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On Integrals of Matrix Coefficients Associated to Spherical Models

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Abstract

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We define a local ingredient of the Ichino–Ikeda conjecture for isometry groups, for representations arising as local components of irreducible automorphic cuspidal representations lying in generic packets, without assuming temperedness everywhere. The representations in consideration are parabolically induced from characters and a tempered representation.

On Integrals Of Matrix Coefficients Associated To Spherical Models

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CHAPTER 1

Introduction

In the 1990s, Gross and Prasad stated a fascinating conjecture relating the vanishing of a period associated to automorphic representations of two special orthogonal groups to the value of the tensor product L-function of these representations at the point $s = \frac{1}{2}$ [13]. Later, in 2009, Ichino and Ikeda stated a beautiful refinement of this conjecture [17]. Their refinement expresses the square of the absolute value of the period as a product of an L-function and a product of certain local periods. Later, in his PhD thesis, Harris stated the analogous conjecture for unitary groups [16]. Since then much progress has been done on the Ichino–Ikeda conjecture for unitary groups, see [35, 5, 4].

In the statement of the Ichino–Ikeda conjectures mentioned above, one assumes that the representations involved are irreducible cuspidal automorphic representations that are tempered at all places. The temperedness assumption is crucial, as it allows one to define the local periods, which are a key ingredient for the statement of the conjecture. The Ramanujan conjecture speculates that irreducible cuspidal automorphic representations that lie in a generic packet are already tempered everywhere, see [24]. However, the Ramanujan conjecture does not seem at reach anywhere in the near future: not much progress has been made since [25].

In this work, we explain how to define the local periods for local components of representations lying in a generic packet, without assuming temperedness. We focus on principal series representations, as we are not able to solve this problem for other cases. We hope to come back to this problem in the future.

The starting point for our work is a result of Moeglin and Waldspurger that explains how one can define the local integral of matrix coefficients using "meromorphic continuation". We show that the quotient of this meromorphic continuation with the remaining factor of the local invariant is well defined in our domain of interest. This domain is guaranteed by the trivial bound for the Ramanujan conjecture for cuspidal automorphic representations.

1.1. The Ichino–Ikeda conjecture

Note: This is the only section where we consider the global setting. In other sections, we will always consider the local setting.

Let F be a number field, and let $E = F$ or E/F be an étale quadratic algebra. Let σ be a generator of $\mathrm{Aut}(E/F)$. Let $(\mathrm{V}_n, \langle\cdot,\cdot\rangle)$ be a non-degenerate σ-sesquilinear space of rank n over E, and let $\mathrm{V}_{n+1} = \mathrm{V}_n \oplus^{\perp} Ee$, where $\langle e, e\rangle \neq 0$. We consider the isometry groups $\mathrm{G}(\mathrm{V}_n)$ and $\mathrm{G}(\mathrm{V}_{n+1})$ of V_n and V_{n+1}, respectively.

Let π_n and π_{n+1} be irreducible cuspidal automorphic representations of $\mathrm{G}(\mathrm{V}_n)$ and $\mathrm{G}(\mathrm{V}_{n+1})$, respectively. We assume that both π_n and π_{n+1} lie in generic packets. The global Gan–Gross–Prasad conjecture [9] considers the period ($\varphi_n \in \pi_n$, $\varphi_{n+1} \in \pi_{n+1}$)

$$\mathcal{P}_{\mathrm{GGP}}(\varphi_n, \varphi_{n+1}) = \int_{\mathrm{G}(\mathrm{V}_n)(F)\backslash\, \mathrm{G}(\mathrm{V}_n)(\mathbb{A}_F)} \varphi_n(g_n)\varphi_{n+1}(g_n)\mathrm{d}g_n,$$

and relates it to the special L-function value $L(\frac{1}{2}, \mathrm{Std}(\pi_n) \times \mathrm{Std}(\pi_{n+1}))$. In particular, it speculates that if the special L-function value is zero, then the period $\mathcal{P}_{\mathrm{GGP}}$ is identically zero.

The Ichino–Ikeda conjecture can be seen as a refinement of this conjecture. It roughly states that the period can be written as a product of certain local periods. To explain the motivation for this conjecture we explain the phenomenon in a special case.

Suppose that $E = F \times F$, then $\mathrm{G}(\mathrm{V}_n) \cong \mathrm{GL}_n$ and $\mathrm{G}(\mathrm{V}_{n+1}) \cong \mathrm{GL}_{n+1}$. In this case, if π_n and π_{n+1} are irreducible cuspidal automorphic representations of GL_n and GL_{n+1} respectively, then π_n and π_{n+1} are unitarizable and generic, and it is known

[23, Theorem 1.4] that

$$\mathcal{P}_{\mathrm{GGP}}(\varphi_n, \varphi_{n+1}) = L(\tfrac{1}{2}, \pi_{n+1} \times \pi_n) \prod_v \frac{\Psi_v(s, W_{n,v}, W_{n+1,v})}{L(s, \pi_{n+1,v} \times \pi_{n,v})} \mid_{s=\frac{1}{2}} .$$

Here, we realize $\pi_{n,v}$ (respectively $\pi_{n+1,v}$) via its Whittaker model with respect to an additive character $\psi : F \to \mathbb{C}^\times$ (respectively, with respect to $\psi^{-1} : F \to \mathbb{C}^\times$). We also assume that $\varphi_n = \bigotimes_v W_{n,v}$ and $\varphi_{n+1} = \bigotimes_v W_{n+1,v}$ are decomposable, and $\Psi_v(s, W_{n,v}, W_{n+1,v})$ are the Rankin–Selberg integrals

$$\Psi_v(s, W_{n,v}, W_{n+1,v}) = \int_{N_n(F_v)\backslash \mathrm{GL}_n(F_v)} W_{n,v}(g_{n,v}) W_{n+1,v}(g_{n,v}) \left|\det g_{n,v}\right|_v^{s-\frac{1}{2}} \mathrm{d}g_{n,v}.$$

The integral $\Psi_v(s, W_{n,v}, W_{n+1,v})$ converges for $\operatorname{Re} s$ large, and admits a meromorphic continuation to the entire plane, which we continue to denote by the same symbol. The quotient $\frac{\Psi_v(s,W_{n,v},W_{n+1,v})}{L(s,\pi_{n+1,v}\times\pi_{n,v})}$ is an entire function, hence we can evaluate it at $s = \frac{1}{2}$. For almost all v, we have that $\pi_{n,v}$ and $\pi_{n+1,v}$ are unramified representations, and that $W_{n,v} = W^\circ_{n,v}$ and $W_{n+1,v} = W^\circ_{n+1,v}$ are the normalized spherical Whittaker functions. Therefore, for almost every v, the quotient $\frac{\Psi_v(s,W_{n,v},W_{n+1,v})}{L(s,\pi_{n+1,v}\times\pi_{n,v})}$ is 1, and hence the product over v is actually a finite product.

The Ichino–Ikeda conjecture suggests that a formula of the following form should hold ($\varphi_n \in \pi_n$, $\varphi_{n+1} \in \pi_{n+1}$, $\varphi_n^\vee \in \pi_n^\vee, \varphi_{n+1}^\vee \in \pi_{n+1}^\vee$):

$$\begin{aligned} &\mathcal{P}_{\mathrm{GGP}}(\varphi_n, \varphi_{n+1}) \mathcal{P}_{\mathrm{GGP}}(\varphi_n^\vee, \varphi_{n+1}^\vee) \\ \sim &L(\tfrac{1}{2}, \pi_{n+1}, \pi_n) \prod_v \mathcal{P}_{\pi_{n,v}, \pi_{n+1,v}}(\varphi_{n,v}, \varphi_{n+1,v}; \varphi_{n,v}^\vee, \varphi_{n+1,v}^\vee), \end{aligned}$$

where $\varphi_n = \bigotimes_v \varphi_{n,v}$, $\varphi_{n+1} = \bigotimes_v \varphi_{n+1,v}$, $\varphi_n^\vee = \bigotimes_v \varphi_{n,v}^\vee$, $\varphi_{n+1}^\vee = \bigotimes_v \varphi_{n+1,v}^\vee$ are decomposable, and $\sim$ means that both sides are identical up to a well understood invertible rational number (independent of $\varphi_n, \varphi_{n+1}, \varphi_n^\vee, \varphi_{n+1}^\vee$). In order for this conjecture to make sense, we need to define $L(\frac{1}{2}, \pi_{n+1}, \pi_n)$ and $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v}, \varphi_{n+1,v}; \varphi_{n,v}^\vee, \varphi_{n+1,v}^\vee)$. We will also have to add some further assumptions on π_n and π_{n+1}.

We move to explain the motivation for the definitions of $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}$ and $L(\frac{1}{2},\pi_{n+1},\pi_n)$ in the Ichino–Ikeda conjecture. We have that the assignment

$$(\varphi_n,\varphi_{n+1};\varphi_n^\vee,\varphi_{n+1}^\vee)\mapsto\mathcal{P}_{\mathrm{GGP}}(\varphi_n,\varphi_{n+1})\mathcal{P}_{\mathrm{GGP}}(\varphi_n^\vee,\varphi_{n+1}^\vee)$$

defines an element of

$$\mathrm{Hom}_{\mathrm{G}(\mathrm{V}_n)(\mathbb{A}_F)}(\pi_n\otimes\pi_{n+1},1)\boxtimes\mathrm{Hom}_{\mathrm{G}(\mathrm{V}_n)(\mathbb{A}_F)}(\pi_n^\vee\otimes\pi_{n+1}^\vee,1).$$

Hence, we should take $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}$ to be in the space

$$\mathrm{Hom}_{\mathrm{G}(\mathrm{V}_n)(F_v)}(\pi_{n,v}\otimes\pi_{n+1,v},1)\boxtimes\mathrm{Hom}_{\mathrm{G}(\mathrm{V}_n)(F_v)}(\pi_{n,v}^\vee\otimes\pi_{n+1,v}^\vee,1). \tag{1.1.1}$$

By [1], the latter space is of dimension at most 1. A distinguished element of the latter space is given, at least formally, by

$$\begin{aligned}&\alpha_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v},\varphi_{n+1,v};\varphi_{n,v}^\vee,\varphi_{n+1,v}^\vee)\\&=\int_{\mathrm{G}(\mathrm{V}_n)(F)}\langle\pi_{n,v}(g_{n,v})\varphi_{n,v},\varphi_{n,v}^\vee\rangle\langle\pi_{n+1,v}(g_{n,v})\varphi_{n+1,v},\varphi_{n+1,v}^\vee\rangle\mathrm{d}g_{n,v}.\end{aligned}$$

The integral defining $\alpha_{\pi_{n,v},\pi_{n+1,v}}$ absolutely converges if $\pi_{n,v}$ and $\pi_{n+1,v}$ are tempered. Under the assumption that $\pi_{n,v}$ and $\pi_{n+1,v}$ are tempered, it is shown by Waldspurger [32, Proposition 5.7] and by Sakellaridis–Venkatesh [30, Theorem 6.4.1] that the space (1.1.1) is spanned by $\alpha_{\pi_{n,v},\pi_{n+1,v}}$ (see also the work by Beuzart-Plessis [3, Theorem 7.2.1]). If $\pi_{n,v}$ and $\pi_{n+1,v}$ are tempered, and if v is an unramified place (with other certain assumptions) and all data is unramified, then

$$\alpha_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v}^\circ,\varphi_{n+1,v}^\circ;\varphi_{n,v}^{\circ\vee},\varphi_{n+1,v}^{\circ\vee})=\Delta_{n+1,v}\frac{L(\frac{1}{2},\mathrm{Std}(\pi_{n,v})\times\mathrm{Std}(\pi_{n+1,v}))}{L(1,\pi_{n,v},\mathrm{Ad})L(1,\pi_{n+1,v},\mathrm{Ad})},$$

where $\Delta_{n+1,v}$ is a product of values of L factors of characters depending on V_{n+1} and v. We denote for every v

$$L(s,\pi_{n+1,v},\pi_{n,v})=\Delta_{n+1,v}\frac{L(\frac{1}{2},\mathrm{Std}(\pi_{n,v})\times\mathrm{Std}(\pi_{n+1,v}))}{L(1,\pi_{n,v},\mathrm{Ad})L(1,\pi_{n+1,v},\mathrm{Ad})}.$$

We define a normalized version of $\alpha_{\pi_{n,v},\pi_{n+1,v}}$, so that it will evaluate to 1 for unramified data:

$$\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v},\varphi_{n+1,v};\varphi_{n,v}^{\vee},\varphi_{n+1,v}^{\vee})=\frac{\alpha_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v},\varphi_{n+1,v};\varphi_{n,v}^{\vee},\varphi_{n+1,v}^{\vee})}{L(\frac{1}{2},\pi_{n+1,v},\pi_{n,v})}.$$

This is done with analogy to the fact that the quotient $\frac{\Psi_v(s,W_{n,v},W_{n+1,v})}{L(s,\pi_{n+1,v}\times\pi_{n,v})}\,|_{s=\frac{1}{2}}$ above equals 1 for unramified data.

Assume from now on that $\varphi_n=\bigotimes_v\varphi_{n,v}$, $\varphi_{n+1}=\bigotimes_v\varphi_{n+1,v}$, $\varphi_n^{\vee}=\bigotimes_v\varphi_{n,v}^{\vee}$, $\varphi_{n+1}^{\vee}=\bigotimes_v\varphi_{n+1,v}^{\vee}$ are decomposable. In light of the result of Waldspurger and Sakellaridis–Venkatesh, we must have that if $\pi_{n,v}$ and $\pi_{n+1,v}$ are tempered everywhere, then

$$\mathcal{P}_{\mathrm{GGP}}(\varphi_n,\varphi_{n+1})\mathcal{P}_{\mathrm{GGP}}(\varphi_n^{\vee},\varphi_{n+1}^{\vee})=C_{\pi_n,\pi_{n+1}}\prod_v\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v},\varphi_{n+1,v};\varphi_{n,v}^{\vee},\varphi_{n+1,v}^{\vee}),$$

where $C_{\pi_n,\pi_{n+1}}$ is a constant. The Ichino–Ikeda conjecture describes the constant $C_{\pi_n,\pi_{n+1}}$.

The current Ichino–Ikeda conjecture [17, 16, 33] asserts that if $\pi_{n,v}$ and $\pi_{n+1,v}$ are tempered for every v, then, with respect to the Tamagawa measure, we have that

$$\begin{aligned}&\mathcal{P}_{\mathrm{GGP}}(\varphi_n,\varphi_{n+1})\mathcal{P}_{\mathrm{GGP}}(\varphi_n^{\vee},\varphi_{n+1}^{\vee})\\=&\frac{1}{2^{\beta}}L(\tfrac{1}{2},\pi_{n+1},\pi_n)\prod_v\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v},\varphi_{n+1,v};\varphi_{n,v}^{\vee},\varphi_{n+1,v}^{\vee}),\end{aligned}$$

where $\beta\geq 0$ is some integer, and where

$$L(s,\pi_{n+1,v},\pi_{n,v})=\Delta_{n+1,v}\frac{L(s,\mathrm{Std}(\pi_{n,v})\times\mathrm{Std}(\pi_{n+1,v}))}{L(s+\frac{1}{2},\pi_{n,v},\mathrm{Ad})L(s+\frac{1}{2},\pi_{n+1,v},\mathrm{Ad})}$$

and $L(\frac{1}{2},\pi_{n+1},\pi_n)$ is given by considering the meromorphic continuation of the Euler product

$$L(s,\pi_{n+1},\pi_n)=\prod_v L(s,\pi_{n+1,v},\pi_{n,v}).$$

We note that the current conjecture can be also stated without having to define $L(s,\pi_{n+1,v},\pi_{n,v})$ at every place: let S be a finite set of places such that for $v\notin S$, we have that v is unramified and that $\varphi_{n,v}, \varphi_{n+1,v}, \varphi_{n,v}^\vee, \varphi_{n+1,v}^\vee$ are spherical vectors as above. Then the Ichino–Ikeda conjecture can be formulated as

$$\begin{aligned}&\mathcal{P}_{\mathrm{GGP}}(\varphi_n, \varphi_{n+1})\mathcal{P}_{\mathrm{GGP}}(\varphi_n^\vee, \varphi_{n+1}^\vee)\\ =&\frac{1}{2^\beta} L^S(\tfrac{1}{2}, \pi_{n+1}, \pi_n) \prod_{v\in S} \alpha_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v}, \varphi_{n+1,v}; \varphi_{n,v}^\vee, \varphi_{n+1,v}^\vee),\end{aligned}$$

where $L^S(s, \pi_{n+1}, \pi_n)$ is the partial Euler product

$$L^S(s, \pi_{n+1}, \pi_n) = \prod_{v\notin S} L(s, \pi_{n+1,v}, \pi_{n,v}).$$

The assumption that $\pi_{n,v}$ and $\pi_{n+1,v}$ are tempered for every v is crucial for the statement of the Ichino–Ikeda conjecture, since otherwise the local periods $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}$ are not defined. The generalized Ramanujan conjecture speculates that if π_n (respectively π_{n+1}) lies in a generic packet, then $\pi_{n,v}$ (respectively $\pi_{n+1,v}$) is already tempered for every v. However, this conjecture is far from being known.

We remark that in [26, Lemme 1.7], Moeglin and Waldspurger provided a meromorphic continuation for $\alpha_{\pi_{n,v},\pi_{n+1,v}}$ that is holomorphic under the assumption that the exponents $(\sigma_{\pi_n}(v,i))_i$ and $(\sigma_{\pi_{n+1}}(v,i))_i$ of π_n and π_{n+1}, respectively, satisfy the inequalities

$$\max_i |\sigma_{\pi_n}(v,i)| < \frac{1}{2}, \quad \max_j \left|\sigma_{\pi_{n+1}}(v,j)\right| < \frac{1}{2}, \tag{1.1.2}$$

$$\max_{i,j} \left|\sigma_{\pi_n}(v,i) \pm \sigma_{\pi_{n+1}}(v,j)\right| < \frac{1}{2}. \tag{1.1.3}$$

While the inequalities in (1.1.2) are known to be true (the trivial bound of Jacquet–Shalika [18, Corollary 2.5]), the inequality in (1.1.3) is not known to be true in general. However, it holds for $n = 2$, see [6, Section 5.2].

We would like to bypass the Ramanujan conjecture and state an Ichino–Ikeda conjecture, given that π_n and π_{n+1} lie in generic packets (equivalently, given that their base change is an isobaric sum of self-dual cuspidal representations of the correct sign). In order to do that, we need to define $L(s, \pi_{n+1}, \pi_n)$ and $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}$ for every

v. The definition of $L(s, \pi_{n+1}, \pi_n)$ is available thanks to the existence of weak base change [2, 27, 19, 7]. In this work, we provide a definition of $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}$ for places v where $\pi_{n,v}$ and $\pi_{n+1,v}$ are given by principal series representations. Our work shows that for places v, where $\pi_{n,v}$ and $\pi_{n+1,v}$ are principal series, a holomorphic extension is possible for $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}$, i.e., for the normalized version of $\alpha_{\pi_{n,v},\pi_{n+1,v}}$, under the assumption that the inequalities in (1.1.2) hold.

1.2. The main result

We state a version of our main result. Let F be a p-adic field and let $E = F$ or E/F be a quadratic field extension. Let σ be a generator of $\mathrm{Aut}(E/F)$.

For every non-negative integer k, we will consider a σ-sesquilinear space V_k of rank k, such that its isometry group $\mathrm{G}(\mathrm{V}_k)$ is quasi-split. For a non-negative even integer $2m$, we set V_{2m} to be a σ-sesquilinear space of rank $2m$ over E, spanned by the orthogonal basis $b_m, \dots, b_1, b_{-1}, \dots, b_{-m}$, where for every $1 \leq i \leq m$,

$$\langle b_i, b_i \rangle = 1,$$

$$\langle b_{-i}, b_{-i} \rangle = -1.$$

We view V_{2m} as a subspace of V_{2m+2} using the obvious identification. For an odd integer $2m+1$, we define V_{2m+1} as the following subspace of V_{2m+2}:

$$\mathrm{V}_{2m+1} = E(b_{m+1}) \oplus \mathrm{V}_{2m}.$$

Denote by $\mathrm{G}(\mathrm{V}_k)$ the isometry group of V_k. For $1 \leq i \leq m$, let $f_i = b_i + b_{-i} \in \mathrm{V}_{2m}$. Then the space

$$\mathrm{span}_E(f_1, \dots, f_m)$$

is a maximal totally isotropic subspace of V_{2m} and of V_{2m+1}.

Consider the following flag:

$$\mathcal{F}_m : Ef_m \subset Ef_m \oplus Ef_{m-1} \subset \cdots \subset Ef_m \oplus \cdots \oplus Ef_1.$$

Denote by $P_{2m} = P_{\mathcal{F}_m,\mathrm{G}(\mathrm{V}_{2m})}$ (respectively, $P_{2m+1} = P_{\mathcal{F}_m,\mathrm{G}(\mathrm{V}_{2m+1})}$) the parabolic subgroup of $\mathrm{G}(\mathrm{V}_{2m})$ (respectively, of $\mathrm{G}(\mathrm{V}_{2m+1})$) stabilizing the flag $\mathcal{F}_m$. Then P_{2m} has Levi part isomorphic to $(\mathrm{Res}_{E/F}\, E^\times)^m$, and P_{2m+1} has Levi part isomorphic to $(\mathrm{Res}_{E/F}\, E^\times)^m \times \mathrm{G}(\mathrm{V}_1)$.

Let $\underline{a} = (a_1, \ldots, a_m)$ be a tuple of complex numbers. Let $\omega_1, \ldots, \omega_m : E^\times \to \mathbb{C}^\times$ be unitary characters. We define the principal series representation of $\mathrm{G}(\mathrm{V}_{2m})$ associated to the characters $\omega_1, \ldots, \omega_m$ with the parameter $\underline{a}$ to be the following (normalized) parabolically induced representation:

$$\pi_{2m}^{\underline{a}} = \Pi^{\underline{a}}(\omega_1, \ldots, \omega_m) = \mathrm{I}_{P_{2m}}^{\mathrm{G}(\mathrm{V}_{2m})}(|\cdot|^{a_m}\, \omega_m \boxtimes \cdots \boxtimes |\cdot|^{a_1}\, \omega_1).$$

Let $\underline{b} = (b_1, \ldots, b_m)$ be a tuple of complex numbers. Let $\mu_1, \ldots, \mu_m : E^\times \to \mathbb{C}^\times$ be unitary characters. Let $\pi_1 : \mathrm{G}(\mathrm{V}_1) \to \mathbb{C}^\times$ be a character of $\mathrm{G}(\mathrm{V}_1)$. We define the principal series representation of $\mathrm{G}(\mathrm{V}_{2m+1})$ associated to the characters $\pi_1, \mu_1, \ldots, \mu_m$ with the parameter $\underline{b}$ to be the following (normalized) parabolically induced representation:

$$\pi_{2m+1}^{\underline{b}} = \Pi^{\underline{b}}(\pi_1, \mu_1, \ldots, \mu_m) = \mathrm{I}_{P_{2m+1}}^{\mathrm{G}(\mathrm{V}_{2m+1})}(|\cdot|^{b_m}\, \mu_m \boxtimes \cdots \boxtimes |\cdot|^{b_1}\, \mu_1 \boxtimes \pi_1).$$

Given a non-negative integer n and admissible representations π_n and π_{n+1} of $\mathrm{G}(\mathrm{V}_n)$ and $\mathrm{G}(\mathrm{V}_{n+1})$, respectively, we define

$$\alpha_{\pi_n,\pi_{n+1}}(f_n, f_{n+1}; f_n^\vee, f_{n+1}^\vee) = \int_{\mathrm{G}(\mathrm{V}_n)} \langle \pi_n(g_n) f_n, f_n^\vee \rangle \left\langle \pi_{n+1}(g_n) f_{n+1}, f_{n+1}^\vee \right\rangle \mathrm{d}g_n,$$

where $f_n \in \pi_n$, $f_{n+1} \in \pi_{n+1}$, $f_n^\vee \in \pi_n^\vee$, $f_{n+1}^\vee \in \pi_{n+1}^\vee$. Then $\alpha_{\pi_n,\pi_{n+1}}$ absolutely converges when π_n and π_{n+1} are tempered representations.

We are now ready to state our main result.

Theorem 1.2.1. *Let n be a non-negative integer. Let $\pi_n^{\underline{a}}$ and $\pi_{n+1}^{\underline{b}}$ be principal series representations of $\mathrm{G}(\mathrm{V}_n)$ and $\mathrm{G}(\mathrm{V}_{n+1})$, with parameters $\underline{a}$ and $\underline{b}$, respectively. Then for every holomorphic sections $f_n^{\underline{a}} \in \pi_n^{\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1}^{\underline{b}}$, $f_n^{\vee\underline{a}} \in \pi_n^{\vee\underline{a}}$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1}^{\vee\underline{b}}$,* *the assignment*

$$(\underline{a}, \underline{b}) \mapsto \frac{\alpha_{\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})}{L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}})},$$

originally defined only for $\underline{a}$ and $\underline{b}$ imaginary, has a holomorphic continutation to the entire plane. Moreover, the holomorphic continuation is a polynomial, i.e., an element of $\mathbb{C}[q^{\pm\underline{a}}, q^{\pm\underline{b}}]$.

Here, $L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}})$ is an L-factor defined in Section 3.6 using the doubling method, which should be thought of as the value $L(\frac{1}{2}, \mathrm{Std}(\pi_{n+1}^{\underline{b}}) \times \mathrm{Std}(\pi_n^{\underline{a}}))$.

CHAPTER 2

Statement of the main result

2.1. Isometry groups notations

Let F be a p-adic field. Let q be the cardinality of the residue field of F. Let $E = F$ or E/F be an étale quadratic algebra, that is, E/F is a quadratic field extension or $E = F \times F$. Let σ be a generator of $\mathrm{Aut}(E/F)$ (if $E = F$, then σ is the identity map).

Let $E^\times$ be the multiplicative group of E, i.e., the group consisting of all invertible elements of E. If E is a field then $E^\times = E \setminus \{0\}$. Otherwise, if $E = F \times F$, then $E^\times = F^\times \times F^\times$, where $F^\times = F \setminus \{0\}$. In both cases, each character of $E^\times$ can be represented as a product of a unitary character and an unramified character. We will often say "Let (s, χ) be a parameter for a character of $E^\times$", "s is imaginary", "$\mathrm{Re}\, s$ is large". By this we mean:

- If E is a field, then $s \in \mathbb{C}$, $\chi : E^\times \to \mathbb{C}^\times$ is a unitary character, and we denote by $|\cdot|^s \chi$ the character $E^\times \to \mathbb{C}^\times$ given by $x \mapsto |x|_E^s \cdot \chi(x)$. We say that s *is imaginary* if $s \in \sqrt{-1} \cdot \mathbb{R}$. We say that $\mathrm{Re}\, s$ *is large* whenever the real part of s is large.
- If $E = F \times F$, then $s = (s_1, s_2) \in \mathbb{C}^2$, $\chi = (\chi_1, \chi_2)$, where $\chi_1, \chi_2 : F^\times \to \mathbb{C}^\times$ are unitary characters, and we denote by $|\cdot|^s \chi$ the character $F^\times \times F^\times \to \mathbb{C}^\times$ given by $(x_1, x_2) \mapsto |x_1|_F^{s_1} |x_2|_F^{s_2} \cdot \chi_1(x_1)\chi_2(x_2)$. We say that s *is imaginary* if $s \in \sqrt{-1} \cdot \mathbb{R}^2$. We say that $\mathrm{Re}\, s$ *is large* whenever both the real parts of s_1 and s_2 are large. If $s = (s_1, s_2) \in \mathbb{C}^2$ and $t = (t_1, t_2) \in \mathbb{C}^2$, we write $\mathrm{Re}\, s > \mathrm{Re}\, t$ if $\mathrm{Re}\, s_1 > \mathrm{Re}\, t_1$ and $\mathrm{Re}\, s_2 > \mathrm{Re}\, t_2$.

By a parameter s of an unramified character of $E^\times$, we mean a parameter (s, χ) for a character of $E^\times$, where χ is the trivial character. We denote in this case $|\cdot|^s = |\cdot|^s \chi$. If $E = F \times F$, we mean $\chi = (1, 1)$, where $1 : F^\times \to \mathbb{C}^\times$ is the trivial character.

Let $\underline{a} = (a_i)_{i=1}^r$ be a tuple of parameters for unramified characters of $E^\times$.

- If E is a field, we denote $\mathbb{C}[q^{\pm\underline{a}}] = \mathbb{C}[q^{a_i}, q^{-a_i}]_{i=1,\dots,r}$, $\operatorname{Re}\underline{a} = (\operatorname{Re} a_i)_{i=1}^r$, and $\|\operatorname{Re}\underline{a}\| = \max_{i=1,\dots,r} |\operatorname{Re} a_i|$.
- If $E = F \times F$, write $a_i = (a_{i1}, a_{i2}) \in \mathbb{C}^2$. We denote $\mathbb{C}[q^{\pm\underline{a}}] = \mathbb{C}[q^{a_i}, q^{-a_i}]_{i=1,\dots,r} = \mathbb{C}[q^{a_{ij}}, q^{-a_{ij}}]_{\substack{i=1,\dots,r,\\ j=1,2}}$, $\operatorname{Re}\underline{a} = (\operatorname{Re} a_{ij})_{\substack{i=1,\dots,r,\\ j=1,2}}$, and $\|\operatorname{Re}\underline{a}\| = \max_{\substack{i=1,\dots,r\\ j=1,2}} |\operatorname{Re} a_{ij}|$.

In both cases we denote by $\mathbb{C}(q^{\pm\underline{a}})$ the ring of fractions of $\mathbb{C}[q^{\pm\underline{a}}]$, which we refer to as *the ring of rational functions in* $q^{-\underline{a}}$.

Suppose that $(\mathbf{V}, \langle\cdot,\cdot\rangle)$ is a non-degenerate σ-sesquilinear space of finite rank over E, with respect to the involution σ. By this we mean that the form

$$\langle\cdot,\cdot\rangle : \mathbf{V} \times \mathbf{V} \to E$$

satisfies for $\alpha_1, \alpha_2 \in E$ and for $v_1, v_2, w \in \mathbf{V}$

$$\langle \alpha_1 v_1 + \alpha_2 v_2, w\rangle = \alpha_1 \langle v_1, w\rangle + \alpha_2 \langle v_2, w\rangle,$$

$$\langle v_1, v_2\rangle = \sigma(\langle v_2, v_1\rangle),$$

and that the form $\langle\cdot,\cdot\rangle$ is non-degenerate, i.e., for any $0 \neq v \in \mathbf{V}$, there exists $w \in \mathbf{V}$, such that $\langle v, w\rangle \neq 0$.

Denote by $\mathrm{G}(\mathbf{V})$ the isometry group of $\mathbf{V}$, that is, the subgroup of $\operatorname{Res}_{E/F} \mathrm{GL}_E(\mathbf{V})$ consisting of elements $g \in \operatorname{Res}_{E/F} \mathrm{GL}_E(\mathbf{V})$, such that $\langle gv, gw\rangle = \langle v, w\rangle$, for every $v, w \in \mathbf{V}$.

A subspace $\mathbf{X} \subset \mathbf{V}$ is called *totally isotropic* if for every $x_1, x_2 \in \mathbf{X}$ we have $\langle x_1, x_2\rangle = 0$. If $\mathbf{X}, \mathbf{Y} \subset \mathbf{V}$ are both totally isotropic subspaces, then we say that $\mathbf{X}$ and $\mathbf{Y}$ are dual if the map $\mathbf{X} \times \mathbf{Y} \to E$, $(x, y) \mapsto \langle x, y\rangle$ is non-degenerate.

If $E = F \times F$, then $\mathbf{V}$ can be written as $\mathbf{V} = \mathbf{X}_{\mathbf{V}} \times \mathbf{X}_{\mathbf{V}}^{\vee}$, where $\mathbf{X}_{\mathbf{V}}$ is a vector space over F, and $\mathbf{X}_{\mathbf{V}}^{\vee}$ is its dual, and the form is given by $\langle (x, x^{\vee}), (y, y^{\vee}) \rangle = (\langle x, y^{\vee} \rangle, \langle y, x^{\vee} \rangle)$. In this case, the isometry group $\mathrm{G}(\mathbf{V})$ is isomorphic to $\mathrm{GL}_F(\mathbf{X}_{\mathbf{V}})$, via the isomorphism sending $g \in \mathrm{GL}_F(\mathbf{X}_{\mathbf{V}})$ to the map $\mathbf{X}_{\mathbf{V}} \times \mathbf{X}_{\mathbf{V}}^{\vee} \to \mathbf{X}_{\mathbf{V}} \times \mathbf{X}_{\mathbf{V}}^{\vee}$, defined by $(x, x^{\vee}) \mapsto (gx, x^{\vee} \circ g^{-1})$.

2.2. Representations of isometry groups

Let $(\mathbf{V}, \langle \cdot, \cdot \rangle)$ be a non-degenerate σ-sesquilinear space. Suppose that we have a decomposition

$$\mathbf{V} = \mathbf{X}_r \oplus \cdots \oplus \mathbf{X}_1 \oplus \mathbf{W} \oplus \mathbf{Y}_1 \oplus \cdots \oplus \mathbf{Y}_r, \tag{2.2.1}$$

where:

(1) $\mathbf{W} \subset \mathbf{V}$ is a non-degenerate subspace.
(2) The spaces $\mathbf{X}_1, \dots, \mathbf{X}_r, \mathbf{Y}_1, \dots, \mathbf{Y}_r$ are totally isotropic and orthogonal to $\mathbf{W}$.
(3) For every $i \neq j$, $\mathbf{X}_i$ (respectively $\mathbf{Y}_i$) is orthogonal to $\mathbf{X}_j$ and $\mathbf{Y}_j$.
(4) For every i, $\mathbf{X}_i$ and $\mathbf{Y}_i$ are dual.

Consider the following flag:

$$\mathcal{F} : \mathbf{X}_r \subset \mathbf{X}_r \oplus \mathbf{X}_{r-1} \subset \cdots \subset \mathbf{X}_r \oplus \cdots \oplus \mathbf{X}_1.$$

Let $P_{\mathcal{F}, \mathrm{G}(\mathbf{V})} \subset \mathrm{G}(\mathbf{V})$ be the parabolic subgroup stabilizing this flag. It has Levi part isomorphic to

$$\mathrm{Res}_{E/F} \mathrm{GL}_E(\mathbf{X}_r) \times \cdots \times \mathrm{Res}_{E/F} \mathrm{GL}_E(\mathbf{X}_1) \times \mathrm{G}(\mathbf{W}).$$

Let $\tau_1, \dots, \tau_r$ be irreducible admissible representations of the F-points of $\mathrm{Res}_{E/F} \mathrm{GL}_E(\mathbf{X}_1)$, $\dots$, $\mathrm{Res}_{E/F} \mathrm{GL}_E(\mathbf{X}_r)$ respectively, and let $\pi_{\mathbf{W}}$ be an irreducible admissible representation of $\mathrm{G}(\mathbf{W})$. Let $\underline{a} = (a_i)_{i=1}^r$ be a tuple of parameters for unramified characters of $E^\times$. We consider the (normalized) parabolic induction

$$\Pi^{\underline{a}} = \Pi^{\underline{a}}(\pi_{\mathbf{W}}, \tau_1, \dots, \tau_r) = \mathrm{I}_{P_{\mathcal{F},\mathrm{G}(\mathbf{V})}}^{\mathrm{G}(\mathbf{V})}(|\det|^{a_r}\,\tau_r \boxtimes \cdots \boxtimes |\det|^{a_1}\,\tau_1 \boxtimes \pi_{\mathbf{W}}). \tag{2.2.2}$$

We have the following classification of irreducible representations of $\mathrm{G}(\mathbf{V})$:

Theorem 2.2.1. [11, Section 8.4]

(1) *Suppose that* $\pi_{\mathbf{W}}, \tau_1, \dots, \tau_r$ *are tempered and* $\operatorname{Re} a_r > \cdots > \operatorname{Re} a_1 > 0$. *Then* $\Pi^{\underline{a}}$ *has a unique irreducible quotient, which we denote by* $J(\Pi^{\underline{a}})$. $J(\Pi^{\underline{a}})$ *is called the Langlands quotient of* $\Pi^{\underline{a}}$.

(2) *Conversely, every irreducible representation of* $\mathrm{G}(\mathbf{V})$ *is isomorphic to a Langlands quotient* $J(\Pi^{\underline{a}}(\pi_{\mathbf{W}}, \tau_1, \dots, \tau_r))$, *for some decomposition as in (2.2.1), some irreducible tempered representations* $\pi_{\mathbf{W}}, \tau_1, \dots, \tau_r$, *and some* $a_1, \dots, a_r$ *with* $\operatorname{Re} a_r > \cdots > \operatorname{Re} a_1 > 0$.

We move to define special sections of the representation $\Pi^{\underline{a}}$.

Let $\mathcal{K} \subset \mathrm{G}(\mathbf{V})$ be a maximal compact subgroup in good position with respect to $P_{\mathcal{F},\mathrm{G}(\mathbf{V})}$. We have the Iwasawa decomposition $\mathrm{G}(\mathbf{V}) = P_{\mathcal{F},\mathrm{G}(\mathbf{V})}\mathcal{K}$. We say that a section $f^{\underline{a}} \in \Pi^{\underline{a}}$ is standard with respect to $\mathcal{K}$ if its restriction to the subgroup $\mathcal{K}$ is independent of $\underline{a}$, i.e., the value $f^{\underline{a}}(k)$ does not depend on $\underline{a}$ for any $k \in \mathcal{K}$. We say that a section $f^{\underline{a}} \in \Pi^{\underline{a}}$ is holomorphic (respectively meromorphic) if for any $g \in \mathrm{G}(\mathbf{V})$, there exist polynomials $(p_{g,i}(q^{-\underline{a}}))_{i=1}^{N_g} \subset \mathbb{C}[q^{\pm\underline{a}}]$ (respectively rational functions $(p_{g,i}(q^{-\underline{a}}))_{i=1}^{N_g} \subset \mathbb{C}(q^{\pm\underline{a}})$) and vectors $(v_{g,i})_{i=1}^{N_g} \subset \tau_r \boxtimes \cdots \boxtimes \tau_1 \boxtimes \pi_{\mathbf{W}}$, such that $f^{\underline{a}}(g) = \sum_{i=1}^{N_g} p_{g,i}(q^{-\underline{a}})v_{g,i}$. The subspace of holomorphic (respectively meromorphic) sections is invariant under the action of $\mathrm{G}(\mathbf{V})$.

2.3. Conjectural standard transfer

Let $(\mathbf{V}, \langle\cdot,\cdot\rangle)$ be a non-degenerate σ-sesquilinear space of finite rank over E, and let π be an irreducible admissible representation of $\mathrm{G}(\mathbf{V})$. Then, conjecturally, there exists an irreducible admissible representation $\mathrm{Std}(\pi)$ of $\mathrm{GL}_N(E)$ for a suitable N

(depending only on the rank of $\mathbf{V}$ and on the type of E) corresponding to the standard transfer (known as base change in the case of unitary groups) of the Langlands parameter of π.

When π is unramified, $\mathrm{Std}(\pi)$ is an unramified representation defined using the Satake parameter of π. In the general case, we expect $\mathrm{Std}(\pi)$ to have the following properties:

(1) If π is tempered, then $\mathrm{Std}(\pi)$ is tempered.

(2) If E/F is a quadratic extension and $\mathbf{V}$ is one-dimensional, then

$$\mathrm{G}(\mathbf{V}) \cong E^1 = \left\{x \in E^\times \mid x\sigma(x) = 1\right\}.$$

Then π is a character $\chi_{\mathbf{V}} : E^1 \to \mathbb{C}^\times$, and its standard transfer is given by the character $\mathrm{Std}(\chi_{\mathbf{V}}) = \chi_{\mathbf{V},E} : E^\times \to \mathbb{C}^\times$, defined by the formula

$$\chi_{\mathbf{V},E}(x) = \chi_{\mathbf{V}}\left(\frac{x}{\sigma(x)}\right).$$

(3) If $E = F \times F$, then $\mathrm{G}(\mathbf{V})$ is isomorphic to $\mathrm{GL}_N(F)$, where N is the rank of $\mathbf{V}$ over E. Then π is an irreducible representation of $\mathrm{GL}_N(F)$, and $\mathrm{Std}(\pi) = \pi \boxtimes \pi^\vee$ is an irreducible admissible representation of $\mathrm{GL}_N(E) = \mathrm{GL}_N(F \times F) = \mathrm{GL}_N(F) \times \mathrm{GL}_N(F)$.

(4) Suppose that $\mathbf{V}$ has a decomposition as in (2.2.1), and let $\pi = J(\Pi^{\underline{a}})$ as in Theorem 2.2.1. Then $\mathrm{Std}(\pi)$ is the Langlands quotient of the representation

$$|\det|^{a_r}\,\tau_r \times \cdots \times |\det|^{a_1}\,\tau_1 \times \mathrm{Std}(\pi_{\mathbf{W}}) \times |\det|^{-a_1}\,\tau_1^{\vee,\sigma} \times \cdots \times |\det|^{-a_r}\,\tau_r^{\vee,\sigma},$$

where $\times$ denotes parabolic induction, and $\tau_i^{\vee,\sigma} = \tau_i^\vee \circ \sigma$, where $\tau_i^\vee$ is the contragredient representation of τ_i.

For our purposes, we extend the notion of $\mathrm{Std}(\Pi^{\underline{a}})$ for $\Pi^{\underline{a}}$ as in (2.2.2) with $\|\mathrm{Re}\,\underline{a}\| < \frac{1}{2}$, even if $\Pi^{\underline{a}}$ is not irreducible. We define a naive standard transfer $\mathrm{Std}(\Pi^{\underline{a}})$ by the formula

$$|\det|^{a_r}\,\tau_r \times \cdots \times |\det|^{a_1}\,\tau_1 \times \mathrm{Std}(\pi_{\mathbf{W}}) \times |\det|^{-a_1}\,\tau_1^{\vee,\sigma} \times \cdots \times |\det|^{-a_r}\,\tau_r^{\vee,\sigma},$$

where again $\|\mathrm{Re}\,\underline{a}\| < \frac{1}{2}$. We have that $\mathrm{Std}(\Pi^{\underline{a}})$ is irreducible.

2.4. Integrals of matrix coefficients

Let $(\mathrm{V}_{n+1}, \langle\cdot,\cdot\rangle)$ be a non-degenerate σ-sesquilinear space, and suppose that $\mathrm{V}_{n+1} = \mathrm{V}_n \oplus Ee$, where $\mathrm{V}_n \subset \mathrm{V}_{n+1}$ is a non-degenerate subspace, and $e \in \mathrm{V}_{n+1}$ is orthogonal to V_n. Let π_n and π_{n+1} be irreducible admissible representations of $\mathrm{G}(\mathrm{V}_n)$ and $\mathrm{G}(\mathrm{V}_{n+1})$, respectively. The local integral considered in the statement of the Ichino–Ikeda conjecture [17, 16, 33] is defined by the formula

$$\alpha_{\pi_n,\pi_{n+1}}(v_n, v_{n+1}; v_n^{\vee}, v_{n+1}^{\vee}) = \int_{\mathrm{G}(\mathrm{V}_n)} \langle \pi_n(g_n)v_n, v_n^{\vee}\rangle \langle \pi_{n+1}(g_n)v_{n+1}, v_{n+1}^{\vee}\rangle \mathrm{d}g_n,$$

where $v_n \in \pi_n$, $v_{n+1} \in \pi_{n+1}$, $v_n^{\vee} \in \pi_n^{\vee}$, $v_{n+1}^{\vee} \in \pi_{n+1}^{\vee}$. This integral absolutely converges whenever the representations π_n and π_{n+1} are tempered [17, Proposition 1.1], [16, Proposition 2.1].

We have the following unramified computation. If E/F is a quadratic field extension, suppose that E/F is unramified. Suppose that π_n and π_{n+1} are unramified representations, then for the data $v_n^{\circ} \in \pi_n$, $v_{n+1}^{\circ} \in \pi_{n+1}$, $v_n^{\vee\circ} \in \pi_n^{\vee}$, $v_{n+1}^{\vee\circ} \in \pi_{n+1}^{\vee}$, where all vectors are spherical, and $\langle v_n^{\circ}, v_n^{\vee\circ}\rangle = \langle v_{n+1}^{\circ}, v_{n+1}^{\vee\circ}\rangle = 1$, we have [16, Section 2.2.3]:

$$\alpha_{\pi_n,\pi_{n+1}}(v_n^{\circ}, v_{n+1}^{\circ}; v_n^{\vee\circ}, v_{n+1}^{\vee\circ}) = \Delta_{n+1} \cdot \frac{L(\frac{1}{2}, \mathrm{Std}(\pi_n) \times \mathrm{Std}(\pi_{n+1}))}{L(1, \pi_n, \mathrm{Ad}) \cdot L(1, \pi_{n+1}, \mathrm{Ad})},$$

where:

- If $E = F$ and $n = 2m$, then $\Delta_{n+1} = \prod_{i=1}^{m} L(2i, 1)$.
- If $E = F$ and $n = 2m-1$, let $F_{\mathrm{V}_{n+1}}$ be the discriminant field of V_{n+1}, and let $\chi_{\mathrm{V}_{n+1}} : F^{\times} \to \mathbb{C}^{\times}$ be the character associated to $F_{\mathrm{V}_{n+1}}/F$ by local class field theory. Then $\Delta_{n+1} = \prod_{i=1}^{m-1} L(2i, 1) \cdot L(m, \chi_{\mathrm{V}_{n+1}})$.

- If E/F is a quadratic field extension, let $\chi_{E/F}$ be the quadratic character associated to the field extension E/F by local class field theory. Then $\Delta_{n+1} = \prod_{j=1}^{n+1} L(j, \chi_{E/F}^j)$.
- If $E = F \times F$, then $\Delta_{n+1} = \prod_{j=1}^{n+1} L(j, 1)$.

2.5. Statement of the problem

The Ichino–Ikeda conjecture considers a normalized version of α, the integral of matrix coefficients from 2.4, normalized so that the unramified computation gives the value 1, i.e., it considers the functional $\mathcal{P}_{\pi_n,\pi_{n+1}}$

$$\mathcal{P}_{\pi_n,\pi_{n+1}}(v_n, v_{n+1}; v_n^\vee, v_{n+1}^\vee) = \Delta_{n+1}^{-1} \cdot \frac{L(1, \pi_n, \mathrm{Ad}) \cdot L(1, \pi_{n+1}, \mathrm{Ad})}{L(\frac{1}{2}, \mathrm{Std}(\pi_n) \times \mathrm{Std}(\pi_{n+1}))} \times \alpha_{\pi_n,\pi_{n+1}}(v_n, v_{n+1}; v_n^\vee, v_{n+1}^\vee).$$

Our goal is to understand how to make sense of $\mathcal{P}_{\pi_n,\pi_{n+1}}$ for non-tempered representations.

Our starting point is a result of Moeglin and Waldspurger [26, Lemme 1.7]. Their result is only stated for representations of special orthogonal groups, but the proof also works for orthogonal and unitary groups [14, Lemma 4.1.11]. In order to state it, we first set up our representations as in Section 2.2.

For $\mathbf{V} = \mathrm{V}_n, \mathrm{V}_{n+1}$, choose decompositions as in Section 2.2:

$$\mathrm{V}_n = \mathrm{X}_l \oplus \cdots \oplus \mathrm{X}_1 \oplus \mathrm{W} \oplus \mathrm{Y}_1 \oplus \cdots \oplus \mathrm{Y}_l,$$

$$\mathrm{V}_{n+1} = \mathrm{X}'_{l'} \oplus \cdots \oplus \mathrm{X}'_1 \oplus \mathrm{W}' \oplus \mathrm{Y}'_1 \oplus \cdots \oplus \mathrm{Y}'_{l'}.$$

Assume that $\mathrm{W} \subset \mathrm{W}'$ or $\mathrm{W}' \subset \mathrm{W}$. Choose flags $\mathcal{F}$ and $\mathcal{F}'$ as in Section 2.2.

$$\mathcal{F} : \mathrm{X}_l \subset \mathrm{X}_l \oplus \mathrm{X}_{l-1} \subset \cdots \subset \mathrm{X}_l \oplus \cdots \oplus \mathrm{X}_1,$$

$$\mathcal{F}' : \mathrm{X}'_{l'} \subset \mathrm{X}'_{l'} \oplus \mathrm{X}'_{l'-1} \subset \cdots \subset \mathrm{X}'_{l'} \oplus \cdots \oplus \mathrm{X}'_1,$$

and let $P_n = P_{\mathcal{F},\mathrm{G}(\mathrm{V}_n)} \subset \mathrm{G}(\mathrm{V}_n)$ and $P_{n+1} = P_{\mathcal{F}',\mathrm{G}(\mathrm{V}_{n+1})} \subset \mathrm{G}(\mathrm{V}_{n+1})$ be the parabolic subgroups stabilizing the flags $\mathcal{F}$ and $\mathcal{F}'$, respectively. For every $1 \leq i \leq l$, let τ_i be an irreducible tempered representation of the F-points of $\mathrm{Res}_{E/F}\,\mathrm{GL}_E(\mathrm{X}_i)$, and let π_{W} be an irreducible tempered representation of $\mathrm{G}(\mathrm{W})$. Similarly, for every $1 \leq j \leq l'$, let τ'_j be an irreducible tempered representation of the F-points of $\mathrm{Res}_{E/F}\,\mathrm{GL}_E(\mathrm{X}'_i)$, and let $\pi_{\mathrm{W}'}$ be an irreducible tempered representation of $\mathrm{G}(\mathrm{W}')$. Let $\underline{a} = (a_i)_{i=1}^{l}$ and $\underline{b} = (b_j)_{j=1}^{l'}$ be tuples of parameters for unramified characters of $E^\times$. We consider the parabolically induced representations

$$\pi_n^{\underline{a}} = \mathrm{I}_{P_n}^{\mathrm{G}(\mathrm{V}_n)}(|\det|^{a_l}\,\tau_l \boxtimes \cdots \boxtimes |\det|^{a_1}\,\tau_1 \boxtimes \pi_{\mathrm{W}})$$

$$\pi_{n+1}^{\underline{b}} = \mathrm{I}_{P_{n+1}}^{\mathrm{G}(\mathrm{V}_{n+1})}(|\det|^{b_{l'}}\,\tau'_{l'} \boxtimes \cdots \boxtimes |\det|^{b_1}\,\tau'_1 \boxtimes \pi_{\mathrm{W}'}).$$

By Theorem 2.2.1, all irreducible representations of $\mathrm{G}(\mathrm{V}_n)$, $\mathrm{G}(\mathrm{V}_{n+1})$ can be realized as quotients of representations of this form. Note that a matrix coefficient of a quotient of a given representation gives rise to a matrix coefficient of the representation, via composition with the quotient map. Hence, it suffices to study the integral of matrix coefficients for matrix coefficients of $\pi_n^{\underline{a}}$ and $\pi_{n+1}^{\underline{b}}$. Let $\mathcal{K}_n$ and $\mathcal{K}_{n+1}$ be maximal compact subgroups of $\mathrm{G}(\mathrm{V}_n)$ and $\mathrm{G}(\mathrm{V}_{n+1})$, respectively, such that $\mathcal{K}_n \subset \mathcal{K}_{n+1}$, and such that P_{n+1} and P_n are in good position with respect to $\mathcal{K}_{n+1}$ and $\mathcal{K}_n$, respectively.

We are now ready to state the result of Moeglin and Waldspurger [26, Lemme 1.7].

Proposition 2.5.1. *For any standard sections $f_n^{\underline{a}} \in \pi_n^{\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1}^{\underline{b}}$, $f_n^{\vee\underline{a}} \in \pi_n^{\vee\underline{a}}$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1}^{\vee\underline{b}}$, the integral $\alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})$ absolutely converges in the domain*

$$\mathcal{D} = \left\{ (\underline{a}, \underline{b}) \;\middle|\; \|\mathrm{Re}\,\underline{a}\|, \|\mathrm{Re}\,\underline{b}\| < \frac{1}{2}, \max_{\substack{u_i \in \mathrm{Re}\,\underline{a} \\ t_j \in \mathrm{Re}\,\underline{b}}} |u_i \pm t_j| < \frac{1}{2} \right\}.$$

Furthermore, there exists a polynomial $D(q^{-\underline{a}}, q^{-\underline{b}}) \in \mathbb{C}[q^{\pm\underline{a}}, q^{\pm\underline{b}}]$ that does not vanish in the domain $\mathcal{D}$, such that for every standard sections $f_n^{\underline{a}} \in \pi_n^{\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1}^{\underline{b}}$, $f_n^{\vee\underline{a}} \in \pi_n^{\vee\underline{a}}$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1}^{\vee\underline{b}}$, there exists a polynomial $L_{f_n^{\underline{a}}, f_{n+1}^{\underline{b}}, f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}}(q^{-\underline{a}}, q^{-\underline{b}}) \in \mathbb{C}[q^{\pm\underline{a}}, q^{\pm\underline{b}}]$, *such that for* $(\underline{a},\underline{b}) \in \mathcal{D}$*,*

$$\alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}) = \frac{L_{f_n^{\underline{a}},f_{n+1}^{\underline{b}},f_n^{\vee\underline{a}},f_{n+1}^{\vee\underline{b}}}(q^{-\underline{a}}, q^{-\underline{b}})}{D(q^{-\underline{a}}, q^{-\underline{b}})}.$$

In particular, the assignment $(\underline{a},\underline{b}) \mapsto \alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})$ *has a meromorphic continuation to the entire plane.*

Proposition 2.5.1 already gives an extension of $\alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}$ for non-tempered representations. The problem is that this meromorphic continuation is only defined in the domain $\mathcal{D}$, which requires the extra condition

$$\max_{\substack{u_i \in \operatorname{Re}\underline{a} \\ t_j \in \operatorname{Re}\underline{b}}} |u_i \pm t_j| < \frac{1}{2}.$$

This condition is not guaranteed to be satisfied by representations arising as local components of cuspidal automorphic representations lying in a generic packet. Our goal then is to try to find a refined version of the denominator polynomial $D(q^{-\underline{a}}, q^{-\underline{b}})$, which will allow us to define the normalized value $\mathcal{P}_{\pi_n,\pi_{n+1}}(v_n, v_{n+1}; v_n^\vee, v_{n+1}^\vee)$ for all $\underline{a}$ and $\underline{b}$ satisfying $\|\operatorname{Re}\underline{a}\|, \|\operatorname{Re}\underline{b}\| < \frac{1}{2}$.

In the region $\|\operatorname{Re}\underline{a}\|, \|\operatorname{Re}\underline{b}\| < \frac{1}{2}$, we have that the assignments $\underline{a} \mapsto L(1, \pi_n^{\underline{a}}, \mathrm{Ad})$, $\underline{b} \mapsto L(1, \pi_{n+1}^{\underline{b}}, \mathrm{Ad})$ are holomorphic. Therefore, it suffices to find a holomorphic extension for the assignment

$$(\underline{a},\underline{b}) \mapsto \frac{\alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})}{L(\frac{1}{2}, \mathrm{Std}(\pi_n^{\underline{a}}) \times \mathrm{Std}(\pi_{n+1}^{\underline{b}}))},$$

for $\underline{a}$ and $\underline{b}$ satisfying $\|\operatorname{Re}\underline{a}\|, \|\operatorname{Re}\underline{b}\| < \frac{1}{2}$.

2.6. Intuition from the split case

In this section, we give a formal (but not rigorous) identity for $\alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}$ in the split case. Let $\mathfrak{o}$ be the ring of integers of F. Fix a non-trivial additive character $\psi : F \to \mathbb{C}^\times$. We normalize the measures, so that the volume of $\mathfrak{o}$ is 1.

In this case, $E = F \times F$, and $\mathrm{G}(\mathrm{V}_n) \cong \mathrm{GL}_n(F)$ and $\mathrm{G}(\mathrm{V}_{n+1}) \cong \mathrm{GL}_{n+1}(F)$. We have that for $\underline{a}$ and $\underline{b}$ outside of a finite union of hyperplanes, $\pi_n^{\underline{a}}$ and $\pi_{n+1}^{\underline{b}}$ are irreducible and generic. We realize $\pi_n^{\underline{a}}$ and $\pi_{n+1}^{\underline{b}}$ via their Whittaker models $\mathcal{W}(\pi_n^{\underline{a}}, \psi)$ and $\mathcal{W}(\pi_{n+1}^{\underline{b}}, \psi^{-1})$, with respect to the corresponding upper triangular unipotent subgroups. See the discussion in [8, Section 3.1]. Assume that $\pi_n^{\underline{a}}$ and $\pi_{n+1}^{\underline{b}}$ are unitarizable.

Let $B_m \subset \mathrm{GL}_m(F)$ be the upper triangular Borel subgroup, $A_m \subset \mathrm{GL}_m(F)$ be the diagonal subgroup, and $N_m \subset \mathrm{GL}_m(F)$ be the upper unipotent subgroup. Let $K_m = \mathrm{GL}_m(\mathfrak{o})$ be the standard maximal compact subgroup of $\mathrm{GL}_m(F)$. We normalize the measures, so that K_m has volume 1. For $a_m = (a_{mi})_{i=1}^m$, let

$$\delta_{B_m}(a_m) = \prod_{1 \le i < j \le m} \left| \frac{a_{mi}}{a_{mj}} \right|$$

be the modular character. For $a_{m-1} \in A_{m-1}$, we have

$$\delta_{B_m}(a_{m-1}) = |\det a_{m-1}| \, \delta_{B_{m-1}}(a_{m-1}),$$

where we realize $A_{m-1} \subset A_m$ via the embedding $a_{m-1} \mapsto \mathrm{diag}(a_{m-1}, 1)$. We have the Iwasawa decomposition: if $f : \mathrm{GL}_m(F) \to \mathbb{C}$ is integrable, then

$$\int_{\mathrm{GL}_m(F)} f(g_m) \mathrm{d}g_m = \int_{N_m} \int_{A_m} \int_{K_m} \delta_{B_m}^{-1}(a_m) f(n_m a_m k_m) \mathrm{d}k_m \mathrm{d}a_m \mathrm{d}n_m.$$

We will use a formula by Lapid and Mao. Let $R_m \subset \mathrm{GL}_m(F)$ be the mirabolic subgroup, consisting of matrices having $(0, \dots, 0, 1)$ as their last row. Let τ_m be an irreducible unitarizable generic representation of $\mathrm{GL}_m(F)$. We have a non-degenerate $\mathrm{GL}_m(F)$-invariant pairing of $\tau_m \times \tau_m^\vee \to \mathbb{C}$, given by the formula

$$[W, W^\vee] = \int_{N_m \backslash R_m} W(g_m) W^\vee(g_m) \mathrm{d}g_m.$$

If τ_m is unramified, and $W^\circ \in \mathcal{W}(\tau_m, \psi)$ and $W^{\circ\vee} \in \mathcal{W}(\tau_m^\vee, \psi^{-1})$ are spherical vectors with $W^\circ(I_m) = W^{\vee\circ}(I_m) = 1$, then by the Iwasawa decomposition

$$\begin{aligned}[W^\circ, W^{\circ\vee}] &= \int_{A_{m-1}} \delta_{B_{m-1}}^{-1}(a_{m-1}) W^\circ(a_{m-1}) W^{\circ\vee}(a_{m-1}) \mathrm{d}a_{m-1} \\ &= \int_{A_m} \delta_{B_m}^{-1}(a_m) W^\circ(a_m) W^{\circ\vee}(a_m) \Phi(a_m e_m) \left|\det a_m\right| \mathrm{d}a_m,\end{aligned}$$

where $\Phi : F^m \to \mathbb{C}$ is the characteristic function of $\mathfrak{o}^m$, and $e_m = (0, \dots, 0, 1) \in F^m$. By the unramified computation of the Rankin–Selberg integrals of $\mathrm{GL}_m \times \mathrm{GL}_m$, we have that

$$[W^\circ, W^{\circ\vee}] = L(1, \tau_m \times \tau_m^\vee) = L(1, \tau_m, \mathrm{Ad}).$$

Therefore, if $\langle \cdot, \cdot \rangle$ is a non-degenerate $\mathrm{GL}_m(F)$-invariant pairing $\tau_m \times \tau_m^\vee \to \mathbb{C}$, satisfying for unramified τ_m that $\langle W^\circ, W^{\circ\vee} \rangle = 1$, then $\langle W, W^\vee \rangle = \frac{1}{L(1,\tau_m,\mathrm{Ad})} \left[W, W^\vee\right]$. We set for any irreducible generic unitarizable τ_m, and every $W \in \mathcal{W}(\tau_m, \psi)$, $W^\vee \in \mathcal{W}(\tau_m^\vee, \psi)$, the pairing

$$\langle W, W^\vee \rangle = \frac{1}{L(1, \tau_m, \mathrm{Ad})} \left[W, W^\vee\right].$$

Theorem 2.6.1 ([21, Lemma 4.7]). *Let $1 \le i \le m$. Let U_i be the unipotent radical of the parabolic subgroup of $\mathrm{GL}_m(F)$ of type $(m-i, 1, \dots, 1)$. Let N_i be the upper unipotent subgroup of $\mathrm{GL}_i(F)$. Then*

$$\begin{aligned}&\int_{U_{i-1}} \left[\tau_m(u_{i-1}) W, W^\vee\right] \psi(u_{i-1}) \mathrm{d}u_{i-1} \\ =&\int_{N_{m-i} \backslash \mathrm{GL}_{m-i}(F)} W(g_{m-i}) W^\vee(g_{m-i}) \left|\det g_{m-i}\right|^{1-i} \mathrm{d}g_{m-i},\end{aligned}$$

for every $W \in \mathcal{W}(\tau_m, \psi)$, $W^\vee \in \mathcal{W}(\tau_m^\vee, \psi^{-1})$. Here, $\mathrm{GL}_{m-i}(F)$ is realized as a subgroup of $\mathrm{GL}_m(F)$ via the embedding $g_{m-i} \mapsto \mathrm{diag}(g_{m-i}, I_i)$.

We now consider the integral

$$\begin{aligned}
&\alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}})\\
&= \int_{\mathrm{GL}_n(F)} \langle \pi_n^{\underline{a}}(h_n) W_n^{\underline{a}}, W_n^{\vee\underline{a}}\rangle \langle \pi_{n+1}^{\underline{b}}(h_n) W_{n+1}^{\underline{b}}, W_{n+1}^{\vee\underline{b}}\rangle \mathrm{d}h_n,
\end{aligned}$$

where $W_n^{\underline{a}} \in \mathcal{W}(\pi_n^{\underline{a}}, \psi)$, $W_{n+1}^{\underline{b}} \in \mathcal{W}(\pi_{n+1}^{\underline{b}}, \psi)$, $W_n^{\vee\underline{a}} \in \mathcal{W}(\pi_n^{\vee\underline{a}}, \psi^{-1})$, $W_{n+1}^{\vee\underline{b}} \in \mathcal{W}(\pi_{n+1}^{\vee\underline{b}}, \psi^{-1})$. Then, by Theorem 2.6.1 with $m = n+1$, $\tau_m = \pi_{n+1}^{\underline{b}}$, and $i = 1$, we have

$$\begin{aligned}
&L(1, \pi_n^{\underline{a}}, \mathrm{Ad}) L(1, \pi_{n+1}^{\underline{b}}, \mathrm{Ad}) \alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}})\\
&= \int_{\mathrm{GL}_n(F)} [\pi_n^{\underline{a}}(h_n) W_n^{\underline{a}}, W_n^{\vee\underline{a}}] \int_{N_n\backslash \mathrm{GL}_n(F)} W_{n+1}^{\underline{b}}(g_n^\vee h_n) W_{n+1}^{\vee\underline{b}}(g_n^\vee) \mathrm{d}g_n^\vee \mathrm{d}h_n.
\end{aligned} \tag{2.6.1}$$

Changing variables, $h_n = (g_n^\vee)^{-1} g_n$, we have that (2.6.1) equals

$$\begin{aligned}
&\int_{\mathrm{GL}_n(F)} \int_{N_n\backslash \mathrm{GL}_n(F)} [\pi_n^{\underline{a}}(g_n) W_n^{\underline{a}}, \pi_n^{\vee\underline{a}}(g_n^\vee) W_n^{\vee\underline{a}}]\, W_{n+1}^{\underline{b}}(g_n) W_{n+1}^{\vee\underline{b}}(g_n^\vee) \mathrm{d}g_n^\vee \mathrm{d}g_n\\
&= \int_{N_n\backslash \mathrm{GL}_n(F)} \int_{N_n\backslash \mathrm{GL}_n(F)} \int_{N_n} [\pi_n^{\underline{a}}(u_n g_n) W_n^{\underline{a}}, \pi_n^{\vee\underline{a}}(g_n^\vee) W_n^{\vee\underline{a}}]\, \psi^{-1}(u_n)\\
&\times W_{n+1}^{\underline{b}}(g_n) W_{n+1}^{\vee\underline{b}}(g_n^\vee) \mathrm{d}u_n \mathrm{d}g_n \mathrm{d}g_n^\vee.
\end{aligned}$$

Using Theorem 2.6.1 again, this time with $m = n$, $\tau_m = \pi_n^{\underline{a}}$, and $i = n$, we get

$$\int_{N_n} [\pi_n^{\underline{a}}(u_n g_n) W_n^{\underline{a}}, \pi_n^{\vee\underline{a}}(g_n^\vee) W_n^{\vee\underline{a}}]\, \psi^{-1}(u_n) \mathrm{d}u_n = W_n^{\underline{a}}(g_n) W_n^{\vee\underline{a}}(g_n^\vee).$$

Hence, we have that

$$\begin{aligned}
&\alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}})\\
&= \frac{1}{L(1, \pi_n^{\underline{a}}, \mathrm{Ad}) L(1, \pi_{n+1}^{\underline{b}}, \mathrm{Ad})} \int_{N_n\backslash \mathrm{GL}_n(F)} W_{n+1}^{\underline{b}}(g_n) W_n^{\underline{a}}(g_n) \mathrm{d}g_n\\
&\times \int_{N_n\backslash \mathrm{GL}_n(F)} W_{n+1}^{\vee\underline{b}}(g_n^\vee) W_n^{\vee\underline{a}}(g_n^\vee) \mathrm{d}g_n^\vee.
\end{aligned} \tag{2.6.2}$$

The integrals in (2.6.2) are the Rankin–Selberg integrals for $\pi_{n+1}^{\underline{b}} \times \pi_n^{\underline{a}}$ and $\pi_{n+1}^{\vee\underline{b}} \times \pi_n^{\vee\underline{a}}$, respectively, evaluated at $s = \frac{1}{2}$. They converge for $\mathrm{Re}\,\underline{a}$ and $\mathrm{Re}\,\underline{b}$ large, and are understood elsewhere using meromorphic continuation. Let $W_n^{\underline{a}}, W_{n+1}^{\underline{b}}, W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}}$ be Whittaker functions that correspond to holomorphic sections. Then the Whittaker functions are also holomorphic at every point of the group, as functions of $\underline{a}$ or $\underline{b}$, see [8, Section 3.1]. For such functions, the quotient

$$\frac{\alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}})}{L(\frac{1}{2}, \mathrm{Std}(\pi_{n+1}^{\underline{b}}) \times \mathrm{Std}(\pi_n^{\underline{a}}))} = \frac{\alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}})}{L(\frac{1}{2}, \pi_{n+1}^{\underline{b}} \times \pi_n^{\underline{a}})L(\frac{1}{2}, \pi_{n+1}^{\vee\underline{b}} \times \pi_n^{\vee\underline{a}})}$$

is holomorphic, as a function of $\underline{a}$ and $\underline{b}$, whenever $\pi_{n+1}^{\underline{b}}$ and $\pi_n^{\underline{a}}$ are irreducible and generic.

We remark that this is only a formal computation. In order to make it rigorous, one needs to take care of convergence issues. However, recall that by [1], the space

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi_{n+1}^{\underline{b}} \otimes \pi_n^{\underline{a}}, 1) \times \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi_{n+1}^{\vee\underline{b}} \otimes \pi_n^{\vee\underline{a}}, 1)$$

is at most one dimensional, whenever $\pi_{n+1}^{\underline{b}}$ and $\pi_n^{\underline{a}}$ are irreducible. We can define a distinguished element of this space by setting

$$\mathcal{P}_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}}) = I_{\frac{1}{2}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}) I_{\frac{1}{2}}^{\vee}(W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}}),$$

where

$$I_s(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}) = \frac{1}{L(s, \pi_{n+1}^{\underline{b}} \times \pi_n^{\underline{a}})} \int_{N_n \backslash \mathrm{GL}_n(F)} W_{n+1}^{\underline{b}}(g_n) W_n^{\underline{a}}(g_n) \left|\det g_n\right|^{s-\frac{1}{2}} \mathrm{d}g_n,$$
$$I_s^{\vee}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}) = \frac{1}{L(s, \pi_{n+1}^{\vee\underline{b}} \times \pi_n^{\vee\underline{a}})} \int_{N_n \backslash \mathrm{GL}_n(F)} W_{n+1}^{\vee\underline{b}}(g_n) W_n^{\vee\underline{a}}(g_n) \left|\det g_n\right|^{s-\frac{1}{2}} \mathrm{d}g_n,$$

where the integrals converge for $\operatorname{Re} s$ large, and are understood elsewhere by holomorphic continuation.

2.7. The main result

We now move to explain our main result. For $\mathbf{V} = \mathrm{V}_n, \mathrm{V}_{n+1}$, assume that there exist decompositions as in Section 2.5:

$$\mathrm{V}_n = \mathrm{X}_l \oplus \cdots \oplus \mathrm{X}_1 \oplus \mathrm{W} \oplus \mathrm{Y}_1 \oplus \cdots \oplus \mathrm{Y}_l,$$

$$\mathrm{V}_{n+1} = \mathrm{X}'_{l'} \oplus \cdots \oplus \mathrm{X}'_1 \oplus \mathrm{W}' \oplus \mathrm{Y}'_1 \oplus \cdots \oplus \mathrm{Y}'_{l'},$$

such that $\mathrm{W} \subset \mathrm{W}'$ or $\mathrm{W}' \subset \mathrm{W}$. Further assume that $|\dim_E \mathrm{W} - \dim_E \mathrm{W}'| = 1$, and that $\dim_E \mathrm{X}_i = \dim_E \mathrm{Y}_i = 1$, for every $1 \leq i \leq l$, and $\dim_E \mathrm{X}'_j = \dim_E \mathrm{Y}'_j = 1$, for every $1 \leq j \leq l'$. For every $1 \leq i \leq l$, choose $0 \neq f_i \in \mathrm{X}_i$ (and then $\mathrm{X}_i = Ef_i$), and for every $1 \leq j \leq l'$, choose $0 \neq f'_j \in \mathrm{X}'_j$ (and then $\mathrm{X}'_j = Ef'_j$). Denote $m = \dim_E \mathrm{W}$ and $\mathrm{V}_m = \mathrm{W}$, and similarly denote $m' = \dim_E \mathrm{W}'$ and $\mathrm{V}_{m'} = \mathrm{W}'$.

As in Section 2.5, choose flags $\mathcal{F} = \mathcal{F}_l$ and $\mathcal{F}' = \mathcal{F}'_{l'}$

$$\mathcal{F}_l : Ef_l \subset Ef_l \oplus Ef_{l-1} \subset \cdots \subset Ef_l \oplus \cdots \oplus Ef_1,$$

$$\mathcal{F}'_{l'} : Ef'_{l'} \subset Ef'_{l'} \oplus Ef'_{l'-1} \subset \cdots \subset Ef'_{l'} \oplus \cdots \oplus Ef'_1.$$

Also, as in Section 2.5, for every $1 \leq i \leq l$, choose a tempered representation ω_i of the F-points of $\mathrm{Res}_{E/F}\,\mathrm{GL}_E(Ef_i)$. In this case, ω_i is a unitary character of $E^\times$. Similarly, for every $1 \leq j \leq l'$, choose a tempered representation μ_j of the F-points of $\mathrm{Res}_{E/F}\,\mathrm{GL}_E(Ef'_j)$, i.e., a unitary character of $E^\times$. Let π_m and $\pi_{m'}$ be irreducible tempered representations of $\mathrm{G}(\mathrm{V}_m)$ and $\mathrm{G}(\mathrm{V}_{m'})$, respectively. Let $\underline{a} = (a_i)_{i=1}^{l}$ and $\underline{b} = (b_j)_{j=1}^{l'}$ be tuples of parameters for unramified characters of $E^\times$. As in Section 2.5, consider the parabolically induced representations

$$\pi_n^{\underline{a}} = \mathrm{I}_{P_n}^{\mathrm{G}(\mathrm{V}_n)}(|\cdot|^{a_l}\,\omega_l \boxtimes \cdots \boxtimes |\cdot|^{a_1}\,\omega_1 \boxtimes \pi_m),$$

$$\pi_{n+1}^{\underline{b}} = \mathrm{I}_{P_{n+1}}^{\mathrm{G}(\mathrm{V}_{n+1})}(|\cdot|^{b_{l'}}\,\mu_{l'} \boxtimes \cdots \boxtimes |\cdot|^{b_1}\,\mu_1 \boxtimes \pi_{m'}).$$

Our main result is the following theorem.

Theorem 2.7.1. *For every holomorphic sections* $f_n^{\underline{a}} \in \pi_n^{\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1}^{\underline{b}}$, $f_n^{\vee\underline{a}} \in \pi_n^{\vee\underline{a}}$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1}^{\vee\underline{b}}$, *the map*

$$(\underline{a},\underline{b}) \mapsto \alpha^{\natural}_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}) = \frac{\alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})}{L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}})},$$

originally defined only for imaginary $\underline{a},\underline{b}$, *has an analytic continuation to the entire plane. The analytic continuation is actually a polynomial, i.e., an element of* $\mathbb{C}[q^{\pm\underline{a}}, q^{\pm\underline{b}}]$.

Here, $L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}})$ is an L-factor defined using the doubling method, which should be thought of as the value $L(\frac{1}{2}, \mathrm{Std}(\pi_{n+1}^{\underline{b}}) \times \mathrm{Std}(\pi_n^{\underline{a}}))$. The precise definition of $L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}})$ will be given in Section 3.6. The next chapter is devoted to the proof of Theorem 2.7.1.

Recall that we are interested in an extension of the normalized functional $\mathcal{P}_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}$. The following corollary provides such an extension in the desired region.

Corollary 2.7.2. *For every holomorphic sections* $f_n^{\underline{a}} \in \pi_n^{\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1}^{\underline{b}}$, $f_n^{\vee\underline{a}} \in \pi_n^{\vee\underline{a}}$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1}^{\vee\underline{b}}$, *the map*

$$(\underline{a},\underline{b}) \mapsto \mathcal{P}_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}),$$

originally defined only for imaginary $\underline{a},\underline{b}$, *has a meromorphic continuation to the entire plane, which is holomorphic when* $\left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}$.

Proof. If $\left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}$, then $\mathrm{Std}(\pi_{n+1}^{\underline{b}})$ and $\mathrm{Std}(\pi_n^{\underline{a}})$ are irreducible. In this case, we have that the assignments $\underline{a} \mapsto L(1, \pi_n^{\underline{a}}, \mathrm{Ad})$, $\underline{b} \mapsto L(1, \pi_{n+1}^{\underline{b}}, \mathrm{Ad})$ are holomorphic. For such $\underline{a}$ and $\underline{b}$, we have by Proposition 3.6.1 that the assignment

$$(\underline{a},\underline{b}) \mapsto \frac{L(\frac{1}{2}, \mathrm{Std}(\pi_{n+1}^{\underline{b}}) \times \mathrm{Std}(\pi_n^{\underline{a}}))}{L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}})}$$

is a non-vanishing holomorphic function. Therefore, the function

$$
\begin{aligned}
\mathcal{P}_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}) = {} & \Delta_{n+1}^{-1} \cdot L(1, \pi_n^{\underline{a}}, \mathrm{Ad}) L(1, \pi_{n+1}^{\underline{b}}, \mathrm{Ad}) \\
& \times \frac{L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}})}{L(\frac{1}{2}, \mathrm{Std}(\pi_{n+1}^{\underline{b}}) \times \mathrm{Std}(\pi_n^{\underline{a}}))} \\
& \times \alpha^{\natural}_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})
\end{aligned}
$$

is a holomorphic function in the variables $\underline{a}$ and $\underline{b}$ in the region $\|\mathrm{Re}\,\underline{a}\|, \|\mathrm{Re}\,\underline{b}\| < \frac{1}{2}$. $\square$

CHAPTER 3

The case of principal series representations

In this chapter, we show how a construction related to the doubling method can be used to regularize the integral of matrix coefficients for the case that π_n and π_{n+1} are given by principal series representations.

3.1. Doubling integrals

In the 1980s, Piatetski-Shapiro and Rallis introduced an integral representation for the tensor product representation of representations of $G \times \mathrm{GL}_1$, where G is a classical group [28, 10]. Their construction relies only on matrix coefficients of G, and does not require the representation of G to have any model (such as a Whittaker model). This construction is now known as the doubling method. In this section, we give a brief overview of the doubling method. We refer the reader to [22] and [34] for standard references about the doubling method.

Let $(\mathbf{V}, \langle \cdot, \cdot \rangle)$ be a non-degenerate σ-sesquilinear space of finite rank over E. Let $\overline{\mathbf{V}}$ be a vector space isomorphic to $\mathbf{V}$, via an isomorphism $\mathbf{V} \to \overline{\mathbf{V}}$, $v \mapsto \overline{v}$, equipped with the σ-sesquilinear product $\langle \overline{v}, \overline{w} \rangle = -\langle v, w \rangle$, for $v, w \in \mathbf{V}$. Let $\mathbf{V}^{\Box} = \mathbf{V} \oplus \overline{\mathbf{V}}$, where we set $\mathbf{V}$ and $\overline{\mathbf{V}}$ to be orthogonal. Let $\mathbf{V}^{\Delta} = \{v + \overline{v} \mid v \in \mathbf{V}\}$. Then $\mathbf{V}^{\Delta}$ is a maximal totally isotropic subspace of $\mathbf{V}^{\Box}$. Let $P_{\mathbf{V}^{\Delta}} \subset \mathrm{G}(\mathbf{V}^{\Box})$ be the parabolic subgroup stabilizing the subspace $\mathbf{V}^{\Delta}$. Then $P_{\mathbf{V}^{\Delta}}$ has Levi part isomorphic to $\mathrm{GL}(\mathbf{V}^{\Delta})$. We have a character $\det_{\Delta} : P_{\mathbf{V}^{\Delta}} \to E^{\times}$ by projection to the Levi part $P_{\mathbf{V}^{\Delta}} \to \mathrm{GL}(\mathbf{V}^{\Delta})$ and composition with det.

We have an embedding $i : \mathrm{G}(\mathbf{V}) \times \mathrm{G}(\mathbf{V}) \to \mathrm{G}(\mathbf{V}^{\square})$ given by

$$i\,(g,h)\,(v) = gv,$$
$$i\,(g,h)\,(\overline{v}) = \overline{hv},$$

where $v \in \mathbf{V}$. We denote by $\Delta : \mathrm{G}(\mathbf{V}) \to \mathrm{G}(\mathbf{V}^{\square})$ the map $\Delta(g) = i(g,g)$. We actually have that the image of Δ is contained in the Levi part of $P_{\mathbf{V}^{\Delta}}$.

Let π be an admissible representation of $\mathrm{G}(\mathbf{V})$ and let (s,χ) be a parameter for a character of $E^{\times}$. We consider the following space of (normalized) parabolic induction

$$I(|\cdot|^{s}\,\chi, \mathbf{V}) = \mathrm{I}_{P_{\mathbf{V}^{\Delta}}}^{\mathrm{G}(\mathbf{V}^{\square})}((|\cdot|^{s}\,\chi) \circ \det_{\Delta}).$$

The doubling zeta integrals are defined via the formula

$$\begin{aligned} &Z\left(f^{s}, v_{\pi}, v_{\pi}^{\vee}\right) \\ &= \int_{\Delta(\mathrm{G}(\mathbf{V}))\backslash\, \mathrm{G}(\mathbf{V})\times \mathrm{G}(\mathbf{V})} f^{s}\left(i(g_1,g_2)\right) \langle \pi(g_1)v, \pi^{\vee}(g_2)v^{\vee}\rangle \chi^{-1}\left(\det g_2\right) \mathrm{d}(g_1,g_2), \end{aligned}$$

where $f^{s} \in I(|\cdot|^{s}\,\chi, \mathbf{V})$ is a holomorphic section, $v_{\pi} \in \pi$ and $v_{\pi}^{\vee} \in \pi^{\vee}$. These integrals are absolutely convergent for $\operatorname{Re} s$ large that depends only on π. In this convergence domain, the integrals converge to holomorphic functions that have a meromorphic continuation to the entire plane. We continue denoting the meromorphic continuation by the same notation.

We move to define L-factors of $\pi \times \chi$. If $E = F \times F$, then $\mathrm{G}(\mathbf{V})$ is a general linear group. In this case, we define for $\chi = (\chi_1, \chi_2)$ and $s = (s_1, s_2)$,

$$L_{\mathrm{PSR}}(s + \tfrac{1}{2}, \pi \times \chi) = L_{\mathrm{GJ}}(s_1 + \tfrac{1}{2}, \pi \times \chi_1) L_{\mathrm{GJ}}(s_2 + \tfrac{1}{2}, \pi^{\vee} \times \chi_2),$$

where L_{GJ} are the L-factors of Godement–Jacquet. By [34, Lemma 5.3], for any holomorphic section $f^{s} \in I(|\cdot|^{s}\,\chi, \mathbf{V})$, any $v_{\pi} \in \pi$, and any $v_{\pi}^{\vee} \in \pi^{\vee}$, the quotient

$$\frac{Z\left(f^{s}, v_{\pi}, v_{\pi}^{\vee}\right)}{L_{\mathrm{PSR}}(s + \frac{1}{2}, \pi \times \chi)}$$

is a polynomial (an element of $\mathbb{C}[q^{\pm s}]$).

If E is a field, the definition goes through the greatest common divisor of a fractional ideal. There exists a notion of "good sections", see [22, 34, 20]. In particular, holomorphic sections are good. Consider the space $I_{\pi,\chi}$ defined as the $\mathbb{C}[q^{\pm s}]$-linear span of the set

$$\left\{ Z\left(f^s, v_\pi, v_\pi^\vee\right) \mid f^s \in I(\left|\cdot\right|^s \chi, \mathbf{V}) \text{ is a good section}, v_\pi \in \pi, v_\pi^\vee \in \pi^\vee \right\}.$$

If π is irreducible, then there exists a "greatest common divisor" for $I_{\pi,\chi}$. To explain this, we first note that $I_{\pi,\chi}$ is a fractional ideal of $\mathbb{C}[q^{\pm s}]$ with $1 \in I_{\pi,\chi}$. There exists a unique polynomial $P(Z) \in \mathbb{C}[Z]$, such that $P\left(0\right) = 1$, and such that $I_{\pi,\chi} = \frac{1}{P(q^{-s})}\mathbb{C}[q^{\pm s}]$. We denote

$$L_{\mathrm{PSR}}(s + \tfrac{1}{2}, \pi \times \chi) = \frac{1}{P\left(q^{-s}\right)}.$$

When π and χ are unramified, we have that $L_{\mathrm{PSR}}(s, \pi \times \chi) = L(s, \mathrm{Std}(\pi) \times \chi)$ [34, Proposition 7.1].

3.2. Rankin–Selberg integrals

Suppose that $(\mathrm{V}_n, \langle \cdot, \cdot \rangle)$ is a non-degenerate σ-sesquilinear space of rank n over E. Let H be a hyperbolic plane. By this we mean a two-dimensional space over E with an orthogonal basis b_+, b_-, such that $\langle b_+, b_+ \rangle = -\langle b_-, b_- \rangle \neq 0$. Let $\mathrm{V}_{n+2} = \mathrm{V}_n \oplus H$, where we set V_n and H to be orthogonal, and let $\mathrm{V}_{n+1} = \mathrm{V}_n \oplus E b_+$.

Let $\mathrm{G}(\mathrm{V}_{n+2})$ be the isometry group of V_{n+2}, and let $\mathrm{G}(\mathrm{V}_{n+1})$ be the isometry group of V_{n+1}, realized as a subgroup of $\mathrm{G}(\mathrm{V}_{n+2})$, consisting of all elements acting trivially on the vector b_-. Similarly, we realize $\mathrm{G}(\mathrm{V}_n)$, the isometry group of V_n, as a subgroup of $\mathrm{G}(\mathrm{V}_{n+1})$, consisting of all elements acting trivially on b_+.

Let $f_+ = b_+ + b_-$, $f_- = b_+ - b_-$. We have that f_+ and f_- are isotropic vectors, that $\langle f_+, f_- \rangle = 2\langle b_+, b_+ \rangle \neq 0$, and that

$$\mathrm{V}_{n+2} = E f_+ \oplus \mathrm{V}_n \oplus E f_-.$$

Let Q be the parabolic subgroup of $\mathrm{G}(\mathrm{V}_{n+2})$ stabilizing the line Ef_+. Then Q has Levi part isomorphic to $\operatorname{Res}_{E/F}\mathrm{GL}_E(Ef_+) \times \mathrm{G}(\mathrm{V}_n) \cong \operatorname{Res}_{E/F} E^\times \times \mathrm{G}(\mathrm{V}_n)$.

Let π_n and π_{n+1} be irreducible representations of $\mathrm{G}(\mathrm{V}_n)$ and $\mathrm{G}(\mathrm{V}_{n+1})$, respectively. Let (s, χ) be a parameter for a character of $E^\times \cong \mathrm{GL}_E(Ef_+)$. Consider the (normalized) parabolic induction

$$\pi_{n+2}^s = |\cdot|^s \chi \times \pi_n = \mathrm{I}_Q^{\mathrm{G}(\mathrm{V}_{n+2})}(|\cdot|^s \chi \boxtimes \pi_n).$$

Suppose $c_{\pi_n,\pi_{n+1}} \in \operatorname{Hom}_{\mathrm{G}(\mathrm{V}_n)}(\pi_n \otimes \pi_{n+1}, 1)$. We consider the Rankin–Selberg integral

$$C_{\pi_{n+1},\pi_{n+2}^s}\left(v_{n+1}, f_{n+2}^s\right) = \int_{\mathrm{G}(\mathrm{V}_n)\backslash \mathrm{G}(\mathrm{V}_{n+1})} c_{\pi_n,\pi_{n+1}}(f_{n+2}^s(g_{n+1}), \pi_{n+1}(g_{n+1})v_{n+1})\mathrm{d}g_{n+1},$$

where $f_{n+2}^s \in \pi_{n+2}^s$ is a holomorphic section, and $v_{n+1} \in \pi_{n+1}$. By [31, Section 3], this integral converges for $\operatorname{Re} s$ large, depending only on π_n and π_{n+1}, and has a meromorphic continuation to the entire plane, which is a rational function in q^{-s}. By its definition, we get that in its convergence domain $C_{\pi_{n+1},\pi_{n+2}^s} \in \operatorname{Hom}_{\mathrm{G}(\mathrm{V}_{n+1})}(\pi_{n+1} \otimes \pi_{n+2}^s, 1)$, and by the uniqueness theorem, this holds for the meromorphic continuation of $C_{\pi_{n+1},\pi_{n+2}^s}$ as well.

3.3. Construction of special sections

In this section, we review a construction of Ginzburg–Piatetski-Shapiro–Rallis for sections of π_{n+2} from Section 3.2. See [12, Chapter 1] and [29, Section 3].

Let $(\mathrm{V}_n, \langle\cdot,\cdot\rangle)$ be a non-degenerate σ-sesquilinear space of rank n over E, and let $\mathrm{V}_{n+1} = \mathrm{V}_n \oplus Eb$, where $\langle b, b\rangle \neq 0$, and we set b to be orthogonal to V_n. Take $\mathbf{V} = \mathrm{V}_{n+1}$ in Section 3.1. We realize V_{n+2} from Section 3.2, as a subspace of $\mathrm{V}_{n+1}^\square$, where $b_+ = b$, and $b_- = \overline{b}$, i.e., we realize $\mathrm{V}_{n+2} = Eb \oplus \mathrm{V}_n \oplus E\overline{b} \subset \mathrm{V}_{n+1}^\square$.

We have an embedding $i : \mathrm{G}(\mathrm{V}_{n+2}) \times \mathrm{G}(\mathrm{V}_n) \to \mathrm{G}(\mathrm{V}_{n+1}^\square)$ given by

$$i\left(g_{n+2}, g_n\right)\left(v_{n+2}\right) = g_{n+2}v_{n+2},$$

$$i\left(g_{n+2}, g_n\right)\left(\overline{v_n}\right) = \overline{g_n v_n},$$

where $g_{n+2} \in \mathrm{G}(\mathrm{V}_{n+2})$, $g_n \in \mathrm{G}(\mathrm{V}_n)$, $v_{n+2} \in \mathrm{V}_{n+2}$, and $v_n \in \mathrm{V}_n$. We also have the embedding $i : \mathrm{G}(\mathrm{V}_{n+1}) \times \mathrm{G}(\mathrm{V}_{n+1}) \to \mathrm{G}(\mathrm{V}^{\square}_{n+1})$ from Section 3.1, given by

$$i(g_{n+1}, h_{n+1})(v_{n+1}) = g_{n+1}v_{n+1},$$

$$i(g_{n+1}, h_{n+1})(\overline{v_{n+1}}) = \overline{h_{n+1}v_{n+1}},$$

where $g_{n+1}, h_{n+1} \in \mathrm{G}(\mathrm{V}_{n+1})$, and $v_{n+1} \in \mathrm{V}_{n+1}$.

Let π_n, π_{n+1}, $|\cdot|^s \chi$, $\pi^s_{n+2} = |\cdot|^s \chi \times \pi_n$ be as in Section 3.2. Let

$$\rho_{\chi,s} = I(|\cdot|^s \chi, \mathrm{V}_{n+1}) = \mathrm{I}_{P_{\mathrm{V}^{\Delta}_{n+1}}}^{\mathrm{G}(\mathrm{V}^{\square}_{n+1})}((|\cdot|^s \chi) \circ \det{}_{\Delta}).$$

Given a holomorphic section $f^s_\rho \in \rho_{\chi,s}$ and $v_n \in \pi_n$, we consider the kernel integral

$$\Lambda_{f^s_\rho, v_n}(g_{n+2}) = \int_{\mathrm{G}(\mathrm{V}_n)} f^s_\rho(i(g_{n+2}, g_n))\chi^{-1}(\det g_n)\pi_n(g_n)v_n \mathrm{d}g_n,$$

where $g_{n+2} \in \mathrm{G}(\mathrm{V}_{n+2})$. This integral converges for $\operatorname{Re} s$ large enough, depending only on π_n (see the discussion in [29, Section 3]). It has a meromorphic continuation to the entire plane, which we continue to denote by the same symbol. We have that $\Lambda_{f^s_\rho, v_n}$ is a meromorphic section that lies in the space of π^s_{n+2}.

Let $c_{\pi_n,\pi_{n+1}} \in \mathrm{Hom}_{\mathrm{G}(\mathrm{V}_n)}(\pi_n \otimes \pi_{n+1}, 1)$, and let $C_{\pi_{n+1},\pi^s_{n+2}} : \pi_{n+1} \otimes \pi^s_{n+2} \to \mathbb{C}$ be the Rankin–Selberg integral introduced in Section 3.2. Then we have the following identity (see [12, Lemma 1.1] and [29, Lemma 4.1]):

$$\begin{aligned}(3.3.1) \qquad & C_{\pi_{n+1},\pi^s_{n+2}}(v_{n+1}, \Lambda_{f^s_\rho, v_n}) \\ &= \int_{\mathrm{G}(\mathrm{V}_{n+1})} f^s_\rho(i(g_{n+1}, \mathrm{id}_{\mathrm{V}_{n+1}}))c_{\pi_n,\pi_{n+1}}(v_n, \pi_{n+1}(g_{n+1})v_{n+1})\mathrm{d}g_{n+1}.\end{aligned}$$

Here, the right hand side converges for $\operatorname{Re} s$ large depending only on π_n and π_{n+1}. It has a meromorphic continuation to the entire plane, that is a rational function in q^{-s}. The identity is then understood as an equality of meromorphic functions.

3.4. Recursive formula for integrals of matrix coefficients

Let us be in the setup of Section 3.2. Suppose that π_n and π_{n+1} are tempered. Consider $\pi^s_{n+2} = |\cdot|^s \chi \times \pi_n$, as in Section 3.2. If s is imaginary, then π^s_{n+2} is tempered, and there exists a choice of Haar measures such that the following identity holds [3, eq. (7.4.9)]:

$$\begin{aligned}(3.4.1) \qquad & \alpha_{\pi_{n+1},\pi^s_{n+2}}(v_{n+1}, f^s_{n+2}; v^\vee_{n+1}, f^{\vee s}_{n+2}) \\ = \quad & \int_{(\mathrm{G}(\mathrm{V}_n)\backslash\,\mathrm{G}(\mathrm{V}_{n+1}))^2} \mathrm{d}(g_{n+1}, g'_{n+1}) \\ & \times \alpha_{\pi_n,\pi_{n+1}}(f^s_{n+2}(g_{n+1}), \pi_{n+1}(g_{n+1})v_{n+1}; f^{\vee s}_{n+2}(g'_{n+1}), \pi^\vee_{n+1}(g'_{n+1})v^\vee_{n+1}),\end{aligned}$$

where $v_{n+1} \in \pi_{n+1}$, $f^s_{n+2} \in \pi^s_{n+2}$, $v^\vee_{n+1} \in \pi^\vee_{n+1}$, $f^{\vee s}_{n+2} \in \pi^{\vee s}_{n+2}$. In this case, the integral in (3.4.1) absolutely converges [3, Claim (7.4.10)].

3.5. The representations considered

Let $(\mathrm{V}_n, \langle\cdot,\cdot\rangle)$ be a non-degenerate σ-sesquilinear space of rank n over E, and let $\mathrm{V}_{n+1} = \mathrm{V}_n \oplus Ee_+$, where e_+ is orthogonal to V_n and $\langle e_+, e_+\rangle \neq 0$.

Suppose that there exists a decomposition

$$\mathrm{V}_n = \mathrm{Z}_{l,+} \oplus \mathrm{V}_m \oplus \mathrm{Z}_{l,-},$$

where V_m is a non-degenerate subspace of rank m, and $\mathrm{Z}_{l,+}$ and $\mathrm{Z}_{l,-}$ are totally isotropic subspaces of rank l, dual to each other. Let

$$\mathcal{F}_l : 0 \subset Ef_l \subset Ef_l \oplus Ef_{l-1} \subset \cdots \subset Ef_l \oplus \cdots \oplus Ef_1 = \mathrm{Z}_{l,+}$$

be a complete flag in $\mathrm{Z}_{l,+}$. Let $P_{\mathcal{F}_l,\mathrm{G}(\mathrm{V}_n)}$ be the parabolic subgroup of $\mathrm{G}(\mathrm{V}_n)$ stabilizing this flag. It has Levi part isomorphic to

$$(\mathrm{Res}_{E/F} E^\times)^l \times \mathrm{G}(\mathrm{V}_m).$$

Similarly, suppose that there exists a decomposition

$$\mathrm{V}_{n+1} = \mathrm{Z}'_{l',+} \oplus \mathrm{V}_{m'} \oplus \mathrm{Z}'_{l',-},$$

where $|m - m'| = 1$, with either $l = l'$ and $\mathrm{V}_m \subset \mathrm{V}_{m'}$, or $l' = l + 1$ and $\mathrm{V}_{m'} \subset \mathrm{V}_m$. Here again $\mathrm{Z}'_{l',+}$, $\mathrm{Z}'_{l',-}$ are totally isotropic subspaces of rank l', dual to each other, and $\mathrm{V}_{m'} \subset \mathrm{V}_{n+1}$ is a non-degenerate subspace of rank m'. Choose a complete flag in $\mathrm{Z}'_{l',+}$:

$$\mathcal{F}'_{l'} : 0 \subset Ef'_{l'} \subset Ef'_{l'} \oplus Ef'_{l'-1} \subset \cdots \subset Ef'_{l'} \oplus \cdots \oplus Ef'_1 = \mathrm{Z}'_{l',+},$$

and let $P_{\mathcal{F}'_{l'},\mathrm{G}(\mathrm{V}_{n+1})}$ be the parabolic subgroup of $\mathrm{G}(\mathrm{V}_{n+1})$ stabilizing the flag $\mathcal{F}'_{l'}$. It has Levi part isomorphic to

$$(\mathrm{Res}_{E/F}\, E^\times)^{l'} \times \mathrm{G}(\mathrm{V}_{m'}).$$

Let π_m and $\pi_{m'}$ be irreducible tempered representations of $\mathrm{G}(\mathrm{V}_m)$ and $\mathrm{G}(\mathrm{V}_{m'})$, respectively. Let $(a_i, \omega_i)_{i=1}^{l}$ and $(b_i, \mu_i)_{i=1}^{l'}$ be tuples of parameters for characters of $E^\times$. We denote $\underline{a} = (a_i)_{i=1}^{l}$ and $\underline{b} = (b_i)_{i=1}^{l'}$. We define

$$\pi_n^{\underline{a}} = \mathrm{I}_{P_{\mathcal{F}_l,\mathrm{G}(\mathrm{V}_n)}}^{\mathrm{G}(\mathrm{V}_n)}(|\cdot|^{a_l}\,\omega_l \boxtimes \cdots \boxtimes |\cdot|^{a_1}\,\omega_1 \boxtimes \pi_m),$$
$$\pi_{n+1}^{\underline{b}} = \mathrm{I}_{P_{\mathcal{F}'_{l'},\mathrm{G}(\mathrm{V}_{n+1})}}^{\mathrm{G}(\mathrm{V}_{n+1})}(|\cdot|^{b_{l'}}\,\mu_{l'} \boxtimes \cdots \boxtimes |\cdot|^{b_1}\,\mu_1 \boxtimes \pi_{m'}).$$

We say that $\underline{a}$ (respectively $\underline{b}$) is imaginary if a_i is imaginary (respectively b_i is imaginary), for every $1 \le i \le l$ (respectively $1 \le i \le l'$).

We will denote sections of these representations by $f_n^{\underline{a}} \in \pi_n^{\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1}^{\underline{b}}$, etc.

For induction purposes, we explain how to construct a space V_{n+2} containing V_{n+1}, so that its isometry group $\mathrm{G}(\mathrm{V}_{n+2})$ will serve for a representation $\pi_{n+2}^{(s,\underline{a})}$. The construction is similar to the one as discussed in Section 3.2. Let $\mathrm{V}_{n+2} = \mathrm{V}_{n+1} \oplus Ee_-$, where e_- is orthogonal to V_{n+1}, and $\langle e_-, e_-\rangle = -\langle e_+, e_+\rangle$. Denote $f_+ = e_+ + e_-$, $f_- = e_+ - e_-$. Then f_+ and f_- are isotropic vectors with $\langle f_+, f_-\rangle = 2\langle e_+, e_+\rangle$.

Consider the flag

$$\mathcal{F}_{l+1} : 0 \subset Ef_+ \subset Ef_+ \oplus Ef_l \subset \cdots \subset Ef_+ \oplus Ef_l \oplus \cdots \oplus Ef_1 = Ef_+ \oplus \mathrm{Z}_{l,+}.$$

Let $P_{\mathcal{F}_{l+1},\mathrm{G}(\mathrm{V}_{n+2})}$ be the parabolic subgroup of $\mathrm{G}(\mathrm{V}_{n+2})$ stabilizing the flag $\mathcal{F}_{l+1}$. It has Levi part isomorphic to $(\mathrm{Res}_{E/F} E^\times)^{l+1} \times \mathrm{G}(\mathrm{V}_m)$. Let (s,χ) be a parameter for a character of $E^\times$. We denote

$$\pi_{n+2}^{(s,\underline{a})} = \mathrm{I}_{P_{\mathcal{F}_{l+1},\mathrm{G}(\mathrm{V}_{n+2})}}^{\mathrm{G}(\mathrm{V}_{n+2})}(|\cdot|^s \chi \boxtimes |\cdot|^{a_l} \omega_l \boxtimes \cdots \boxtimes |\cdot|^{a_1} \omega_1 \boxtimes \pi_m).$$

On the other hand, we have the decomposition

$$\mathrm{V}_{n+2} = Ef_+ \oplus \mathrm{V}_n \oplus Ef_-,$$

where the subspaces Ef_+, Ef_- are isotropic lines, dual to each other. Let $P_{Ef_+,\mathrm{G}(\mathrm{V}_{n+2})}$ be the parabolic subgroup of $\mathrm{G}(\mathrm{V}_{n+2})$, stabilizing the subspace Ef_+. It has Levi part isomorphic to $\mathrm{Res}_{E/F}(E^\times) \times \mathrm{G}(\mathrm{V}_n)$.

We realize $\pi_{n+2}^{(s,\underline{a})}$ via the (normalized) parabolic induction $\mathrm{I}_{P_{Ef_+,\mathrm{G}(\mathrm{V}_{n+2})}}^{\mathrm{G}(\mathrm{V}_{n+2})}(|\cdot|^s \chi \boxtimes \pi_n^{\underline{a}})$. This is done using transitivity of induction: we map a section $F_{n+2}^{(s,\underline{a})} \in \mathrm{I}_{P_{Ef_+,\mathrm{G}(\mathrm{V}_{n+2})}}^{\mathrm{G}(\mathrm{V}_{n+2})}(|\cdot|^s \chi \boxtimes \pi_n^{\underline{a}})$ to the section $f_{n+2}^{(s,\underline{a})} \in \pi_{n+2}^{(s,\underline{a})}$ defined by $f_{n+2}^{(s,\underline{a})}(g_{n+2}) = F_{n+2}^{(s,\underline{a})}(g_{n+2})(\mathrm{id}_{\mathrm{V}_n})$. Similarly, for a parameter t for the unramified character $|\cdot|^{-t}$ of $E^\times$, we realize $\pi_{n+2}^{\vee(t,\underline{a})}$ via the (normalized) parabolic induction $\mathrm{I}_{P_{Ef_+,\mathrm{G}(\mathrm{V}_{n+2})}}^{\mathrm{G}(\mathrm{V}_{n+2})}(|\cdot|^{-t} \chi^{-1} \boxtimes \pi_n^{\vee\underline{a}})$.

3.6. Mixed L-factors

Before stating our main result, we define a formal naive mixed L-factor, which will serve as a candidate of the denominator for our statement. The reason for defining this naive factor is that for some values of $\underline{a}$ and $\underline{b}$, the representations $\pi_n^{\underline{a}}$ and $\pi_{n+1}^{\underline{b}}$ might not be irreducible.

For a unitary character $\chi : E^\times \to \mathbb{C}^\times$ and $\varepsilon = \pm 1$, we denote

$$
{}^{\varepsilon}\chi = \begin{cases} \chi & \varepsilon = 1, \\ \chi^{-1} \circ \sigma & \varepsilon = -1. \end{cases}
$$

We define

$$
\begin{aligned}
L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}) = & \prod_{i=1}^{l} \prod_{j=1}^{l'} \prod_{\varepsilon, \varepsilon' \in \{\pm 1\}} L(\tfrac{1}{2}, |\cdot|^{\varepsilon a_i + \varepsilon' b_j} \cdot {}^{\varepsilon}\omega_i \cdot {}^{\varepsilon'}\mu_j) \\
& \times \prod_{i=1}^{l} \prod_{\varepsilon \in \{\pm 1\}} L_{\mathrm{PSR}}(\tfrac{1}{2}, \pi_{m'} \times |\cdot|^{\varepsilon a_i} \cdot {}^{\varepsilon}\omega_i) \\
& \times \prod_{j=1}^{l'} \prod_{\varepsilon' \in \{\pm 1\}} L_{\mathrm{PSR}}(\tfrac{1}{2}, \pi_m \times |\cdot|^{\varepsilon' b_j} \cdot {}^{\varepsilon'}\mu_j).
\end{aligned}
$$

This should be thought of as the value $L(\frac{1}{2}, \mathrm{Std}(\pi_n^{\underline{a}}) \times \mathrm{Std}(\pi_{n+1}^{\underline{b}}))$. See also the discussion below.

By properties of the L-factors of the doubling method [22, Theorem 4, (1) & (5)], we have that

$$
\begin{aligned}
L_{\mathrm{PSR}}(\tfrac{1}{2}, \pi_{m'} \times |\cdot|^{\varepsilon s} \cdot {}^{\varepsilon}\chi) &= L_{\mathrm{PSR}}(\varepsilon s + \tfrac{1}{2}, \pi_{m'} \times {}^{\varepsilon}\chi), \\
L_{\mathrm{PSR}}(-s + \tfrac{1}{2}, \pi_{m'} \times \chi^{-1} \circ \sigma) &= L_{\mathrm{PSR}}(-s + \tfrac{1}{2}, \pi_{m'}^{\vee} \times \chi^{-1}).
\end{aligned}
$$

Hence, we may rewrite $L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}})$ as the product

$$
\begin{aligned}
L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}) = & \prod_{i=1}^{l} \prod_{j=1}^{l'} \prod_{\varepsilon, \varepsilon' \in \{\pm 1\}} L(\tfrac{1}{2}, |\cdot|^{\varepsilon a_i + \varepsilon' b_j} \cdot {}^{\varepsilon}\omega_i \cdot {}^{\varepsilon'}\mu_j) \\
& \times \prod_{i=1}^{l} L_{\mathrm{PSR}}(a_i + \tfrac{1}{2}, \pi_{m'} \times \omega_i) L_{\mathrm{PSR}}(-a_i + \tfrac{1}{2}, \pi_{m'}^{\vee} \times \omega_i^{-1}) \\
& \times \prod_{j=1}^{l'} L_{\mathrm{PSR}}(b_j + \tfrac{1}{2}, \pi_m \times \mu_j) L_{\mathrm{PSR}}(-b_j + \tfrac{1}{2}, \pi_m^{\vee} \times \mu_j^{-1}).
\end{aligned}
$$

By using [34, Proposition 7.1] repeatedly, we have that if we consider irreducible unramified representations $\pi_n^{\underline{a}}$ and $\pi_{n+1}^{\underline{b}}$ (where we take $\max(m, m') = 1$ and $\pi_{\max(m,m')}$

is an unramified representation), then

$$L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}) = L(\tfrac{1}{2}, \mathrm{Std}(\pi_n^{\underline{a}}) \times \mathrm{Std}(\pi_{n+1}^{\underline{b}})).$$

Let (s, χ) be a parameter for a character of $E^\times$, and let $\pi_{n+2}^{(s,\underline{a})} = |\cdot|^s \chi \times \pi_n^{\underline{a}}$ as in Section 3.2. Denote

$$\begin{aligned}
&L(s, \chi, \pi_{n+1}^{\underline{b}}) \\
&= L_{\mathrm{PSR}}(s + \tfrac{1}{2}, \pi_{m'} \times \chi) \prod_{j=1}^{l} \prod_{\varepsilon' \in \{\pm 1\}} L(s + \tfrac{1}{2}, \chi \cdot |\cdot|^{\varepsilon' b_j} \cdot {}^{\varepsilon'}\mu_j), \\
&L(-s, \chi^{-1} \circ \sigma, \pi_{n+1}^{\vee \underline{b}}) \\
&= L_{\mathrm{PSR}}(-s + \tfrac{1}{2}, \pi_{m'}^\vee \times \chi^{-1}) \prod_{j=1}^{l} \prod_{\varepsilon' \in \{\pm 1\}} L(-s + \tfrac{1}{2}, \chi^{-1} \circ \sigma \cdot |\cdot|^{\varepsilon' b_j} \cdot {}^{\varepsilon'}\mu_j).
\end{aligned}$$

Then we have

$$L(\pi_{n+1}^{\underline{b}}, \pi_{n+2}^{(s,\underline{a})}) = L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}) L(s, \chi, \pi_{n+1}^{\underline{b}}) L(-s, \chi^{-1} \circ \sigma, \pi_{n+1}^{\vee \underline{b}}). \tag{3.6.1}$$

Consider the doubling integrals discussed in Section 3.1 for $\mathbf{V} = \mathrm{V}_{n+1}$ and $\pi = \pi_{n+1}^{\underline{b}}$. By [34, Section 6], for every fixed $\underline{b}$, $f_\rho^s \in I(|\cdot|^s \chi, \mathrm{V}_{n+1})$, $f_{n+1}^{\underline{b}} \in \pi_{n+1}^{\underline{b}}$, $f_{n+1}^{\vee \underline{b}} \in \pi_{n+1}^{\vee \underline{b}}$, we have that

$$\frac{Z(f_\rho^s, f_{n+1}^{\underline{b}}, f_{n+1}^{\vee \underline{b}})}{L(s, \chi, \pi_{n+1}^{\underline{b}})} \in \mathbb{C}[q^{\pm s}].$$

Suppose that t is a parameter for an unramified character of $E^\times$. Similarly to the discussion above, by considering the doubling integrals for $\mathbf{V} = \mathrm{V}_{n+1}$, $\pi = \pi_{n+1}^{\vee \underline{b}}$, we have that for every fixed $\underline{b}$, $f_{\rho^\vee}^{-t} \in I(|\cdot|^{-t} \chi^{-1}, \mathrm{V}_{n+1})$, $f_{n+1}^{\vee \underline{b}} \in \pi_{n+1}^{\vee \underline{b}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1}^{\underline{b}}$, the following quotient is polynomial:

$$\frac{Z(f_{\rho^\vee}^{-t}, f_{n+1}^{\vee \underline{b}}, f_{n+1}^{\underline{b}})}{L(-t, \chi^{-1} \circ \sigma, \pi_{n+1}^{\vee \underline{b}})} \in \mathbb{C}[q^{\pm t}].$$

We now compare analytic properties of the factors $L(s+\frac{1}{2}, \mathrm{Std}(\pi^{b}_{n+1}) \times \mathrm{Std}(\pi^{a}_{n}))$ and $L(\pi^{a}_{n}, \pi^{b}_{n+1})$. Suppose that $\|\mathrm{Re}\,\underline{a}\|, \|\mathrm{Re}\,\underline{b}\| < \frac{1}{2}$. By Section 2.3,

$$\begin{aligned}
\mathrm{Std}(\pi^{a}_{n}) &= |\cdot|^{a_l}\,\omega_l \times \cdots \times |\cdot|^{a_1}\,\omega_1 \times \mathrm{Std}(\pi_m) \\
&\qquad \times |\cdot|^{-a_1}\,\omega_1^{-1,\sigma} \cdots \times |\cdot|^{-a_l}\,\omega_l^{-1,\sigma}, \\
\mathrm{Std}(\pi^{b}_{n+1}) &= |\cdot|^{b_{l'}}\,\mu_{l'} \times \cdots \times |\cdot|^{b_1}\,\mu_1 \times \mathrm{Std}(\pi_{m'}) \\
&\qquad \times |\cdot|^{-b_1}\,\mu_1^{-1,\sigma} \times \cdots \times |\cdot|^{-b_{l'}}\,\mu_{l'}^{-1,\sigma}.
\end{aligned}$$

By [8], we have that

$$\begin{aligned}
L(s, \mathrm{Std}(\pi^{b}_{n+1}) \times \mathrm{Std}(\pi^{a}_{n})) = &\prod_{i=1}^{l}\prod_{j=1}^{l'}\prod_{\varepsilon,\varepsilon'\in\{\pm 1\}} L(s, |\cdot|^{\varepsilon a_i+\varepsilon' b_j}\cdot {}^{\varepsilon}\omega_i \cdot {}^{\varepsilon'}\mu_j) \\
&\times \prod_{i=1}^{l}\prod_{\varepsilon\in\{\pm 1\}} L(s, \mathrm{Std}(\pi_{m'}) \times |\cdot|^{\varepsilon a_i}\cdot {}^{\varepsilon}\omega_i) \\
&\times \prod_{j=1}^{l'}\prod_{\varepsilon'\in\{\pm 1\}} L(s, \mathrm{Std}(\pi_{m}) \times |\cdot|^{\varepsilon' b_j}\cdot {}^{\varepsilon'}\mu_j) \\
&\times L(s, \mathrm{Std}(\pi_m) \times \mathrm{Std}(\pi_{m'})).
\end{aligned}$$

Conjecturally, for any character $|\cdot|^{s}\chi$ of $E^{\times}$ we have

$$\begin{aligned}
L(\tfrac{1}{2}, \mathrm{Std}(\pi_m) \times |\cdot|^{s}\chi) &= L_{\mathrm{PSR}}(s+\tfrac{1}{2}, \pi_m \times \chi), \\
L(\tfrac{1}{2}, \mathrm{Std}(\pi_{m'}) \times |\cdot|^{s}\chi) &= L_{\mathrm{PSR}}(s+\tfrac{1}{2}, \pi_{m'} \times \chi).
\end{aligned}$$

This is not known to be true, except for some certain cases. See for example [15, Proposition 8.4] and [34, Proposition 7.1 and Theorems 7.1-7.2].

The following proposition shows that $L(\frac{1}{2}, \mathrm{Std}(\pi^{b}_{n+1}) \times \mathrm{Std}(\pi^{a}_{n}))$ and $L(\pi^{a}_{n}, \pi^{b}_{n+1})$ are the same up to a rational function that is holomorphic and non-vanishing in the region $\|\mathrm{Re}\,\underline{a}\|, \|\mathrm{Re}\,\underline{b}\| < \frac{1}{2}$.

Proposition 3.6.1. *The assignment*

$$(\underline{a},\underline{b}) \mapsto \frac{L(\frac{1}{2}, \mathrm{Std}(\pi^{\underline{b}}_{n+1}) \times \mathrm{Std}(\pi^{\underline{a}}_{n}))}{L(\pi^{\underline{a}}_{n}, \pi^{\underline{b}}_{n+1})}$$

is a non-vanishing holomorphic function the region $\left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}$.

Proof. When $\left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}$, we have that $\mathrm{Std}(\pi^{\underline{a}}_{n})$ and $\mathrm{Std}(\pi^{\underline{b}}_{n+1})$ are irreducible. Both $L(\frac{1}{2}, \mathrm{Std}(\pi^{\underline{b}}_{n+1}) \times \mathrm{Std}(\pi^{\underline{a}}_{n}))$ and $L(\pi^{\underline{a}}_{n}, \pi^{\underline{b}}_{n+1})$ contain the product

$$\prod_{i=1}^{l}\prod_{j=1}^{l'}\prod_{\varepsilon,\varepsilon'\in\{\pm 1\}} L(\tfrac{1}{2}, |\cdot|^{\varepsilon a_i + \varepsilon' b_j} \cdot {}^{\varepsilon}\omega_i \cdot {}^{\varepsilon'}\mu_j).$$

It suffices to show that all factors in $L(\frac{1}{2}, \mathrm{Std}(\pi^{\underline{b}}_{n+1}) \times \mathrm{Std}(\pi^{\underline{a}}_{n}))$ and $L(\pi^{\underline{a}}_{n}, \pi^{\underline{b}}_{n+1})$, except for this product, are holomorphic and not vanishing in the required region. Since π_m and $\pi_{m'}$ are tempered, we have that $\mathrm{Std}(\pi_m)$ and $\mathrm{Std}(\pi_{m'})$ are tempered.

We begin with analyzing the poles of the relevant factors of $L(\frac{1}{2}, \mathrm{Std}(\pi^{\underline{b}}_{n+1}) \times \mathrm{Std}(\pi^{\underline{a}}_{n}))$. These are

$$L(s, \mathrm{Std}(\pi_m) \times \mathrm{Std}(\pi_{m'})),$$

$$L(s, \mathrm{Std}(\pi_{m'}) \times |\cdot|^{\varepsilon a_i} \cdot {}^{\varepsilon}\omega_i) = L(\varepsilon a_i + s, \mathrm{Std}(\pi_{m'}) \times {}^{\varepsilon}\omega_i),$$

$$L(s, \mathrm{Std}(\pi_m) \times |\cdot|^{\varepsilon' b_j} \cdot {}^{\varepsilon'}\mu_j) = L(\varepsilon' b_j + s, \mathrm{Std}(\pi_m) \times {}^{\varepsilon'}\mu_j).$$

Recall that for tempered representations τ_{n_1} and τ_{n_2} of $\mathrm{GL}_{n_1}(E)$ and $\mathrm{GL}_{n_2}(E)$, the L-factor $L(s, \tau_{n_1} \times \tau_{n_2})$ can only have poles if $\mathrm{Re}\, s = 0$. Since $\left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}$, we have that $\varepsilon a_i + \frac{1}{2}$, $\varepsilon' b_j + \frac{1}{2}$ have real part different than zero, and therefore the function $L(\frac{1}{2}, \mathrm{Std}(\pi^{\underline{b}}_{n+1}) \times \mathrm{Std}(\pi^{\underline{a}}_{n}))$ is holomorphic in the given region. Recall that $L(s, \mathrm{Std}(\pi^{\underline{b}}_{n+1}) \times \mathrm{Std}(\pi^{\underline{a}}_{n}))$ is a reciprocal of an element of the ring $\mathbb{C}[q^{\pm s}, q^{\pm \underline{a}}, q^{\pm \underline{b}}]$, and therefore wherever it is holomorphic, it is non-zero.

We proceed to analyze the poles of the factors of $L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}})$ of the form

$$L_{\mathrm{PSR}}(a_i + \tfrac{1}{2}, \pi_{m'} \times \omega_i), L_{\mathrm{PSR}}(-a_i + \tfrac{1}{2}, \pi_{m'}^\vee \times \omega_i^{-1}),$$

$$L_{\mathrm{PSR}}(b_j + \tfrac{1}{2}, \pi_m \times \mu_j), L_{\mathrm{PSR}}(-b_j + \tfrac{1}{2}, \pi_m^\vee \times \mu_j^{-1}).$$

By [34, Lemma 7.2], since $\pi_{m'}$ and π_m are tempered and ω_i and μ_j are unitary, the above factors are holomorphic in the region $\|\mathrm{Re}\,\underline{a}\| < \frac{1}{2}$, $\|\mathrm{Re}\,\underline{b}\| < \frac{1}{2}$. As before, these factors of the form L_{PSR} are reciprocals of elements of the ring $\mathbb{C}[q^{\pm\underline{a}}, q^{\pm\underline{b}}]$, and therefore wherever they are holomorphic, they are non-zero. $\square$

We remark that by [34, Proposition 7.1], when $\pi_n^{\underline{a}}$ and $\pi_{n+1}^{\underline{b}}$ are unramified irreducible representations then

$$L(\tfrac{1}{2}, \mathrm{Std}(\pi_{n+1}^{\underline{b}}) \times \mathrm{Std}(\pi_n^{\underline{a}})) = L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}).$$

3.7. A lemma about $\Lambda_{f_\rho^s, v_{n+1}}$

In this section, we relate the Rankin–Selberg integrals from Section 3.2 to the L-factors arising from the doubling method from Section 3.1. This relation will be essential for the proof of our main result.

We use the same notations as in Section 3.3. Let $\mathcal{K}_{n+2} \subset \mathrm{G}(\mathrm{V}_{n+2})$ be a maximal compact subgroup in good position with respect to Q.

Lemma 3.7.1. *For $g_{n+2} \in \mathrm{G}(\mathrm{V}_{n+2})$, the following are equivalent:*

(1) $i(g_{n+2}, \mathrm{id}_{\mathrm{V}_n}) \in P_{\mathrm{V}_{n+1}^\triangle}$.
(2) $g_{n+2} \in Q$ and g_{n+2} has trivial $\mathrm{G}(\mathrm{V}_n)$ Levi part.

Proof. Let $p = i(g_{n+2}, \mathrm{id}_{\mathrm{V}_n})$. Write $g_{n+2}(b + \overline{b}) = w_n + \lambda b + \lambda' \overline{b}$, where $w_n \in \mathrm{V}_n$, $\lambda, \lambda' \in E$. Then

$$p(b + \overline{b}) = g_{n+2}(b + \overline{b}) = w_n + \lambda b + \lambda' \overline{b}.$$

We have that $p(b+\overline{b}) \in \mathrm{V}^{\Delta}_{n+1}$ if and only if $w_n = 0$ and $\lambda = \lambda'$. This is equivalent to $g_{n+2}(b+\overline{b}) = \lambda(b+\overline{b})$, for some $\lambda \in E^\times$, which by definition is equivalent to $g_{n+2} \in Q$.

Let $v_n \in \mathrm{V}_n$. Then

$$p(v_n + \overline{v}_n) = g_{n+2}v_n + \overline{v}_n.$$

By writing $g_{n+2}v_n = v'_n + \mu b + \mu'\overline{b}$, where $v'_n \in \mathrm{V}_n$, and $\mu, \mu' \in E$, we get that $p(v_n + \overline{v}_n) \in \mathrm{V}^{\Delta}_{n+1}$, if and only if $v'_n + \mu b = v_n + \mu' b$, which is equivalent to $v_n = v'_n$ and $\mu' = \mu$. Hence, we get that $p(v_n + \overline{v}_n) \in \mathrm{V}^{\Delta}_{n+1}$ if and only if $g_{n+2}v_n = v_n + \mu(b+\overline{b})$, for some $\mu \in E$. This is equivalent to saying that the $\mathrm{G}(\mathrm{V}_n)$ Levi part of g_{n+2} is trivial.

Therefore, we have shown that $i(g_{n+2}, g_n)$ preserves the subspace

$$E(b+\overline{b}) \oplus \mathrm{span}_E \left\{ v_n + \overline{v_n} \mid v_n \in \mathrm{V}_{n+1} \right\} = \mathrm{V}^{\Delta}_{n+1}$$

if and only if $g_{n+2} \in Q$ and g_{n+2} has trivial $\mathrm{G}(\mathrm{V}_n)$ Levi part. $\square$

As an immediate consequence of the lemma, we get the following corollary:

Corollary 3.7.2. *Let $K_0 \subset \mathrm{G}(\mathrm{V}_{n+2})$ be a compact open subgroup. Let $g_{n+2} \in \mathrm{G}(\mathrm{V}_{n+2})$, such that $i(g_{n+2}, \mathrm{id}_{\mathrm{V}_n}) \in P_{\mathrm{V}^{\Delta}_{n+1}} i(K_0 \times \{\mathrm{id}_{\mathrm{V}_n}\})$. Then $g_{n+2} = q \cdot k_0$ for some $q \in Q$ that has trivial $\mathrm{G}(\mathrm{V}_n)$ Levi part, and some $k_0 \in K_0$.*

The following lemma shows that holomorphic sections of π^s_{n+2} can be represented as finite sums of elements of the form $\Lambda_{f^s_\rho, v_n}$.

Lemma 3.7.3. *For every standard section (with respect to $\mathcal{K}_{n+2}$) $f^s_{n+2} \in \pi^s_{n+2}$, there exist holomorphic sections $(f^s_{\rho,i})_{i=1}^N \subset \rho_{\chi,s}$ and vectors $(v_{n,i})_{i=1}^N \subset \pi_n$, such that*

$$f^s_{n+2} = \sum_{j=1}^{N} \Lambda_{f^s_{\rho,j}, v_{n,j}}.$$

Proof. Denote by $\ell_{Q,E^\times} : Q \to E^\times$, $\ell_{Q,\mathrm{G}(\mathrm{V}_n)} : Q \to \mathrm{G}(\mathrm{V}_n)$ the projections of Q on its Levi parts. Let $K_0 \subset \mathcal{K}_{n+2}$ be a normal compact open subgroup, such that

$|\cdot|^s\chi$ is trivial on $\ell_{Q,E^\times}(Q\cap K_0)$, and let $v_0\in\pi_n$, such that v_0 is invariant under the π_n action of $\ell_{Q,\mathrm{G}(\mathrm{V}_n)}(Q\cap K_0)$. Consider the section $f_{n+2}^{K_0,v_0,s}$ defined by ($q\in Q$, $k\in\mathcal{K}_{n+2}$):

$$f_{n+2}^{K_0,v_0,s}(q\cdot k)=\begin{cases}\delta_Q^{\frac{1}{2}}(q)(|\cdot|^s\chi\boxtimes\pi_n)(q)v_0 & k\in K_0,\\ 0 & k\notin K_0.\end{cases}$$

Every standard section of π_{n+2}^s is a finite sum of $\mathcal{K}_{n+2}$ right translations of sections of the form $f_{n+2}^{K_0,v_0,s}$. We notice that for $h_{n+2}\in\mathrm{G}(\mathrm{V}_{n+2})$

$$\Lambda_{\rho_{\chi,s}(i(h_{n+2},1))f_\rho^s,v_0}=\pi_{n+2}^s(h_{n+2})\Lambda_{f_\rho^s,v_0}.$$

Hence, it suffices to prove that every section of the form $f_{n+2}^{K_0,v_0,s}$ can be represented as some $\Lambda_{\rho_{\chi,s}f_\rho^s,v_0'}$, for some holomorphic section $f_\rho^s\in\rho_{\chi,s}$ and some $v_0'\in\pi_n$.

Let $f_{n+2}^{K_0,v_0,s}$ be a section as above. Since the orbit

$$P_{\mathrm{V}_{n+1}^\Delta}\cdot i(\mathrm{G}(\mathrm{V}_{n+1})\times\mathrm{G}(\mathrm{V}_{n+1}))=P_{\mathrm{V}_{n+1}^\Delta}\cdot i(\mathrm{G}(\mathrm{V}_{n+1})\times\{\mathrm{id}_{\mathrm{V}_{n+1}}\})$$

is open, it follows that $P_{\mathrm{V}_{n+1}^\Delta}\cdot i(K_0\times\{\mathrm{id}_{\mathrm{V}_n}\})$ is open. Let f_ρ^s be the section of $\rho_{\chi,s}$, which is supported on $P_{\mathrm{V}_{n+1}^\Delta}\cdot i(K_0\times\{\mathrm{id}_{\mathrm{V}_n}\})$, whose value on $i(K_0\times\{\mathrm{id}_{\mathrm{V}_n}\})$ is 1. Then for $g_{n+2}\in\mathrm{G}(\mathrm{V}_{n+2})$ we have

$$\Lambda_{f_\rho^s,v_0}(g_{n+2})=\int_{\mathrm{G}(\mathrm{V}_n)}f_\rho^s(i(g_n^{-1}g_{n+2},\mathrm{id}_{\mathrm{V}_n}))\pi_n(g_n)v_0\mathrm{d}g_n. \tag{3.7.1}$$

In order for this integral not to vanish, we must have that

$$i(g_n^{-1}g_{n+2},\mathrm{id}_{\mathrm{V}_n})\in P_{\mathrm{V}_{n+1}^\Delta}\cdot i(K_0\times\{\mathrm{id}_{\mathrm{V}_n}\}),$$

for some $g_n\in\mathrm{G}(\mathrm{V}_n)$. By Corollary 3.7.2, this is equivalent to $g_n^{-1}g_{n+2}=q\cdot k_0$ for some $q\in Q$ having trivial $\mathrm{G}(\mathrm{V}_n)$ Levi part, and some $k_0\in K_0$. Since $\mathrm{G}(\mathrm{V}_n)\subset Q$, we get that if the integral in (3.7.1) does not vanish, then $g_{n+2}=q'\cdot k_0$, where $q'\in Q$ and $k_0\in K_0$.

Now suppose $g_{n+2} = q \cdot k_0$, where $q \in Q$, and $k_0 \in K_0$. Since $\Lambda_{f^s_\rho, v_0} \in \pi^s_{n+2}$, we have

$$\Lambda_{f^s_\rho, v_0}(g_{n+2}) = \Lambda_{f^s_\rho, v_0}(q \cdot k_0) = \delta_Q^{\frac{1}{2}}(q)(|\cdot|^s \chi \boxtimes \pi_n)(q)\Lambda_{f^s_\rho, v_0}(k_0).$$

Write

$$\Lambda_{f^s_\rho, v_0}(k_0) = \int_{\mathrm{G}(\mathrm{V}_n)} f^s_\rho(i(g_n^{-1} \cdot k_0, \mathrm{id}_{\mathrm{V}_n}))\pi_n(g_n)v_0 \mathrm{d}g_n. \tag{3.7.2}$$

By its construction, f^s_ρ is invariant under right translations of $i(K_0 \times \{\mathrm{id}_{\mathrm{V}_n}\})$. Therefore, we may assume without loss of generality that $k_0 = \mathrm{id}_{\mathrm{V}_{n+2}}$.

The integrand in (3.7.2) is supported on $g_n \in \mathrm{G}(\mathrm{V}_n)$, such that $i(g_n^{-1}, \mathrm{id}_{\mathrm{V}_n}) \in P_{\mathrm{V}^\Delta_{n+1}} \cdot i(K_0 \times \{\mathrm{id}_{\mathrm{V}_n}\})$. By Corollary 3.7.2, the last condition is equivalent to

$$g_n^{-1} = q' \cdot k'_0,$$

where $q' \in Q$ has trivial $\mathrm{G}(\mathrm{V}_n)$ Levi part, and $k'_0 \in K_0$. This implies $(k'_0)^{-1} = g_n q'$, which implies that $g_n q' \in Q \cap K_0$, and hence g_n is in $\ell_{Q,\mathrm{G}(\mathrm{V}_n)}(Q \cap K_0)$.

Conversely, suppose that $g_n \in \ell_{Q,\mathrm{G}(\mathrm{V}_n)}(Q \cap K_0)$, then there exists $q' \in Q$ with trivial $\mathrm{G}(\mathrm{V}_n)$ Levi part and $k'_0 \in K_0$, such that $g_n q' = k'_0$. Since $g_n q' \in Q \cap K_0$, by the choice of K_0 and v_0, we have

$$\delta_Q^{\frac{1}{2}}(g_n q') \cdot (|\cdot|^s \chi \boxtimes \pi_n)(g_n q')v_0 = v_0,$$

which implies that $\pi_n(g_n)v_0 = v_0$ and $(\delta_Q^{\frac{1}{2}} |\cdot|^s \chi)(\ell_{Q,E^\times}(q')) = 1$. This implies

$$f^s_\rho(i(g_n^{-1}, \mathrm{id}_{\mathrm{V}_n}))\pi_n(g_n)v_0 = f^s_\rho(i(q' \cdot (k'_0)^{-1}, \mathrm{id}_{\mathrm{V}_n}))v_0 = v_0,$$

where we used the fact that f^s_ρ is right invariant under $i(K_0 \times \{\mathrm{id}_{\mathrm{V}_n}\})$, that $\det_\Delta(q') = \ell_{Q,E^\times}(q')$, and that $\delta_Q(q') = \delta_{P_{\mathrm{V}^\Delta_{n+1}}}(i(q', \mathrm{id}_{\mathrm{V}_n}))$.

To summarize, we get that the integrand in (3.7.2) is supported on $g_n \in \ell_{Q,\mathrm{G}(\mathrm{V}_n)}(Q \cap K_0)$, and that for such g_n the integrand equals v_0. Therefore, we get that $\Lambda_{f^s_\rho, v_0}(k_0) = \Lambda_{f^s_\rho, v_0}(\mathrm{id}_{\mathrm{V}_{n+2}}) = C_{K_0} \cdot v_0$, where $C_{K_0} = \mathrm{Vol}(\ell_{Q,\mathrm{G}(\mathrm{V}_n)}(Q \cap K_0))$ is the volume of $\ell_{Q,\mathrm{G}(\mathrm{V}_n)}(Q \cap K_0)$ in $\mathrm{G}(\mathrm{V}_n)$. Hence, we showed that

$$\Lambda_{f^s_\rho, v_0} = C_{K_0} \cdot f^{K_0, v_0, s}_{n+2}.$$

Since $\ell_{Q,\mathrm{G}(\mathrm{V}_n)}(Q \cap K_0) \subset \mathrm{G}(\mathrm{V}_n)$ is compact and open, we get that $C_{K_0} > 0$, and therefore we have that

$$C^{-1}_{K_0} \cdot \Lambda_{f^s_\rho, v_0} = f^{K_0, v_0, s}_{n+2},$$

as required. □

As a result of the lemma, we get the following corollary:

Corollary 3.7.4. *For every holomorphic section $f^s_{n+2} \in \pi^s_{n+2}$, and every $v_{n+1} \in \pi_{n+1}$, there exist holomorphic sections $(f^s_{\rho,j})^N_{j=1} \subset \rho_{\chi,s}$ and $(v^\vee_{n+1,j})^N_{j=1} \subset \pi^\vee_{n+1}$, such that*

$$C_{\pi_{n+1}, \pi^s_{n+2}}(v_{n+1}, f^s_{n+2}) = \sum_{j=1}^{N} Z(f^s_{\rho,j}, v_{n+1}, v^\vee_{n+1,j}),$$

where $C_{\pi_{n+1}, \pi^s_{n+2}}$ is defined in Section 3.2.

Proof. By Lemma 3.7.3, there exist holomorphic sections $(f^s_{\rho,j})^N_{j=1} \subset \rho_{\chi,s}$ and vectors $(v_{n,j})^N_{j=1} \subset \pi_n$, such that $f^s_{n+2} = \sum_{j=1}^N \Lambda_{f^s_{\rho,j}, v_{n,j}}$, and then

$$C_{\pi_{n+1}, \pi^s_{n+2}}(v_{n+1}, f^s_{n+2}) = \sum_{j=1}^{N} C_{\pi_{n+1}, \pi^s_{n+2}}(v_{n+1}, \Lambda_{f^s_{\rho,j}, v_{n,j}}).$$

By the identity in (3.3.1), we have that

$$\begin{aligned} &C_{\pi_{n+1}, \pi^s_{n+2}}(v_{n+1}, \Lambda_{f^s_{\rho,j}, v_{n,j}}) \\ &= \int_{\mathrm{G}(\mathrm{V}_{n+1})} f^s_{\rho,j}(i(g_{n+1}, \mathrm{id}_{\mathrm{V}_{n+1}})) c_{\pi_n, \pi_{n+1}}(v_{n,j}, \pi_{n+1}(g_{n+1}) v_{n+1}) \mathrm{d}g_{n+1}. \end{aligned} \tag{3.7.3}$$

By [29, Lemma 4.3], for every j, there exists $v^\vee_{n+1,j} \in \pi^\vee_{n+1}$, such that $Z(f^s_{\rho,j}, v_{n+1}, v^\vee_{n+1,j})$ equals the right hand side of (3.7.3), as required. □

Combining this corollary with the discussion in Section 3.6, we get the following result for the case $\pi_n = \pi^{\underline{a}}_n$, $\pi_{n+1} = \pi^{\underline{b}}_{n+1}$, $\pi^s_{n+2} = \pi^{(s,\underline{a})}_{n+2}$:

PROPOSITION 3.7.5. *Let $\underline{a}, \underline{b}$ be fixed. Then for any holomorphic section* $f_{n+2}^{(s,\underline{a})} \in \pi_{n+2}^{(s,\underline{a})}$, *any* $f_{n+1}^{\underline{b}} \in \pi_{n+1}^{\underline{b}}$, *and any* $c_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}} \in \mathrm{Hom}_{\mathrm{G}(\mathrm{V}_n)}(\pi_n^{\underline{a}} \otimes \pi_{n+1}^{\underline{b}}, 1)$, *we have that the following quotient is a polynomial (an element of* $\mathbb{C}[q^{\pm s}]$*):*

$$\frac{C_{\pi_{n+1}^{\underline{b}},\pi_{n+2}^{(s,\underline{a})}}(f_{n+1}^{\underline{b}}, f_{n+2}^{(s,\underline{a})})}{L(s,\chi,\pi_{n+1}^{\underline{b}})}.$$

Similarly, for any holomorphic sections $f_{n+2}^{\vee(t,\underline{a})} \in \pi_{n+2}^{\vee(t,\underline{a})}$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1}^{\vee\underline{b}}$, *and any* $c_{\pi_n^{\vee\underline{a}},\pi_{n+1}^{\vee\underline{b}}} \in \mathrm{Hom}_{\mathrm{G}(\mathrm{V}_n)}(\pi_n^{\vee\underline{a}} \otimes \pi_{n+1}^{\vee\underline{b}}, 1)$, *the following quotient is polynomial (an element of* $\mathbb{C}[q^{\pm t}]$*):*

$$\frac{C_{\pi_n^{\vee\underline{a}},\pi_{n+1}^{\vee\underline{b}}}(f_{n+1}^{\vee\underline{b}}, f_{n+2}^{\vee(t,\underline{a})})}{L(-t,\chi^{-1}\circ\sigma,\pi_{n+1}^{\vee\underline{b}})}.$$

3.8. Proof of the main result

In this section, we prove Theorem 2.7.1. Our proof is by induction, and our construction is based on the identity in (3.4.1). Let $f_n^{\underline{a}} \in \pi_n^{\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1}^{\underline{b}}$, $f_n^{\vee\underline{a}} \in \pi_n^{\vee\underline{a}}$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1}^{\vee\underline{b}}$ be holomorphic sections. Recall that we are looking for a holomorphic extension for the assignment

$$(\underline{a},\underline{b}) \mapsto \alpha^{\natural}_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}) = \frac{\alpha_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})}{L(\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}})}.$$

We will show that such extension exists. Furthermore, we will show that

$$\alpha^{\natural}_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}) = c^{\natural}_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}) \cdot c^{\natural}_{\pi_n^{\vee\underline{a}},\pi_{n+1}^{\vee\underline{b}}}(f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}),$$

where

$$c^{\natural}_{\pi_n^{\underline{a}},\pi_{n+1}^{\underline{b}}} \in \mathrm{Hom}_{\mathrm{G}(\mathrm{V}_n)}(\pi_n^{\underline{a}} \otimes \pi_{n+1}^{\underline{b}}, 1),$$
$$c^{\natural}_{\pi_n^{\vee\underline{a}},\pi_{n+1}^{\vee\underline{b}}} \in \mathrm{Hom}_{\mathrm{G}(\mathrm{V}_n)}(\pi_n^{\vee\underline{a}} \otimes \pi_{n+1}^{\vee\underline{b}}, 1),$$

so that the assignments

$$(\underline{a}, \underline{b}) \mapsto c^{\natural}_{\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}),$$

$$(\underline{a}, \underline{b}) \mapsto c^{\natural}_{\pi_n^{\vee\underline{a}}, \pi_{n+1}^{\vee\underline{b}}}(f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})$$

are holomorphic. We write for short $\alpha^{\natural}_{\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}} = c^{\natural}_{\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}} \boxtimes c^{\natural}_{\pi_n^{\vee\underline{a}}, \pi_{n+1}^{\vee\underline{b}}}$.

If $n = \min(m, m')$ and $n+1 = \max(m, m')$, then $\pi_n^{\underline{a}} = \pi_n$ and $\pi_{n+1}^{\underline{b}} = \pi_{n+1}$, and therefore we do not have parameters, and $L(\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}) = 1$. Since π_m and $\pi_{m'}$ are both tempered, the assignment $\alpha^{\natural}_{\pi_n, \pi_{n+1}} = \alpha_{\pi_n, \pi_{n+1}}$ is defined for all vectors. By [1], we have

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathrm{G}(\mathrm{V}_n)}(\pi_n \otimes \pi_{n+1}, 1) \leq 1,$$

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathrm{G}(\mathrm{V}_n)}(\pi_n^{\vee} \otimes \pi_{n+1}^{\vee}, 1) \leq 1,$$

and therefore there exist $c_{\pi_n, \pi_{n+1}}$ and $c_{\pi_n^{\vee}, \pi_{n+1}^{\vee}}$, such that

$$\alpha^{\natural}_{\pi_n, \pi_{n+1}} = \alpha_{\pi_n, \pi_{n+1}} = c_{\pi_n, \pi_{n+1}} \boxtimes c_{\pi_n^{\vee}, \pi_{n+1}^{\vee}}.$$

Since we do not have parameters in this case, we do not need to prove anything regarding holomorphicity.

Suppose that $\alpha^{\natural}_{\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}} = c^{\natural}_{\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}} \boxtimes c^{\natural}_{\pi_n^{\vee\underline{a}}, \pi_{n+1}^{\vee\underline{b}}}$ is already defined. We move to construct $\alpha^{\natural}_{\pi_{n+1}^{\underline{b}}, \pi_{n+2}^{(s,\underline{a})}} = c^{\natural}_{\pi_{n+1}^{\underline{b}}, \pi_{n+2}^{(s,\underline{a})}} \boxtimes c^{\natural}_{\pi_{n+1}^{\vee\underline{b}}, \pi_{n+2}^{\vee(s,\underline{a})}}$. Let $f_{n+1}^{\underline{b}}$, $f_{n+2}^{(s,\underline{a})}$, $f_{n+1}^{\vee\underline{b}}$, $f_{n+2}^{\vee(s,\underline{a})}$ be holomorphic sections of $\pi_{n+1}^{\underline{b}}$, $\pi_{n+2}^{(s,\underline{a})}$, $\pi_{n+1}^{\vee\underline{b}}$, $\pi_{n+2}^{\vee(s,\underline{a})}$, respectively. If $\underline{a}$, $\underline{b}$, s are imaginary and fixed, we can use the identity in (3.4.1), which reads

$$\begin{aligned}
\text{(3.8.1)} \qquad & \alpha_{\pi_{n+1}^{\underline{b}}, \pi_{n+2}^{(s,\underline{a})}}(f_{n+1}^{\underline{b}}, f_{n+2}^{(s,\underline{a})}; f_{n+1}^{\vee\underline{b}}, f_{n+2}^{\vee(s,\underline{a})}) \\
&= \int_{(\mathrm{G}(\mathrm{V}_n)\backslash \mathrm{G}(\mathrm{V}_{n+1}))^2} \mathrm{d}(g_{n+1}, g'_{n+1}) \\
&\quad \times \alpha_{\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}}(f_{n+2}^{(s,\underline{a})}(g_{n+1}), \pi_{n+1}^{\underline{b}}(g_{n+1}) f_{n+1}^{\underline{b}}; f_{n+2}^{\vee(s,\underline{a})}(g'_{n+1}), \pi_{n+1}^{\vee\underline{b}}(g'_{n+1}) f_{n+1}^{\vee\underline{b}}).
\end{aligned}$$

Using the fact that $\alpha^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}} = c^\natural_{\pi^{\underline{a}}_n,\pi^{\underline{b}}_{n+1}} \boxtimes c^\natural_{\pi^{\vee\underline{a}}_n,\pi^{\vee\underline{b}}_{n+1}}$ and using (3.6.1), we get

(3.8.2)

$$\begin{aligned}
&\alpha^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}; f^{\vee\underline{b}}_{n+1}, f^{\vee(s,\underline{a})}_{n+2})\\
&= \frac{1}{L(s,\chi,\pi^{\underline{b}}_{n+1})}\int_{\mathrm{G}(\mathrm{V}_n)\backslash\,\mathrm{G}(\mathrm{V}_{n+1})} c^\natural_{\pi^{\underline{a}}_n,\pi^{\underline{b}}_{n+1}}(f^{(s,\underline{a})}_{n+2}(g_{n+1}), \pi^{\underline{b}}_{n+1}(g_{n+1})f^{\underline{b}}_{n+1})\mathrm{d}g_{n+1}\\
&\quad\times \frac{1}{L(-s,\chi^{-1,\sigma},\pi^{\vee\underline{b}}_{n+1})}\int_{\mathrm{G}(\mathrm{V}_n)\backslash\,\mathrm{G}(\mathrm{V}_{n+1})} c^\natural_{\pi^{\vee\underline{a}}_n,\pi^{\vee\underline{b}}_{n+1}}(f^{\vee(s,\underline{a})}_{n+2}(g'_{n+1}), \pi^{\vee\underline{b}}_{n+1}(g'_{n+1})f^{\vee\underline{b}}_{n+1})\mathrm{d}g'_{n+1}.
\end{aligned}$$

We are going to use (3.8.2) in order to define $c^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}$ and $c^\natural_{\pi^{\vee\underline{b}}_{n+1},\pi^{\vee(s,\underline{a})}_{n+2}}$, and use them to extend the definition of $\alpha^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}$.

We introduce a new variable t, which is a parameter for an unramified character of $E^\times$, and consider the integrals

(3.8.3)

$$\begin{aligned}
&c^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2})\\
&= \frac{1}{L(s,\chi,\pi^{\underline{b}}_{n+1})}\int_{\mathrm{G}(\mathrm{V}_n)\backslash\,\mathrm{G}(\mathrm{V}_{n+1})} c^\natural_{\pi^{\underline{a}}_n,\pi^{\underline{b}}_{n+1}}(f^{(s,\underline{a})}_{n+2}(g_{n+1}), \pi^{\underline{b}}_{n+1}(g_{n+1})f^{\underline{b}}_{n+1})\mathrm{d}g_{n+1},\\
&c^\natural_{\pi^{\vee\underline{b}}_{n+1},\pi^{\vee(t,\underline{a})}_{n+2}}(f^{\vee\underline{b}}_{n+1}, f^{\vee(t,\underline{a})}_{n+2})\\
&= \frac{1}{L(-t,\chi^{-1}\circ\sigma,\pi^{\vee\underline{b}}_{n+1})}\int_{\mathrm{G}(\mathrm{V}_n)\backslash\,\mathrm{G}(\mathrm{V}_{n+1})} c^\natural_{\pi^{\vee\underline{a}}_n,\pi^{\vee\underline{b}}_{n+1}}(f^{\vee(t,\underline{a})}_{n+2}(g'_{n+1}), \pi^{\vee\underline{b}}_{n+1}(g'_{n+1})f^{\vee\underline{b}}_{n+1})\mathrm{d}g'_{n+1},
\end{aligned}$$

where $f^{\underline{b}}_{n+1} \in \pi^{\underline{b}}_{n+1}$, $f^{\vee\underline{b}}_{n+1} \in \pi^{\vee\underline{b}}_{n+1}$, $f^{(s,\underline{a})}_{n+2} \in \pi^{(s,\underline{a})}_{n+2}$, $f^{\vee(t,\underline{a})}_{n+2} \in \pi^{\vee(t,\underline{a})}_{n+2}$ are holomorphic sections.

Proposition 3.8.1. *For fixed $\underline{a}$ and $\underline{b}$, the integrals in (3.8.3) absolutely converge for* $\operatorname{Re} s$ *large and* $\operatorname{Re}(-t)$ *large (both depending on $\pi^{\underline{a}}_n$ and $\pi^{\underline{b}}_{n+1}$), respectively. $c^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}$ and $c^\natural_{\pi^{\vee\underline{b}}_{n+1},\pi^{\vee(t,\underline{a})}_{n+2}}$ have meromorphic continuations to the entire plane, which we continue to denote by the same symbols. These meromorphic continuations are* *actually polynomial, i.e.,*

$$c^{\natural}_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}) \in \mathbb{C}[q^{\pm s}],$$

$$c^{\natural}_{\pi^{\vee\underline{b}}_{n+1},\pi^{\vee(t,\underline{a})}_{n+2}}(f^{\vee\underline{b}}_{n+1}, f^{\vee(t,\underline{a})}_{n+2}) \in \mathbb{C}[q^{\pm t}].$$

Proof. The integrals in (3.8.3) are Rankin–Selberg integrals, as in Section 3.2. We have that these integrals converge absolutely for $\operatorname{Re} s$ large and $\operatorname{Re}(-t)$ large, respectively. In addition, they have meromorphic continuations to the entire plane, which are rational functions in q^{-s} and q^{-t}, respectively. By Proposition 3.7.5, since we divide by the appropriate L-factors, the meromorphic continuations are polynomial. □

We denote

$$\beta^{\natural}_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}; f^{\vee\underline{b}}_{n+1}, f^{\vee(s,\underline{a})}_{n+2}) = c^{\natural}_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}) c^{\natural}_{\pi^{\vee\underline{b}}_{n+1},\pi^{\vee(s,\underline{a})}_{n+2}}(f^{\vee\underline{b}}_{n+1}, f^{\vee(s,\underline{a})}_{n+2}).$$

Our next task to is to show that $\beta^{\natural}_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}$ is an extension of $\alpha^{\natural}_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}$. In order to show this, we need the following lemma:

Lemma 3.8.2. *Suppose that there exist holomorphic sections $\varphi^{\underline{a}}_n \in \pi^{\underline{a}}_n$, $\varphi^{\underline{b}}_{n+1} \in \pi^{\underline{b}}_{n+1}$, $\varphi^{\vee\underline{a}}_n \in \pi^{\vee\underline{a}}_n$, $\varphi^{\vee\underline{b}}_{n+1} \in \pi^{\vee\underline{b}}_{n+1}$, such that $\alpha_{\pi^{\underline{a}}_n,\pi^{\underline{b}}_{n+1}}(\varphi^{\underline{a}}_n, \varphi^{\underline{b}}_{n+1}; \varphi^{\vee\underline{a}}_n, \varphi^{\vee\underline{b}}_{n+1}) = 1$, for every $\underline{a}$ and $\underline{b}$ imaginary. Then there exist holomorphic sections $\varphi^{(s,\underline{a})}_{n+2} \in \pi^{(s,\underline{a})}_{n+2}$ and $\varphi^{\vee(s,\underline{a})}_{n+2} \in \pi^{\vee(s,\underline{a})}_{n+2}$, such that:*

1. *$\alpha_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) = 1$, for all $\underline{a}$, $\underline{b}$ and s imaginary.*
2. *$\beta^{\natural}_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) = L(\pi^{\underline{b}}_{n+1}, \pi^{(s,\underline{a})}_{n+2})^{-1}$, for all $\underline{a}$ and $\underline{b}$ imaginary and every s.*

Proof. We realize $\pi^{(s,\underline{a})}_{n+2}$ and $\pi^{\vee(s,\underline{a})}_{n+2}$ as in Section 3.5.

When all parameters are imaginary, all representations are tempered, and we can use the identity in (3.8.1). The integral in (3.8.1) absolutely converges in this case [3, Claim (7.4.10)].

Let $K_0 \subset \mathrm{G}(\mathrm{V}_{n+1})$ be a compact open subgroup, such that $\varphi_{n+1}^{\underline{b}}$, $\varphi_{n+1}^{\vee\underline{b}}$ are invariant under its action, and so that $\varphi_n^{\underline{a}}$ (respectively $\varphi_n^{\vee\underline{a}}$) is invariant under the $|\cdot|^s \chi \boxtimes \pi_n$ action (respectively, the $\chi^{-1} |\cdot|^{-s} \boxtimes \pi_n^\vee$ action) of $K_0 \cap P_{Ef_+,\mathrm{G}(\mathrm{V}_{n+2})}$. We have that $\mathrm{G}(\mathrm{V}_{n+1}) P_{Ef_+,\mathrm{G}(\mathrm{V}_{n+2})}$ is a dense open subset of $\mathrm{G}(\mathrm{V}_{n+2})$, and hence so is $P_{Ef_+,\mathrm{G}(\mathrm{V}_{n+2})}\, \mathrm{G}(\mathrm{V}_{n+1})$. This implies that $P_{Ef_+,\mathrm{G}(\mathrm{V}_{n+2})} K_0$ is an open subset of $\mathrm{G}(\mathrm{V}_{n+2})$. Let $f_{n+2}^{(s,\underline{a})}$ and $f_{n+2}^{\vee(s,\underline{a})}$ be sections of $\pi_{n+2}^{(s,\underline{a})}$ and $\pi_{n+2}^{\vee(s,\underline{a})}$, supported on the open subset $P_{Ef_+,\mathrm{G}(\mathrm{V}_{n+2})} K_0$, such that their restrictions to K_0 are $\varphi_n^{\underline{a}}$ and $\varphi_n^{\vee\underline{a}}$, respectively. Then we have that $f_{n+2}^{(s,\underline{a})}$ and $f_{n+2}^{\vee(s,\underline{a})}$ are holomorphic sections, and by (3.8.1), we get that

$$\alpha_{\pi_{n+1}^{\underline{b}}, \pi_{n+2}^{(s,\underline{a})}} (\varphi_{n+1}^{\underline{b}}, f_{n+2}^{(s,\underline{a})}; \varphi_{n+1}^{\vee\underline{b}}, f_{n+2}^{\vee(s,\underline{a})}) = C_{K_0}^2 \cdot \alpha_{\pi_n^{\underline{a}}, \pi_{n+1}^{\underline{b}}} (\varphi_n^{\underline{a}}, \varphi_{n+1}^{\underline{b}}; \varphi_n^{\vee\underline{a}}, \varphi_{n+1}^{\vee\underline{b}}) = C_{K_0}^2,$$

where $C_{K_0} = \mathrm{Vol}(\mathrm{G}(\mathrm{V}_n) \backslash \mathrm{G}(\mathrm{V}_n) K_0)$ is the volume of $\mathrm{G}(\mathrm{V}_n) \backslash \mathrm{G}(\mathrm{V}_n) K_0$ in $\mathrm{G}(\mathrm{V}_n) \backslash \mathrm{G}(\mathrm{V}_{n+1})$. Therefore, by choosing $\varphi_{n+2}^{(s,\underline{a})} = \frac{1}{C_{K_0}} f_{n+2}^{(s,\underline{a})}$ and $\varphi_{n+2}^{\vee(s,\underline{a})} = \frac{1}{C_{K_0}} f_{n+2}^{\vee(s,\underline{a})}$, we get the desired equality as in (1).

Similarly, we have that $c^\natural_{\pi_{n+1}^{\underline{b}}, \pi_{n+2}^{(s,\underline{a})}} (\varphi_{n+1}^{\underline{b}}, f_{n+2}^{(s,\underline{a})})$ and $c^\natural_{\pi_{n+1}^{\vee\underline{b}}, \pi_{n+2}^{\vee(s,\underline{a})}} (f_{n+1}^{\vee\underline{b}}, f_{n+2}^{\vee(s,\underline{a})})$ are given by (3.8.3), for $\mathrm{Re}\, s$ large and $\mathrm{Re}(-s)$ large, respectively. We get from the construction of $f_{n+2}^{(s,\underline{a})}$ and $f_{n+2}^{\vee(s,\underline{a})}$ that

$$\begin{aligned} c^\natural_{\pi_{n+1}^{\underline{b}}, \pi_{n+2}^{(s,\underline{a})}} (\varphi_{n+1}^{\underline{b}}, f_{n+2}^{(s,\underline{a})}) &= \frac{1}{L(s, \chi, \pi_{n+1}^{\underline{b}})} C_{K_0}, \\ c^\natural_{\pi_{n+1}^{\vee\underline{b}}, \pi_{n+2}^{\vee(s,\underline{a})}} (f_{n+1}^{\vee\underline{b}}, f_{n+2}^{\vee(s,\underline{a})}) &= \frac{1}{L(-s, \chi^{-1} \circ \sigma, \pi_{n+1}^{\vee\underline{b}})} C_{K_0}. \end{aligned}$$

This implies that for all s,

$$\beta^\natural_{\pi_{n+1}^{\underline{b}}, \pi_{n+2}^{(s,\underline{a})}} (\varphi_{n+1}^{\underline{b}}, \varphi_{n+2}^{(s,\underline{a})}; \varphi_{n+1}^{\vee\underline{b}}, \varphi_{n+2}^{\vee(s,\underline{a})}) = L(\pi_{n+1}^{\underline{b}}, \pi_{n+2}^{(s,\underline{a})})^{-1}.$$

This shows (2). $\square$

Using the last lemma repeatedly, we are able to conclude the following:

Corollary 3.8.3. *Let $m_0 = \min(m, m')$ and $m_0 + 1 = \max(m, m')$.*

(1) *Suppose that* $\underline{a}$, $\underline{b}$ *and* s *are imaginary. Then* $\alpha^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}} = \beta^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}$. *Furthermore,* $\alpha^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}$ *is not identically zero if and only if* $\alpha_{\pi_{m_0},\pi_{m_0+1}} \neq 0$.

(2) *Let* $f^{\underline{b}}_{n+1} \in \pi^{\underline{b}}_{n+1}$, $f^{(s,\underline{a})}_{n+2} \in \pi^{(s,\underline{a})}_{n+2}$, $f^{\vee\underline{b}}_{n+1} \in \pi^{\vee\underline{b}}_{n+1}$, $f^{\vee(s,\underline{a})}_{n+2} \in \pi^{\vee(s,\underline{a})}_{n+2}$ *be holomorphic sections. Then we have that* $\beta^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}, f^{\vee\underline{b}}_{n+1}, f^{\vee(s,\underline{a})}_{n+2}) \in \mathbb{C}[q^{\pm\underline{a}}, q^{\pm\underline{b}}, q^{\pm s}]$.

Proof. If $\alpha_{\pi_{m_0},\pi_{m_0+1}} = 0$, then we get by repeatedly using the recursive formula in (3.8.1) that $\alpha^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}} = 0$, and by repeatedly using (3.8.3) we get that $\beta^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}} = 0$. Suppose that $\alpha_{\pi_{m_0},\pi_{m_0+1}} \neq 0$. Then we can find vectors $\varphi_{m_0} \in \pi_{m_0}$, $\varphi_{m_0+1} \in \pi_{m_0+1}$, $\varphi^\vee_{m_0} \in \pi^\vee_{m_0}$, $\varphi^\vee_{m_0+1} \in \pi^\vee_{m_0+1}$, such that $\alpha_{\pi_{m_0},\pi_{m_0+1}}(\varphi_{m_0}, \varphi_{m_0+1}; \varphi^\vee_{m_0}, \varphi^\vee_{m_0+1}) = 1$. Using Lemma 3.8.2 repeatedly, we get that we can find holomorphic sections $\varphi^{\underline{b}}_{n+1} \in \pi^{\underline{b}}_{n+1}$, $\varphi^{\vee\underline{b}}_{n+1} \in \pi^{\vee\underline{b}}_{n+1}$, $\varphi^{(s,\underline{a})}_{n+2} \in \pi^{(s,\underline{a})}_{n+2}$, $\varphi^{\vee(s,\underline{a})}_{n+2} \in \pi^{\vee(s,\underline{a})}_{n+2}$, such that for every $\underline{a}, \underline{b}$ imaginary,

$$\begin{aligned}\alpha^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) &= \beta^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) \\ &= L(\pi^{\underline{b}}_{n+1}, \pi^{(s,\underline{a})}_{n+2})^{-1}.\end{aligned}$$

We have that both $\alpha^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}$ and $\beta^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}$ define elements of

$$\mathrm{Hom}_{\mathrm{G}(\mathrm{V}_n)}(\pi^{\underline{b}}_{n+1} \otimes \pi^{(s,\underline{a})}_{n+2}, 1) \boxtimes \mathrm{Hom}_{\mathrm{G}(\mathrm{V}_n)}(\pi^{\vee\underline{b}}_{n+1} \otimes \pi^{\vee(s,\underline{a})}_{n+2}, 1). \tag{3.8.4}$$

The representations $\pi^{\underline{b}}_{n+1}$ and $\pi^{(s,\underline{a})}_{n+2}$ are irreducible for $\underline{a}, \underline{b}, s$ outside of a finite union of hyperplanes, and in that case, by [1], we have that the space in (3.8.4) is at most one dimensional. Therefore, we must have that if $\underline{a}$, $\underline{b}$ and s are imaginary, then $\alpha^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}} = \beta^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}$. Therefore we have shown (1).

By Bernstein's rationality theorem (see for example [8, Section 3.2]), the above discussion shows that

$$\beta^\natural_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) \in \mathbb{C}(q^{\pm\underline{a}}, q^{\pm\underline{b}}, q^{\pm s}).$$

Since for every fixed $\underline{a}$ and $\underline{b}$, we have that

$$\beta^{\natural}_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(\varphi^{\underline{b}}_{n+1},\varphi^{(s,\underline{a})}_{n+2};\varphi^{\vee\underline{b}}_{n+1},\varphi^{\vee(s,\underline{a})}_{n+2}) \in \mathbb{C}[q^{\pm s}],$$

we must have that

$$\beta^{\natural}_{\pi^{\underline{b}}_{n+1},\pi^{(s,\underline{a})}_{n+2}}(\varphi^{\underline{b}}_{n+1},\varphi^{(s,\underline{a})}_{n+2};\varphi^{\vee\underline{b}}_{n+1},\varphi^{\vee(s,\underline{a})}_{n+2}) \in \mathbb{C}[q^{\pm\underline{a}},q^{\pm\underline{b}},q^{\pm s}].$$

This shows (2). $\square$

Bibliography

[1] Avraham Aizenbud, Dmitry Gourevitch, Stephen Rallis, and Gérard Schiffmann. Multiplicity one theorems. *Ann. of Math. (2)*, 172(2):1407–1434, 2010. 4, 22, 44, 48

[2] James Arthur. *The endoscopic classification of representations*, volume 61 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups. 7

[3] Raphaël Beuzart-Plessis. A local trace formula for the Gan-Gross-Prasad conjecture for unitary groups: the archimedean case. 2015. arXiv:1506.01452. 4, 31, 46

[4] Raphaël Beuzart-Plessis, Pierre-Henri Chaudouard, and Michał Zydor. The global Gan-Gross-Prasad conjecture for unitary groups: the endoscopic case. 2020. arXiv:2007.05601. 1

[5] Raphaël Beuzart-Plessis, Yifeng Liu, Wei Zhang, and Xinwen Zhu. Isolation of cuspidal spectrum, with application to the Gan-Gross-Prasad conjecture. *Ann. of Math. (2)*, 194(2):519–584, 2021. 1

[6] Valentin Blomer and Farrell Brumley. The role of the Ramanujan conjecture in analytic number theory. *Bull. Amer. Math. Soc. (N.S.)*, 50(2):267–320, 2013. 6

[7] Yuanqing Cai, Solomon Friedberg, and Eyal Kaplan. Doubling constructions: local and global theory, with an application to global functoriality for non-generic cuspidal representations. 2018. arXiv:1802.02637. 7

[8] J. W. Cogdell and I. I. Piatetski-Shapiro. Derivatives and L-functions for GL_n. In *Representation theory, number theory, and invariant theory*, volume 323 of *Progr. Math.*, pages 115–173. Birkhäuser/Springer, Cham, 2017. 19, 22, 36, 48

[9] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad. Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups. Number 346, pages 1–109. 2012. Sur les conjectures de Gross et Prasad. I. 2

[10] Stephen Gelbart, Ilya Piatetski-Shapiro, and Stephen Rallis. *Explicit constructions of automorphic L-functions*, volume 1254 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987. 26

[11] Jayce R Getz and Heekyoung Hahn. An introduction to automorphic representations. *Available at website: https://www. math. duke. edu/˜ hahn/GTM. pdf*, 2019. 13

[12] D. Ginzburg, I. Piatetski-Shapiro, and S. Rallis. L functions for the orthogonal group. *Mem. Amer. Math. Soc.*, 128(611):viii+218, 1997. 29, 30

[13] Benedict H. Gross and Dipendra Prasad. On the decomposition of a representation of SO_n when restricted to SO_{n-1}. *Canad. J. Math.*, 44(5):974–1002, 1992. 1

[14] Michael Harris. L-functions and periods of adjoint motives. *Algebra Number Theory*, 7(1):117–155, 2013. 16

[15] Michael Harris, Stephen S. Kudla, and William J. Sweet. Theta dichotomy for unitary groups. *J. Amer. Math. Soc.*, 9(4):941–1004, 1996. 36

[16] R. Neal Harris. The refined Gross-Prasad conjecture for unitary groups. *Int. Math. Res. Not. IMRN*, (2):303–389, 2014. 1, 5, 15

[17] Atsushi Ichino and Tamutsu Ikeda. On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture. *Geom. Funct. Anal.*, 19(5):1378–1425, 2010. 1, 5, 15

[18] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic representations. I. *Amer. J. Math.*, 103(3):499–558, 1981. 6

[19] Tasho Kaletha, Alberto Minguez, Sug Woo Shin, and Paul-James White. Endoscopic Classification of Representations: Inner Forms of Unitary Groups. 2014. arXiv:1409.3731. 7

[20] Eyal Kaplan. On the gcd of local Rankin-Selberg integrals for even orthogonal groups. *Compos. Math.*, 149(4):587–636, 2013. 28

[21] Erez Lapid and Zhengyu Mao. A conjecture on Whittaker-Fourier coefficients of cusp forms. *J. Number Theory*, 146:448–505, 2015. 20

[22] Erez M. Lapid and Stephen Rallis. On the local factors of representations of classical groups. In *Automorphic representations, L-functions and applications: progress and prospects*, volume 11 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 309–359. de Gruyter, Berlin, 2005. 26, 28, 34

[23] Yifeng Liu. Relative trace formulae toward Bessel and Fourier-Jacobi periods on unitary groups. *Manuscripta Math.*, 145(1-2):1–69, 2014. 3

[24] Luis Alberto Lomelí. Langlands program and Ramanujan conjecture: a survey. 2018. arXiv:1812.05203. 1

[25] Wenzhi Luo, Zeév Rudnick, and Peter Sarnak. On the generalized Ramanujan conjecture for $\mathrm{GL}(n)$. In *Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996)*, volume 66 of *Proc. Sympos. Pure Math.*, pages 301–310. Amer. Math. Soc., Providence, RI, 1999. 1

[26] Colette Mœ glin and Jean-Loup Waldspurger. La conjecture locale de Gross-Prasad pour les groupes spéciaux orthogonaux: le cas général. Number 347, pages 167–216. 2012. Sur les conjectures de Gross et Prasad. II. 6, 16, 17

[27] Chung Pang Mok. Endoscopic classification of representations of quasi-split unitary groups. *Mem. Amer. Math. Soc.*, 235(1108):vi+248, 2015. 7

[28] I. Piatetski-Shapiro and S. Rallis. ϵ factor of representations of classical groups. *Proc. Nat. Acad. Sci. U.S.A.*, 83(13):4589–4593, 1986. 26

[29] S. Rallis and D. Soudry. On local gamma factors for orthogonal groups and unitary groups. *Bull. Iranian Math. Soc.*, 43(4):387–403, 2017. 29, 30, 42

[30] Yiannis Sakellaridis and Akshay Venkatesh. Periods and harmonic analysis on spherical varieties. *Astérisque*, (396):viii+360, 2017. 4

[31] David Soudry. The unramified computation of Rankin-Selberg integrals expressed in terms of Bessel models for split orthogonal groups: Part I. *Israel J. Math.*, 222(2):711–786, 2017. 29

[32] Jean-Loup Waldspurger. Une formule intégrale reliée à la conjecture locale de Gross-Prasad, 2e partie: extension aux représentations tempérées. Number 346, pages 171–312. 2012. Sur les conjectures de Gross et Prasad. I. 4

[33] Hang Xue. Fourier-Jacobi periods and the central value of Rankin-Selberg L-functions. *Israel J. Math.*, 212(2):547–633, 2016. 5, 15

[34] Shunsuke Yamana. L-functions and theta correspondence for classical groups. *Invent. Math.*, 196(3):651–732, 2014. 26, 27, 28, 34, 35, 36, 38

[35] Wei Zhang. Automorphic period and the central value of Rankin-Selberg L-function. *J. Amer. Math. Soc.*, 27(2):541–612, 2014. 1