-1}) \approx -6DN^2$ rather than the criterion being simply $\chi_D < 0$. The observed probability distribution of the propensity for this positive feedback (the factor F) does not show any tendency for an increased frequency of occurrence near $F = 1$ nor in the range near Γ^{-1}.

If $F < \Gamma^{-1}$, the GPE feedback process does not lead to unlimited growth of the diapycnal diffusivity, but it can cause some modest growth in this turbulent mixing, proportional to the amount of lateral cabbeling, lateral thermobaricity and double-diffusive convection. In this case the revised feedback hypothesis bounds the vertical diffusivity to be no less than the value given by equality in (22). This is not a growth in D that can be characterized as a traditional positive feedback process, and it is not able to limit the vertical stability of the water column, but rather it provides a modest background level of diapycnal mixing that is proportional to the double-diffusive and epineutral mixing processes.

If we were to take this localized GPE feedback mechanism seriously as a source of mixing activity and as a way of limiting the buoyancy frequency, would it be expected to occur as a sudden onset whenever the relevant criterion was exceeded? One suggestion might be that the ocean is not 100% efficient at focusing its response at the exact location where the mixing activity first occurred and that one should perhaps expect that F should exceed Γ^{-1} by some significant factor before the feedback process was in fact effective. For example, if the ocean were able to focus only 10% of the energy back to the original mixing location, the vertical stratification would need to obey $F > 10\Gamma^{-1} \approx 70$ in order for the GPE positive feedback process to be operative. These values are certainly not observed.

The conservation equation of total energy can be written as (Griffies, 2004)

$$(\rho E)_t + \nabla \cdot (\rho \mathbf{u} E) = -\nabla \cdot (p\mathbf{u}) - \nabla \cdot \mathbf{F}_Q + \nabla \cdot (\mu \nabla (\tfrac{1}{2}\mathbf{u} \cdot \mathbf{u})) \tag{24}$$

where

$$E = I + \Phi + \tfrac{1}{2}\mathbf{u} \cdot \mathbf{u} \tag{25}$$

is the sum of the specific internal energy, I, the geopotential, $\Phi = gz$, and the kinetic energy per unit mass, $\mathbf{F}_Q$ is the molecular and radiative fluxes of heat, and μ is the dynamic viscosity. On volume integrating (24) over the whole ocean it is apparent that when kinetic energy is dissipated (i.e. a reduction in volume integrated $\frac{1}{2}\rho\mathbf{u} \cdot \mathbf{u}$) it must appear as an increase of the volume integral of $\rho(I + \Phi)$. Similarly, if the nonlinear nature of the equation of state or double-diffusive convection cause a reduction in the volume integrated GPE, this could appear as either an increase in the volume integrated internal energy or as an increase in the volume integrated kinetic energy. Fofonoff has assumed that 100% of this energy appears as extra kinetic energy that is subsequently dissipated. However, this is only one assumption and an extreme one at that. A counter example is provided by considering a strictly one-dimensional ocean. Then there is no physical mechanism that can channel any excess gravitational potential energy into kinetic energy and hence the lost GPE will directly increase the local internal energy without any possibility of a positive feedback process involving kinetic energy and its dissipation.

Note from (24) that there is no requirement that a decrease in the depth-integrated GPE on a particular cast should appear as an increase in kinetic energy on that cast, nor indeed at the depth of the original mixing. Even if this reduction in depth-integrated GPE led to a rather localized increase in *kinetic energy,* there is no guarantee that all of this increased kinetic energy would be *dissipated* locally and so cause the increased local diapycnal mixing that is part of the assumed feedback mechanism.

10. Summary

We have examined Fofonoff's gravitational potential energy feedback hypothesis and have

- eliminated the unknown dilution factor that appears in Fofonoff's parcel mixing approach,
- we have shown that since the feedback process involves an increase in turbulent mixing, it is not consistent to regard molecular diffusion as an initiator of the process,
- because turbulent mixing does not change the local vertical stability ratio of the water column, the feedback hypothesis is best regarded as providing an upper bound on the buoyancy frequency rather than on the vertical temperature gradient,
- the rather small mixing efficiency with which turbulent dissipation is converted into an increase in local gravitational potential energy implies that the criterion on the vertical stratification that is required for positive feedback is a factor of seven larger than that advanced by Fofonoff,
- we have found no hint of the positive feedback process in oceanic atlas data, and
- disturbingly, we can think of no reason why the reduction in a water column's gravitational potential energy would appear as increased turbulent mixing at the

precise location where the initial mixing occurred, especially as this does not occur in a horizontally uniform ocean.

There seems no reason to assume that the ocean's response to a depth-integrated reduction in potential energy would (a) appear as an increase in kinetic energy rather than an increase in internal energy (as it must in a purely one-dimensional situation), (b) that it would be localized in three-dimensional space, and (c) that this localized increased kinetic energy would be dissipated locally. These are three very significant assumptions that need to be satisfied simultaneously in order for the GPE positive feedback process to occur. The very pin-pointed feedback whereby a change of gravitational potential energy that is felt over a large part of the water column feeds back and affects the mixing intensity at a specific depth seems unrealistic, and in the absence of a physical mechanism to provide this pin-pointed feedback, we are inclined to discount the hypothesis. The assumed localization of the ocean's response is particularly difficult to envisage as the slumping of the water column and the consequent change to the horizontal pressure gradient occurs at all heights above the mixing location.

We have searched hydrographic data of the world ocean and have found no indication of a critical value of the stratification beyond which large diapycnal mixing would be expected. While the appropriate factor F often exceeds unity, it almost never approaches seven which is the critical value of F for Fofonoff's feedback process when corrected for the relatively inefficient nature of small-scale mixing at achieving diapycnal buoyancy fluxes. The absence of a peak in the frequency distribution of F at $F \approx 7$ means that there is no observational support for the positive feedback GPE hypothesis from oceanic data.

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