Journal of Marine Research, 61, 693–702, 2003

Comments on "A generic length-scale equation for geophysical turbulence models" by L. Umlauf and H. Burchard

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1. Introduction

Umlauf and Burchard (2003) present a generic length-scale equation for the prognostic quantity $\psi = (c_\mu^0)^p k^m l^n$ for use in two-equation models of turbulence. However, because of the traditional form used for the diffusion term, it can be used at present for only negative values of n. We show that a simple modification of the diffusion term is sufficient to insure a more universal generic length-scale equation that is valid for all values of m and n, including positive values of n. We also show that an appropriate combination of exponents m and n will enable the generic length-scale equation to simulate any desired length-scale equation including those in the k-ε, k-ω, k-kl, k-$k\tau$, k-l and k-τ models.

2. UB generic length-scale equation and polymorphism

Umlauf and Burchard (2003, hereafter UB) have recently presented a generic length-scale equation for use in two-equation turbulence models. It is a conservation equation for the quantity $\psi = (c_\mu^0)^p k^m l^n$ involving the macro-length scale of turbulence l, and is of the form

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x_k}(U_k \psi) = \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_\psi}\frac{\partial \psi}{\partial z}\right) + \frac{\psi}{k}(c_{\psi 1} P + c_{\psi 3} G - c_{\psi 2} \varepsilon) \tag{1}$$

where the first term on the r.h.s. is the term denoting down-the-gradient turbulent diffusion of ψ; P is the shear production, G is the buoyancy production/destruction, and ε is the dissipation of turbulence kinetic energy (TKE). The quantities σ_ψ, c_μ^0, $c_{\psi 1}$, $c_{\psi 2}$ and $c_{\psi 3}$ are closure constants and ν_t is the turbulent viscosity. The quantity k, the TKE, is given by the conservation equation:

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x_k}(U_k k) = \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_k^\psi}\frac{\partial k}{\partial z}\right) + (P + G - \varepsilon) \tag{2}$$

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where once again, the first term on the r.h.s. is the turbulent diffusion of k. The quantity σ_k^ψ is another closure constant. σ_ψ and σ_k^ψ are Schmidt numbers. Along with the stability functions that control the value of turbulent viscosity, Eqs. (1) and (2) constitute a two-equation model of turbulence. We will follow the UB notation throughout, for simplicity, even though we prefer the notation of Kantha (2003, hereafter K3).

The exponents p, m and n take particular values for the different turbulence length-scale equations that have been used in the past: (i) k-ε model: $p = 3$, $m = \frac{3}{2}$, $n = -1$; (ii) k-ω model: $p = -1$, $m = \frac{1}{2}$, $n = -1$; (iii) k-$k\tau$ model: $p = -3$, $m = \frac{1}{2}$, $n = 1$ and (iv) k-kl model: $p = 0$, $m = 1$, $n = 1$. Quantity ω is the turbulence frequency and $\tau = k/\varepsilon$ is the turbulence time scale. Thus, it must be possible to derive any of these length-scale equations as a subset of a generic length-scale equation. However, the UB generic equation fails for positive values of the exponent n (when m is chosen to be a nonzero, positive integer, as UB did) and hence cannot simulate the k-kl and k-$k\tau$ models. More importantly, this generic length-scale equation, which is supposed to be a pseudo-length-scale equation cannot be reduced to an equation for the turbulence macroscale, l, for which $m = 0$ and $n = 1$! Moreover, the manner in which UB determine the closure constants leads to a generic quantity devoid of any physical meaning.

The authors are part of the team (that includes Karsten Bolding) that has done a great service to the ocean turbulence modeling community by developing and making freely available the General Ocean Turbulence Model (GOTM, see *www.gotm.net* for the downloadable version of the latest revision 3.0). The code is written in modern Fortran, allows for easy testing of new ideas in turbulence modeling, and is readily included in 3-D ocean models. However, the authors have overlooked the fact that, by a judicious choice of the exponents m and n, their generic length-scale equation (included in the code at present) can be made to simulate any length-scale equation used in the past, as outlined in detail below.

The traditional way of determining the closure constants is to appeal to simple turbulent flows, where turbulence quantities are known with a high degree of certainty (for example, see Mellor and Yamada, 1982). The model closure constants can be and were selected by UB by applying the TKE and the generic length-scale equations to simple flow situations such as the logarithmic law-of-the-wall region of the turbulent boundary layer near a wall (see UB for details; see also K3), where the behavior of turbulence is known with a good degree of certainty. Thus,

$$(c_\mu^0)^2 = 0.3,\ \sigma_\psi = \frac{n^2\kappa^2}{(c_\mu^0)^2(c_{\psi 2} - c_{\psi 1})},\ c_{\psi 1} = m,\ d = \frac{-2n}{2m + n - 2c_{\psi 2}}. \tag{3}$$

The first two conditions are obtained by appealing to the logarithmic law of the wall, the third one uses the Tennekes hypothesis in homogeneous shear flow and the last one refers to the decay of homogeneous turbulence where d is the decay rate with time of TKE (see K3 or UB for details). The quantity κ is the von Karman constant, with a traditional value of 0.4, although aerodynamicists use a value of 0.41. Wind tunnel experiments show that d

has a value slightly lower than -1, whereas theoretical considerations dictate that $d = -1$ be the asymptotic value for high Reynolds number turbulence. UB chose $d = -1.2$, while K3 chooses $d = -1$. From Eq. (3),

$$c_{\psi 2} = m + n\left(\frac{1}{2} + \frac{1}{d}\right), \quad \sigma_\psi = \frac{0.533n}{\left(\frac{1}{2} + \frac{1}{d}\right)} \tag{4}$$

where the numerical values of κ and c_μ^0, which are known with a great degree of certainty, have been substituted. This leaves only σ_k^ψ to be determined ($c_{\psi 3}$ is irrelevant to the thrust of these comments).

UB then appeal to the experiments on spatial decay of turbulence in a tank away from a stirring grid that generates the turbulence at one end of the tank, as did Kantha (1988; 2003) and Kantha and Clayson (2003). The spatial change of TKE and the length scale are known from these experiments to be $k = k_0(z - z_0)^\alpha$ and $l = L(z - z_0)$, with $\alpha = -2$ (a value that can also be derived from theoretical arguments for asymptotic turbulence) and L around 0.2 (z is the distance from the grid and z_0 is the virtual origin). Eqs. (1) and (2) give for this flow situation, where the principal balance is between dissipation and diffusion terms (see UB, K3 or Kantha and Clayson (2003) for details):

$$(\alpha L)^2 = \frac{2}{3}(c_\mu^0)^2 R \sigma_k^\psi, \quad (c_\mu^0)^2 R \sigma_\psi c_{\psi 2} = (\alpha m + n)\left[\left(m + \frac{1}{2}\right)\alpha + n\right] L^2 \tag{5}$$

where $R = c_\mu^0 / c_\mu$ with c_μ determined from the specific algebraic closure model used.

It is here that UB deviate from the traditional approaches. They argue that since the stirring grid experiments provide two constraints but there is only one closure constant σ_k^ψ left undetermined, one of the exponents (either m or n, they choose n) must be determined from Eq. (5), even if it means that the exponent must have a fractional value. They call this approach "polymorphism," and justify this approach by pointing out that the k-ε model provides an unjustifiably high value for α for traditional values for the closure constants. They do concede that the resulting quantity, ψ, has no definite physical meaning and may just be a mathematical concoction. This approach is in sharp contrast to traditional approaches where the length-scale equation is always written for a quantity that has a well-known physical interpretation: the dissipation rate ε in k-ε models, the turbulence frequency ω in k-ω models, the turbulence viscosity in k-$k\tau$ models and the two-point correlation in k-kl models (see K3).

We believe that a length-scale equation for a quantity without a well-known physical meaning is undesirable, and most importantly, unnecessary. To show this, following K3, we eliminate L, which is after all not known as precisely as α. (The experimental values exhibit a large scatter around 0.2; see UB; see also Kantha, 1988.) This yields an equation for the last undetermined constant σ_k^ψ:

$$\sigma_k^\psi = \frac{3\alpha^2\sigma_\psi c_{\psi 2}}{2(\alpha m + n)\left[\left(m + \frac{1}{2}\right)\alpha + n\right]} = \frac{6\sigma_\psi c_{\psi 2}}{(2m - n)(2m - n + 1)} \tag{6}$$

after substitution of the well-known numerical value of -2 for α. Eqs. (4) and (6) provide numerical values for the three constants σ_ψ, $c_{\psi 2}$ and σ_k^ψ.

If we choose the asymptotic value of -1 for d, we get:

$$\sigma_\psi = -1.067n, \quad c_{\psi 2} = m - n/2, \tag{7}$$

$$\sigma_k^\psi = \frac{-3.2n}{(2m - n + 1)}.$$

For the k-ε model, $m = \frac{3}{2}$ and $n = -1$ so that $\sigma_\psi = 1.067$, $c_{\psi 2} = 2$, and $\sigma_k^\psi = 0.64$. These values give $L = 0.18$, not too far from the mean experimental value of around 0.2. For the k-ω model, $m = \frac{1}{2}$, $n = -1$ so that $\sigma_\psi = 1.067$, $c_{\psi 2} = 1$, and $\sigma_k^\psi = 1.067$. These values give $L = 0.23$, once again not too far from the experimental value of around 0.2. The traditional values for the closure constants are $\sigma_\psi = 1.3$, $c_{\psi 2} = 1.92$, and $\sigma_k^\psi = 1.0$ for the k-ε model, and $\sigma_\psi = 2$, $c_{\psi 2} = 0.833$, and $\sigma_k^\psi = 2$ for the k-ω model (see the tables in UB). The generic equation (1) is not valid for positive values of n and so the equivalent values cannot be determined for k-$k\tau$ and k-kl models. Nor is it possible to put $m = 0$ and $n = 1$, and derive the constants in an equation for the length scale l itself.

On the other hand, the value of -1.2 chosen by UB for d gives:

$$\sigma_\psi = -1.6n, c_{\psi 2} = m - n/3, \quad \sigma_k^\psi = \frac{-3.2n(3m - n)}{(2m - n)(2m - n + 1)}. \tag{8}$$

For the k-ε model, this gives $\sigma_\psi = 1.6$, $c_{\psi 2} = 1.833$, $\sigma_k^\psi = 0.88$, and $L = 0.21$, whereas for the k-ω model, $\sigma_\psi = 1.6$, $c_{\psi 2} = 0.833$, $\sigma_k^\psi = 1.333$, and $L = 0.26$. Once again L values are not too far from the experimental value of around 0.2.

Clearly, by a readjustment of one of the closure constants, we can achieve conformity with *all* the above experimental results in both the k-ε and k-ω models. Also, if we do not insist on L being exactly 0.2, there is no need for polymorphism and nonphysical quantity ψ. A downward readjustment of the constant σ_k^ψ is enough to assure that the k-ε model will work for stirring grid turbulence and hence can simulate the effect of wave breaking in oceanic mixed layer models. The inability of the traditional k-ε model to simulate wave breaking was cited by UB to be the principal motivation for the adoption of "polymorphism."

3. Form of the diffusion term in the UB generic equation

The generic length-scale equation of UB cannot be used when the exponent n is positive (m is a positive integer for traditional length-scale models) since this leads to negative values for some constants, which must remain positive-definite. This deficiency can be traced directly to the form of the diffusion term used (see K3 for details). Briefly, the diffusion term must be modified. The form of the additional terms needed in the generic equation can be derived by appealing to the length-scale equation in k-ε models. Admittedly, this makes the generic equation mathematically equivalent to the k-ε model, but the utility of the former lies in the need in some applications for an equation with positive values for the exponent n. It is well known that it is difficult to extend k-ε models to the boundary and the k-ω model has difficulty satisfying the boundary conditions at the outer edge of a turbulent layer, and for both these models n is negative. We will not use up space here for describing the general form of the diffusion term needed in a generic equation of general validity, but instead refer the reader to K3. We do feel that the introduction of "polymorphism" to make the generic length-scale equation with *traditional* diffusion term "work," when the same equation with *appropriate* modification of the diffusion term will work quite well with any values for the exponents m and n (including positive values for n), is unnecessary.

UB state that their generic model works best for $m = 1$ and $n = -0.67$. Then $\psi \sim kl^{-2/3} = (k^{3/2}l^{-1})^{2/3} = \varepsilon^{2/3}$. In other words, their generic equation works best for a quantity ε^q when $q = \frac{2}{3}$. It can be shown that an equation for ε^q with the traditional form of the diffusion term is equivalent to a modified equation for ε itself:

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_k}(U_k \varepsilon) = \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial z}\right) + \frac{\nu_t}{\sigma_\varepsilon}\frac{q-1}{\varepsilon}\left(\frac{\partial \varepsilon}{\partial z}\right)^2 + \frac{\varepsilon}{k}(c^*_{\varepsilon 1}P + c^*_{\varepsilon 3}G - c^*_{\varepsilon 2}\varepsilon) \tag{9}$$

where $c^*_{\varepsilon 1} = c_{\varepsilon 1}/q$, $c^*_{\varepsilon 2} = c_{\varepsilon 2}/q$, $c^*_{\varepsilon 3} = c_{\varepsilon 3}/q$, and the second term on the r.h.s. is an additional diffusion term. Clearly, the UB generic equation, in this case at least, is an equation for ε but with a modified diffusion term! This is in perfect agreement with the conclusions of K3, who argues that the diffusion term must be modified to make a generic length-scale equation work for all values of exponents m and n. Note however, that even if we put q equal to 1, Eq. (9) will still simulate the rate of decay in stirring grid turbulence, as long as σ_k^ψ is chosen appropriately, as was shown in Section 2.

Because of the additional term in Eq. (9) the constraints on closure constants now become

$$\begin{gathered}
(c_\mu^0)^2 = 0.3,\ c^*_{\varepsilon 1} = \frac{3}{2}, \quad c^*_{\varepsilon 2} = \frac{3}{2} - \left(\frac{1}{2} + \frac{1}{d}\right) \\
\sigma_\varepsilon = \frac{-\kappa^2 q}{(c_\mu^0)^2(c^*_{\varepsilon 2} - c^*_{\varepsilon 1})} = \frac{-\kappa^2 q}{(c_\mu^0)^2\left(\frac{1}{2} + \frac{1}{d}\right)} \\
(\alpha L)^2 = \frac{2}{3}(c_\mu^0)^2 R \sigma_k \\
(c_\mu^0)^2 R \sigma_\varepsilon c^*_{\varepsilon 2} = \left(\frac{3}{2}\alpha - 1\right)\left[(2\alpha - 1) + (q-1)\left(\frac{3}{2}\alpha - 1\right)\right]L^2.
\end{gathered} \tag{10}$$

Following UB, we put $R = 1$ here. Using the value of $d = -1.2$ used by UB, we get $c^*_{\varepsilon 1} = 1.5$, $c^*_{\varepsilon 2} = 1.833$ and $\sigma_\varepsilon = 1.6q$. Using $\alpha = -2$ and $L = 0.2$, we get $\sigma_k = 0.8$. The last relationship in Eq. (10) then gives $q = \frac{2}{3}$. This is precisely the value used by UB in their best-performing generic equation. Note that the value of σ_ε is, therefore, 1.067 and the corresponding values of $c_{\varepsilon 1}$ and $c_{\varepsilon 2}$ in the unmodified k-ε length-scale equation Eq. (1) are 1.0 and 1.22, in agreement with the tabulated values for the generic UB equation with $m = 1$ and $n = -0.67$.

4. A more universal generic length-scale equation

If we accept the notion that a generic length-scale equation must be valid for all values of exponents m and n (including positive values for n), and should be written for a quantity that has some physical meaning, then the UB form is unacceptable, and we must examine the form of the diffusion term in Eq. (1) as indeed K3 did. However, while the form of the diffusion terms suggested by K3 works, one could argue that the generic equation derived by K3 is nothing but a transformed k-ε equation. This raises the question: what is the simplest modification of the diffusion term that will achieve the same goal? Clearly, the traditional form does not work for positive values for the exponent n. Eq. (9) provides a clue. We could generalize the form of the additional term in Eq. (9) and write the generic length-scale equation as:

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x_k}(U_k \psi) = \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_\psi}\frac{\partial \psi}{\partial z}\right) + \frac{\nu_t}{\sigma_\psi}\frac{\xi}{\psi}\left(\frac{\partial \psi}{\partial z}\right)^2 + \frac{\psi}{k}(c_{\psi 1}P + c_{\psi 3}G - c_{\psi 2}\varepsilon). \tag{11}$$

This equation represents a necessary and sufficient modification of the traditional form of the length-scale equation to make it applicable to positive values of n. (It can be shown that terms involving just $\partial k/\partial z$ or $\partial l/\partial z$ are not sufficient.) In particular, it works for the length-scale l itself ($m = 0$, $n = 1$). The new constraints on the closure constants can be shown to be:

$$\begin{gathered}
(c^0_\mu)^2 = 0.3,\; c_{\psi 1} = m, \quad c_{\psi 2} = m + n\left(\frac{1}{2} + \frac{1}{d}\right) \\
\sigma_\psi = \frac{n^2\kappa^2(1+\xi)}{(c^0_\mu)^2(c_{\psi 2} - c_{\psi 1})} = \frac{n\kappa^2(1+\xi)}{(c^0_\mu)^2\left(\frac{1}{2} + \frac{1}{d}\right)} \\
(\alpha L)^2 = \frac{2}{3}(c^0_\mu)^2 R \sigma^\psi_k \\
(c^0_\mu)^2 R \sigma_\psi c_{\psi 2} = (m\alpha + n)\left[\left(m + \frac{1}{2}\right)\alpha + n + \xi(m\alpha + n)\right]L^2.
\end{gathered} \tag{12}$$

From Eq. (12), an expression can be derived for ξ:

Table 1. The model constants for different values of L and d.

Model	L	d	$c_{\psi 1}$	$c_{\psi 2}$	ξ	σ_ψ	σ_k^ψ
k-ε	0.2	−1.2	1.5	1.833	−0.333	1.0667	0.8
k-ω	0.2	−1.2	0.5	0.833	−0.667	0.5333	0.8
k-kl	0.2	−1.2	1.0	0.667	−1.111	0.18	0.8
k-$k\tau$	0.2	−1.2	0.5	0	< -1	$-1.6(1+\xi)$	0.8
k-l	0.2	−1.2	0	−0.333	−1.333	0.5333	0.8
k-τ	0.2	−1.2	−0.5	−0.833	−1.333	0.5333	0.8
k-ε	0.2	−1	1.5	2	—	—	0.8
k-ω	0.2	−1	0.5	1	−0.5	0.5333	0.8
k-kl	0.2	−1	1.0	0.5	−1.2	0.2133	0.8
k-$k\tau$	0.2	−1	0.5	0	< -1	$-1.067(1+\xi)$	0.8
k-l	0.2	−1	0	−0.5	−1.333	0.356	0.8
k-τ	0.2	−1	−0.5	−1	−1.5	0.5333	0.8
k-ε	0.224	−1.2	1.5	1.833	−0.07	1.486	1.0
k-ω	0.224	−1.2	0.5	0.833	−0.5	0.8	1.0
k-kl	0.224	−1.2	1.0	0.667	−1.135	0.216	1.0
k-$k\tau$	0.224	−1.2	0.5	0	< -1	$-1.6(1+\xi)$	1.0
k-l	0.224	−1.2	0	−0.333	−1.455	0.728	1.0
k-τ	0.224	−1.2	−0.5	−0.833	−1.5	0.8	1.0
k-ε	0.224	−1	1.5	2	−2.25	1.3333	1.0
k-ω	0.224	−1	0.5	1	−0.167	0.889	1.0
k-kl	0.224	−1	1.0	0.5	−1.238	0.254	1.0
k-$k\tau$	0.224	−1	0.5	0	< -1	$-1.067(1+\xi)$	1.0
k-l	0.224	−1	0	−0.5	−1.455	0.485	1.0
k-τ	0.224	−1	−0.5	−1	−1.833	0.889	1.0

$$\xi = \frac{-Rn\left[m + n\left(\frac{1}{2}+\frac{1}{d}\right)\right] + (m\alpha+n)\left[\left(m+\frac{1}{2}\right)\alpha + n\right]\left(\frac{1}{2}+\frac{1}{d}\right)\left(\frac{L}{\kappa}\right)^2}{Rn\left[m + n\left(\frac{1}{2}+\frac{1}{d}\right)\right] - (m\alpha+n)^2\left(\frac{1}{2}+\frac{1}{d}\right)\left(\frac{L}{\kappa}\right)^2} \tag{13}$$

where $R = 1$, $\alpha = -2$ and $L = 0.2$, $\kappa = 0.4$ can be substituted. $\sigma_k^\psi = 0.8$ irrespective of the value of m and n. For the k-ε model, if we use $d = -1.2$, we recover Eq. (9), since ξ becomes $-\frac{1}{3}$ and $\sigma_\psi = 1.067$.

Table 1 shows the resulting values for ξ and σ_ψ for $d = -1.2$ and -1, and for two values of L (0.2 and 0.224) or equivalently two different values for σ_k^ψ (0.8 and 1.0). Note that for the k-$k\tau$ model, the length-scale equation is trivially satisfied for the stirring grid turbulence ($\partial k/\partial z = 0$ and $c_{\psi 2} = 0$, and hence both sides of the length-scale equation become zero) and hence adds no new information, and the last relation in Eq. (12) cannot be used. Since $\sigma_\psi \sim -(1+\xi)$, any reasonable value of ξ can be used as long as $\xi < -1$. If, for example, we use $\xi = -2$, $\sigma_\psi = 1.067$. For the k-ε model, for $\sigma_k^\psi = 0.8$ (equivalently

Table 2. The model constants for an arbitrary value of the parameter ξ.

Model	d	$c_{\psi 1}$	$c_{\psi 2}$	ξ	σ_ψ	σ_k^ψ	L
k-ε	-1.2	1.5	1.833	> -1	$1.6(1 + \xi)$	$0.888(1 + \xi)/(1 + 0.8\xi)$	$(0.05\sigma_k^\psi)^{1/2}$
k-ω	-1.2	0.5	0.833	> -1	$1.6(1 + \xi)$	$1.333(1 + \xi)/(1 + 0.67\xi)$	$(0.05\sigma_k^\psi)^{1/2}$
k-kl	-1.2	1.0	0.667	< -1	$-1.6(1 + \xi)$	$-3.2(1 + \xi)/(1 + 0.5\xi)$	$(0.05\sigma_k^\psi)^{1/2}$
k-$k\tau$	-1.2	0.5	0	< -1	$-1.6(1 + \xi)$	Arbitrary	$(0.05\sigma_k^\psi)^{1/2}$
k-l	-1.2	0	-0.333	< -1	$-1.6(1 + \xi)$	$3.2(1 + \xi)/\xi$	$(0.05\sigma_k^\psi)^{1/2}$
k-τ	-1.2	-0.5	-0.833	< -1	$-1.6(1 + \xi)$	$4.0(1 + \xi)/(1 + 2\xi)$	$(0.05\sigma_k^\psi)^{1/2}$
k-ε	-1	1.5	2	> -1	$1.067(1 + \xi)$	$0.640(1 + \xi)/(1 + 0.8\xi)$	$(0.05\sigma_k^\psi)^{1/2}$
k-ω	-1	0.5	1	> -1	$1.067(1 + \xi)$	$1.067(1 + \xi)/(1 + 0.67\xi)$	$(0.05\sigma_k^\psi)^{1/2}$
k-kl	-1	1.0	0.5	< -1	$-1.067(1 + \xi)$	$-1.6(1 + \xi)/(1 + 0.8\xi)$	$(0.05\sigma_k^\psi)^{1/2}$
k-$k\tau$	-1	0.5	0	< -1	$-1.067(1 + \xi)$	Arbitrary	$(0.05\sigma_k^\psi)^{1/2}$
k-l	-1	0	-0.5	< -1	$-1.067(1 + \xi)$	$3.2(1 + \xi)/\xi$	$(0.05\sigma_k^\psi)^{1/2}$
k-τ	-1	-0.5	-1	< -1	$-1.067(1 + \xi)$	$3.2(1 + \xi)/(1 + 2\xi)$	$(0.05\sigma_k^\psi)^{1/2}$

$L = 0.2$), the denominator in Eq. (13) vanishes. This arises simply from the fact that we insist that L be 0.2, when we do not of course know its value precisely.

However, since there is no reason to insist that L be precisely 0.2, and some variance around this value can be tolerated, we can eliminate L from the last two relations in Eq. (12) and obtain expressions for σ_ψ and σ_k^ψ:

$$\sigma_\psi = \frac{n\kappa^2(1 + \xi)}{(c_\mu^0)^2\left(\frac{1}{2} + \frac{1}{d}\right)}, \quad \sigma_k^\psi = \frac{1.5\alpha^2\sigma_\psi\left[m + n\left(\frac{1}{2} + \frac{1}{d}\right)\right]}{(m\alpha + n)\left[\left(m + \frac{1}{2}\right)\alpha + n + \xi(m\alpha + n)\right]} \tag{14}$$

where $\alpha = -2$. Note that these values are independent of R. Table 2 shows the dependence of σ_ψ and σ_k^ψ (and consequently L) on parameter ξ. Clearly, when n is positive, the value of ξ must be nonzero, whereas it is possible to have $\xi = 0$ for negative n values. In any case, the generic equation must have a modified diffusion term if it is to be applicable for all values of m and n. An added advantage of this new approach is that an equation can be written just for the macro-length scale of turbulence l ($m = 0$, $n = 1$), which has not been possible hitherto, since the traditional form of the diffusion term will not work for positive n values!

The UB generic equation in GOTM should therefore be replaced by Eq. (11), with the parameter ξ chosen appropriately (which will in turn determine the values of the Schmidt numbers—see Table 2), in which case, ψ can represent a quantity with physical meaning, and any length-scale equation used in the past can be simulated. Alternatively, Eq. (11) is equivalent to a conservation equation with the traditional form for the diffusion term, but for the quantity ψ^q, where $q = 1 + \xi$ must be chosen from Table 2, according to the value desired for σ_k^ψ. In other words, in the UB generic equation, m and n can be chosen such that $k^m = (k^{m'})^q$, $l^n = (l^{n'})^q$, where m' and n' correspond to values used in traditional

Table 3. UB model parameter values.

Model	p	m	n	q	$c_{\psi 1}$	$c_{\psi 2}$	σ_k^{ψ}	σ_{ψ}
k-ε	2	1	− 2/3	2/3	1	11/9	0.8	1.0667
k-ω	− 1/3	1/6	− 1/3	1/3	1/6	5/18	0.8	0.5333
k-kl	0	− 1/9	− 1/9	− 1/9	− 1/9	− 2/27	0.8	0.178
k-$k\tau$	3/2	− 1/4	− 1/2	− 1/2	− 1/4	− 1/12	0.8	0.8
k-l	0	0	− 1/3	− 1/3	0	1/9	0.8	0.5333
k-τ	0	1/6	− 1/3	− 1/3	1/6	5/18	0.8	0.5333

length-scale models (for example, for the k-ε model, $m' = \frac{3}{2}$, $n' = -1$, so that if we choose $\xi = -\frac{1}{3}$ so that $\sigma_k^{\psi} = 0.8$, then $m = 1$, $n = -\frac{2}{3}$ and $q = \frac{2}{3}$). In this manner, any traditional length-scale equation can be simulated. Table 3 shows the parameters that are needed in the UB generic equation to simulate the desired length-scale equation (for their chosen values of -1.2 for d, 0.8 for σ_k^{ψ} or equivalently 0.2 for L).

5. Concluding remarks

We have shown that returning of a closure constant in the traditional k-ε model can make it simulate the spatial decay of stirring grid turbulence. Not insisting that L be precisely 0.2 obviates both the need for the "polymorphic" approach and a general length-scale equation for a quantity without physical meaning. Also, the generic length-scale equation proposed by UB, which can now be used only for negative values of the exponent n can be made more universally applicable by an appropriate modification of the diffusion term. We suggest that the utility of GOTM will be greatly enhanced if the model parameters are chosen according to Table 3.

It would also be useful to have Langmuir turbulence included. Earlier work by Kantha and Clayson (2003) suggests that modified TKE and generic length-scale equations of the form

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x_k}(U_k k) = \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_k^{\psi}}\frac{\partial k}{\partial z}\right) + (P + P_L + G - \varepsilon) \tag{15}$$

$$\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x_k}(U_k \varphi) = \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varphi}{\partial z}\right) + \frac{\varphi}{k}(c_{\varphi 1}P + c_{\varphi 4}P_L + c_{\varphi 3}G - c_{\varphi 2}\varepsilon) \tag{16}$$

where the Langmuir turbulence production term P_L is given by

$$P_L = \nu_t\left(\frac{\partial U_1}{\partial z}\frac{\partial U_{S1}}{\partial z} + \frac{\partial U_2}{\partial z}\frac{\partial U_{S2}}{\partial z}\right), \tag{17}$$

U_{S1} and U_{S2} being the two components of Stokes drift velocity of the surface gravity wave, are needed. The quantity $\varphi = \psi^q = [(c_\mu^0)^p k^{m'} l^{n'}]^q = (c_\mu^0)^p k^m l^n$. The values of p, m, n and q are shown in Table 3. Note that $c_{\varphi 1} = q c_{\psi 1}$ . . . Eq. (16) is of course equivalent to

$$\frac{\partial\psi}{\partial t} + \frac{\partial}{\partial x_k}(U_k\psi) = \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial\psi}{\partial z}\right) + \frac{\nu_t}{\sigma_\varepsilon}\frac{q-1}{\psi}\left(\frac{\partial\psi}{\partial z}\right)^2 + \frac{\psi}{k}(c_{\psi 1}P + c_{\psi 4}P_L + c_{\psi 3}G - c_{\psi 2}\varepsilon) \tag{18}$$

where $c_{\psi 1} = m'$, $c_{\psi 2} = m' - n'/3$, $c_{\psi 4} = m' + 3n'$; the value of $c_{\psi 3}$ depends on the ratio of the stability functions for momentum and scalar diffusivities and hence is a function of specific closure (such as Mellor and Yamada, 1982 and Kantha and Clayson, 1994).

Acknowledgment. LK thanks ONR and Dr. Manuel Fiadeiro for the partial support of this work through ONR grant N00014-03-1-0488. The hospitality of Venice CNR-ISMAR during LK's visit and work on this paper is also acknowledged with pleasure.

REFERENCES

Burchard, H. and H. Baumert. 1995. On the performance of a mixed-layer model based on the k-ε turbulence closure. J. Geophys. Res., *100,* 8523–8540.

Kantha, L. H. 1988. On some aspects and applications of second moment closure. Atmospheric and Oceanic Sciences Program Report, Princeton University, Princeton, 167 pp.

—— 2003. The length scale equation in turbulence models. Nonlinear Proc. in Geophys. (in press).

Kantha, L. H. and C. A. Clayson. 1994. An improved mixed layer model for geophysical applications. J. Geophys. Res., *99,* 25,235–25,266.

—— 2003. On the effect of surface gravity waves on mixing in the oceanic mixed layer. Ocean Model, *6,* 101–124.

Mellor, G. L. and T. Yamada. 1982. Development of a turbulence closure model for geophysical fluid problems. Rev. Geophys. Space Phys., *20,* 851–875.

Umlauf, L. and H. Burchard. 2003. A generic length-scale equation for geophysical turbulence models. J. Mar. Res., *61,* 235–265.

Received: *16 July, 2003;* revised: *17 September, 2003.*