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Essays on Information Economics

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Abstract

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This dissertation consists of three independent essays that examine how to improve information transmission and how to incentivize learning.

In Chapter 2, I study the role of a recommender's career concerns in his relationship with a consumer when the recommender has a private type in expertise. An informed type has valuable expertise for the consumer, whereas an uninformed type does not. The uninformed type cannot mimic the informed type, suggesting that the informed type can build a reputation for competence. However, I find that the relationship breaks down completely if the recommender is sufficiently patient.

In Chapter 3, which is co-authored with Florian Ederer, we embed probabilistic lie detection in a standard model of Bayesian persuasion. We show that the Sender lies more when the lie detection probability increases. Moreover, the Sender's and the Receiver's equilibrium payoffs are unaffected by a weak lie detection technology because the Sender compensates by lying more.

In Chapter 4, I analyze optimal contracting for experimentation when the agent who experiments and the principal who provides incentives agree to disagree over the quality of the project. If efforts are contractible, the principal prefers to reward good outcomes (efforts) exclusively for a more (less) optimistic agent. Moreover, longer experimentation is sustained with non-common prior. If efforts are not contractible, the optimal duration of experimentation is increasing in the agent's confidence.

Essays on Information Economics

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In dedication to my dear parents, Yuangen and Pinghong, for their unspoken love while I fly and heartfelt hugs while I cry.

谁言寸草心，报得三春晖。

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Chapter 1

Introduction

Information and learning are at the heart of any economic problem with an uncertain nature. People could obtain information by at least two approaches: learning from potentially more informed people or actively generating information through experimentation. In the first case, information transmission is often limited by strategic incentives. In the second case, experimentation is typically costly and thus requires external incentives. This dissertation consists of three independent essays with a unified objective of improving learning in distinct environments. The first two essays examine how to improve information transmission subject to strategic incentives. The last chapter explores how to incentivize costly experimentation.

Chapter 2 considers the interaction between a recommender and a consumer whose purchase decisions rely on the recommender's expertise. The recommender has a private type in his expertise. An informed type's expertise is valuable for the consumer, whereas an uninformed type lacks such expertise. In a one-shot game, information transmission is possible only if the consumer sufficiently trusts the recommender. This chapter analyzes the role of the recommender's career concerns (dynamic incentives) in improving information transmission. The uninformed type cannot mimic the informed type, suggesting that the informed type can build a reputation for competence and that an equilibrium in the repeated game should exhibit more information transmission. However, I find a contrary result. The relationship breaks down completely, and no information is transmitted if the recommender is sufficiently patient. Moreover, this occurs despite an arbitrarily small uncertainty in the expertise.

In Chapter 3, which is co-authored with Florian Ederer, we consider an environment in which a Sender passes information via a message to a Receiver who then takes an action. The Sender commits to a reporting strategy, and the Receiver can detect lies with positive probability whenever the reported message is inconsistent with the true state. In this model, we ask whether such a lie detection technology improves information transmission and makes the Receiver better off. We show that the Sender lies more in response to a higher lie detection probability. Due to this strategic effect, the Sender's and the Receiver's equilibrium payoffs are unaffected by a weak lie detection technology. However, a sufficiently strong lie detection technology strictly benefits the Receiver and hurts the Sender.

In Chapter 4, I consider learning through costly experimentation instead. I employ a principal-agent framework in which the principal incentivizes the agent to experiment on a project. I am interested in how differential beliefs can be exploited to provide incentives. Specifically, the two parties are allowed to disagree over the quality of the project. I study optimal contracting, particularly the optimal length of experimentation when the two parties agree to disagree. In the absence of moral hazard, the contract can be contingent on efforts and outcomes. If the agent is more optimistic than the principal, the optimal contract rewards good outcomes exclusively. Otherwise, the principal finds it more appealing to reward efforts exclusively. In either case, longer experimentation is sustained than in the common-prior benchmark. When there is moral hazard, the contract can only depend on outcomes. In the optimal contract, both the length of experimentation and the principal's payoff are increasing in the agent's confidence.

Chapter 2

Expert Recommenders and Reputation Failure

2.1 Introduction

Recommenders, people in various roles ranging from financial advisors and insurance sales agents to social media influencers, are supposed to be experts in their respective fields and recommend suitable products to their audience, or more generally consumers.[1] However, not all recommenders may have sufficient expertise. For example, many influencers on social media claiming to be experts actually lack formal training or comprehensive knowledge in their fields of expertise, rendering their blogs, testimonials, and recommendations of limited informational value (Sabbagh et al., 2020). Some influencers accounts are even fake (Ross, 2015) and engage in fraudulent activities to boost and inflate influencer metrics.[2]

Consumers are not experts themselves and thus cannot distinguish between competent and incompetent recommenders. So, incompetent recommenders undermine

[1]Strictly speaking, sometimes influencers are followed not because they have potential expertise, but simply because their followers like them. This type of influencer, including most celebrities, is not the object of interest of this chapter.

[2]See Influencity (2021) for different categories and examples of influencer fraud.

the trust of consumers. For example, the Edelman (2021) shows that social media and financial services are the least trustworthy sectors in 2021. In light of the trust issues, the value of competent recommenders is limited because they cannot transmit valuable information to consumers. Thus, it entails substantial benefit if the relationship between consumers and recommenders is enhanced.[3]

Specifically, this chapter studies how the reputational concerns of recommenders affect this relationship. There is a large literature on the effects of reputation, including the influential work of Ely and Välimäki (2003), who show that reputational concerns are not always good.[4] Their "bad reputation" result, however, relies on "bad types" being able to mimic "good types." In the case of recommenders, incompetent types simply do not have the necessary information to mimic competent types. One may expect then that reputational concerns would benefit the relationship between consumers and recommenders. Intuitively, if unsuitable products are recommended, consumers' negative feedback would hurt the recommenders' reputation. Provided that recommenders care enough about their reputation, recommenders will thus be disciplined to behave in consumers' best interest, and incompetent recommenders should eventually be driven out of the market.

However, I show that the above intuition is incorrect. This chapter highlights and analyzes a new mechanism of reputation failure, suggesting reputational concerns can be detrimental to the relationship even though incompetent recommenders cannot mimic competent recommenders.

In particular, I examine an infinitely repeated game between a long-lived recommender (he) and a long-lived consumer (she). In each period, the recommender has access to a new product and either recommends it to the consumer or does not. Given a recommendation, the consumer then decides whether to buy the product.

[3] See Prior (2015), Zingales (2015), and Mannheim (2021) for further evidence on low trust in these sectors and related professions.

[4] A detailed literature review is deferred to the end of this section.

Otherwise, the consumer is not even aware of the product and makes no choice.

Two types of uncertainty arise in my model: the product may be suitable or unsuitable to the consumer, and the recommender may be either informed or uninformed. An informed recommender privately knows the product suitability, while an uninformed recommender only knows its distribution. At the end of each period, the suitability is publicly revealed if and only if the consumer buys the product. Hence, the consumer draws an inference about the recommender's type from both the recommender's actions and the feedback about suitability.

The consumer derives a value from the product and pays a fixed price if she buys it. The recommender receives a commission when his recommendation is followed but receives a penalty for making unsuitable recommendations. The two types of recommender (informed or uninformed) have identical preferences ex-post, but the discrepancy in their information results in a difference in their ex-ante preferences. In the baseline model, I consider the most interesting scenario in which the informed type's static incentives are aligned with those of the consumer but the uninformed type has misaligned static incentives.

There are two natural benchmarks in this repeated game with one-sided incomplete information. In the static benchmark, the Pareto-optimal equilibrium has an intuitive cutoff structure. If the consumer is sufficiently confident in the recommender's expertise, she follows his recommendations. As a best response, the informed type recommends honestly, and the uninformed type always recommends. If instead, the consumer's confidence in the recommender's expertise is inadequate, the consumer does not follow the recommendation and thus no information is transmitted. The second benchmark considers the repeated game with complete information. If the customer knows the recommender is uninformed, then the consumer does not follow any recommendation in the unique equilibrium outcome. If instead she knows the recommender is informed, there is a multiplicity of equilibria. However, in the unique

Pareto-optimal equilibrium, the consumer always buys and the informed type always recommends honestly.

In the full model, I analyze the Markov perfect equilibria. The state variable is the consumer's belief that the recommender is informed, or, equivalently, the recommender's reputation. In addition, I impose a property requiring that once the recommender is believed to be informed with certainty, the Pareto-optimal equilibrium described in the complete information benchmark is played.

Under this solution concept, a trivial equilibrium is the complete relationship breakdown: the consumer never follows any recommendation whenever there is uncertainty in expertise. My main result shows this inefficient outcome is the unique equilibrium outcome when the recommender is sufficiently patient, even when the consumer is virtually certain that the recommender is informed. This result is intriguing for two reasons. First, the uninformed type lacks the information to mimic the informed type, ensuring the latter can distinguish himself. Second, the consumer is forward-looking and thus has experimentation incentives to screen the recommender's type. Both ingredients should help the informed type to build a reputation and transmit information. Nonetheless, the relationship still breaks down and thus no information is transmitted.

The culprits of the relationship breakdown are two countervailing forces. On the one hand, the recommender receives a positive flow payoff only if he recommends and the consumer buys. If the relationship does *not* break down, the uninformed type must sometimes strictly prefer to recommend. On the other hand, if he strictly prefers to recommend, the possible revelation of an unsuitable product accelerates the separation of types and reduces his continuation payoff. In contrast, if he deviates to not recommend, the consumer infers that he is certainly informed, after which he receives a lucrative continuation payoff. Thus, when the recommender is sufficiently patient, there is a profitable deviation from the putative action, implying the uninformed type always at least weakly prefers not to recommend. As the implications of these two forces are contradictory, an informative equilibrium must not exist.

This result suggests one explanation regarding low trust in people in roles such as social media influencers and financial advisors. The relationship between consumers and these people are hard to sustain even though reputational incentives for being honest are present. Moreover, this result remains unchanged if the long-lived consumer is replaced with a sequence of short-lived consumers who engage in observational learning. Thus, my model also speaks to applications that feature infrequent interactions. For example, doctors prescribe new drugs to different patients, real estate brokers recommend new properties to potential buyers, and management consultants provide expert opinions to organizations or individuals.

I next identify the exact threshold of the recommender's patience level above which the relationship breaks down and find that this threshold only depends on the expected product suitability and monotonically increases over it. In other words, after fixing the recommender's patience level, products that are ex-ante more likely to be suitable are more likely to be recommended successfully. This comparative static suggests that established big brands are more likely to succeed through influencer marketing relative to less-known brands (e.g., brands of new entrants). Given this prediction, less-known brands should be more cautious in adopting influencer marketing.

I then explore the main result's robustness to alternative solution concepts and modeling assumptions. While the main result's proof relies on the solution concept that I use, the general message that reputational concerns can harm the relationship is still obtained in some special cases even under weaker solution concepts. Furthermore, the main insight also carries over to settings where the consumer's inference is noisy and where the consumer can buy products without recommendations.

Finally, I discuss several approaches to restore the relationship. First, the recommender's commission is often predetermined in contracts, and thus the relationship breakdown can be avoided if the commission level is chosen appropriately so that the two types of recommender's incentives align with the consumer's. Second, in some applications, the product space is large and the recommender may be able to examine multiple products in a period. Due to the difference in expertise, the informed type can select the best product, whereas the uninformed type cannot. It is therefore as if the informed type has access to products with a higher expected suitability. If the difference is sufficiently large, the relationship can be restored.

This chapter naturally belongs to the literature on bad reputation. The influential paper by Ely and Välimäki (2003) studies the interaction between a long-lived agent (the counterpart of the recommender) and a sequence of short-lived principals (the counterpart of the consumer). The agent either has biased or an unbiased preferences relative to the consumer's preference. They find that when the agent's reputational incentives are strong, the unbiased agent is too eager to be separated from the biased agent, and thus chooses an action that hurts the short-lived principal. Anticipating this, the principal finds it in her best interest not to participate, and the market fails altogether. Ely et al. (2008) generalize this insight in multiple directions and identify a class of games where the mechanism applies.

My model differs from theirs in three aspects. First, I consider an information/strategic type instead of a payoff type, which distinguishes the payoff structure between my model and theirs. More importantly, this distinction also implies that the "bad" type cannot mimic the "good" type in my model, whereas such mimicking is possible in theirs. Second, my reputation failure result is unaffected with a patient consumer, but their result disappears if the principal is patient. Last, they focus on participation games in which once the principal stops participating, no information is generated henceforth. In contrast, the consumer moves after the recommender in my model. Even if the consumer does not buy the product at some period, the recommender's action still reveals some information. In fact, if the order of moves is reversed in their models, the bad reputation result disappears again.

A recent paper Deb et al. (forthcoming) also establish a bad reputation result in a model with two long-lived players, one of whom has a strategic type. However, their result applies only when the principal is impatient, and does not depend on the agent's discount factor. In contrast, this chapter's result applies when the recommender is patient and does not depend on the consumer's discount factor. The difference in the payoff structure is one crucial factor that drives the difference in results. The agent in their model only cares about extending the relationship, but the recommender in my model has direct concerns for the consumer's suitability, making it easier to screen the recommender's type. Roughly speaking, their payoff structure corresponds to a special case in this chapter in which both types of recommender have a conflict of interest with the consumer. In this case, the results are consistent with theirs.[5]

This chapter is also related to a strand of literature on expert reputation, which studies experts who strive to appear informed in different settings (e.g., Scharfstein and Stein (1990); Trueman (1994); Ottaviani and Sørensen (2006a,b,c)).[6] In those models, the experts are evaluated based on the expert's actions and underlying states. There are two substantial differences compared to this chapter. First, the states are realized independent of actions in those models. In contrast, the product's suitability, which is the counterpart of the state in my model, is revealed only if the product is recommended and purchased. Second, those models are mostly in a cheap talk framework. In comparison, the recommender's action, particularly the action "not recommending," does affect the payoff directly, and thus my model does not fit in the cheap talk framework.

Last, as an important application of this chapter, influencer marketing has caught

[5]Other papers that concern reputation failure include Schmidt (1993), Cripps and Thomas (1997), Chan (2000), Morris (2001), Pei (2022), and Deb and Ishii (2021).

[6]See Marinovic et al. (2013) and recent papers such as Backus and Little (2020) and Vong (2022) for a detailed survey of references.

increased attention from researchers, yet the existing literature takes different perspectives. Pei and Mayzlin (forthcoming) model how firms choose compensation and affiliation levels with influencers. Fainmesser and Galeotti (2021) and Mitchell (2021) both study how influencers strategically choose content composition. Most of the literature uses a static model and thus cannot capture reputation effects.

The rest of the chapter is organized as follows. Section 2.2 describes the model. Section 2.3 defines the equilibrium and analyzes the two benchmarks. In Section 2.4, I establish the main result, and in Section 2.5 I consider comparative statics for the expected suitability of products and the recommender's payoff structure. Section 2.6 contains various robustness checks and extensions. Section 2.7 concludes. All omitted proofs are in Appendix 2.A.

2.2 Model

The model considers a discrete-time infinitely repeated game between two long-lived players: a recommender (he) and a consumer (she). I first introduce the stage game in the order of actions, information, and payoffs. Then I describe the ingredients that connect the repeated game. Last, I discuss some assumptions that are less standard.

2.2.1 Stage Game

Actions: In each period, a new product becomes accessible to the recommender but not to the consumer. The recommender moves first, and chooses to either recommend (R) the product to the consumer or not (NR). Given a recommendation, the consumer then chooses to buy the product (B) or not (NB). Given no recommendation, the stage game ends immediately, and the consumer does not make a decision. Basically, if no recommendation is provided, the consumer is not even aware of the product.

Information: Two types of uncertainty are present in the model. First, the product is either suitable (S) or unsuitable (NS) for the consumer, but the consumer only knows it is suitable with probability q. Second, the recommender has uncertain expertise and is either informed or uninformed. An informed recommender (I) receives a perfect signal of the suitability, whereas an uninformed recommender (U) receives no signal. The recommender is informed with ex-ante probability θ_0. In summary, the informed recommender knows both the product's suitability and his expertise, the uninformed recommender only knows his expertise, and the consumer knows neither.

Payoffs: There are four possible outcomes in the stage game: the product is recommended but the consumer does not buy it; the product is not recommended; the product is recommended, the consumer buys it, and it turns out to be suitable; and the product is recommended, the consumer buys it, and it turns out to be unsuitable. Denote the four outcomes by NB, NR, S, and NS, respectively.

If the consumer buys the product, she earns a payoff $v - p$, where v is the value of the product,

$$v = \begin{cases} 1 & \text{if product is suitable,} \\ 0 & \text{if product is unsuitable.} \end{cases}$$

and p is the product's fixed price. Meanwhile, denote the recommender's payoff by $\overline{u}$ if the product is suitable and $\underline{u} \leq \overline{u}$ if the product is unsuitable. Here, $\overline{u}$ can be interpreted as the commission paid to the recommender, whereas $\overline{u} - \underline{u}$ can be interpreted as the penalty for making an unsuitable recommendation. Section 2.2.3 discusses the interpretations in more details.

If instead the consumer does not buy the product, either because the recommender does not recommend or the consumer does not follow the recommendation, both players' payoffs are normalized to 0.[7] Table 2.1 summarizes the payoff structure.

	S	NS	NB	NR
Consumer	$1-p$	$-p$	0	0
Recommender	$\overline{u}$	$\underline{u}$	0	0

Table 2.1: The payoff structure in the stage game.

Notice that, if $p \leq q$, then even in absence of the recommender, the consumer finds it optimal to buy the product. On the other hand, if $p \geq 1$, then it is so costly that the consumer never buys it. Therefore, I restrict attention to parameters satisfying the following assumption.

Assumption 2.1. $q < p < 1$.

The two types of recommender have an identical ex-post preference. Yet, the discrepancy in their expertise may result in distinct ex-ante preferences, which are determined by the parameters $\overline{u}$, $\underline{u}$, and q. The following definitions clarify when the recommender's static incentives are aligned with the consumer's.

Definition 2.1. *(Aligned Static Incentives)*

- *The informed recommender has aligned static incentives if $\overline{u} > 0$ and $\underline{u} < 0$. He has misaligned static incentives if $\overline{u} < 0$ or $\underline{u} > 0$.*
- *The uninformed recommender has aligned static incentives if $q\overline{u} + (1-q)\underline{u} < 0$. He has misaligned static incentives if $q\overline{u} + (1-q)\underline{u} > 0$.*

Essentially, a recommender with aligned static incentives would find it strictly optimal to behave in the consumer's interest in a one-shot game. This means that the informed type recommends honestly and the uninformed type does not recommend.

In the baseline model, I focus on the most interesting case in which the informed type has aligned static incentives and the uninformed type has misaligned static

[7] In principle, recommendation may incur a cost even though the consumer does not buy. However, introducing a negligible cost does not change the main result.

incentives. This restriction is captured by Assumption 2.2. The remaining cases are more straightforward and are considered in Appendix 2.A.4.

Assumption 2.2. $\overline{u} > 0$, $\underline{u} < 0$, $q\overline{u} + (1-q)\underline{u} > 0$,

To simplify the notation, denote

$$\Delta_g = \overline{u}, \quad \Delta_b = \underline{u}, \quad \Delta_\emptyset = q\overline{u} + (1-q)\underline{u}.$$

as the expected flow payoff of the recommender conditional on the signal g, b, $\emptyset$, respectively. Then, Assumption 2.2 is equivalent to $\Delta_b < 0 < \Delta_\emptyset < \Delta_g$.

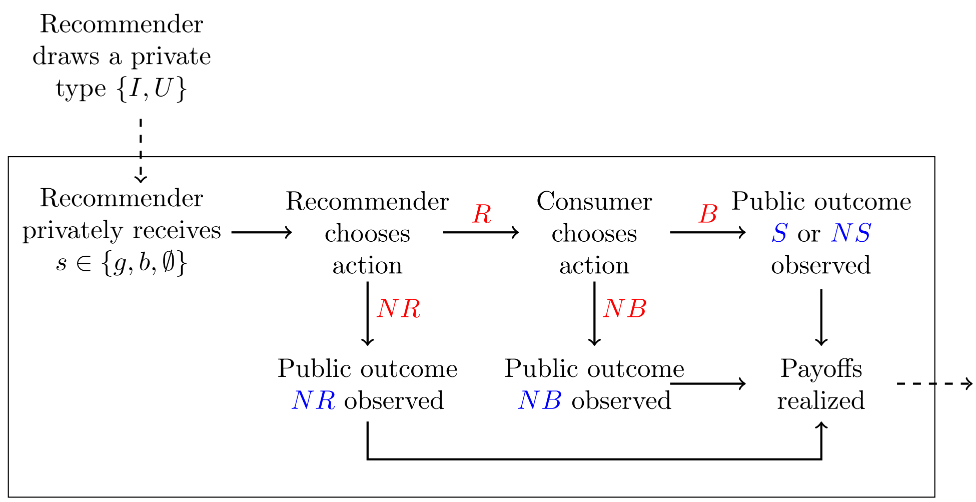


Figure 2.1: The flow chart of the model.

2.2.2 Repeated Game

The recommender's type is persistent and drawn at the beginning of the game. Hence, I exclude the possibility that an uninformed recommender learns to be informed through gradual interactions with the consumer. In contrast, the value v is i.i.d. across periods. The suitability is publicly revealed at the end of each period if, and only if, the consumer buys the product at that period. Last, both the consumer and the recommender are forward-looking. Unless otherwise stated, they share the same discount factor $\delta \in [0, 1)$. Figure 2.1 summarizes the game, where the rectangle contains a typical stage game.[8]

2.2.3 Discussion of Assumptions

Most of assumptions of the model are standard, though some of them deserve further discussion. First, the consumer is not allowed to purchase the product if no recommendation is given. This assumption is imposed to simplify the analysis, and does not drive the main result. See Section 2.6.4 for an extension in which the consumer can buy the product without any recommendation.

Moreover, this choice is also natural in some circumstances. The products in some categories, including beauty, fashion, and food, are very heterogeneous and updating quickly. It is nearly impossible for the consumer to track all new products. Hence, if a recommender does not recommend in a period, the consumer would not know which product has arrived. In addition, even though the consumer knows which product has arrived, she might not be able to purchase it. For example, patients cannot buy certain medications without a prescription from doctors.

As to the payoff parameters, it is natural to interpret $\overline{u}$ as the commission because it is a common form of compensation for financial advisors, social media influencers, and doctors who prescribe drugs. The penalty $\overline{u} - \underline{u}$ can be motivated in several ways. First, an unsuitable recommendation hurts the consumer, and as a result, she may report it to a professional association or market regulator, who then fines the recommender. In fact, financial advisors can even be sued for making unsuitable recommendations. Second, an unsuitable recommendation may also indirectly hurt the firm that provides the product if the consumer leaves a negative review for the

[8] This figure is inspired by the flow chart in Deb et al. (forthcoming). The comparison between two flow charts conveniently clarifies the models' distinctions.

product, and so the firm may transfer this cost to the recommender via explicit contracts (the firm's behavior, however, is not modeled in this chapter). Last, the recommender may care about the consumer's welfare for ethical reasons and thus develops an intrinsic preference for recommending the correct product. This last motivation is appropriate for health-related occupations such as doctors.[9]

In addition, the two types of recommender are modeled in an extreme way such that the informed type is fully informed and the uninformed type knows nothing. On top of tractability issues, this modeling choice also maximizes the difference between the two types, rendering it harder for the uninformed type to mimic the informed type. If no information can be transmitted with this maximal difference, there is little hope when the two types differ less. Section 2.6.3 considers the extension when the informed type's signal is noisy.

2.3 Equilibrium

The baseline model restricts attention to Markov perfect equilibiria, where the natural state variable is the consumer's belief that the recommender is informed. The state variable, denoted by θ, can be naturally interpreted as the recommender's reputation.

The uninformed type's Markov strategy is represented by $r_\emptyset : [0,1] \longrightarrow [0,1]$, where $r_\emptyset(\theta)$ specifies the probability that the uninformed type recommends a product when the state is θ. Due to additional information, the informed type's Markov strategy is represented by a pair of functions $r_g : [0,1] \longrightarrow [0,1]$ and $r_b : [0,1] \longrightarrow [0,1]$, where $r_s(\theta)$ specifies the probability that the informed type recommends a product when the state is θ and he observes a signal $s \in \{g, b\}$. A good signal g implies the product is suitable, and a bad signal b implies the product is unsuitable. Last, the consumer's Markov strategy is given by $b : [0,1] \longrightarrow [0,1]$, where $b(\theta)$ specifies the probability that she buys a recommended product when the state is θ.

[9]See Inderst and Ottaviani (2012) for more justifications.

Let $\theta|_{NB}$, $\theta|_{NR}$, $\theta|_S$, $\theta|_{NS}$ be the posterior beliefs followed by each corresponding outcome. Whenever Bayes' rule is applicable, the posterior beliefs evolve as follows.

$$\begin{aligned}
\theta|_{NB} &= \frac{\theta\left[qr_g(\theta) + (1-q)r_b(\theta)\right]}{\theta\left[qr_g(\theta) + (1-q)r_b(\theta)\right] + (1-\theta)r_\emptyset(\theta)},\\
\theta|_{NR} &= \frac{\theta\left[1 - qr_g(\theta) - (1-q)r_b(\theta)\right]}{\theta\left[1 - qr_g(\theta) - (1-q)r_b(\theta)\right] + (1-\theta)[1 - r_\emptyset(\theta)]},\\
\theta|_S &= \frac{\theta r_g(\theta)}{\theta r_g(\theta) + (1-\theta)r_\emptyset(\theta)},\\
\theta|_{NS} &= \frac{\theta r_b(\theta)}{\theta r_b(\theta) + (1-\theta)r_\emptyset(\theta)}.
\end{aligned}$$

Denote V_U, V_I, V_C as the average discounted payoff of each player. Given a strategy profile $(r_g,\, r_b,\, r_\emptyset,\, b)$, let θ_t denote the public posterior belief after t periods. Then the payoffs are defined accordingly.[10]

$$\begin{aligned}
V_U(\theta_0) &= (1-\delta)\mathbb{E}_{\theta_t}\sum_{t=0}^{\infty}\delta^t b(\theta_t) r_\emptyset(\theta_t)\Delta_\emptyset,\\
V_I(\theta_0) &= (1-\delta)\mathbb{E}_{\theta_t}\sum_{t=0}^{\infty}\delta^t b(\theta_t)[qr_g(\theta_t)\Delta_g + (1-q)r_b(\theta_t)\Delta_b],\\
V_C(\theta_0) &= (1-\delta)\mathbb{E}_{\theta_t}\sum_{t=0}^{\infty}\delta^t b(\theta_t)\theta_t[q(1-p)r_g(\theta_t) - p(1-q)r_b(\theta_t)]\\
&\quad + (1-\delta)\mathbb{E}_{\theta_t}\sum_{t=0}^{\infty}\delta^t b(\theta_t)(1-\theta_t)(q-p)r_\emptyset(\theta_t).
\end{aligned}$$

Formally, the Markov perfect equilibrium is defined as follows.

Definition 2.2. *A strategy profile* $(r_\emptyset,\, r_g,\, r_b,\, b)$ *and a belief system* $(\theta|_{NB},\, \theta|_{NR},\, \theta|_S,\, \theta|_{NS})$ *constitute a Markov perfect equilibrium if*

(a) there is no unilateral profitable deviation for $\forall i \in \{I,\, U,\, C\}$*;*

(b) on-path beliefs are derived according to Bayes rule; and

[10] Be cautious that the three expectation signs have distinct meanings because the each (type of) player is facing with a different distribution of posteriors.

(c) if the belief reaches one, then any off-path belief stays at one.[11]

In what follows, I first characterize the Pareto-optimal equilibrium in the static benchmark. Then, I analyze the complete information benchmark and introduce a further refinement on the continuation play at $\theta = 1$. Last, I establish the existence of an equilibrium that survives the refinement in the full model. For brevity, I drop the qualifiers in the solution concept whenever there is no confusion, and use the term "equilibrium" instead.

2.3.1 Static Benchmark

In the one-shot game, the consumer has no experimentation incentive and the recommender does not need to worry about building his reputation. At any belief θ, there are two possibilities. In the first case, the consumer does not buy the product: $b(\theta) = 0$. It follows immediately that everyone gets 0 payoff. This payoff can be sustained by a plethora of equilibria; for example, the two types pool on recommending. Given such play, the recommendation contains no information, and thus the consumer refuses to buy the product. Such equilibrium is reminiscent of the "babbling equilibrium" in the cheap talk literature. Strictly speaking, this game is not a standard cheap talk model, but nonetheless, I use their terminology in this chapter. Henceforth, whenever every player receives zero payoff at a belief, the equilibrium outcome at this belief is called babbling. Clearly, the babbling outcome can be sustained at any belief.

In the second case, the consumer buys the product with a positive probability: $b(\theta) > 0$. Then by Assumption 2.2, the informed type strictly prefers to recommend honestly: $r_g(\theta) = 1$, $r_b(\theta) = 0$, and the uninformed type strictly prefers to recommend: $r_\emptyset(\theta) = 1$. Given the recommender's best responses, the consumer indeed

[11]This off-path restriction is commonly used in the literature (See Rubinstein (1985) and Vong (2022)), which essentially identifies the subgame at $\theta = 1$ as one of complete information.

prefers to purchase the product if and only if the conditional value of the recommended product exceeds its price.

$$p \leq \mathbb{E}[v|R] = \frac{q}{q + (1-q)(1-\theta)} \Longleftrightarrow \theta \geq \frac{p-q}{p(1-q)} \equiv \theta^S. \tag{2.1}$$

It is straightforward that this condition is equivalent to a lower bound on the reputation. The second possible outcome dominates the babbling outcome, but is sustainable only if the state θ is higher than threshold θ^S.

Since the two cases cover all possibilities, the Pareto-optimal equilibrium takes a cutoff structure. As demonstrated in Figure 2.2, there is a no-purchase region for $\theta < \theta^S$ and a purchase region for $\theta \geq \theta^S$, where the strategies in each region are given below. Henceforth, this Pareto-optimal equilibrium is referred to as the static equilibrium and θ^S is referred to as the static threshold.

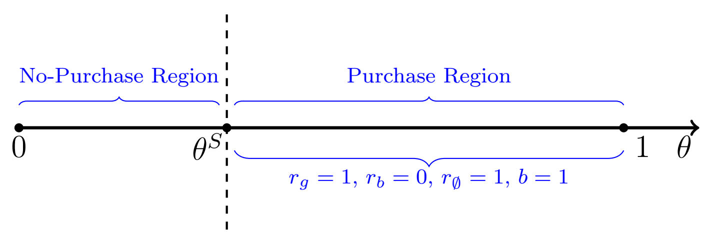


Figure 2.2: Illustration of the Pareto-optimal equilibrium in the static benchmark.

- No-Purchase Region $(\theta < \theta^S) : r_g(\theta) = r_b(\theta) = r_\emptyset(\theta) = 1, \ b(\theta) = 0.$
- Purchase Region $(\theta \geq \theta^S) : r_g(\theta) = r_\emptyset(\theta) = b(\theta) = 1, \ r_b(\theta) = 0.$

The cutoff increases in p and decreases in q. Intuitively, when the product is less likely to be suitable ex-ante or the product's price is high, the recommendation has to be more informative to attract the consumer. But this is possible only if the recommender has a higher reputation.

2.3.2 Complete Information Benchmark

There are two cases in the complete information benchmark. If the recommender is known to be uninformed, the unique equilibrium outcome features no purchase because an uninformed recommender never benefits the consumer regardless of his strategy. If the recommender is known to be informed, there are two possible equilibrium outcomes.[12]

First, the "no purchase" outcome is still sustainable if the recommender ignores his signal and always recommends. Conditional on the recommender's strategy, it is indeed optimal for the consumer not to purchase since the recommendation is uninformative. Conversely, given that the consumer does not buy, the recommender is indifferent in everything.

However, the other equilibrium outcome is much better, where the recommender recommends a product if and only if the signal is good, and the consumer always follows the recommendation. The recommender's strategy and the consumer's strategy constitute mutual best responses. Conditional on the recommender's strategy, the consumer figures that a recommended product must be suitable and thus buys it without a doubt. Conditional on the consumer's strategy, the recommender has no reputational concern and simply acts on the flow payoff. It follows from Assumption 2.2 that it is indeed optimal for him to recommend honestly. In this equilibrium, information is fully embedded in the recommendation, rendering the equilibrium efficient.

Due to efficiency, the second outcome is naturally favored over the first outcome. Thus, I exclude the unfavorable babbling outcome and focus on the efficient outcome. Such a choice is implied by many renegotiation-proofness refinements.[13]

[12] If $b > 0$, then we obtain the second equilibrium outcome. Otherwise we get the first equilibrium outcome.

[13] Examples include the strongly renegotiation-proof requirement in Farrell and Maskin (1989), the strong consistency requirement in Bernheim and Ray (1989) and the consistent bargaining equilibrium in Abreu et al. (1993).

2.3.3 Equilibrium Existence

By Definition 2.2(c), selecting the efficient outcome in the complete information benchmark is equivalent to imposing an additional property on the solution concept in the full model. This property, referred to as ““*Efficiency at the Top*"”, states that once the recommender achieves full reputation, the efficient continuation equilibrium described in the complete information benchmark is played. Thus, the solution concept used in the baseline model is Markov perfect equilibrium that satisfies ““*Efficiency at the Top*"”. Again, I use the term “equilibrium” instead whenever there is no confusion.

Mathematically, ““*Efficiency at the Top*"” restricts the strategies at the top belief to be $r_\emptyset(1) = 1$, $r_g(1) = 1$, $r_b(1) = 0$, $b(1) = 1$,[14] which gives rise to the following payoffs:

$$V_U(1) = \Delta_\emptyset, \quad V_I(1) = q\Delta_g, \quad V_C(1) = q(1-p).$$

While the strategy at $\theta = 1$ is completely determined, there is no restriction in the strategies at $\theta < 1$. Thus, the equilibrium existence is easily ensured. One trivial equilibrium is described as follows: at any belief $\theta < 1$, the consumer never buys the product and the two types pool on recommending, whereas at $\theta = 1$, they play the efficient equilibrium. Abusing the notation, this equilibrium is still referred to as a babbling equilibrium even though the play at $\theta = 1$ is non-babbling. Perhaps surprisingly, this babbling equilibrium turns out to be the unique equilibrium outcome provided the recommender is sufficiently patient. In other words, the recommender's reputational concerns destroy all the surplus.

[14]Conditional on the consumer's strategy, the uninformed type strictly prefers to recommend because the flow payoff $\Delta_\emptyset$ is positive by Assumption 2.2.

2.4 Main Result: Relationship Breakdown

The main result states that an informative equilibrium exists if and only if the recommender's discount factor is lower than a threshold $\underline{\delta} < 1$.

Proposition 2.1. *Let* $\underline{\delta} = \frac{1}{1+\sqrt{1-q}}$.

(a) If $\delta \leq \underline{\delta}$*, there exists an equilibrium such that* $V_U(\theta), V_I(\theta), V_C(\theta) > 0$*,* $\forall \theta \in [\theta^*, 1)$ *for some* $\theta^* < 1$.

(b) If $\delta > \underline{\delta}$*, then in any equilibrium,* $V_U(\theta) = V_I(\theta) = V_C(\theta) = 0$*,* $\forall \theta < 1$.

The second part of Proposition 2.1 suggests an extreme relationship breakdown, which applies even for a prior belief arbitrarily close to one. Moreover, this result is not a limit result because it does not require an arbitrarily patient recommender. The two parts combined suggest a stark discontinuity of equilibrium payoffs in the recommender's discount factor. In what follows, I first show part (a) by explicitly constructing an informative equilibrium with a low discount factor. Then, I sketch the proof of part (b) and discuss the underlying mechanism.

2.4.1 Informative Equilibrium with Impatient Players

The construction of the informative equilibrium, summarized in Claim 2.1, has an identical cutoff structure as the static equilibrium. When the belief is lower than some threshold, the outcome is babbling. When the belief is higher than the threshold, the informed type recommends honestly, the uninformed type recommends with probability one, and the consumer buys the product with probability one. The only difference lies in the cutoff value.[15] Due to the consumer's experimentation incentive, the cutoff here, denoted by θ^*, is strictly smaller than the static cutoff θ^S. Essentially, even when the consumer's flow payoff is negative, she might purchase the product to

[15]Similarly, the cutoff is not unique. Instead, I focus on the minimal cutoff that preserves this type of equilibrium.

discern the recommender better.

Claim 2.1. *Let* $\theta^* = \frac{p-q}{(1-q)p+\frac{\delta^2 q^2(1-q)(1-p)}{1-\delta}}$. *Then the following strategy profile and belief profile constitute an equilibrium if* $\delta \leq \underline{\delta}$.

- ***Purchase Region :*** $\theta \in [\theta^*, 1]$

  Strategies: $r_g(\theta) = 1$, $r_b(\theta) = 0$, $r_\emptyset(\theta) = 1$, $b(\theta) = 1$.

  Beliefs: $\theta|_S = \theta$, $\theta|_{NS} = 0$, $\theta|_{NR} = 1$, $\theta|_{NB} = \frac{q\theta}{q\theta+1-\theta}$.

- ***No-Purchase Region :*** $\theta \in [0, \theta^*)$

  Strategies: $r_g(\theta) = 1$, $r_b(\theta) = 1$, $r_\emptyset(\theta) = 1$, $b(\theta) = 0$.

  Beliefs: $\theta|_{NB} = \theta$ *(on path),* $\theta|_S = \theta|_{NS} = \theta|_{NR} = \theta$ *(off path).*

Proof. The verification is deferred to Appendix 2.A.2.

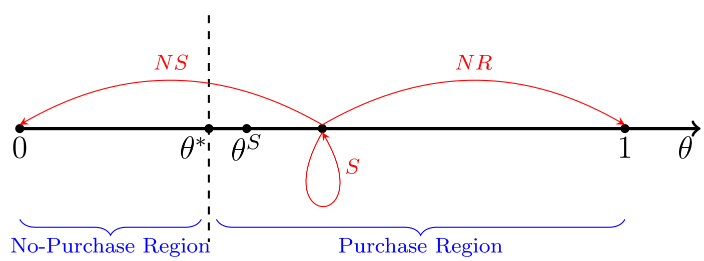


Figure 2.3: Illustration of the cutoff equilibrium.

In the no-purchase region, the recommender's strategies leave both the recommender's action and the feedback on suitability uninformative. So, at a generic belief θ in this region,

$$V_i(\theta) = 0, \ i \in \{I, U, C\}.$$

In contrast, for a generic belief in the purchase region, the red curves in Figure

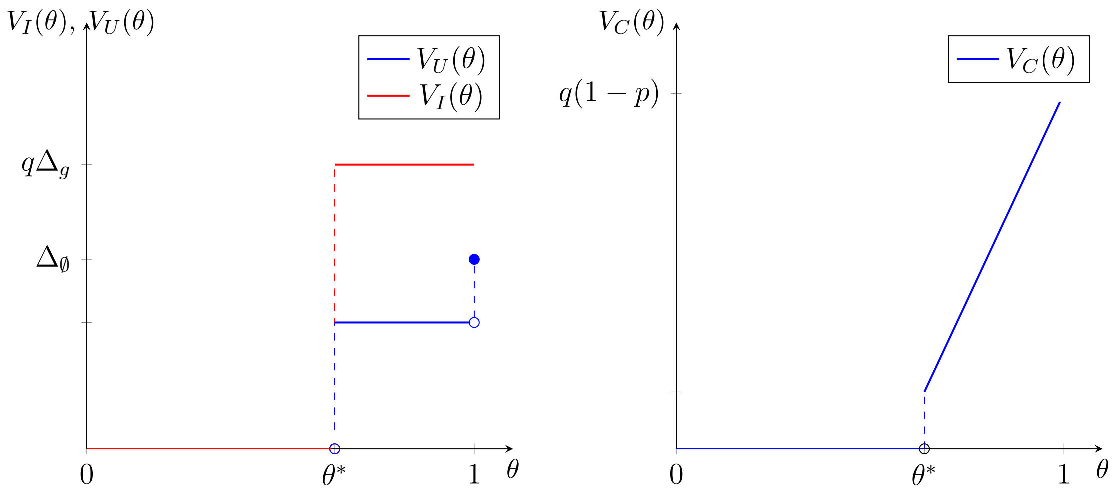
2.3 reflect the dynamics of posteriors.[16] Specifically, the belief drops to 0 whenever the consumer buys an unsuitable product. On the other hand, the belief jumps to 1 if no recommendation is given. Last, the belief does not move if the consumer buys a suitable product, and it can then be computed for $\theta < \theta^*$,

$$\begin{aligned} V_I(\theta) &= q\Delta_g, \\ V_U(\theta) &= \frac{(1-\delta)\Delta_\emptyset}{1-q\delta} \cdot \mathbb{1}_{\{\theta<1\}} + \Delta_\emptyset \cdot \mathbb{1}_{\{\theta=1\}}, \\ V_C(\theta) &= \frac{(1-\delta)[q-p+p(1-q)\theta] + \delta q\theta(1-q)(1-p)}{1-q\delta}. \end{aligned}$$

Figure 2.4 depicts the payoffs of this cutoff equilibrium, and there are several immediate observations. First, everyone's payoff is weakly increasing in the reputation. Second, the uninformed type's payoff is discontinuous at $\theta = 1$ due to the requirement (c) of the solution concept. Specifically, at an interior belief, an unsuitable recommendation results in a complete loss of reputation, whereas an unsuitable recommendation at the extreme belief is off path and thus does not lead to any change in reputation.

Third, the informed type achieves a higher payoff than the uninformed type. In fact, this property holds in any equilibrium because they share the same ex-post preference but the former's strategy space contains the latter's. Last, the consumer's payoff is piece-wise linear and discontinuous at the cutoff θ^*. To see this, remember that the consumer's payoff is a weighted average of the payoff conditional on a recommendation and the payoff conditional on no recommendation. While the first part is equal to 0 at the cutoff θ^*, the second part is always strictly positive because the efficient continuation equilibrium is played with positive probability.

[16] The posterior $\theta|_{NB}$ is not drawn in Figure 2.3 because it is irrelevant for the recommender's incentives, but it should lie between $\theta|_S$ and $\theta|_{NS}$ by the martingale property of belief updating.



(a) The Recommender's Equilibrium Payoffs **(b)** The Consumer's Equilibrium Payoff

Figure 2.4: Equilibrium payoffs of the cutoff equilibrium.

2.4.2 Babbling Equilibrium with Patient Players

Observe that the cutoff equilibrium fails to exist for $\delta > \underline{\delta}$ because when the consumer's belief falls in the purchase region, the uninformed type's reputational concerns would be so strong that he finds it profitable to deviate to not recommend. In particular, his incentive constraint requires

$$V_U(\theta) \geq \delta V_U(1) \iff \frac{1-\delta}{1-q\delta} \geq \delta \iff \delta \leq \frac{1}{1+\sqrt{1-q}} = \underline{\delta}.$$

Proposition 2.1(b) generalizes this observation. If $\delta > \underline{\delta}$, not only does this cutoff equilibrium fail to exist, but, in fact, any informative equilibrium fails to exist. To this end, it is equivalent to show that the consumer never follows any recommendation, at any belief $\theta < 1$.

Suppose the consumer buys with positive probability at some belief $\theta < 1$ in some equilibrium. The proof's sketch can be broken down into two steps. In the first step, I prove the existence of a belief at which the consumer buys with positive probability and the uninformed type strictly prefers to recommend. If no such belief exists, the uninformed type always prefers to not recommend when the consumer buys, suggesting he receives zero payoff at any interior belief. However, this contradicts the presumption that the consumer buys with positive probability at some belief.

The second step, on the contrary, argues that such a belief cannot exist, thereby concluding the proof. Suppose the uninformed type recommends with probability one and the consumer buys with positive probability at some belief. Then, the outcome NR must be on path; otherwise, the two types pool on recommending, rendering the recommendation uninformative. But this is impossible because the consumer would find it profitable to deviate and not buy. Now that the outcome NR is on path, it leads to a full reputation by Bayes's rule, and therefore the static incentives and dynamic incentives are aligned for an informed type who observes a bad signal. He strictly prefers to not recommend, further implying that the informed type never recommends unsuitable products. Hence, the outcome NS is conclusive evidence of being uninformed. Due to a lack of expertise, the uninformed type inevitably recommends an unsuitable product sometimes, in which case the continuation payoff is zero. But if he deviates to not recommend, the continuation payoff is lucrative. It then follows that he has a profitable deviation from the putative equilibrium when he is sufficiently patient.

The sketch of the proof highlights two countervailing forces. On the one hand, the uninformed type gets a positive flow payoff only if he recommends and the consumer buys. This feature ensures that he gets a positive payoff and recommends with probability one at some interior belief. On the other hand, whenever he recommends with probability one, the possible revelation of an unsuitable product accelerates the separation of types and thus lowers his continuation payoff. In contrast, the deviation to not recommend yields him the highest reputation and a lucrative continuation payoff. Hence, if his reputational concerns are strong, the second force kills the possibility that he gets a positive payoff while recommending with probability one at an interior belief.

The culprit of the relationship breakdown is the uninformed recommender's reputational incentives. Hypothetically, if we let both the informed recommender and the consumer be myopic, the arguments above remain valid. This is in stark contrast with Ely and Välimäki (2003) and Deb et al. (forthcoming). In the former, the unbiased agent's reputational incentives generate a bad reputation result. In the latter, the agent's discount factor does not play an important role. Instead, their insights require that the principal is long-lived but impatient.

Several elements of the model are crucial for this result. First, suitable products only arrive stochastically so that not recommending is sometimes necessary. Second, not recommending immediately ends the game without generating any payoff. These two features combined give the uninformed type a leeway to camouflage himself by playing NR. This result fails if each option of the recommender must reveal information regarding the type. To understand this, consider the following variant of the original model. One suitable product and one unsuitable product simultaneously arrive in each period. An informed recommender can tell the two products apart, whereas the uninformed recommender and the consumer cannot. The recommender's task here is to select which product to recommend and does not have the option to not recommend. In this scenario, the uninformed type gets the same flow payoff with either choice so that the first force identified above disappears. As a result, it is possible to sustain an informative equilibrium.[17]

I end this section by commenting on the informed type's signal structure. Having a binary and perfect signal simplifies the dynamics of posteriors and is necessary for the construction of equilibrium in Claim 2.1. However, it is not so necessary for the reputation failure result. More broadly, the proof can be adapted for any signal

[17]For example, the informed type always chooses the suitable product, the uninformed recommender strictly mixes over two products, and the consumer follows the recommendation if the recommender's reputation is sufficiently high.

structure that satisfies the following property: when the product is unsuitable, then it is believed to be unsuitable with sufficiently high probabilities for any possible signal realizations. The binary and perfect signal structure is one obvious example that satisfies this property, but it is not the unique one. See Section 2.6.3 for more details.

2.5 Comparative Statics

In this section, I examine comparative statics with respect to the expected product suitability q, and the payoff parameters $\overline{u}$, $\underline{u}$.

2.5.1 Expected Product Suitability

The threshold of the discount factor in Proposition 2.1 takes the following form.

$$\underline{\delta}(q) = \frac{1}{1 + \sqrt{1-q}},$$

where $q \in (\frac{-\underline{u}}{\overline{u}-\underline{u}}, p)$ by Assumption 2.1 and 2.2. As expected, the threshold is increasing in q. If the product is ex-ante more likely to be suitable, the uninformed type makes less unsuitable recommendations, on average. As a result, the second force that pushes the uninformed type to not recommend gets weaker. Hence, it is easier to sustain an informative equilibrium for a fixed discount factor.

What if the expected suitability lies outside the range in which Proposition 2.1 applies? If $q \geq p$, it is trivial to construct an efficient equilibrium because even an uninformed type is beneficial for the consumer. In particular, the following efficient equilibrium outcome can be sustained for any discount factor $\delta < 1$: at any belief, the informed type recommends honestly, the uninformed type always recommends, and the consumer always follows the recommendation.

If $q \leq \frac{-\underline{u}}{\overline{u}-\underline{u}}$, one may expect a bad outcome to occur because more information is required to attract the consumer. However, this logic ignores the fact that q also affects the uninformed type's static incentive. When the expected suitability is sufficiently small, recommending is exceptionally risky for him since it accelerates the separation of types and results in a negative expected flow payoff. Indeed, the uninformed type must always get zero payoffs in any equilibrium, and an obvious way to achieve this payoff is to never recommend. Using this fact, an efficient equilibrium is constructed as follows: at any $\theta \in (0, 1]$, the informed type recommends honestly, the uninformed type never recommends, and the consumer always follows the recommendation. Again, this efficient equilibrium can be sustained independent of the discount factor.

Taken together, a non-monotonicity emerges. For products that are either very likely to be suitable or very likely to be unsuitable, the uncertainty in expertise is not an issue at all. However, when products are of intermediate expected suitability, even arbitrarily small uncertainty in expertise causes a relationship breakdown.

2.5.2 Recommender's Payoff Structure

The baseline results are derived under the assumption that the (un)informed type's static incentives (does not) align with the consumer's. This is ensured by Assumption 2.2: $\Delta_b < 0 < \Delta_\emptyset < \Delta_g$. In this extension, I explore alternative payoff structures. While doing so, I maintain that $\overline{u} \geq \underline{u}$, which preserves the ranking among $\Delta_g, \Delta_\emptyset, \Delta_b$. This is a natural assumption to impose because $\overline{u} - \underline{u}$ can be viewed as a penalty for making unsuitable recommendations. A negative penalty means rewarding for unsuitable recommendations, which is hard to justify. In addition, I ignore knife-edging cases where any of $\Delta_g, \Delta_b, \Delta_\emptyset$ equals 0.

In the first case, assume $0 < \Delta_b \leq \Delta_\emptyset \leq \Delta_g$. This payoff structure suggests the informed type has a perverse static incentive to recommend even when he observes a bad signal. Given such a perverse incentive, the consumer never follows any recommendation in the static benchmark. If she does, then any type of recommender strictly prefers to recommend, leaving the recommendation completely uninformative.[18] Perhaps surprisingly, patience does not overturn the inefficiency in the static benchmark. This perverse static incentive is so strong that it precludes any informative equilibrium for any discount factor $\delta < 1$, and therefore the relationship breakdown is even stronger than in the baseline model. Formally,

Proposition 2.2. *Assume $0 < \Delta_b \leq \Delta_\emptyset \leq \Delta_g$. Then for any $\delta < 1$, the unique equilibrium outcome is $V_U(\theta) = V_I(\theta) = V_C(\theta) = 0$, $\forall \theta \in [0, 1]$.*

Proof. See Appendix 2.A.4.

Roughly speaking, while the informed type possesses useful information, his perverse static incentive prevents the information from being transmitted to the consumer. Thus, it is as if he is an uninformed type.

In the second case, $\Delta_b < 0 < \Delta_\emptyset < \Delta_g$. This is an ideal scenario in which the static incentives of both types of recommender align with the consumer's. Similar to the last case in Section 2.5.1, an efficient equilibrium exists for arbitrary $\delta < 1$.

Proposition 2.3. *Assume $\Delta_b < \Delta_\emptyset < 0 < \Delta_g$. Then the following strategy profile and beliefs constitute an efficient equilibrium for any $\delta < 1$.*

- *Strategies:* $r_g(\theta) = 1$, $r_b(\theta) = 0$, $r_\emptyset(\theta) = 0$, $b(\theta) = 1$.
- *Beliefs:* $\theta|_S = \theta|_{NS} = \theta|_{NB} = 1, \theta|_{NR} = \frac{\theta(1-q)}{1-q\theta}$.

In the last case, $\Delta_b \leq \Delta_\emptyset \leq \Delta_g < 0$, and in fact, this case may appear unreasonable because it requires the commission $\overline{u}$ to be negative. However, this is unreasonable only because the informed type has a perfect signal. If instead he receives a noisy signal, then the expected flow payoff conditional on a good signal (Δ_g)

[18] For a fair comparison, I employ the same solution concept as in the baseline results. However, "*Efficiency at the Top*" has no bite here because the unique equilibrium outcome at the top belief is the babbling outcome.

would be a weighted average of $\overline{u}$ and $\underline{u}$. It is then possible to have a negative Δ_g while simultaneously keeping $\overline{u}$ positive. Nevertheless, for the purpose of consistency, I maintain the perfect signal assumption.

In this payoff structure, the informed type's static incentive is perverse in a different way because he prefers to not recommend even after observing a good signal. Intuitively, recommendation is too costly for both types of recommender because any purchase would yield them a loss in expectation. On the other hand, they could secure zero payoff by never recommending. Hence, the unique equilibrium outcome must again be babbling. Likewise, this result does not hinge on the discount factor either. The proof is straightforward and thus omitted.

Proposition 2.4. *Assume $\Delta_b \leq \Delta_\emptyset \leq \Delta_g < 0$. Then for any $\delta < 1$, the unique equilibrium outcome is $V_U(\theta) = V_I(\theta) = V_C(\theta) = 0$, $\forall \theta \in [0,1]$.*

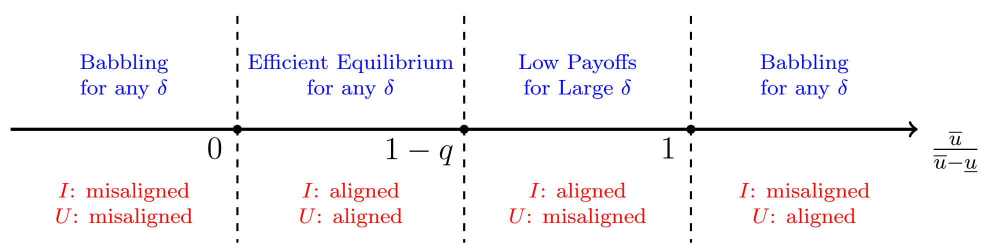


Figure 2.5: Summary of equilibrium outcomes under different payoff structures.

Conditional on $\overline{u} \geq \underline{u}$, the recommender's payoff structure can be summarized by a single ratio $\frac{\overline{u}}{\overline{u}-\underline{u}}$. For example, the payoff structure in the baseline model (Assumption 2.2) is equivalent to $\frac{\overline{u}}{\overline{u}-\underline{u}} \in (1-q, 1)$. Making use of this observation, Figure 2.5 compactly presents a summary of the equilibrium outcomes for all four different payoff structures. The comparison across payoff structures suggests that it is possible to avoid relationship breakdown if the static incentives of both types of recommender align with the consumer's. Moreover, this condition is also necessary. If either type has misaligned static incentives, the relationship cannot be sustained when the recommender is sufficiently patient.

2.6 Extensions

In this section, I explore the robustness of the main result to weaker solution concepts and alternative modelling assumptions.

2.6.1 Alternative Solution Concepts

In the baseline model, I consider two refinements—Markov perfection and "*Efficiency at the Top*". In this section, I weaken the solution concept in two ways. First, I drop "*Efficiency at the Top*" while maintaining the Markov restriction. I show that when either the uninformed type is a commitment type who plays a pure strategy, or the informed type is a commitment type who recommends honestly, the equilibrium payoff of each player vanishes as the recommender gets patient. Then, conversely, I drop the Markov restriction while maintaining "*Efficiency at the Top*". I find that as long as the consumer is not so patient, the main result applies.

Babbling at the Top

As previously mentioned, the implication of "*Efficiency at the Top*" is that once the recommender is believed to be informed with certainty, the continuation play is efficient. If the continuation play is inefficient, then it must be babbling according to Section 2.3.2. Thus, it suffices to analyze equilibria that satisfies a property called ""*Babbling at the Top*"".

The first observation is that, by coordinating on the worst equilibrium once the uncertainty is resolved, it is actually possible to construct an informative equilibrium for arbitrary discount factor $\delta < 1$.[19] The construction is identical to the cutoff equilibrium in Section 2.4 except that I impose ""*Babbling at the Top*"". The equilibrium is described in Claim 2.2 and is henceforth referred to as the modified cutoff

[19]This is similar in spirit to Deb et al. (forthcoming), where the main insight is that reputational incentives can be exceedingly strong unless the players coordinate on maximally inefficient strategies.

equilibrium.

Claim 2.2. *Let* $\theta^{**} = \frac{p-q}{p(1-q)}$.[20] *Then, the following strategy profile and belief system constitute a Markov perfect equilibrium for any* $\delta \in [0, 1)$.

- *Purchase Region:* $\theta \in [\theta^{**}, 1)$

  Strategies: $r_g(\theta) = 1$, $r_b(\theta) = 0$, $r_\emptyset(\theta) = 1$, $b(\theta) = 1$.

  Beliefs: $\theta|_S = \theta$, $\theta|_{NS} = 0$, $\theta|_{NR} = 1$, $\theta|_{NB} = \frac{q\theta}{q\theta+1-\theta}$.

- *No-purchase Region:* $\theta \in [0, \theta^{**}) \cup \{1\}$

  Strategies: $r_g(\theta) = 1$, $r_b(\theta) = 1$, $r_\emptyset(\theta) = 1$, $b(\theta) = 0$.

  Beliefs: $\theta|_{NB} = \theta$ *(on path),* $\theta|_S = \theta|_{NS} = \theta|_{NR} = \theta$ *(off path).*

Proof. The verification is deferred to Appendix 2.A.5.

This equilibrium gives rise to the following payoffs. When $\theta \in [\theta^{**}, 1)$,

$$V_U(\theta) = \frac{1-\delta}{1-q\delta}\Delta_\emptyset,\ V_I(\theta) = \frac{q(1-\delta)}{1-q\delta}\Delta_g,\ V_C(\theta) = \frac{(1-\delta)[q-p+p(1-q)\theta]}{1-q\delta}. \quad (2.2)$$

Otherwise,

$$V_i(\theta) = 0,\ \forall i \in \{U, I, C\}.$$

Why does this equilibrium survive when the recommender is patient? In the mechanism of a relationship breakdown, a key step is that whenever the uninformed type recommends aggressively, there is a force that pushes him toward a deviation to not recommend. Strictly speaking, such a force has two parts. First, the revelation of

[20]As usual, θ^{**} is the minimal cutoff such that an equilibrium with this cutoff structure exists. It is higher than θ^*, the cutoff of the equilibrium in Section 2.4. ““*Babbling at the Top*"" drives the difference. It can be computed that the consumer's payoff is 0 at the cutoff θ^{**}, conditional on both recommendation and no recommendation. Since $\theta^{**}|_S = \theta^{**}$ and $\theta^{**}|_{NS}$, $\theta^{**}|_{NB} < \theta^{**}$, experimentation essentially brings no value at θ^{**} as both buying and not buying lead to zero continuation payoff. It then follows that only the static incentives matter at the cutoff, implying θ^{**} equals the static cutoff θ^S. In contrast, the consumer's payoff is strictly positive at θ^* in the original cutoff equilibrium because of "*Efficiency at the Top*". This ensures that experimentation still has a value at θ^*.

the product suitability lowers his continuation payoff if he recommends. Second, a deviation leads to the highest reputation and thus the highest continuation payoff by "*Efficiency at the Top*". Yet, under ""*Babbling at the Top*"", being identified as informed is the worst thing that can happen. Hence, the second part of the force fails so that the uninformed type has no incentive to deviate to not recommend.

However, the first part of the force remains in effect, and as a result, the uninformed type cannot get a high payoff. Moreover, the more patient the recommender, the larger amount of continuation payoff is reduced due to the revelation of the product suitability. This intuition is formalized in Proposition 2.5. Before the result, denote E_δ to be the set of Markov perfect equilibrium associated with an arbitrary discount factor $\delta < 1$.

Proposition 2.5. *Assume ""Babbling at the Top"". Then*

$$\lim_{\delta \to 1} \sup_{e \in E_\delta} \sup_{\theta \in [0,1]} V_U(\theta; e) = 0.$$

Proof. See Appendix 2.A.6.

This result might be natural for many readers because whenever the uninformed type recommends, he is further separated from the informed type. But on the other hand, the only way to get a positive flow payoff is to recommend. Hence, from a long-run perspective, the types are eventually separated. It then follows that the uninformed type's payoff must vanish as he gets patient.

How about the informed type and the consumer? If full separation occurs almost surely, it seems that their payoffs should also vanish as δ increases as the outcomes at two extreme beliefs are both bad. In fact, Equation (2.2) suggests that their payoffs indeed go to 0 in the modified cutoff equilibrium as δ goes to 1. If an analogous result of Proposition 2.5 with the informed type and the consumer is shown, it can be viewed as a weaker version of reputation failure. Basically, reputational incentives are still harmful and can destroy all the surplus as the recommender gets arbitrarily patient.

While these results are desirable and intuitive, it is difficult to establish them. Instead, I provide two orthogonal sets of restrictions, under which the informed type's payoff and the consumer's payoff both vanish as the recommender gets patient. In the first type of restriction, the uninformed type is a commitment type who always plays a pure strategy.[21] In the second type of restrictions, the informed type is a commitment type who always recommends honestly, and the consumer is myopic. Essentially, both types of restrictions play the same role and make it impossible for the uninformed type to mimic the informed type. As a result, the types are separated at a faster speed. But due to ""*Babbling at the Top*"", full separation hurts everyone, and it follows, then, that neither type can obtain a high payoff.

Proposition 2.6 formalizes the result under the first type of restriction. Denote E^U_δ as the set of Markov perfect equilibrium associated with any discount factor $\delta < 1$ such that $r_\emptyset(\theta) \in \{0, 1\}$, $\forall \theta \in [0, 1]$. It is not empty as the modified cutoff equilibrium, described in Claim 2.2, belongs to E^U_δ.

Proposition 2.6. *Assume ""Babbling at the Top"". Then for $i \in \{C, I\}$,*

$$\lim_{\delta \to 1} \sup_{e \in E^U_\delta} \sup_{\theta \in [0,1]} V_i(\theta; e) = 0.$$

Proof. See Appendix 2.A.7.

Proposition 2.7 formalizes the result under the second type of restriction. Denote

[21] Ely and Välimäki (2003) also assume that one of the type is a commitment type. Otherwise, they impose a renegotiation-proof refinement which plays the same role as the "*Efficiency at the Top*" property.

$E_\delta^{I,C}$ as the set of Markov perfect equilibrium associated with any discount factor $\delta < 1$ such that $r_g(\theta) = 1$, $r_b(\theta) = 0$, $\forall\theta \in [0, 1]$ and the consumer is myopic. Abusing notation, let $V_C(\theta)$ be the consumer's non-discounted payoff.

Proposition 2.7. *Assume "“Babbling at the Top"". Then,*

(a) $\lim_{\delta\to 1} \sup_{e\in E_\delta^{I,C}} \sup_{\theta\in[0,1]} V_I(\theta; e) = 0,$

(b) $\lim_{T\to\infty} \mathbb{E}[V_C(\theta^T)] = 0$, *where* θ^T *is the consumer's posterior after* T *periods, and* $\theta^0 = \theta$.

Proof. See Appendix 2.A.9.

To demonstrate that $E_\delta^{I,C}$ is non-empty, I explicitly construct an equilibrium, summarized in Claim 2.3. In fact, it Pareto dominates any equilibrium within this class point-wise.[22]

Claim 2.3. *Define* $\theta_k = \frac{p-q}{1-q} \cdot \frac{1}{1-(1-p)(1-q)^k}$, $k \in \mathbb{N}$ *and* $\theta_\infty = \frac{p-q}{1-q}$.[23] *The following strategy profile and belief system constitute a Markov perfect equilibrium.*

- *Region* I*:* $\theta \in [\theta_0, 1]$

  Strategies: $r_g(\theta) = r_\emptyset(\theta) = b(\theta) = 1,\ r_b(\theta) = 0$

  Beliefs: $\theta|_S = \theta,\ \theta|_{NS} = 0,\ \theta|_{NR} = 1,\ \theta|_{NB} = \frac{q\theta}{q\theta+1-\theta}$.

- *Region* II$_k$ *for* $k \in \mathbb{N}_+$*:* $\theta \in [\theta_k, \theta_{k-1})$

  Strategies: $r_g(\theta) = 1,\ r_b(\theta) = 0,\ r_\emptyset(\theta) = \frac{q(1-p)}{p-q} \cdot \frac{\theta}{1-\theta},\ b(\theta) = \delta^k$.

  Beliefs: $\theta|_S = \theta_0,\ \theta|_{NS} = 0,\ \theta|_{NB} = \theta_\infty,\ \theta|_{NR} = \frac{(p-q)(1-q)\theta}{p-q-p(1-q)\theta}$.

- *Region* III*:* $\theta \in [0, \theta_\infty]$

  Strategies: $r_g(\theta) = r_\emptyset(\theta) = 1,\ r_b(\theta) = b(\theta) = 0$.

  Beliefs: $\theta|_S = \theta,\ \theta|_{NS} = 0,\ \theta|_{NR} = 1,\ \theta|_{NB} = \frac{q\theta}{q\theta+1-\theta}$.

[22] This equilibrium still exists even when the consumer is long-lived and the informed type is not committed to recommend honestly. In that case, the equilibrium payoffs vanish as δ goes to 1 as well, but it is unclear whether Proposition 2.7 holds generally after dropping these restrictions.

[23] It can be easily verified that $\theta_\infty < \theta_k$ and $\theta_{k+1} < \theta_k$ for $\forall k \in \mathbb{N}$.

Proof. The verification is deferred to Appendix 2.A.8.

In addition to a no-purchase region and a purchase region, there is a new region for intermediate beliefs where both the consumer and the uninformed type strictly mix. The new region can be further partitioned into countable sub-regions. Due to the complicated cutoff structure, this equilibrium is referred to as the general cutoff equilibrium, illustrated in Figure 2.6.

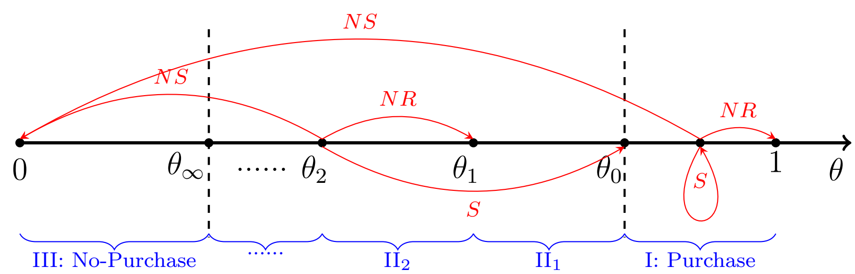


Figure 2.6: Illustration of the general cutoff equilibrium.

Rename I and III, respectively, to II_0 and II_∞, and denote the payoff of player $i \in \{U, I\}$ restricted in the sub-region II_k by V_i^k for $k \in \mathbb{N} \cup \{\infty\}$. Then, the recommender's payoff can be compactly represented by

$$V_U^k(\theta) = \frac{1-\delta}{1-q\delta}\delta^k \Delta_\emptyset, \quad V_I^k(\theta) = \frac{1-\delta}{1-q\delta}\delta^k \left[1 - (1-q)^{k+1}\right] \Delta_g. \tag{2.3}$$

Meanwhile, the consumer's payoff is a piece-wise linear function:

$$V_C(\theta) = \begin{cases} q - p + p(1-q)\theta & \text{if } \theta \in [\theta_0, 1). \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.4}$$

According to Equation 2.3, $\sup_{\theta \in [0,1]} V_I(\theta)$ is bounded above by $\frac{1-\delta}{1-q\delta}\Delta_g$, which converges to zero as δ goes to one. Since this equilibrium is point-wise Pareto dominant within $E_\delta^{I,C}$, the first part of Proposition 2.7 follows naturally.

As for the second part, note that there is no flexibility in the consumer's payoff within $E_\delta^{I,C}$. For any equilibrium in this class, the consumer's payoff must take the form described by Equation 2.4. For a generic belief θ, the consumer's payoff after T periods is positive only if the posterior lives in region I. But even for a belief in this region, there is a constant attrition in the sense that the posterior jumps up to 1 or down to 0 with a total probability $1-q$, in which cases the consumer's payoff equals 0 henceforth. Thus, after a sufficiently long period, the posterior is almost surely at one of the extreme beliefs, establishing the second part of Proposition 2.7.

Non-Markov Equilibria

When the Markov restriction is relaxed, the strategies must be redefined. Recall that in each period, there can be only four outcomes: NR, NB, S, NS. Then, the public history at the beginning of period t, given below, contains all previous actions and outcomes observed by the consumer and both types of recommender.

$$h^{t-1} \in \{NR, NB, S, NS\}^{t-1}.$$

with the usual convention that $h^0 = \emptyset$. Both the consumer and the uninformed type have no additional private history. The informed type, on the other hand, additionally observes a signal $s \in \{g, b\}$ of the product's value in every period. It follows that his private history at the beginning of period t is given by

$$h_I^{t-1} \in \{NR, NB, S, NS\}^{t-1} \times \{g, b\}^{t-1}.$$

However, when the signal is perfect, as is the case here, some histories are not plausible. If the public outcome at some period was S (NS), then the signal must be $g(b)$ to be consistent. Henceforth, denote $\mathcal{H}$ as the set of public histories, and denote $\mathcal{H}^I$

as the set of the informed type's private histories that are consistent with the public history.

The strategies are defined as follows. The strategy of the consumer $b(h^{t-1}) : \mathcal{H} \longrightarrow [0,1]$ specifies the probability of buying the product at the history h^{t-1}.[24] The uninformed type's strategy $r_\emptyset(h^{t-1}) : \mathcal{H} \to [0,1]$ specifies the probability that he makes a recommendation at the history h^{t-1}. Last, the informed type's strategy is again represented by a pair of functions: $r_g(h^{t-1}, s^{t-1}) : \mathcal{H}^I \to [0,1]$ and $r_b(h^{t-1}, s^{t-1}) : \mathcal{H}^I \to [0,1]$, which, respectively, specifies the probability that he makes a recommendation at his private history (h^{t-1}, s^{t-1}) when the current period signal is good/bad.

After dropping the Markov restriction, there is a plethora of equilibria when the recommender is believed to be informed with certainty. However, "*Efficiency at the Top*" still uniquely selects the efficient equilibrium in which the consumer always buys the product. Further, the Markov restriction entails an additional implication, referred to as the "*No-Information-No-Purchase*" property. It states that, at any belief, it is impossible that both types of recommender pool on recommending, yet the consumer buys the product with positive probability. Formally, there is no Markov perfect equilibrium such that $r_\emptyset(\theta) = r_g(\theta) = r_b(\theta) = 1$, $b(\theta) > 0$ for some belief $\theta \in [0,1]$.

Thus, the main result (Proposition 2.1(*b*)) can be decomposed into two steps. First, when the recommender is sufficiently patient, the unique Markov perfect equilibrium that satisfies "*Efficiency at the Top*" and "*No-Information-No-Purchase*" is the babbling equilibrium. Second, every Markov perfect equilibrium that satisfies "*Efficiency at the Top*" also satisfies "*No-Information-No-Purchase*". The first step can be generalized to non-Markov equilibria without additional difficulty.[25] Analogously,

[24] The consumer moves after the recommender and also observes whether the recommender recommends in the current period. But such information is vacuous because she makes a decision only if the recommender recommends.

[25] To accommodate non-Markov equilibria, replace the belief with the history in defining "*No-*

the uninformed type receives zero payoff at any on-path public history. It follows, then, that the consumer cannot buy the product at any on-path public history. The second step can also be generalized if we disentangle the consumer's discount factor from the recommender's and consider an impatient consumer. Let the consumer's and the recommender's discount factor, respectively, be δ_c and δ_r. The following assumption provides a sufficient condition for "*No-Information-No-Purchase*".

Assumption 2.3. $\delta_c \leq \frac{p-q}{p(1-q)}$.

Intuitively, by following an uninformative recommendation, the consumer bears a loss $(q-p)(1-\delta)$ in the flow payoff but at most gains $q(1-p)\delta$ in the continuation payoff. When she is sufficiently impatient, the loss outweighs the gain so that she strictly prefers to not follow the recommendation. Since both steps can be generalized, the main insight that reputational concerns can destroy the surplus carries over to this extension as well.

Proposition 2.8. *Assume* $\delta_c \leq \frac{p-q}{p(1-q)}$ *and* $\delta_r > \frac{1}{1+\sqrt{1-q}}$. *Then any equilibrium that satisfies "Efficiency at the Top" is babbling.*

Proof. See Appendix 2.A.10.

An alternative sufficient condition to generalize the second step is to impose an off-path condition. Specifically, at any histories such that NR is off path, let the off-path belief be equal to 1. This off-path condition ensures "*No-Information-No-Purchase*"; otherwise, the informed type would then strictly prefer to deviate to not recommend when he receives a bad signal as it yields both a higher flow payoff and a continuation payoff.

Information-No-Purchase".

2.6.2 Product Selection

An important element of the baseline model is that the two types face the same products and suitable products only arrive stochastically. However, in real life, an informed type may be able to actively select products. For example, there may be multiple new products of independent values, and the informed type knows each product's value and decides whether to recommend the best one. In contrast, the uninformed type does not know the value of any product and behaves as if he only sees one product.

The probability that at least one of the products is suitable clearly exceeds the probability that a randomly drawn product is suitable, and therefore the two types are essentially facing products with different expected suitability. Thus, the selection can be conveniently incorporated into the model without changing the underlying product structure. In each period, let the product that arrives at the informed (uninformed) type be suitable with probability q' (q), where $q' > q$. The next proposition suggests that the main insight carries over to this setting if the selection is not too strong. The arguments are repeating the main result's proof and are thus omitted.

Proposition 2.9. *If $q' < p$ and $\delta > \underline{\delta} = \frac{1}{1+\sqrt{1-q}}$, the unique equilibrium is babbling.*

The assumption $q' < p$ ensures "*No-Information-No-Purchase*", which is a key ingredient of the main result. Intuitively, if the selection is sufficiently strong, i.e., q' is sufficiently high, then even when the two types pool on recommending, the consumer may want to buy the product as long as she believes the recommender is informed with high probabilities. This comparison implies that it is more likely to retain consumers' trust when the product space is large because the informed type could then demonstrate his competence through product selection. This prediction matches the fact that the most successful categories in influencer marketing, including beauty and apparel, are often the categories with countless new products.

2.6.3 Noisy Inference

In the baseline model, the consumer's inference of the recommender's type is not interfered with any noise. This extension considers noisy inference, which may arise in two different settings: 1) the informed type is not perfectly informed about the consumer's preference, and 2) the product's suitability is revealed with a noise. The second setting is more plausible if we consider a sequence of short-lived consumers, where subsequent consumers may not perfectly know the payoffs of previous consumers. Mathematically, these two settings are equivalent, but it is easier to describe the first setting. In what follows, I explore whether the main insight extends to alternative signal structures.

First, as briefly mentioned in Section 2.4, the proof of Proposition 2.1(b) does not require the signal to be perfect. The essential property is that when the product is unsuitable, it is believed to be unsuitable with sufficiently high probabilities for any possible signal realizations. This property ensures that when the uninformed type recommends with probability one, NS is conclusive evidence of being uninformed so that the type's continuation payoff from recommending is bounded away from the continuation payoff if he deviates to not recommend. In the context of a binary signal $s \in \{g, b\}$, this property is satisfied for any signal structure that satisfies Assumption 2.4.[26]

Assumption 2.4. $\Pr(s = b|v = 0) = 1$ *and* $\Pr(s = g|v = 1) = \mu \in (\frac{q\overline{u}+(1-q)\underline{u}}{q\overline{u}}, 1]$.

As the previous discussion suggests, the proof of the main result goes through under any binary signal structure, satisfying Assumption 2.4.

Proposition 2.10. *Suppose the informed type's signal structure satisfies Assumption 2.4. Then there exists $\underline{\delta} < 1$ such that, for any $\delta > \underline{\delta} = \frac{1}{1+\sqrt{1-q}}$, the unique equilibrium*

[26]The signal needs not to be binary. Generally, let $\mathcal{S}$ be the set of possible realizations, and $f(s|v)$ is the conditional probability that s is realized given the value is v. Then a counterpart of Assumption 2.4 is $\frac{f(s|v=0)}{f(s|v=1)} > -\frac{q\overline{u}}{(1-q)\underline{u}}$ for $\forall s \in \mathcal{S}$ such that $f(s|v = 0) > 0$.

outcome is babbling.

Proof. See Appendix 2.A.11.

Note that Assumption 2.4 still requires the good signal to be perfect. If both signals are imperfect, the main result's proof does not go through anymore. However, the main message is consistent. For convenience, let the signal structure be symmetric: $\Pr(s = b|v = 0) = \Pr(s = g|v = 1) = \lambda \in (p, 1)$. Moreover, let the uninformed type be a commitment type who always plays a pure strategy. The latter restriction is imposed not just for tractability, and more importantly, the restriction renders it impossible for the uninformed type to mimic the informed type. Thus, it echoes with the motivation that even if such mimicking is impossible, reputational concerns may be detrimental to the relationship.

Proposition 2.11. *Suppose the uninformed type commits to play a pure strategy, then for $\delta > \frac{1}{2}$, the unique equilibrium is babbling.*

Proof. See Appendix 2.A.12.

The following is a sketch of the proof. I first establish a claim that whenever the consumer buys with positive probability, the uninformed type recommends. Using this claim, I show that the consumer does not buy in any equilibrium if her belief is lower than some threshold. Equivalently, I construct an initial babbling interval and then look at beliefs slightly higher than the threshold. Suppose the consumer buys with positive probability at those beliefs; then, again, the uninformed type must recommend. This implies that the posteriors following S, NS, NB all drift down and fall in the initial babbling interval. The uninformed type's continuation payoff from recommending is therefore zero, whereas he receives a lucrative continuation payoff if he deviates to not recommend. Thus, he has a profitable deviation when he is sufficiently patient, suggesting the consumer cannot buy at these beliefs. These arguments essentially expand the initial babbling interval. After sufficiently many repetitions, the babbling interval eventually covers $[0, 1)$.

2.6.4 Purchase without Recommendations

In some scenarios, the consumer knows what the product is even in absence of recommendation. For example, it is easy for a consumer to keep track of new movie releases since producing a movie is very costly, limiting the number of releases. In this case, the consumer should have the option to watch a movie even if she does not see a recommendation. However, if she watches a new movie without a recommendation, the recommender should not be held accountable if this movie turns out to be a bad match for the consumer. In other words, the recommender receives no flow payoff when he chooses to not recommend regardless of the product's suitability.

While allowing this alternative possibility to buy complicates the analysis by introducing two more posteriors, the two forces in the main result are still present. Similar arguments imply that in any equilibrium, the consumer never follows any recommendation, and both types of recommender receive zero payoff. However, the consumer may still be able to obtain a positive payoff from buying the product without a recommendation. This can happen because the recommender receive a zero payoff anyway, and thus may as well behave in such a way that the action NR signals the product to be suitable with high probabilities. As a response, the consumer strictly finds it optimal to buy the product without recommendations.

Nevertheless, I deem this type of equilibrium implausible because it greatly hinges on the recommender's indifference between two actions. Introducing an arbitrarily small cost $\epsilon > 0$ for recommending would break the indifference immediately because both types then strictly prefer to not recommend, rendering the action NR uninformative. Moreover, the arguments used in the main result are immune to a negligible cost for recommendation. Thus, even though the consumer can potentially get additional feedback in this extension, the relationship still breaks down as the recommender gets patient.

Proposition 2.12. *Assume* $\delta > \underline{\delta} = \frac{1}{1+\sqrt{1-q}}$. *There exists* $\underline{\epsilon} > 0$ *such that for any* $\epsilon \in (0, \underline{\epsilon})$*, the unique equilibrium is babbling.*

Proof. The arguments follows from the discussion, and are thus omitted.

2.7 Concluding Remarks

Consumers often rely on the information of recommenders to make decisions. If consumers are convinced the recommenders are genuine experts, there is no trust issue and the information can be fully transmitted to consumers. It is tempting to expect that most information can be transmitted to consumers when there is a tiny chance that recommenders are incompetent. However, this chapter suggests a stark discontinuity. I find that provided sufficiently patient recommenders, no information can be transmitted even with arbitrarily small uncertainty, suggesting reputational concerns cannot enhance the relationship but can be detrimental. Moreover, it may be particularly difficult for new entrants in these markets because consumers most likely have uncertainty over the expertise of new entrants.

Technically speaking, the current arguments for reputation failure either require imposing "*Efficiency at the Top*" or restricting the recommender's strategies. However, neither of these restrictions may be necessary. In general, I conjecture that the informed type's payoff in any Markov perfect equilibrium converges to 0 as he gets patient, which is supported by the following crude intuition. Suppose "*Efficiency at the Top*" is not imposed, and the informed type gets a nontrivial amount of payoff even if the recommender is arbitrarily patient, then the posterior belief must stay interior even after a long horizon with a large probability. But then, even the uninformed type may be able to obtain a non-trivial amount of payoff in the limit, which can not happen in any equilibrium. In the future, I wish to further pursue this direction and prove this conjecture.

Last, I comment on several approaches to restore the relationship. First, the relationship breakdown does not occur if the commission level is chosen appropriately so that the two types of recommender's incentives align with the consumer's. Second, the relationship can be restored if the informed type has access to products with a higher expected suitability relative to the uninformed type and the difference is sufficiently large. This naturally occurs if there is a large set of products so that the informed recommender can apply his expertise and select the best product. Last, in the application of social media influencers, there is a unique feature that may help sustain the relationship. Social media provides followers a platform to communicate with other followers who have no reputational concerns. If the comments from other followers about the products are sufficiently reliable, then one may buy the product even though the influencers' recommendations are uninformative.

2.A Omitted Proofs

2.A.1 Prerequisite Lemmas

The following lemmas summarize some elementary properties of the equilibrium payoffs, and are repetitively used to prove results in both the main body and the extensions. To avoid redundancy, I state them in the most general form. In particular I allow for a noisy signal structure: $\Pr(s = g|S) = \Pr(s = b|NS) = \lambda \in (p, 1]$.

The first lemma simplifies the definition of babbling outcome. Instead of checking that everyone gets 0 payoff, it suffices to show that the uninformed type gets 0 payoff. After all, the recommender gets a positive payoff at a belief θ if and only if the consumer purchases the product at some posterior belief that can be reached from θ with a positive probability (including the prior itself).

Lemma 2.1. *Fix $\delta < 1$ and any corresponding Markov perfect equilibrium. At any $\theta \in [0, 1]$, if $V_U(\theta) = 0$, then $V_C(\theta) = V_I(\theta) = 0$.*

Proof. Suppose $V_C(\theta) > 0$, that means at some future contingency (including the current period), the consumer makes a purchase with some probability. However, given this, the uninformed type could have secured a positive payoff by recommending with positive probability at the same contingency. Contradiction. So, $V_C(\theta) = 0$.

Similarly, if $V_I(\theta) > 0$, then the consumer must purchase the product at some future contingency, contradiction. So, $V_I(\theta) = 0$ as well. □

The next lemma gives upper bounds on the equilibrium payoffs, and indicates that some of the bounds cannot be attained at any belief $\theta < 1$.

Lemma 2.2. *Fix any $\delta < 1$ and any corresponding Markov perfect equilibrium. Then,*

(a) At any $\theta \in [0, 1]$, $V_U(\theta) \leq \Delta_\emptyset$, $V_I(\theta) \leq q\Delta_g$, $V_C(\theta) \leq q(\lambda - p)$.

(b) At any $\theta < 1$, $V_U(\theta) < \Delta_\emptyset$, $V_C(\theta) < q(\lambda - p)$.[27]

Proof. Part (a) is almost straightforward, because those bounds are exactly the bounds of the (undiscounted) flow payoffs. Hence, focus on the part (b) instead.

For the consumer, the upper bound is achieved only if at all future contingencies, the consumer buys the product with probability one, given which the dynamic incentives of both types of recommenders do not play a role anymore. It follows by their static incentives that at all future contingencies, the informed type recommends if and only if the signal is good, while the uninformed type always recommends.

But conditional on the recommender's strategies, the posterior following NS keeps drifting down. The belief is eventually so low that the consumer's flow payoff from buying the product is negative. Since she also does not care about the continuation payoff anymore, it then follows that she has a strict incentive to deviate. Contradiction. Hence, it is impossible that $V_C(\theta) = q(\lambda - p)$ for $\theta < 1$.

Now, suppose the upper bound of the uninformed type's equilibrium payoff is attained at some $\theta < 1$, a necessary condition is that the consumer buys the product with probability one in each period. It then follows that it is impossible $V_U(\theta) = \Delta_\emptyset$ for $\theta < 1$ either. □

Intuitively, whenever there is some chance that the recommender is uninformed, the lack of expertise renders it impossible for the uninformed type to keep the consumer buying forever.

The third lemma illustrates an intuitive relationship between V_U and V_I. In any circumstance, the informed type must get a higher payoff than the uninformed type because having additional information is beneficial.

[27] Notice that the result in part (b) does not apply to V_I. In fact, the threshold equilibrium constructed in Section 2.4 suggests the upper bound of V_I in part (a) is attainable even for interior beliefs. However, that crucially relies on $\lambda = 1$.

Lemma 2.3. *Fix $\delta < 1$ and any corresponding Markov perfect equilibrium. Then $V_U(\theta) \leq V_I(\theta)$ at any $\theta \in [0,1]$. If in addition $b(\theta) > 0$, then the inequality is strict.*

Proof. Since the two types of recommenders share the same payoff function, the informed type could play the following mimicking strategy to perfectly imitate the uninformed type: $r_g(\theta) = r_b(\theta) = r_\emptyset(\theta)$, which guarantees him at least the payoff of the uninformed type. If in addition $b(\theta) > 0$, then the informed recomender could choose to play a different strategy $r_g(\theta) = r_\emptyset(\theta)$, $r_b(\theta) = 0$ for one period, and then play the mimicking strategy henceforth. Under the composite strategy, the informed type gets a higher flow payoff than the uninformed type in the first period because

$$q(1-\delta)b(\theta)\Delta_g > (1-\delta)b(\theta)\Delta_\emptyset$$

by Assumption 2.2. So, this composite strategy breaks the difference and makes the inequality strict. □

The last lemma enables us to apply the Bayes' rule whenever the outcome is not babbling, so that we need not worry about off-path beliefs in this case.

Lemma 2.4. *Fix any $\delta < 1$ and any corresponding Markov perfect equilibrium. For any $\theta \in [0,1]$, if $V_I(\theta) > 0$ or $V_U(\theta) > 0$ or $V_C(\theta) > 0$, then both actions R and NR are on path at θ.*

Proof. It is easier to see the contrapositive statement. At a belief $\theta \in [0,1]$, if R is off-path, then the consumer cannot buy the product. In addition, she does not update the belief as the action NR is not informative about the type of the recommender. Hence,

$$V_i(\theta) = \delta V_i(\theta), \quad i = I, U, C,$$

so that $V_I(\theta) = V_U(\theta) = V_C(\theta) = 0$. Similarly, if NR is off-path, then the action R

is uninformative, which results in a negative flow payoff for the consumer. Moreover, the posterior beliefs $\theta|_S, \theta|_{NS}, \theta|_{NB}$ all equal to θ. As a consequence, the consumer does not buy the product. It then follows that again,

$$V_i(\theta) = \delta V_i(\theta), \quad i = I, U, C.$$

so that $V_I(\theta) = V_U(\theta) = V_C(\theta) = 0$. □

2.A.2 Proof of Claim 2.1 and Proposition 2.1(a)

The outcome in the no-purchase region is purely babbling, it must be that $V_U(\theta) = V_I(\theta) = V_C(\theta) = 0$, $\theta < \theta^*$. The incentive constraints are trivially satisfied. Instead, focus on incentives for beliefs in the purchase region $[\theta^*, 1]$. The strategy profile and beliefs are given by

- $r_g(\theta) = 1$, $r_b(\theta) = 0$, $r_\emptyset(\theta) = 1$, $b(\theta) = 1$.
- $\theta|_S = \theta$, $\theta|_{NS} = 0$, $\theta|_{NR} = 1$, $\theta|_{NB} = \frac{q\theta}{q\theta+1-\theta}$.

For each player, I first compute the equilibrium payoffs conditional on the strategy profile and beliefs, and then verify that no one has profitable deviation.

- Given the profile of strategies and beliefs, the uninformed type's payoff satisfies

$$\begin{aligned} V_U(\theta) &= (1-\delta)\Delta_\emptyset + q\delta V_U(\theta) + (1-q)\delta V_U(0) \\ \Longrightarrow V_U(\theta) &= \frac{1-\delta}{1-q\delta}\Delta_\emptyset. \end{aligned}$$

  He has no incentive to deviate if

$$V_U(\theta) \geq \delta V_U(\theta|_{NR}) = \delta V_U(1) = \delta\Delta_\emptyset.$$

This inequality is guaranteed by $\delta \leq \underline{\delta} = \frac{1}{1+\sqrt{1-q}}$.

- Given the profile of strategies and beliefs, the informed type's payoff satisfies

$$\begin{aligned} &V_I(\theta) = q[(1-\delta)\Delta_g + \delta V_I(\theta)] + (1-q)\delta V_I(1) \\ \Longrightarrow &V_I(\theta) = q\Delta_g. \end{aligned}$$

He has no incentive to deviate after observing a good signal because

$$(1-\delta)\Delta_g + \delta V_I(\theta|_S) = (1-\delta+q\delta)\,\Delta_g > q\Delta_g = \delta V_I(\theta|_{NR}).$$

He has no incentive to deviate after observing a bad signal because

$$(1-\delta)\Delta_b + \delta V_I(\theta|_{NS}) = (1-\delta)\Delta_b < \delta V_I(\theta|_{NR}).$$

- Given the profile of strategies and beliefs, the probability of recommendation is

$$\Pr(R) = \theta \cdot q + (1-\theta) \cdot 1.$$

Conditional on a recommendation, the expected value of the product is

$$\mathbb{E}[v|R] = \frac{q}{q\theta + 1 - \theta}.$$

So, the consumer's payoff satisfies

$$\begin{aligned} V_C(\theta) = (q\theta + 1 - \theta)\Bigg[&(1-\delta)\left(\frac{q}{q\theta+1-\theta} - p\right) + \frac{q}{q\theta+1-\theta}\delta V_C(\theta) \\ &+ \frac{(1-q)(1-\theta)}{q\theta+1-\theta}\delta V_C(0)\Bigg] + \theta(1-q)\delta V_C(1), \end{aligned}$$

which gives

$$V_C(\theta) = \frac{(1-\delta)[q-p+p\theta(1-q)] + \delta q\theta(1-q)(1-p)}{1-\delta q}.$$

She has no incentive to deviate if

$$(1-\delta)\left(\frac{q}{q\theta+1-\theta} - p\right) + \delta\frac{q}{q\theta+1-\theta}V_C(\theta) \geq \delta V_C\left(\theta|_{NB}\right). \tag{1.A.1}$$

which can be further simplified to

$$\frac{(1-\delta)[q-p+p\theta(1-q)] + \delta^2 q^2\theta(1-q)(1-p)}{(q\theta+1-\theta)(1-\delta q)} \geq \delta V_C\left(\frac{q\theta}{q\theta+1-\theta}\right).$$

To verify her incentive constraint, consider two cases depending on whether $\frac{q\theta}{q\theta+1-\theta} \geq \theta^* = \frac{p-q}{(1-q)p+\frac{\delta^2 q^2(1-q)(1-p)}{1-\delta}}$.

(i) If $\frac{q\theta}{q\theta+1-\theta} < \theta^*$, then $V_C\left(\frac{q\theta}{q\theta+1-\theta}\right) = 0$. Equation 1.A.1 is satisfied.

(ii) If $\frac{q\theta}{q\theta+1-\theta} \geq \theta^*$, then $V_C\left(\frac{q\theta}{q\theta+1-\theta}\right)$ is equal to

$$\frac{\delta(1-\delta)[q-p+(q^2-q-pq^2+p)\theta] + \delta^2 q^2\theta(1-q)(1-p)}{(q\theta+1-\theta)(1-\delta q)}.$$

Substitute it into Equation 1.A.1, we eventually get

$$\theta \geq \frac{(1-\delta)(p-q)}{[\delta(q-p-pq)+p](1-q)}.$$

This condition is always satisfied since $\frac{q\theta}{q\theta+1-\theta} \geq \theta^*$ implies

$$\begin{aligned}\theta \geq & \frac{(1-\delta)(p-q)}{(1-\delta)(p-q+pq)(1-q)+\delta^2 q^3(1-q)(1-p)} \\ \geq & \frac{(1-\delta)(p-q)}{(1-\delta)(p-q+pq)(1-q)+q(1-q)(1-p)} \\ = & \frac{(1-\delta)(p-q)}{[\delta(q-p-pq)+p](1-q)}.\end{aligned}$$

Thus, the consumer has no incentive to deviate at any belief.

This concludes the verification of the cutoff equilibrium.

2.A.3 Proof of Proposition 2.1(b)

Fix $\delta > \frac{1}{1+\sqrt{1-q}}$ and an associated equilibrium. By Lemma 2.1, it suffices to prove that $V_U(\theta) = 0$, $\forall \theta \in [0, 1)$. To this end, Let $\overline{V}_U = \sup_{\theta<1} V_U(\theta)$ be the supremum of the uninformed type's equilibrium payoff over all beliefs $\theta \in [0, 1)$. For the purpose of contradiction, suppose $\overline{V}_U$ is positive. Then $\exists \tilde{\theta} \in (0, 1)$ such that $V_U(\tilde{\theta}) > \delta \overline{V}_U > 0$. At this belief, the uninformed type's payoff satisfies

$$\begin{aligned}V_U(\tilde{\theta}) = \max \Big\{ & b(\tilde{\theta}) \left[(1-\delta)\Delta_\emptyset + q\delta V_U(\tilde{\theta}|_S) + (1-q)\delta V_U(\tilde{\theta}|_{NS}) \right] \\ & +[1-b(\tilde{\theta})]\delta V_U(\tilde{\theta}|_{NB}),\, \delta V_U(\tilde{\theta}|_{NR}) \Big\}. \qquad (1.A.2)\end{aligned}$$

The next two claims partially characterize of the strategies and posteriors at $\tilde{\theta}$.

Claim 2.4. *At $\tilde{\theta}$, the uninformed type recommends with probability one: $r_\emptyset(\tilde{\theta}) = 1$. In addition, the posterior following NR jumps to one: $\tilde{\theta}|_{NR} = 1$.*

Proof. Suppose $r_\emptyset(\tilde{\theta}) < 1$. It then follows that $\tilde{\theta}|_{NR} < 1$ by Bayes' rule. Hence,

$$V_U(\tilde{\theta}) = \delta V_U(\tilde{\theta}|_{NR}) \leq \delta \overline{V}_U < V_U(\tilde{\theta}).$$

Contradiction. On the other hand, since $V_U(\tilde{\theta}) > 0$, Lemma 2.4 implies NR is on path. Hence, $\tilde{\theta}|_{NR} = 1$ by Bayes' rule .□

As a direct implication of Claim 2.4, Equation 1.A.2 is reduced to

$$\begin{aligned} V_U(\tilde{\theta}) &= b(\tilde{\theta}) \left[(1-\delta)\Delta_\emptyset + q\delta V_U(\tilde{\theta}|_S) + (1-q)\delta V_U(\tilde{\theta}|_{NS}) \right] + [1-b(\tilde{\theta})]\delta V_U(\tilde{\theta}|_{NB}) \\ &\geq \delta V_U(\tilde{\theta}|_{NR}) = \delta \Delta_\emptyset. \end{aligned} \tag{1.A.3}$$

Claim 2.5. *At the belief $\tilde{\theta}$, the consumer purchases the product with positive probability: $b(\tilde{\theta}) > 0$.*

Proof. Suppose instead $b(\tilde{\theta}) = 0$, then

$$V_U(\tilde{\theta}) = \delta V_U(\tilde{\theta}|_{NB}) \leq \delta \overline{V}_U < V_U(\tilde{\theta}).$$

Contradiction. □

Now that $\tilde{\theta}|_{NR} = 1$ and $b(\tilde{\theta}) > 0$, the informed type strictly prefer to play NR after observing a bad signal: $r_b(\tilde{\theta}) = 0$, because relative to R, NR both leads to a higher continuation payoff and a strictly higher flow payoff. By Bayes' rule, $\tilde{\theta}|_{NS} = 0$ and $\tilde{\theta}|_S, \tilde{\theta}|_{NB} < 1$. It follows that $V_U(\tilde{\theta}|_{NS}) = 0$ and $V_U(\tilde{\theta}|_S), V_U(\tilde{\theta}|_{NB}) \leq \overline{V}_U$. Substitute all these into the first line of Equation 1.A.3, we obtain that

$$\begin{aligned} V_U(\tilde{\theta}) &\leq b(\tilde{\theta}) \left[(1-\delta)\Delta_\emptyset + q\delta \overline{V}_U \right] + [1-b(\tilde{\theta})]\delta \overline{V}_U \\ &= \delta \overline{V}_U + b(\tilde{\theta}) \left[(1-\delta)\Delta_\emptyset - (1-q)\delta \overline{V}_U \right]. \end{aligned}$$

By construction, $V_U(\tilde{\theta}) > \delta \overline{V}_U$. So, the expression in the bracket must be strictly positive, and we could further relax the inequality by letting $b(\tilde{\theta}) = 1$.

$$V_U(\tilde{\theta}) \le \delta\overline{V}_U + \left[(1-\delta)\Delta_\emptyset - (1-q)\delta\overline{V}_U\right] = (1-\delta)\Delta_\emptyset + q\delta\overline{V}_U. \tag{1.A.4}$$

Equation 1.A.4 holds for all beliefs $\tilde{\theta} < 1$ such that $V_U(\tilde{\theta}) > \delta\overline{V}_U$. It then follows that

$$\overline{V}_U \le (1-\delta)\Delta_\emptyset + q\delta\overline{V}_U \Longrightarrow \overline{V}_U \le \frac{1-\delta}{1-q\delta}\Delta_\emptyset.$$

We have found an upper bound on $\overline{V}_U$. When δ is sufficiently large: $\delta > \frac{1}{1+\sqrt{1-q}}$,

$$\frac{1-\delta}{1-q\delta}\Delta_\emptyset < \delta\Delta_\emptyset.$$

so that

$$V_U(\tilde{\theta}) \le \overline{V}_U < \delta\Delta_\emptyset.$$

But this clearly contradicts with the second line of Equation 1.A.3. Therefore, it must be that $\overline{V}_U = 0$, or equivalently, $V_U(\theta) = 0$, $\forall\theta \in [0,1)$. □

2.A.4 Proof of Proposition 2.2

First, it can be easily checked that Lemma 2.1, 2.2(b), 2.3, and 2.4 still hold under this alternative payoff structure. Observe that, even at the best belief, $V_i(1) = 0$, $i \in \{I, C, U\}$.

For the purpose of contradiction, suppose there exists $\theta \in [0,1]$ such that $V_U(\theta) > 0$, then there exists $\tilde{\theta} \in [0,1]$ such that $V_U(\tilde{\theta}) > 0$ and $r_\emptyset(\tilde{\theta}) = 1$. Otherwise $V_U(\tilde{\theta}|_{NR,\ldots,NR})$ explodes as the number of NR increases. So, it follows from Lemma 2.4 that NR is on path and $\theta|_{NR} = 1$. By Lemma 2.3, $V_I(\theta) > 0$, the informed type thus has unambiguous incentives to recommend: $r_g(\theta) = r_b(\theta) = 1$. However, the profile of strategies would leave NR off-path, contradiction. So, there does not exist

$\theta \in [0,1]$ such that $V_U(\theta) > 0$. Lastly, by Lemma 2.1, there does not exist $\theta \in [0,1]$ such that $V_I(\theta) > 0$ or $V_C(\theta) > 0$. □

2.A.5 Proof of Claim 2.2

Again, the outcome in the no-purchase region is purely babbling. Nobody has a strict incentive to deviate. Hence, it suffices to check the incentives in the purchase region $[\theta^{**}, 1)$, where the strategies and beliefs are given by $r_g(\theta) = r_\emptyset(\theta) = b(\theta) = 1,\ r_b(\theta) = 0$ and $\theta|_S = \theta,\ \theta|_{NS} = 0,\ \theta|_{NR} = 1,\ \theta|_{NB} = \frac{q\theta}{q\theta+1-\theta}$.

- Given the profile of strategies and beliefs, the uninformed type's payoff satisfies

$$V_U(\theta) = (1-\delta)\Delta_\emptyset + q\delta V_U(\theta) + (1-q)\delta V_U(0).$$

$$\Longrightarrow V_U(\theta) = \frac{1-\delta}{1-q\delta}\Delta_\emptyset.$$

He has no incentive to deviate because

$$V_U(\theta) > \delta V_U(\theta|_{NR}) = 0.$$

- Given the profile of strategies and beliefs, the informed type's payoff satisfies

$$V_I(\theta) = q[(1-\delta)\Delta_g + \delta V_I(\theta)] + (1-q)\delta V_I(1).$$

$$\Longrightarrow V_I(\theta) = \frac{q(1-\delta)}{1-q\delta}\Delta_g.$$

He has no incentive to deviate given a good signal because

$$(1-\delta)\Delta_g + \delta V_I(\theta) > 0 = \delta V_I(\theta|_{NR}).$$

He also has no incentive to deviate given a bad signal because

$$(1-\delta)\Delta_b + \delta V_I(\theta|_{NS}) = (1-\delta)\Delta_b < 0 = \delta V_I(\theta|_{NR}).$$

- Given the profile of strategies and beliefs, the probability of recommendation is

$$\Pr(R) = q\theta + 1 - \theta.$$

Conditional on a recommendation, the expected value of the product is

$$\mathbb{E}[v|R] = \frac{q}{q\theta + 1 - \theta}.$$

Lastly, the consumer's payoff satisfies

$$\begin{aligned} V_C(\theta) = (q\theta+1-\theta)\Bigg[(1-\delta)\left(\frac{q}{q\theta+1-\theta} - p\right) \\ +\delta\frac{q}{q\theta+1-\theta}V_C(\theta) + \delta\frac{(1-q)(1-\theta)}{q\theta+1-\theta}V_C(0)\Bigg] \\ +(1-q)\theta\delta V_C(1). \end{aligned}$$

which gives

$$V_C(\theta) = \frac{1-\delta}{1-q\delta}[q - p + (1-q)p\theta].$$

She has no incentive to deviate if

$$\begin{aligned} &(1-\delta)\left(\frac{q}{q\theta+1-\theta} - p\right) + \delta\frac{q}{q\theta+1-\theta}V_C(\theta) \geq \delta V_C\left(\theta|_{NB}\right) \\ \Longleftrightarrow &\frac{1-\delta}{1-q\delta}\left(\frac{q}{q\theta+1-\theta} - p\right) \geq \delta V_C\left(\frac{q\theta}{q\theta+1-\theta}\right). \end{aligned} \tag{1.A.5}$$

In order to check the incentive constraint, we need to compute $V_C\left(\frac{q\theta}{q\theta+1-\theta}\right)$ first. There are two cases depending on whether $\frac{q\theta}{q\theta+1-\theta} \geq \theta^{**} = \frac{p-q}{p(1-q)}$.

(i) If $\frac{q\theta}{q\theta+1-\theta} \geq \frac{p-q}{p(1-q)}$, then Equation 1.A.5 is equivalent to

$$\begin{aligned} &\frac{1-\delta}{1-q\delta}\left(\frac{q}{q\theta+1-\theta}-p\right) \geq \delta\frac{1-\delta}{1-q\delta}\left[q-p+(1-q)p\frac{q\theta}{q\theta+1-\theta}\right] \\ \Longleftrightarrow &\frac{q}{q\theta+1-\theta}-p \geq \delta\left[q-p+(1-q)p\frac{q\theta}{q\theta+1-\theta}\right]. \end{aligned}$$

This inequality is always satisfied because

$$\frac{q}{q\theta+1-\theta}-p = q-p+(1-q)\frac{q\theta}{q\theta+1-\theta} > q-p+(1-q)p\frac{q\theta}{q\theta+1-\theta}.$$

(ii) If $\frac{q\theta}{q\theta+1-\theta} < \frac{p-q}{p(1-q)}$, then $V_C(\frac{q\theta}{q\theta+1-\theta}) = 0$. Equation 1.A.5 is trivially satisfied.

Thus, the consumer has no profitable deviation.

This concludes the verification of the modified cutoff equilibrium. □

2.A.6 Proof of Proposition 2.5

Fix $\delta < 1$ and a corresponding equilibrium. Suppose $\overline{V}_U = \sup_{\theta\in[0,1]} V_U(\theta)$. Let $\tilde{\theta}$ be a belief such that $V_U(\tilde{\theta}) > \delta\overline{V}_U$. Such a belief exist for sure unless V_U is equal to 0 everywhere, in which case the result is proved trivially.

At this belief, we must have $r_\emptyset(\tilde{\theta}) = 1$ and $b(\tilde{\theta}) > 0$. Otherwise, we would have $V_U(\tilde{\theta}|_{NR}) > \overline{V}_U$ or $V_U(\tilde{\theta}|_{NB}) > \overline{V}_U$, contradiction. In addition, NR must be on path. Otherwise, the action R is uninformative and thus the consumer does not buy the product, contradicting with $b(\tilde{\theta}) > 0$.

Now that NR is on path, $\tilde{\theta}|_{NR} = 1$ by Bayes' rule. By ““*Babbling at the Top*"", being identified as informed leads to 0 payoff for the informed type. It then follows that $r_g(\tilde{\theta}) = 1$, $r_b(\tilde{\theta}) < 1$ (otherwise NR is off path), and $\tilde{\theta}|_S = \tilde{\theta}$. Hence, playing R

at most gives the informed type a 0 payoff conditional on a bad signal.

$$0 \geq b(\tilde{\theta})[(1-\delta)\Delta_b + \delta V_I(\tilde{\theta}|_{NS})] + [1-b(\tilde{\theta})]\delta V_I(\tilde{\theta}|_{NB}). \tag{1.A.6}$$

Equation 1.A.6 gives an upper bound on $V_I(\tilde{\theta}|_{NS})$ and thus $V_U(\tilde{\theta}|_{NS})$ by Lemma 2.3.

$$V_U(\tilde{\theta}|_{NS}) \leq V_I(\tilde{\theta}|_{NS}) \leq \frac{-(1-\delta)\Delta_b}{\delta}.$$

Now, substitute this upper bound, $\tilde{\theta}|_S = \tilde{\theta}$, and $V_U(\tilde{\theta}|_{NB}) \leq \overline{V}_U$ into the uninformed type's incentive constraint,

$$V_U(\tilde{\theta}) = b(\tilde{\theta})\left[(1-\delta)\Delta_\emptyset + q\delta V_U(\tilde{\theta}|_S) + (1-q)\delta V_U(\tilde{\theta}|_{NS})\right] + [1-b(\tilde{\theta})]\delta V_U(\tilde{\theta}|_{NB}).$$

we then attain an upper bound on $V_U(\tilde{\theta})$.

$$V_U(\tilde{\theta}) \leq \frac{1-\delta}{1-q\delta}[\Delta_\emptyset - (1-q)\Delta_b] = \frac{q(1-\delta)}{1-q\delta}\Delta_g.$$

This bound is valid for any $\tilde{\theta}$ such that $V_U(\tilde{\theta}) > \delta\overline{V}_U$. It follows that $\overline{V}_U$ is bounded by the same expression. Besides, this bound holds for all equilibria and δ. Therefore, $\lim_{\delta\to 1}\sup_{e\in E_\delta}\sup_{\theta\in[0,1]} V_U(\theta; e) = 0$, establishing the desired result. □

2.A.7 Proof of Proposition 2.6

Fix $\delta < 1$, and any corresponding equilibrium such that the uninformed type plays a pure strategy. In part (a), I prove the result for the informed type's payoff. In part (b), I prove the result for the consumer's payoff.

(a) At any belief $\theta \in [0,1]$, if $V_U(\theta) = 0$, then $V_I(\theta) = V_C(\theta) = 0$ by Lemma 2.1. Hence, without loss, consider θ such that $V_U(\theta) > 0$. By the restriction, either $r_\emptyset(\theta) = 0$ or $r_\emptyset(\theta) = 1$.

- If $r_\emptyset(\theta) = 1$, then by the same arguments in the proof of Proposition 2.5, $\theta|_{NR} = 1$, $r_g(\theta) = 1$ and $r_b(\theta) < 1$. Further, $\theta|_S = \theta$ by Bayes' rule. The informed type's equilibrium payoff satisfies:

$$V_I(\theta) = q\{b(\theta)[(1-\delta)\Delta_g + \delta V_I(\theta)] + [1-b(\theta)]\delta V_I(\theta|_{NB})\}. \tag{1.A.7}$$

Conditional a bad signal, he weakly prefers to not recommend. So,

$$b(\theta)[(1-\delta)\Delta_b + \delta V_I(\theta|_{NS})] + [1-b(\theta)]\delta V_I(\theta|_{NB}) = 0.$$

which implies

$$[1-b(\theta)]\delta V_I(\theta|_{NB}) \leq -b(\theta)(1-\delta)\Delta_b.$$

Substitute the inequality back to Equation 1.A.7, we get

$$V_I(\theta) \leq \frac{q(1-\delta)}{1-q\delta b(\theta)} b(\theta)(\Delta_g - \Delta_b) \leq \frac{q(1-\delta)}{1-q\delta}(\Delta_g - \Delta_b).$$

- If $r_\emptyset(\theta) = 0$, then by Lemma 2.4, $\theta|_{NB} = 1$ so that $V_I(\theta|_{NB}) = 0$. Furthermore, $\theta|_S = \theta|_{NS} = 1$. On the other hand, the uninformed type's payoff satisfies

$$V_U(\theta) = \delta V_U(\theta|_{NR}).$$

Since $V_U(\theta) > 0$, it follows that

$$V_I(\theta|_{NR}) \geq V_U(\theta|_{NR}) > 0.$$

Hence, the informed type plays NR for sure upon a bad signal because it yields both a higher flow payoff and a higher continuation payoff. Since R

is on path, it must be that $r_g(\theta) > 0$, so that

$$b(\theta)(1-\delta)\Delta_g \geq \delta V_I(\theta|_{NR}).$$

So, the informed type's equilibrium payoff satisfies

$$V_I(\theta) = qb(\theta)(1-\delta)\Delta_g + (1-q)\delta V_I(\theta|_{NR}) \leq (1-\delta)\Delta_g.$$

In all cases, we obtain a uniform upper bound on $V_I(\theta)$ that does not depend on θ or the particular equilibrium. Moreover, each upper bound goes to 0 as δ goes to 1. So, $\lim_{\delta \to 1} \sup_{e \in E^U_\delta} \sup_{\theta \in [0,1]} V_I(\theta; e) = 0$.

(b) Next, we focus on the consumer's payoff. Suppose $\tilde{V}_C = \sup_{\theta \in [0,1]} V_C(\theta) > 0$. Let $\tilde{\theta}$ be a belief such that $V_C(\tilde{\theta}) > \delta \tilde{V}_C$. I first claim that $b(\tilde{\theta}) = 1$. Otherwise, the payoff $V_C(\tilde{\theta})$ is an average of $\delta V_C(\tilde{\theta}|_{NB})$ and $\delta V_C(\tilde{\theta}|_{NR})$, which is less than $V_C(\tilde{\theta})$, contradiction. Again, either $r_\emptyset(\tilde{\theta}) = 0$ or $r_\emptyset(\tilde{\theta}) = 1$.

- If $r_\emptyset(\tilde{\theta}) = 1$, then $\tilde{\theta}|_{NR} = 1$ and $r_g(\tilde{\theta}) = 1$ by the arguments used in part (a). The consumer's payoff satisfies

$$\begin{aligned} V_C(\tilde{\theta}) = \Pr(R) \cdot \Big\{ & [\mathbb{E}[v|R] - p](1-\delta) \\ & + \delta \mathbb{E}[v|R] V_C(\tilde{\theta}|_S) + \delta(1 - \mathbb{E}[v|R]) V_C(\tilde{\theta}|_{NS}) \Big\}. \end{aligned} \tag{1.A.8}$$

where

$$\begin{aligned} \Pr(R) &= \tilde{\theta}[q + (1-q) r_b(\tilde{\theta})] + 1 - \tilde{\theta}, \\ \mathbb{E}[v|R] &= \frac{q}{\Pr(R)}. \end{aligned}$$

Notice that, $\mathbb{E}[v|R] - p$ must be non-negative. Otherwise,

$$V_C(\tilde{\theta}) \leq \Pr(R) \cdot \left[\mathbb{E}[v|R] - p + V_C(\tilde{\theta})\right] < \Pr(R) \cdot V_C(\tilde{\theta}).$$

Contradiction. But if it is positive, then $\Pr(R)$ is bounded away from 1.

$$\Pr(R) = \frac{q}{(\mathbb{E}[v|R] - p) + p} \leq \frac{q}{p} < 1.$$

Substitute this bound into Equation 1.A.8, and relax the inequality, we obtain

$$V_C(\tilde{\theta}) \leq \frac{q}{p}[(1-p)(1-\delta) + V_C(\tilde{\theta})] \Longrightarrow V_C(\tilde{\theta}) \leq \frac{q(1-p)}{p-q}(1-\delta).$$

This applies for all $\tilde{\theta}$ such that $V_C(\tilde{\theta}) > \delta \tilde{V}_C$. So,

$$\tilde{V}_C \leq \frac{q(1-p)}{p-q}(1-\delta).$$

- If $r_\emptyset(\tilde{\theta}) = 0$, again, $\tilde{\theta}|_{NB} = \tilde{\theta}|_S = \tilde{\theta}|_{NS} = 1$ by the arguments in part (a). Thus $V_C(\tilde{\theta}|_{NB}) = V_C(\tilde{\theta}|_S) = V_C(\tilde{\theta}|_{NS}) = 0$. The consumer's payoff then satisfies

$$V_C(\tilde{\theta}) = \Pr(R) \cdot (\mathbb{E}[v|R] - p)(1-\delta) + \Pr(NR) \cdot \delta V_C(\tilde{\theta}|_{NR}).$$

Because $V_C(\tilde{\theta}) > \delta V_C(\tilde{\theta}|_{NR})$ by construction, it follows that

$$V_C(\tilde{\theta}) \leq [\mathbb{E}[v|R] - p](1-\delta) \leq (1-p)(1-\delta).$$

This applies for all $\tilde{\theta}$ such that $V_C(\tilde{\theta}) > \delta \tilde{V}_C$. So,

$$\tilde{V}_C \leq (1-p)(1-\delta).$$

Now, in both cases, we obtain a uniform bound on $\tilde{V}_C$ that does not depend on the particular equilibrium. Moreover, both bounds go to 0 as δ goes to 1. Hence, $\lim_{\delta \to 1} \sup_{e \in E_\delta^U} \sup_{\theta \in [0,1]} V_C(\theta; e) = 0$.

□

2.A.8 Proof of Claim 2.3

In this proof, I demonstrate that the general cutoff equilibrium still holds when the informed type is not committed to recommend honestly. As usual, I compute the equilibrium payoffs and verify the incentive constraints in the order of Region III, I, II.

Region III

Although the informed type is recommending honestly, the consumer's belief is so low that she finds it optimal to not trust the recommendation. So, the outcome is effectively babbling in Region III, and $V_U(\theta) = V_I(\theta) = V_C(\theta) = 0$ for $\theta \in [0, \theta_\infty] \cup \{1\}$. So, the verification is trivial.

Region I

Observe that, the purchase region in the modified cutoff equilibrium is the same as Region I here. The strategies, beliefs and even the region per se are all identical. It follows that the equilibrium payoffs are also identical. In summary,

$$V_U(\theta) = \frac{1-\delta}{1-q\delta}\Delta_\emptyset, \quad V_I(\theta) = \frac{q(1-\delta)}{1-q\delta}\Delta_g, \quad V_C(\theta) = \frac{1-\delta}{1-q\delta}[q - p + p(1-q)\theta].$$

Lastly, the incentive constraints are also identical. So, both types of recommender have no incentive to deviate. As for the consumer, she indeed finds it optimal to buy because

$$V_C(\theta) \geq \frac{1-\delta}{1-q\delta}[q-p+p(1-q)\theta_0] = 0.$$

Region II$_k$, $k \in \mathbb{N}_+$

For $\theta \in [\theta_k, \theta_{k-1})$, the strategies and beliefs are given by

- $r_g(\theta) = 1$, $r_b(\theta) = 0$, $r_\emptyset(\theta) = \frac{q(1-p)}{p-q} \cdot \frac{\theta}{1-\theta}$, $b(\theta) = \delta^k$.
- $\theta|_S = \theta_0$, $\theta|_{NS} = 0$, $\theta|_{NB} = \theta_\infty$, $\theta|_{NR} = \frac{(p-q)(1-q)\theta}{p-q-p(1-q)\theta}$.

θ_0 and θ_∞ are two important beliefs. Since $\theta_0 \in$ I and $\theta_\infty \in$ III. It follows that

$$V_U(\theta_0) = \frac{1-\delta}{1-q\delta}\Delta_\emptyset,\ V_I(\theta_0) = \frac{q(1-\delta)}{1-q\delta}\Delta_g,\ V_C(\theta_0) = 0.$$

and $V_i(\theta_\infty) = 0$, $i = C, I, U$.

Temporarily, rename I to II$_0$. Denote the payoff of player $i \in \{C, I, U\}$ restricted in the sub-region II$_k$ by V_i^k.

- Given the profile of strategies and beliefs, the uninformed type's payoff satisfies

  $$V_U^k(\theta) = \delta^k(1-\delta)\Delta + q\delta V_U(\theta_0) + (1-q)\delta V_U(0) + (1-\delta^k)\delta V_U(\theta_\infty).$$

  which gives

  $$V_U^k(\theta) = \frac{1-\delta}{1-q\delta}\delta^k\Delta_\emptyset.$$

  The indifference constraint is satisfied because $\theta|_{NR} \in$ II$_{k-1}$, so that

  $$V_U^k(\theta) = \delta V_U^{k-1}(\theta|_{NR}) = \delta \cdot \frac{1-\delta}{1-q\delta}\delta^{k-1}\Delta_\emptyset.$$

- Given the profile of strategies and beliefs, the informed type's payoff satisfies

$$V_I^k(\theta) = q[\delta^k[(1-\delta)\Delta_g + \delta V_I(\theta_0)] + (1-\delta^k)\delta V_I(\theta_\infty)] + (1-q)\delta V_I^{k-1}(\theta|_{NR}).$$

Since V_I is constant over Region I, and $\text{II}_k|_{NR} = \text{II}_{k-1}$, it follows that V_I is constant in each sub-region. So, there is an iterative relationship between V_I^k and V_I^{k-1}. After inserting the payoffs at θ_0 and θ_∞, we obtain

$$V_I^k = q\delta^k \frac{1-\delta}{1-q\delta}\Delta_g + (1-q)\delta V_I^{k-1}.$$

which gives

$$V_I^k(\theta) = \frac{1-\delta}{1-q\delta}\delta^k \left[1-(1-q)^{k+1}\right]\Delta_g.$$

He has no incentive to deviate given a good signal because

$$\begin{aligned}\delta^k[(1-\delta)\Delta_g + \delta V_I(\theta_0)] + (1-\delta^k)\delta V_I(\theta_\infty) &= \frac{1-\delta}{1-q\delta}\delta^k\Delta_g \\ &> \frac{1-\delta}{1-q\delta}\delta^k\left[1-(1-q)^k\right]\Delta_g = \delta V_I(\theta|_{NR}).\end{aligned}$$

He also has no incentive to deviate given a bad signal because

$$\delta^k[(1-\delta)\Delta_b + \delta V_I(\theta|_{NS})] + (1-\delta^k)\delta V_I(\theta_\infty) = (1-\delta)\delta^k\Delta_b < \delta V_I(\theta|_{NR}).$$

- Given the profile of strategies and beliefs, the expected value of the product conditional on a recommendation is

$$\mathbb{E}[v|p] = \frac{q[\theta + (1-\theta)r_\emptyset(\theta)]}{q\theta + (1-\theta)r_\emptyset(\theta)}.$$

which is exactly equal to p by construction of $r_\emptyset(\theta)$. Thus, the consumer is willing to mix. This concludes the verification of the general cutoff equilibrium.

□

2.A.9 Proof of Proposition 2.7

Fix $\delta < 1$, I first characterize the class of equilibrium in which the informed type always does the right thing, and the consumer is myopic. Using the characterizations, I then prove the reputation failure result.

Property 1: For any $\theta \leq \theta_\infty$, $V_i(\theta) = 0$, $\forall i \in \{C, I, U\}$.

For the purpose of contradiction, suppose there exists some belief $\tilde{\theta} \in (0, \theta_\infty]$ such that $V_U(\tilde{\theta}) > 0$. Then both actions are on path, so that $\tilde{\theta}|_R$ and $\tilde{\theta}|_{NR}$ can be derived by Bayes' rule. Next, I prove a key claim.

Claim 2.6. *For all $\theta \in (0, \theta_\infty]$, $V_U(\theta) = \delta V_U(\min\{\theta|_R, \theta|_{NR}\})$.*

Proof. This claim is proved case by case.

(1) If $b(\theta) > 0$, it can be shown that $r_\emptyset(\theta) \leq q$. Suppose the inequality is violated. In this case,

$$\mathbb{E}[v|R] = \frac{q[\theta + (1-\theta)r_\emptyset(\theta)]}{q\theta + (1-\theta)r_\emptyset(\theta)}.$$

Clearly, it decreases in $r_\emptyset(\theta)$ and increase in θ. So,

$$\mathbb{E}[v|R] < \frac{q[\theta_\infty + q(1-\theta_\infty)]}{q\theta_\infty + q(1-\theta_\infty)} = p.$$

Contradicting with $b(\theta) > 0$. By Bayes' rule, $\theta|_{NR} \leq \theta \leq \theta|_R$. Meanwhile, since the uninformed type puts positive probability on NR, the incentive constraint implies

$$V_U(\theta) = \delta V_U(\theta|_{NR}).$$

Or equivalently,

$$V_U(\theta) = \delta V_U(\min\{\theta|_R, \theta|_{NR}\}).$$

(2) If $b(\theta) = 0$ and $r_\emptyset(\theta) = 1$, the incentive constraint implies $V_U(\theta) = \delta V_U(\theta|_R)$. On the other hand, by Bayes' rule, $\theta|_{NR} = 1 > \theta \geq \theta|_R$. So, we again have

$$V_U(\theta) = \delta V_U(\min\{\theta|_R, \theta|_{NR}\}).$$

(3) If $b(\theta) = 0$ and $r_\emptyset(\theta) < 1$, the incentive constraint implies $V_U(\theta) = \delta V_U(\theta|_R) = \delta_U V_U(\theta|_{NR})$. So, we also have

$$V_U(\theta) = \delta V_U(\min\{\theta|_R, \theta|_{NR}\}).$$

The three cases cover all possibilities, hence Claim 2.6 is established. □

Apply this claim to $\tilde{\theta}$, and notice that $\min\{\tilde{\theta}|_R, \tilde{\theta}|_{NR}\} < \theta$ by the martingale property and $V_U(\tilde{\theta}) > 0$. So, we could reapply the claim. Formally, define a sequence as follows. Let $\theta_0 = \tilde{\theta}$, and $\theta_k = \min\{\theta_{k-1}|_R, \theta_{k-1}|_{NR}\}$ for $k \in \mathbb{N}_+$. Applying Claim 2.6 repeatedly, we obtain that $\theta_k \in (0, \theta_{k-1})$ and

$$0 < V_U(\tilde{\theta}) = \delta^k V_U(\theta_k).$$

for any $k \in \mathbb{N}_+$. Let k goes to infinity, then U's payoff explodes to infinity along the sequence of beliefs. This leads to contradiction as $V_U(\theta)$ is bounded above by $\Delta_\emptyset < \infty$. Therefore, it must be that $V_U(\theta) = 0, \forall \theta \leq \theta_\infty$ in any equilibrium. It follows from Lemma 2.1 that any equilibrium must feature babbling when the belief is sufficiently low.

Property 2: For any $\theta > \theta_0$, $b(\theta) = 1$, $r_\emptyset(\theta) = 1$. The payoffs are $V_U(\theta) = \frac{1-\delta}{1-q\delta}\Delta_\emptyset$, $V_I(\theta) = \frac{q(1-\delta)}{1-q\delta}\Delta_g$, and $V_C(\theta) = q - p + p(1-q)\theta$.

The informed type always embeds the information in the recommendation. When the reputation is sufficiently high, regardless of the uninformed type's strategy, the conditional value is higher than the price.

$$\mathbb{E}[v|R] > \frac{q[\theta_0 + (1-\theta_0)r_\emptyset(\theta_0)]}{q\theta_0 + (1-\theta_0)r_\emptyset(\theta_0)} \geq \frac{q[\theta_0 + (1-\theta_0)\cdot 1]}{q\theta_0 + (1-\theta_0)\cdot 1} = p.$$

It then follows that the consumer strictly prefers to buy the product.

For any $\theta > \theta_0$, a feasible strategy for the uninformed type is to always recommend both today and in the future. Because $r_g(\theta) = 1$, it then follows that $\theta|_S \geq \theta$ so that $\theta|_S > \theta_0$. Since the consumer always buys in this range of belief, it follows that the recommender gets at least a flow payoff $(1-\delta)\Delta_\emptyset$ in each period provided that the outcome in the last period was S. So, such a strategy yields at least

$$V_U(\theta) \geq (1-\delta)\Delta_g + q\delta(1-\delta)\Delta_g + q^2\delta^2(1-\delta)\Delta_g + ... = \frac{1-\delta}{1-q\delta}\Delta_\emptyset.$$

Meanwhile, the maximal payoff $\overline{V}_U = \max_{\theta\in[0,1]} V_U(\theta)$ is bounded above by the same expression $\frac{1-\delta}{1-q\delta}\Delta_\emptyset$. To see this, note that at the maximizer $\tilde{\theta}$, it must be that $r_\emptyset = 1$. But then, $\tilde{\theta}|_S = \theta, \tilde{\theta}|_{NS} = 0$. So,

$$\overline{V}_U = V_U(\tilde{\theta}) \leq (1-\delta)\Delta_\emptyset + \delta\overline{V}_U \Longrightarrow \overline{V}_U \leq \frac{1-\delta}{1-q\delta}\Delta_\emptyset.$$

Taken together, the inequality is attained. At the same time, $r_\emptyset(\theta) = 1$, $V_U(\theta) = \frac{1-\delta}{1-q\delta}\Delta_\emptyset$, for any $\theta > \theta_0$. It can be computed that the informed type's payoff is equal to

$$V_I(\theta) = \frac{q(1-\delta)}{1-q\delta}\Delta_g.$$

Lastly, the consumer's payoff is equal to

$$V_C(\theta) = \Pr(R)(\mathbb{E}[v|R] - p) = q - p + p(1-q)\theta.$$

Property 3: At θ_0, $r_\emptyset(\theta_0) = 1$ and $b(\theta_0)$ can take any value in $[0,1]$. In addition, $V_C(\theta_0) = 0$, $V_U(\theta_0) = \frac{b(\theta_0)(1-\delta)}{1-b(\theta_0)q\delta}\Delta_\emptyset$, and $V_I(\theta_0) = \frac{b(\theta_0)q(1-\delta)}{1-b(\theta_0)q\delta}\Delta_g$.

If $r_\emptyset(\theta_0) < 1$, then by the consumer's decision problem, $b(\theta_0) = 1$. In that case, $\theta_0|_S > \theta_0$, so that playing R gives at least

$$(1-\delta)\Delta_\emptyset + q\delta\frac{1-\delta}{1-q\delta}\Delta_\emptyset = \frac{1-\delta}{1-q\delta}\Delta_\emptyset = \overline{V}_U.$$

whereas playing NR gives at most $\delta\overline{V}_U$, contradicting the indifference condition. As a result, $r_\emptyset(\theta_0) = 1$. However, then the consumer is absolutely indifferent, so there is no restriction on $b(\theta_0)$ and she receives zero payoff. For the uninformed type, $\theta_0|_{NB} = \theta_\infty$, so that his payoff equals

$$V_U(\theta_0) = b(\theta_0)[(1-\delta)\Delta_\emptyset + q\delta V_U(\theta_0)] \Longrightarrow V_U(\theta_0) = \frac{b(\theta_0)(1-\delta)\Delta_\emptyset}{1-b(\theta_0)q\delta}.$$

Property 4: For $\theta \in (\theta_\infty, \theta_0)$, either $r_\emptyset(\theta) = \frac{q(1-p)}{p-q} \cdot \frac{\theta}{1-\theta}$ or $V_U(\theta) = V_I(\theta) = 0$. But either way, $V_C(\theta) = 0$.

There are three potential cases.

- If $r_\emptyset(\theta) < \frac{q(1-p)}{p-q} \cdot \frac{\theta}{1-\theta}$, then by the consumer's decision problem, $b(\theta) = 1$ so that $\theta|_S > \theta_0$. A similar argument in the proof of Property 3 suggests that playing R is more profitable than playing NR, contradiction.

- If $r_\emptyset(\theta) > \frac{q(1-p)}{p-q} \cdot \frac{\theta}{1-\theta}$, then by the consumer's decision problem, $b(\theta) = 0$ and

$\theta|_{NB} < \theta_\infty$. So, the uninformed type gets

$$V_U(\theta) = \delta V_U(\theta|_{NB}) = 0.$$

By Lemma 2.1, the informed type and the consumer get zero payoff as well.

- If $r_\emptyset(\theta) = \frac{q(1-p)}{p-q} \cdot \frac{\theta}{1-\theta}$, the consumer is indifferent and thus gets zero payoff.

These properties suggest that any equilibrium in $E^{I,C}$ is Pareto-dominated by the general cutoff equilibrium, described in Claim 2.3. First, Property $1-3$ proves the statement for beliefs either weakly above θ_0 or weakly below θ_∞. So, it suffices to look at $\theta \in (\theta_\infty, \theta_0)$. Moreover, it suffices to look at the recommender because the consumer's payoff is always equal to 0 over this interval by Property 4.

If $V_U(\theta) = V_I(\theta) = 0$, it is clearly dominated. So consider $V_U(\theta) > 0, V_I(\theta) > 0$. Then by Property 4, $r_\emptyset(\theta) = \frac{q(1-p)}{p-q} \cdot \frac{\theta}{1-\theta}$. Without loss, let $\theta \in [\theta_k, \theta_{k-1})$ for some $k \in \mathbb{N}_+$. Then $\theta|_S = \theta_0, \theta|_{NS} = 0, \theta|_{NB} = \theta_\infty, \theta|_{NR} \in [\theta_{k-1}, \theta_{k-2})$. The uninformed type's indifference condition implies $V_U(\theta|_{NR})) > 0$. If the posterior $\theta|_{NR}$ remains in $(\theta_\infty, \theta_0)$, then by Property 4 again, $r_\emptyset(\theta|_{NR}) = \frac{q(1-p)}{p-q} \cdot \frac{\theta|_{NR}}{1-\theta|_{NR}}$. Repeat the procedure. It follows that the posteriors evolution and the strategies at those posteriors are exactly the same as in the general cutoff equilibrium. So, it also generates the same payoffs.

Since $\sup_{\theta\in[0,1]} V_I(\theta)$ vanishes as δ goes to one in the general cutoff equilibrium, it follows that the first part of Proposition 2.7 is proved. For the second part, it suffices to restrict attention to a prior $\theta \in (\theta_0, 1)$ because $V_C(\theta)$ is positive if and only if θ falls in this region. Pick a generic belief $\theta \in (\theta_0, 1)$, the distribution of the posterior θ' from the consumer's perspective is

$$\theta' = \begin{cases} \theta & w.p. \quad q, \\ 0 & w.p. \quad (1-\theta)(1-q), \\ 1 & w.p. \quad \theta(1-q), \end{cases}$$

Because $V_C(0) = V_C(1) = 0$, and the extreme beliefs are absorbing, it follows that

$$\lim_{T\to\infty} \mathbb{E}[V_C(\theta^T)] = \lim_{T\to\infty} q^T V_C(\theta) = 0.$$

□

2.A.10 Proof of Proposition 2.8

The proof is very similar to the proof of Proposition 2.1, which can be found in Appendix 2.A.3. Hence, some details are omitted.

First, Lemma 2.1 continues to hold for non-Markov equilibrium. So, it suffices to show the uninformed type receives a zero payoff. Fix any equilibrium with $\delta_c < \frac{p-q}{p(1-q)}$ and $\delta_r > \underline{\delta}_r$. Let $\overline{V}_U$ be the supremum of the uninformed type's equilibrium payoff over all on-path histories that induce an interior posterior.

We again proceed via contradiction, and suppose $\overline{V}_U > 0$. Then there exists some on-path history $\tilde{h}$ such that it induces an interior posterior and $V_U(\tilde{h}) > \delta_r \overline{V}_U$. By the same arguments in Claim 2.4 and 2.5, we must have $r_\emptyset(\tilde{h}) = 1$ and $b(\tilde{h}) > 0$.

Next, NR must be on-path. Otherwise, a deviation to not buy the product is profitable for the consumer because if she buys the product, she receives at most $(q-p)(1-\delta_c) + \delta_c q(1-p) < 0$. Now that NR is on-path, the posterior belief followed by NR jumps up to one jumps to one. Thus, at any private history that is consistent with $\tilde{h}$, the informed type strictly prefers to not recommend if he observes a bad signal.

The rest of arguments are identical to those in Appendix 2.A.3. In short, I show that the belief followed by NS goes down to zero. Thus, the uninformed type's continuation payoff is low if he recommends. On the other hand, since he receives a lucrative continuation payoff if he deviates to not recommend. It then follows that $r_\emptyset(\tilde{h}) < 1$ provided δ_r is large. Contradiction. □

2.A.11 Proof of Proposition 2.10

It is sufficient to prove that $\theta|_{NS} = 0$ whenever $\theta|_{NR} = 1$ and $b(\theta) > 0$. Once this step is established, the rest of proof is identical to the proof of Proposition 2.1(b). Since the good signal is generated if the product is suitable, the formula of $\theta|_{NS}$ remains unchanged.

$$\theta|_{NS} = \frac{\theta r_b(\theta)}{\theta r_b(\theta) + (1-\theta) r_\emptyset(\theta)}. \tag{1.A.9}$$

However, the definition of Δ_b is changed to

$$\Pr(v = 1|s = b) \cdot \overline{u} + \Pr(v = 0|s = b) \cdot \underline{u} = \mu\overline{u} + (1-\mu)\underline{u}.$$

Assumption 2.4 ensures Δ_b to be strictly negative. Hence, for the informed type with a bad signal, NR still strictly dominates R as it yields a strictly higher flow payoff and a weakly higher continuation payoff. It then follows that $r_b(\theta) = 0$ and thus $\theta|_{NS} = 0$ by Equation 1.A.9. □

2.A.12 Proof of Proposition 2.11

In the two claims below, I first characterize the recommender's strategy as much as possible respectively when the consumer buys with positive probability, and when she does not buy at all.

Claim 2.7. *If* $b(\theta) > 0$*, then* $r_\emptyset(\theta) = 1$*,* $r_b(\theta) = 0$*,* $\theta|_{NS} < \theta|_{NB} < \theta|_S \leq \theta < \theta|_{NR} = 1$*,* $V_U(\theta) \geq \delta\Delta_\emptyset$ *and* $V_C(\theta) > 0$*.*

Proof. To this end, first note that $V_U(\theta) > 0$ whenever $b(\theta) > 0$ because the uninformed type could always secure a minimum payoff $(1-\delta)b(\theta)\Delta_\emptyset$ by recommending the product. Then by Lemma 2.4, R is on path. Now, for the purpose of contradiction, suppose $r_\emptyset(\theta) = 0$, it follows by Bayes' rule that $\theta|_{NB} = 1$. But then, the uninformed type has unambiguous incentives to deviate to recommend because it both yields a higher flow payoff and continuation payoff. Contradiction.

Now that $r_\emptyset(\theta) = 1$, similar arguments imply that $\theta|_{NR} = 1$ and thus $r_b(\theta) = 0$. It follows by Bayes' rule then $\theta|_{NS} < \theta|_{NB} < \theta|_S \leq \theta < \theta|_{NR} = 1$. Moreover, by the incentive constraints,

$$V_U(\theta) \geq \delta V_U(\theta|_{NR}) = \delta\Delta_\emptyset.$$

and

$$V_C(\theta) \geq \Pr(NR) \cdot \delta V_C(\theta|_{NR}) = q(1-q)(\lambda - p).$$

□

Claim 2.8. *If $b(\theta) = 0$, then $V_U(\theta) = V_C(\theta) = V_I(\theta) = 0$.*

Proof. It suffices to prove that $V_U(\theta) = 0$ by Lemma 2.1. If $b(\theta) = 0$, the uninformed type's incentive constraint is reduced to

$$V_U(\theta) = \max\{\delta V_U(\theta|_{NB}), \delta V_U(\theta|_{NR})\}.$$

For the purpose of contradiction, suppose $V_U(\theta) > 0$, then there are two cases.

- If $r_\emptyset(\theta) = 0$, then $V_U(\theta|_{NB}) \leq V_U(\theta|_{NR})$. In addition, both R and NR are on path by Lemma 2.4, so that $\theta|_{NB} = 1$ and $\theta|_{NR} < 1$ by Bayes' rule. Now by Lemma 2.2,

  $$V_U(\theta|_{NR}) < \Delta_\emptyset = V_U(1) = V_U(\theta|_{NB}).$$

  Contradiction.

- If $r_\emptyset(\theta) = 1$, then $V_U(\theta|_{NB}) \geq V_U(\theta|_{NR})$. In addition, both R and NR are on path by Lemma 2.4, so that $\theta|_{NB} < 1$ and $\theta|_{NR} = 1$ by Bayes' rule. Now by Lemma 2.2,

  $$V_U(\theta|_{NB}) < \Delta_\emptyset = V_U(1) = V_U(\theta|_{NR}).$$

Contradiction.

In either case, we reach contradiction. Hence, $V_U(\theta)$ must equal to 0. Moreover, it follows that $V_C(\theta) = 0$. If not, that means the consumer must purchase at some point in the future. $\square$

Now, partition $[0, 1)$ into two regions.

$$R_1 = \{\theta \in [0,1) \,|\, b(\theta) = 0,\, V_U(\theta) = 0\},$$

$$R_2 = \{\theta \in [0,1) \,|\, b(\theta) > 0,\, V_U(\theta) \geq \delta\Delta_\emptyset\}.$$

The goal is to prove $R_2 = \emptyset$ when δ is sufficiently large. To this end, I first show in Claim 2.9 that the region R_1 includes beliefs that are sufficiently low. Then I prove by contradiction that R_1 includes the whole interval $[0, 1)$.

Claim 2.9. *Fix $\delta < 1$, then $\exists \underline{\theta} > 0$ such that in any equilibrium, $b(\theta) = 0$, $\forall \theta \leq \underline{\theta}$.*

Proof. Fix any equilibrium given δ. Suppose $b(\theta) > 0$ for some $0 < \theta \leq \underline{\theta}$, then $V_C(\theta) > 0$ by Claim 2.7. Hence, the consumer's incentive constraint suggests

$$V_C(\theta) \leq \Pr(R)(\mathbb{E}[v|R] - p)(1-\delta) + \Pr(NR)\delta V_C(\theta|_{NR}) + \delta \max\{V_C(\theta|_S),\, V_C(\theta|_{NS})\}.$$

For arbitrary $r_g(\theta)$, as θ decreases, the first term is eventually negative and does not vanish because $\Pr(R)$ goes to one and $\mathbb{E}[v|R] - p$ converges to $q - p < 0$. Meanwhile, the second term is positive but vanishes because $\Pr(NR)$ goes to zero. So, there exists a $\underline{\theta} > 0$ such that the sum of the first two terms is negative for any $\theta \leq \underline{\theta}$. It then follows that

$$V_C(\theta) \leq \delta \max\{V_C(\theta|_S),\, V_C(\theta|_{NS})\}.$$

Let $f(\theta) = \min\{\theta' \in \{\theta|_S,\, \theta|_{NS}\} \,|\, V_C(\theta') = \max\{V_C(\theta|_S),\, V_C(\theta|_{NS})\}$. Then by definition,

$$V(f(\theta)) \geq \frac{1}{\delta} V_C(\theta) > 0.$$

By Claim 2.7, we have $b(f(\theta)) > 0$ and $\theta|_{NS} < \theta|_S < 1$ so that

$$0 < f(\theta) < \theta.$$

Therefore, we could repeat the above procedure and conclude that $V(f^k(\theta)) = \frac{1}{\delta^k} V_C(\theta) \to \infty$ as k goes to infinity, contradiction. As a consequence, it must be that $b(\theta) = 0$ for $\forall \theta \leq \underline{\theta}$. □

Now, suppose $R_2 \neq \emptyset$, then $\underline{\theta} = \inf R_2 \geq \underline{\theta} > 0$ by Claim 2.9. Choose $\tilde{\theta} \in R_2$ that is arbitrarily close to $\underline{\theta}$ such that $\tilde{\theta}|_S < \underline{\theta}$.[28] Then by Claim 2.9, $V_U(\theta|_S) = V_U(\theta|_{NS}) = V_U(\theta|_{NB}) = 0$ so that the uninformed type's incentive constraint implies

$$V_U(\theta) = b(\theta)\Delta_\emptyset(1 - \delta) \geq \delta V_U(\theta|_{NR}) = \delta \Delta_\emptyset$$

However, this is impossible for $\delta > \frac{1}{2}$. It then follows that whenever $\delta > \frac{1}{2}$, $R_2 = \emptyset$, so that $V_U(\theta) = V_C(\theta) = V_I(\theta) = 0$, $\forall \theta < 1$. □

[28]Such $\tilde{\theta}$ is possible because for any $\theta \in R_2$, $\theta|_S = \frac{\theta \lambda r_g(\theta)}{\theta \lambda r_g(\theta) + 1 - \theta} < \theta$.

Chapter 3

Baysian Persuasion and Lie Detection

(joint with Florian Ederer, Yale School of Management)

3.1 Introduction

Lies are a pervasive feature of communication, even when communication is subject to intense public and media scrutiny. For example, during his tenure as US President, Donald Trump has made over 20,000 false or misleading claims.[1] However, such lies are also often detectable. Monitoring and fact-checking should constrain how much license a sender of communication has when making false statements. But, interestingly, in the face of increased fact-checking and media focus, the rate of Trump's lying increased rather than decreased—a development that runs counter to this intuition.

In this chapter, we incorporate probabilistic lie detection in an otherwise standard model of Bayesian persuasion (Kamenica and Gentzkow, 2011; Kamenica, 2019). Two players, a Sender and a Receiver, engage in one round of communication. The Sender observes the binary state of nature and sends a message to the Receiver. To clearly

[1] See https://www.washingtonpost.com/politics/2020/07/13/president-trump-has-made-more-than-20000-false-or-misleading-claims/ for a comprehensive analysis of this behavior.

define whether a message is a lie or not, we assume that the message space and the state space are the same. The Receiver observes the message, and if the message is a lie, it is flagged as such with some probability. The Receiver then takes an action. Whereas the Sender prefers the Receiver to take the "favorable" action regardless of the state of nature, the Receiver wants to match the action to the underlying state. Finally, payoffs (or payoffs which we use interchangeably) are realized for both parties.

Our main assumption that lies are detectable is a natural one in many applications. Ekman and Frank (1993) argue that there are two basic reasons why lies fail due to detection: facts and emotions. First, facts may surface that contradict the message of the Sender. These facts do not necessarily tell all the information about the state of the world, but they reveal that the Sender's message was a lie. Second, lies can be detected because emotions and physical reactions such as blushing or sweating provide strong clues that the Sender is lying.

Our model delivers the following set of results. First, the Sender lies more frequently when the lie detection technology improves. Second, as long as the lie detection probability is sufficiently small, the equilibrium payoffs of both players are unaffected by the lie detection technology because the Sender simply compensates by lying more frequently in the unfavorable state of nature by claiming that the state is favorable. That is to say, the lie detection technology changes the Sender's message strategy but does not have an impact on the payoffs of both players. Third, when the lie detection technology is sufficiently reliable, any further increase in the lie detection probability causes the Sender to lie more frequently in the favorable state of nature and the Sender's (Receiver's) equilibrium payoff decreases (increases) with the lie detection probability.

Our framework is sufficiently tractable to analyze a number of extensions. First, we consider alternative detection technologies such as truth detection and show that the central insights of our model continue to hold. Second, we analyze the (nontrivial) case in which the default action coincides with the Sender's preferred action and show that the main results are analogous to those in the baseline model. Third, we show that our results do not rely on the fully revealing nature of a lie in our model. They continue to hold even when the state is not binary and thus the detection of a lie does not fully reveal the state of nature.

Two recent papers (Balbuzanov, 2019; Dziuda and Salas, 2018) also investigate the role of lie detectability in communication. The most significant difference with respect to this chapter lies in the commitment assumption of the Sender. In all those papers, the communication game takes the form of cheap talk (Crawford and Sobel, 1982) rather than Bayesian persuasion as in this chapter. We defer a detailed comparison between these papers and our work to Section 3.4. Jehiel (2021) considers a setting with two rounds of communication á la Crawford and Sobel (1982), but includes the innovative feature that a Sender who lied in the first period cannot remember what exact lies she told. However, the potential inconsistency of messages never arises in any pure strategy equilibrium. As a result, no lies are ever detected in equilibrium.

Related theoretical work on lying in communication games also includes Kartik et al. (2007) and Kartik (2009) who do not consider lie detection but instead introduce an exogenous cost of lying tied to the size of the lie in a cheap talk setting. They find that most types inflate their messages, but only up to a point. In contrast to our results, they obtain full information revelation for some or all types depending on the bounds of the type and message space.

A large and growing experimental literature (Gneezy, 2005; Hurkens and Kartik, 2009; Sánchez-Pagés and Vorsatz, 2009; Ederer and Fehr, 2017; Gneezy et al., 2018) examines lying in a variety of communication games. Most closely related to our work is Fréchette et al. (forthcoming) who investigate models of cheap talk, information disclosure, and Bayesian persuasion, in a unified experimental framework. Their experiments provide general support for the strategic rationale behind the role of commitment and, more specifically, for the Bayesian persuasion model of Kamenica and Gentzkow (2011).

Finally, this chapter is related to recent work on communication in political science. Whereas we focus on an improvement of the Receiver's communication technology (i.e., lie detection), Gehlbach et al. (2022) analyze how improvements that benefit the Sender (e.g., censorship and propaganda) impact communication under Bayesian persuasion. In a related framework that can be recast as Bayesian persuasion, Luo and Rozenas (2018) study how the electoral mechanism performs when the government (the Sender) can rig elections by manipulating the electoral process ex ante and falsifying election returns ex post.

3.2 Model

Consider the following simple model of Bayesian persuasion in the presence of lie detection. Let $w \in \{0, 1\}$ denote the state of the world and $\Pr(w = 1) = \mu \in (0, 1)$. The Sender ($S$, he) observes w and sends a message $m \in \{0, 1\}$ to the Receiver (R, she).

Lie Detection Technology

If the Sender lies (i.e., $m \neq w$), the Receiver is informed with probability $q \in [0, 1]$ that it is a lie and thus learns w perfectly. With remaining probability $1 - q$, she is not informed. If the Sender does not lie (i.e., $m = w$), the message is never flagged as a lie so and the Receiver is not informed. Formally, the detection technology can be described by the following relation[2]

$$d(m,w) = \begin{cases} lie, & \text{with probability } q \text{ if } m \neq w, \\ \neg lie, & \text{with probability } 1-q \text{ if } m \neq w, \\ \neg lie, & \text{with probability } 1 \text{ if } m = w. \end{cases}$$

With a slight abuse of notation we denote $d = \{lie, \neg lie\}$ as the outcome of the detection result. The detection technology is common knowledge. In a standard Bayesian persuasion setup this detection probability q is equal to 0, giving us an immediately comparable benchmark.

Note that lie detection here is different from state detection. While the former would inform the Receiver the true state conditional on a lie, the latter would inform her the true state independently of the message. Section 3.4 discusses their differences in more detail.

Messages in our model are defined to have literal meanings and thus they are classified as lies if they do not match the true state of nature. An alternative definition of messages and lies views a message as a lie if, in equilibrium, this message induces an action that is inconsistent with the true state of nature. This alternative definition is more complicated and involves calculating a fixed point. Essentially, one starts with an arbitrary lying set $L \subset \{(m=1, \omega=1), (m=1, \omega=0)\} \times \{(m=0, \omega=1), (m=0, \omega=0)\}$ and then solves for the Sender's optimal solution when any pair in L triggers a lie detection with probability q. This provisional solution generates a new lying set L'. A consistency condition $L = L'$ is thus required to close the model. We do not adopt this alternative definition because it leads to a multiplicity of equilibria which hinders the comparative statics.

[2]Note that, the lie detection here is different from state detection. While the former would inform the Receiver the true state conditional on a lie, the latter would inform her the true state independently of the message. Section 3.4.1 discusses their differences in more detail.

Payoffs

Given both m and d, the Receiver takes an action $a \in \{0, 1\}$, and the payoffs are realized. The payoffs are defined as follows.

$$u_S(a, w) = \mathbb{1}_{\{a=1\}},$$
$$u_R(a, w) = (1-t) \times \mathbb{1}_{\{a=w=1\}} + t \times \mathbb{1}_{\{a=w=0\}}, \quad 0 < t < 1.$$

That is, the Sender wants the Receiver to always take the action $a = 1$ regardless of the state, while the Receiver wants to match the state. The payoff from matching the state 0 may differ from the payoff from matching the state 1. Given the payoff function, the Receiver takes action $a = 1$ if and only if

$$\Pr(w = 1 \mid m, d) \geq t.$$

Therefore, one could also interpret t as the threshold of the Receiver's posterior belief above which she takes $a = 1$. Note that if $t \leq \mu$, there is no need to persuade because the Receiver will choose the Sender's preferred action $a = 1$ even without a message. Therefore, we assume $t \in (\mu, 1)$.

Strategies

We assume that the Sender has full commitment power as is common in the Bayesian persuasion framework.[3] Specifically, the strategy of the Sender is a mapping $m : \{0, 1\} \longrightarrow \Delta(\{0, 1\})$, and the strategy of the Receiver is a mapping $a : \{0, 1\} \times$

[3]For a detailed discussion and relaxation of this assumption see Min (2017), Fréchette et al. (forthcoming), Lipnowski et al. (forthcoming), and Nguyen and Tan (2021). Titova (2021) shows that with binary actions and a sufficiently rich enough state space verifiable disclosure enables the Sender's commitment solution as an equilibrium.

$\{lie, \neg lie\} \longrightarrow \Delta(\{0,1\})$. Formally, the Sender is choosing $m(\cdot)$ to maximize

$$\mathbb{E}_{w,d,m}[u_S(a\left(m(w), d(m(w), w)\right), w)],$$

where $a(m, d)$ maximizes

$$\mathbb{E}_w[u_R(a, w) \mid m, d].$$

The two expectation signs are taken with respect to different variables. The expectation sign in the Sender's utility is taken with respect to both w, d, and perhaps m if the strategy is mixed, whereas the (conditional) expectation sign in the Receiver's utility is only taken with respect to w. Due to the simple structure of the model, it is without loss of generality to assume that the Sender chooses only two parameters $p_0 = \Pr(m = 0 \mid w = 0)$ and $p_1 = \Pr(m = 1 \mid w = 1)$ to maximize $\Pr(a(m, d) = 1)$ which we write as $\Pr(a = 1)$ henceforth for brevity of notation. We denote the optimal reporting probabilities of the Sender by p_0^* and p_1^*, and the ex-ante payoffs under this reporting probabilities as U_S and U_R.

3.3 Analysis

3.3.1 Optimal Messages

Given the Sender's reporting strategy, the Receiver could potentially see four types of events to which she needs to react when choosing action a.

First, the Receiver could observe the event $(m = 0,\ d = lie)$ which occurs with probability $\mu(1-p_1)q$. Given the lie detection technology, the Receiver is certain that the message $m = 0$ is a lie. Therefore, the state of the world w must be equal to 1, that is

$$\Pr(w = 1 \mid m = 0, d = lie) = 1.$$

As a result, the Receiver optimally chooses $a = 1$.

Second, the event $(m = 0, d = \neg lie)$ could occur with probability $\mu(1 - p_1)(1 - q) + (1 - \mu)p_0$. In that case, the Receiver is uncertain about w because she does not know whether the Sender lied or not. Her posterior probability is given by

$$\Pr(w = 1 \mid m = 0, d = \neg lie) = \frac{\mu(1 - p_1)(1 - q)}{\mu(1 - p_1)(1 - q) + (1 - \mu)p_0} \equiv \mu_0.$$

Hence, the Receiver takes action $a = 1$ if and only if $\mu_0 \geq t$. We denote the posterior following this event by μ_0 (and thus omitting the lie detection outcome $d = \neg lie$) for brevity of notation. When $p_0 = 0,\ p_1 = 1$, this event occurs with 0 probability, so the belief is off-path and not restricted by Bayesian updating. However, the off-path belief does not matter for the Sender, because if the Sender chooses the strategy that renders $(m = 0, d = \neg lie)$ a zero probability event, he does not care about how the Receiver responds to that event. For expositional convenience, define $\mu_0 = 0$ when $p_0 = 0,\ p_1 = 1$.

Third, $(m = 1,\ d = lie)$ occurs with probability $(1 - \mu)(1 - p_0)q$. Because a lie was detected, the Receiver is again certain about w and therefore her posterior probability is given by

$$\Pr(w = 1 \mid m = 1, d = lie) = 0,$$

which immediately implies the action $a = 0$.

Fourth, $(m = 1, d = \neg lie)$ occurs with probability $\mu p_1 + (1 - \mu)(1 - p_0)(1 - q)$. The Receiver is again uncertain about w. Her posterior is given by

$$\Pr(w = 1 \mid m = 1, d = \neg lie) = \frac{\mu p_1}{\mu p_1 + (1 - \mu)(1 - p_0)(1 - q)} \equiv \mu_1.$$

The Receiver takes action $a = 1$ if and only if $\mu_1 \geq t$. Analogously, for brevity of notation, we denote the posterior following this event by μ_1 (and thus omitting the lie detection outcome $d = \neg lie$). Similarly, if $p_0 = 1,\ p_1 = 0$, this event occurs with 0 probability, and the belief μ_1 is not well-defined, but again this does not matter for the Sender. For simplicity, define $\mu_1 = 0$ when $p_0 = 1,\ p_1 = 0$.

Given these optimal responses by the Receiver, the relationships between the posteriors μ_0, μ_1 and the posterior threshold t divide up the strategy space into four different types of strategies which we denote by I, II, III, and IV respectively. For each strategy type, the Receiver's response as a function of (m, d) is the same, making it then easy to find the specific optimal strategy. We are then left to pick the best strategy out of the four candidates. These types of strategies are defined as follows:

I. $\mu_0 < t,\ \mu_1 < t$: For this type of strategy, the Receiver only chooses $a = 1$ if $(m = 0, d = lie)$ and $a = 0$ otherwise because the posteriors μ_0 and μ_1 are insufficiently high to persuade her to choose S's preferred action. Only if the Sender lies in state $w = 1$ and his message is detected as a lie, is the Receiver sufficiently convinced that $a = 1$ is the right action. The maximal probability that the Receiver chooses $a = 1$[4] is given by

$$\Pr_{\text{I}}(a = 1) = \sup_{p_0, p_1 \in [0,1]} \mu(1 - p_1)q \quad \text{s.t.} \quad \mu_0 < t,\ \mu_1 < t.$$

II. $\mu_0 \geq t,\ \mu_1 < t$: The Receiver chooses $a = 1$ if $(m = 0, d = lie)$ or $(m = 0, d = \neg lie)$ and $a = 0$ otherwise. The maximal probability that the Receiver chooses $a = 1$ is given by

$$\Pr_{\text{II}}(a = 1) = \sup_{p_0, p_1 \in [0,1]} \mu(1 - p_1) + (1 - \mu)p_0 \quad \text{s.t.} \quad \mu_0 \geq t,\ \mu_1 < t.$$

III. $\mu_0 < t,\ \mu_1 \geq t$: The Receiver chooses $a = 1$ if $(m = 0, d = lie)$ or $(m = 1, d =$

[4] The choice set of the maximization problem is not closed, so the maximum may not be achieved.

$\neg lie$) and $a = 0$ otherwise. The maximal probability that the Receiver chooses $a = 1$ is given by

$$\Pr_{\text{III}}(a = 1) = \sup_{p_0, p_1 \in [0,1]} \quad \mu p_1 + \mu(1 - p_1)q + (1 - \mu)(1 - q)(1 - p_0)$$
$$\text{s.t.} \quad \mu_0 < t, \ \mu_1 \geq t.$$

IV. $\mu_0 \geq t$, $\mu_1 \geq t$: The Receiver chooses $a = 1$ if $(m = 0, d = lie)$, $(m = 0, d = \neg lie)$ or $(m = 1, d = \neg lie)$ and $a = 0$ otherwise. The maximal probability that the Receiver chooses $a = 1$ is given by

$$\Pr_{\text{IV}}(a = 1) = \sup_{p_0, p_1 \in [0,1]} \quad 1 - (1 - \mu)(1 - p_0)q \quad \text{s.t.} \quad \mu_0 \geq t, \ \mu_1 \geq t.$$

Table 3.1 summarizes when the Receiver chooses $a = 1$ under different types of strategies. Notably, by the definition of off-path beliefs, $(0, 1)$ is a type III strategy and $(1, 0)$ is a type I strategy. We are now ready to state the main proposition.

	$d = lie$	$d = \neg lie$
$m = 0$	I, II, III, IV	II, IV
$m = 1$		III, IV

Table 3.1: Cases where the Receiver chooses $a = 1$ under I, II, III, and IV.

Proposition 3.1. *Let $\overline{q} = 1 - \frac{\mu(1-t)}{t(1-\mu)} \in (0, 1)$. If $q \leq \overline{q}$, the Sender's optimal strategy is a type* III *strategy, in which the Sender always tells the truth under $w = 1$, but lies with positive probability under $w = 0$. If $q > \overline{q}$, the Sender's optimal strategy is a type* IV *strategy, in which the Sender lies with positive probability under both states.*

Proof. See Appendix 3.A.1.

In Figure 3.1 we graphically illustrate how these four strategy types are divided. The proof involves sequential comparisons between the four type-optimal strategies. First, there exists *some* type II strategy that is better than *all* type I strategies. Consider a particular strategy $p_0 = p_1 = 0$ of type II (i.e., the Sender totally misreports the state). Following this strategy, the Receiver takes action $a = 1$ if and only if $w = 1$, which occurs with probability μ. This strategy may not be optimal among all type II strategies, but it is sufficient to beat all strategies of type I since for those strategies the Receiver takes action $a = 1$ only if $w = 1$ and $(m = 0,\ d = lie)$, which occurs with a probability less than μ.

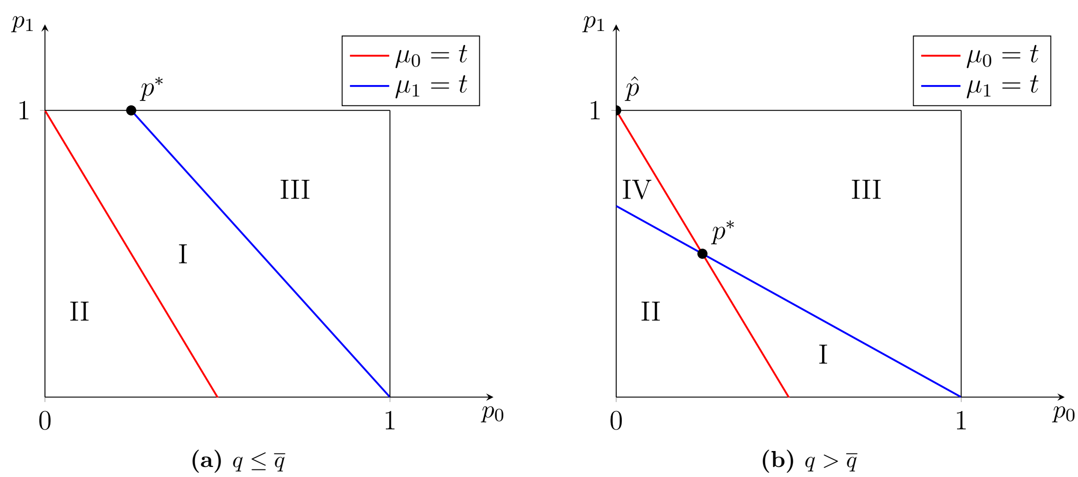


Figure 3.1: Equilibrium strategies for different detection probabilities q.

Second, there exists *some* type III strategy that is better than *all* type II strategies. Within type II strategies, we just need to focus on the ones with $p_1 = 0$ because lying more under state $w = 1$ relaxes both constraints and is beneficial for the Sender. Now, for any type II strategy of the form $(p_0, 0)$, consider a strategy $(\tilde{p}_0, 1)$ such that $p_0 = (1 - \tilde{p}_0)(1 - q)$. It can then be verified that this is a type III strategy. Moreover, this new strategy is equally good as $(p_0, 0)$ for the Sender by construction.

To see the intuition for this result, note that the type II and III strategies are totally symmetric if the lie detection technology is not available $(q = 0)$ since in that case the messages have no intrinsic meaning and we could always rename the messages. However, the introduction of a lie detection technology ($q > 0$) generates an intrinsic meaning for the message the Sender uses. In particular, an on-path message that was not detected as a lie always carries some credibility for the state to which it corresponds. Now, this additional source of credibility breaks the symmetry. By definition, type II strategies are such that ($m = 0,\ d = \neg lie$) suggests $w = 1$ with a sufficiently high probability, while ($m = 1,\ d = \neg lie$) suggests $w = 0$ with a sufficiently high probability. Loosely speaking, it is harder to persuade the Receiver to take $a = 1$ using type II strategies since the Sender needs to counter the intrinsic credibility of messages.

By transitivity, both type I and type II strategies are suboptimal relative to type III strategies, and we only need to focus on the comparison between type III and type IV strategies. Interestingly, as suggested by Figure 3.1 (a), type IV strategies do not exist when q is small. The proof is given in Appendix 3.A. Intuitively, when $q = 0$ our setup yields the standard Bayesian Persuasion benchmark, which essentially only involves two events ($m = 0, d = \neg lie$) and ($m = 1, d = \neg lie$). In that case, we know it is impossible to induce $a = 1$ under both events because by the martingale property, the posteriors following two events must average to the prior, suggesting some posterior is lower than the prior and must induce $a = 0$. However, the presence of lie detection extends the information from m to a couple ($m,\ d$), and the martingale property only requires the four posteriors' average over the prior. Furthermore, the posterior following ($m = 1,\ d = lie$) is 0. Therefore if q is sufficiently large, it is possible to support the two posteriors following ($m = 1,\ d = \neg lie$) and ($m = 0,\ d = \neg lie$) to be both higher than the prior and even higher than the threshold t.

In addition, as shown by Figure 3.1 (a), the constraint $\mu_0 < t$ is implied by the constraint $\mu_1 \geq t$. Hence, the set of type III strategies is compact, and the associated maximization problem admits a solution. Combining this observation with the previous arguments, we immediately obtain the first half of Proposition 3.1, *i.e.*, the Sender's optimal strategy is a type III strategy if $q \leq \overline{q}$. In particular, the optimal strategy takes the following form:

$$p_0^* = \frac{\overline{q} - q}{1 - q} \quad \text{and} \quad p_1^* = 1.$$

This is reminiscent of Kamenica and Gentzkow (2011), where the Receiver is indifferent between two actions when she takes the preferred action $a = 1$, and certain of the state when she takes the less preferred action $a = 0$.

If the detection probability q is larger than $\overline{q}$, the two lines that characterize the constraints in the right panel of Figure 3.1 intersects, implying the set of type III strategies is not closed anymore. However, the associated maximization problem still admits a solution: $(p_0, p_1) = (0, 1)$, which is a type III strategy according to off path beliefs specified earlier. This strategy can be shown to be optimal within type III strategies in two steps. First, increasing p_1 relaxes both constraints and improves the Sender's expected payoff at the same time (i.e., being more sincere in the favorable state benefits the Sender unambiguously). Thus, the optimal type III strategy, if it exists, must be of the form $(p_0, 1)$. Second, the whole segment from $(0, 1)$ to $(1, 1)$ are type III strategies when $q > \overline{q}$. Hence, the optimal strategy on this segment is the leftmost point $(0, 1)$ as it involves sending the persuasive message $m = 1$ as frequently as possible.

Yet, this optimal type III strategy, denoted as $\hat{p}$ in Figure 3.1 (b), is no longer globally optimal because the set of type IV strategies is non-empty, and the optimal type IV strategy is better than $\hat{p}$. In fact, we can prove a stronger statement that $\hat{p}$ is worse than any type IV strategy p whenever the latter is feasible. To this end, we decompose the value of a strategy for the Sender into two parts: the expected payoff in the favorable state $w = 1$ and the expected payoff in the unfavorable state

$w = 0$. The strategy $\hat{p}$ induces $a = 1$ for sure when $w = 1$ because the Sender always truthfully sends $m = 1$, which is credible and is never flagged as a lie. Meanwhile, any strategy p of type IV also induces $a = 1$ for sure. Such a strategy could induce three different events: $(m = 1,\ d = \neg lie)$, $(m = 0,\ d = \neg lie)$, $(m = 0,\ d = lie)$. The first two events successfully persuade the Receiver to take $a = 1$ by definition of type IV strategies. The last event directly informs the Receiver that $w = 1$, so it also induces $a = 1$. Hence, the strategy $\hat{p}$ and p agree in the expected payoff in the favorable state $w = 1$. However, they differ in the expected payoff in the unfavorable state $w = 0$. Given $\hat{p}$, the Sender always lies and sends the message $m = 1$ when $w = 0$, which induces $a = 1$ only if the lie is not detected. Given p, the Sender sometimes tells the truth by sending the message $m = 0$ as well, but by definition of type IV strategies, $m = 0$ is now a risk-free way to induce $a = 1$ since it will never be flagged as a lie in the unfavorable state $w = 0$. Hence, the strategy p results in a higher expected payoff for Sender in the unfavorable state as well as overall. Mathematically,

$$U_S(\hat{p}) = \underbrace{\mu}_{\Pr(w=1)} \times \overbrace{1 \times 1}^{\Pr(a=1|w=1;\hat{p}_1)} + \underbrace{(1-\mu)}_{\Pr(w=0)} \times \overbrace{1 \times (1-q)}^{\Pr(a=1|w=0;\hat{p}_0)},$$

and

$$U_S(p) = \underbrace{\mu}_{\Pr(w=1)} \times \overbrace{[p_1 \times 1 + (1-p_1) \times (1-q) + (1-p_1) \times q]}^{\Pr(a=1|w=1;p_1)}$$
$$+ \underbrace{(1-\mu)}_{\Pr(w=0)} \times \overbrace{[p_0 \times 1 + (1-p_0) \times (1-q)]}^{\Pr(a=1|w=0;p_0)}.$$

where the first term $(\mu \times 1)$ is the same for the two expressions, but the second term is larger for $U_S(p)$ since p_0 is not multiplied by $1 - q$ but instead by 1. As we argued above, the main benefit of p relative to $\hat{p}$ is that the "safer" message $m = 0$ is sent more frequently in p. Thus, the optimal type IV strategy must involve the highest p_0, or the least lying in the unfavorable state. Such a strategy, given by p^* in Figure 3.1 (b), is also globally optimal by the previous arguments provided that $q > \overline{q}$. The expressions are given by

$$p_0^* = \frac{1-q}{(2-q)q}(q-\overline{q}) \quad \text{and} \quad p_1^* = \frac{1-q}{(2-q)q}\left[\frac{1}{1-\overline{q}} - (1-q)\right].$$

Although the optimal strategy features partial lying under both states, the Sender still lies more in the unfavorable state than in the favorable state ($p_0^* < p_1^*$).

Interestingly, the difference between the Sender's payoffs of the strategy $\hat{p}$ and p^* is non-monotone in the detection probability q. When $q = \overline{q}$, $\hat{p}$ coincides with p^*, so they are equally good. When $q = 1$, it is as if the Receiver is informed about the state with probability 1, so any strategy results in the same payoff for the Sender. Only when $q \in (\overline{q}, 1)$, p^* yields a strictly higher payoff than $\hat{p}$.

Finally, the threshold $\overline{q}$ where the optimal strategy switches from a type III to a type IV strategy, is decreasing in μ and increasing in t. To see the intuition for this result, fix the lie detection probability $q \in (0, 1)$. If a weak signal is sufficient to persuade the Receiver (i.e., the prior μ is already close to the threshold t), a type IV strategy is optimal for the Sender. On the other hand, if the signal has to be very convincing to persuade the Receiver (i.e., the threshold t is much larger than the prior μ), a type III strategy is optimal for the Sender.

3.3.2 Comparative Statics

We now consider the comparative statics of our model with respect to the central parameter of the lie detection probability q to show how the optimal communication and the payoffs of the communicating parties changes as the lie detection technology improves.

Optimal Messages

Proposition 3.2 describes how the structure of the optimal message strategy $(p_0^*,\ p_1^*)$ changes as the detection probability varies. Figure 3.2 plots these optimal reporting probabilities as a function of q. For comparison, the probabilities p_0^{BP} and p_1^{BP} are the equilibrium reporting probabilities that would result in a standard Bayesian persuasion setup without lie detection.

Proposition 3.2. *As the lie detection probability q increases,*

1. $p_0^* = Pr(m = 0 \mid w = 0)$ *is decreasing over* $[0, \overline{q}]$*, and has an inverse U shape over* $(\overline{q}, 1]$.

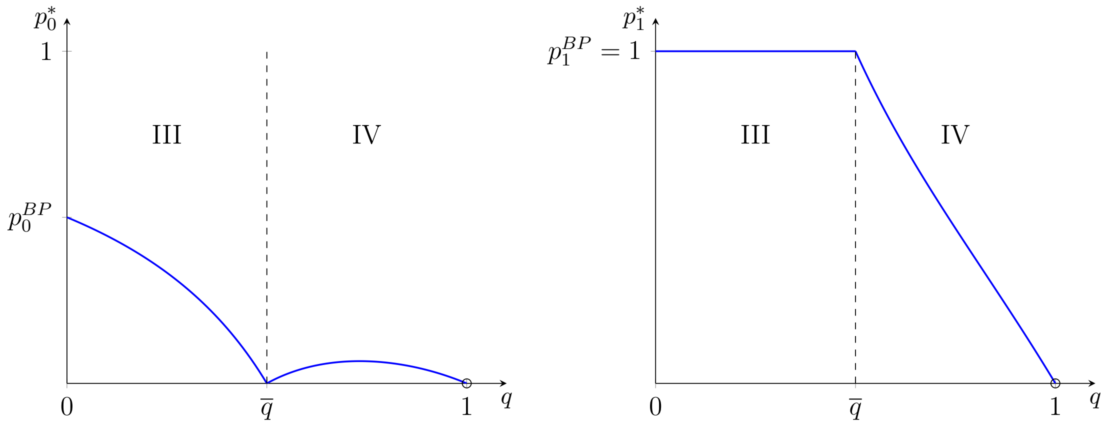
2. $p_1^* = Pr(m = 1 \mid w = 1)$ *is constant over* $[0, \overline{q}]$*, and decreases over* $(\overline{q}, 1]$.

Proof. See Appendix 3.A.2.

If $q \leq \overline{q} = 1 - \frac{\mu(1-t)}{t(1-\mu)}$, p_0^* is decreasing in q and p_1^* is constant at 1. In this range of q, the Sender's optimal strategy lies in III, which involves truthfully reporting the state $w = 1$ (i.e., $p_1 = 1$), but progressively misreporting the state $w = 0$ as the lie detection technology improves (i.e., $p_0 < 1$ and decreasing with q).

If $q > \overline{q}$, p_0^* initially increases and then decreases. In contrast, p_1^* decreases over the entire range of $[\overline{q}, 1]$. In this range, the Sender's optimal strategy lies in IV which involves misreporting both states of the world.

For $q = 0$ we have the Bayesian benchmark. Recall from Kamenica and Gentzkow (2011) that if an optimal signal induces a belief that leads to the worst action for the Sender ($a = 0$ in our case), the Receiver is certain of her action at this belief. In addition, if the optimal signal induces a belief that leads to the best action for the Sender ($a = 1$ in our case), the Receiver is indifferent between the two actions at this belief.



(a) $p_0^* \equiv \Pr(m = 0 \mid w = 0)$ **(b)** $p_1^* \equiv \Pr(m = 1 \mid w = 1)$

Figure 3.2: Equilibrium strategies p_0^* and p_1^* as a function of q for $\mu = \frac{1}{3}$ and $t = \frac{1}{2}$

Now consider the addition of a lie detection technology. As the lie detection probability q increases, $(m = 1, d = \neg lie)$ becomes more indicative of the favorable state $w = 1$, and therefore the Receiver would strictly prefer to take the favorable action $a = 1$. As a response, the Sender would like to send the message $m = 1$ more often while still maintaining that $(m = 1,\ d = \neg lie)$ sufficiently persuades the Receiver to take the action $a = 1$. Because the Sender already sends the message $m = 1$ with probability 1 under $w = 1$, the only way to increase the frequency of $m = 1$ is to send such a message more often in the unfavorable state $w = 0$ (i.e., lie more frequently if $w = 0$). In other words, the Sender increases the frequency of lying just enough about the unfavorable state $(w = 0)$ to make the Receiver indifferent when choosing the favorable action $a = 1$.

Recall that in the canonical Bayesian persuasion setup, the Receiver is held to her outside utility of getting no information whatsoever. Thus, when the lie detection probability q increases, the Receiver is more certain that $(m = 1,\ d = \neg lie)$ means $w = 1$ and would obtain a larger surplus from the improvement in the lie detection technology. However, as long as p_0^* is greater than 0 the Sender can simply undo this improvement by lying more about $w = 0$ (i.e., reduce p_0^* even further), thereby “signal-jamming” the information obtained by the Receiver.

However, once the detection probability q rises above $\overline{q}$ it is no longer possible for the Sender to just lie about the unfavorable state because he already maximally lies about it at $\overline{q}$. His optimal messaging strategy is now a type IV strategy when $q > \overline{q}$. Under a type IV strategy, the Receiver only takes the unfavorable action $a = 0$ if he receives a message $m = 1$ that is flagged as a lie. This is because with a type IV strategy the Receiver has access to such a reliable lie detection technology that a lie involving the message $m = 1$ is sufficiently likely to be detected as a lie and will then induce the unfavorable action $a = 0$. At the same time, the Receiver is also very likely to be notified of a lie involving the message $m = 0$ which the Sender can use to his advantage to ensure that the Receiver chooses the favorable action $a = 1$. Therefore at $q = \overline{q}$, the Sender wants to increase the frequency of the message $m = 0$ which he achieves by both increasing p_0 and decreasing p_1. However, when the detection probability is close to 1, (i.e., the lie detection technology is almost perfect) p_1 is close to 0 and any message $m = 1$ is very likely to be a lie. To make sure that a message $m = 1$ which is not detected as a lie still sufficiently persuades the Receiver to choose $a = 1$ (i.e., does not violate the constraints $\mu_0 \geq t$ and $\mu_1 \geq t$ required for a type IV strategy), the Sender also has to decrease p_0 while decreasing p_1.

These perhaps surprising comparative statics, especially those of the type IV strategy, are partly due to the asymmetric nature of the signal structure (as in Engers et al. (1999)) which in our case only detects lies rather than detecting both lies and truths, and partly due to the persuasion game leading to a mixed strategy equilibrium. Such mixed strategy equilibria often have counterintuitive comparative statics properties, as Crawford and Smallwood (1984) point out.

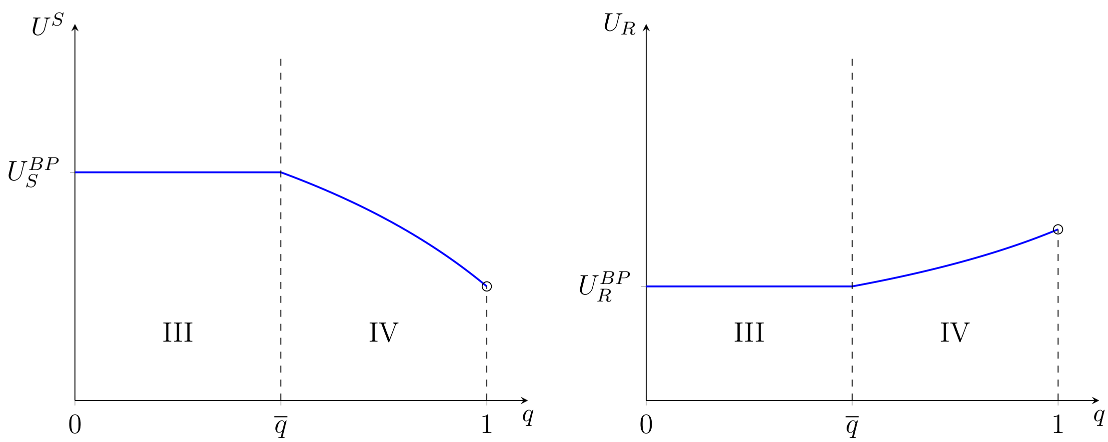
Payoffs

Recall that U_S and U_R denote the equilibrium payoffs of the Sender and the Receiver. We now investigate how U_S and U_R are affected by improvements in the lie detection technology. The results are summarized in Proposition 3.3 and graphically depicted in Figure 3.3. For comparison, U_S^{BP} and U_R^{BP} are the equilibrium payoffs that would result in a standard Bayesian persuasion setup without lie detection.

Proposition 3.3. *As the lie detection probability q increases,*

1. *U_S is constant over $[0, \overline{q}]$, and decreases over $(\overline{q}, 1]$.*
2. *U_R is constant over $[0, \overline{q}]$, and increases over $(\overline{q}, 1]$.*

Proof. See Appendix 3.A.3.



(a) The Sender's Equilibrium Payoff

(b) The Receiver's Equilibrium Payoff

Figure 3.3: Equilibrium payoffs as a function of q for $\mu = \frac{1}{3}$, $t = \frac{1}{2}$

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The Sender's equilibrium payoff does not change for $q \leq \overline{q}$ and decreases with q for $q > \overline{q}$. As long as $q \leq \overline{q}$ the Sender receives exactly the same utility that he would receive under the Bayesian Persuasion benchmark. Any marginal improvement in the lie detection technology (i.e., increase in q) is completely offset by less truthful reporting when $w = 0$ (i.e., decrease in p_0^*). However, for $q > \overline{q}$ any further improvements reduce the Sender's utility. In the limit case where $q = 1$ the Sender has no influence anymore and the action $a = 1$ in only implemented when the state is $w = 1$ which occurs with probability μ.

Analogously for the case of the Sender's utility, the Receiver's utility is also constant at the Bayesian persuasion benchmark as long as $q \leq \overline{q}$ and then increases with q for $q > \overline{q}$ as the lie detection technology starts to bite. If having access to the lie detection technology required any costly investment, the Receiver would only ever want to invest in improving lie detection if it raised q above the threshold $\overline{q}$. In the limit, the Receiver is just as well off as she would be under perfect information.

3.4 Discussion

Our baseline model considers the role of lie detection in a simple setting with binary states. We now investigate how alternative assumptions modify our analysis.

3.4.1 Detection Technology

First, consider a different detection technology that informs the Receiver with probability r that a message is truthful. That is to say, rather than being able to (probabilistically) detect a lie the Receiver can (probabilistically) detect that a message is truthful. Truth detection is perhaps a less realistic assumption as it is often easier to detect whether the Sender has lied than whether he has sent a truthful message.

In our setting, truth detection turns out to be payoff-equivalent to lie detection. Therefore, all of our insights about the equilibrium payoffs as a function of the lie detection probability q in Figure 3.3 also hold for the truth detection probability r. However, under truth detection the Sender's optimal message is completely flipped and has some unnatural features. When the truth detection probability r is low but positive, it is optimal for the Sender to always lie in the favorable state (i.e., $p_1 = 0$)

and to choose p_0 such that the Receiver is indifferent between $a = 0$ and $a = 1$ upon a message $m = 0$ that is not marked as truth.

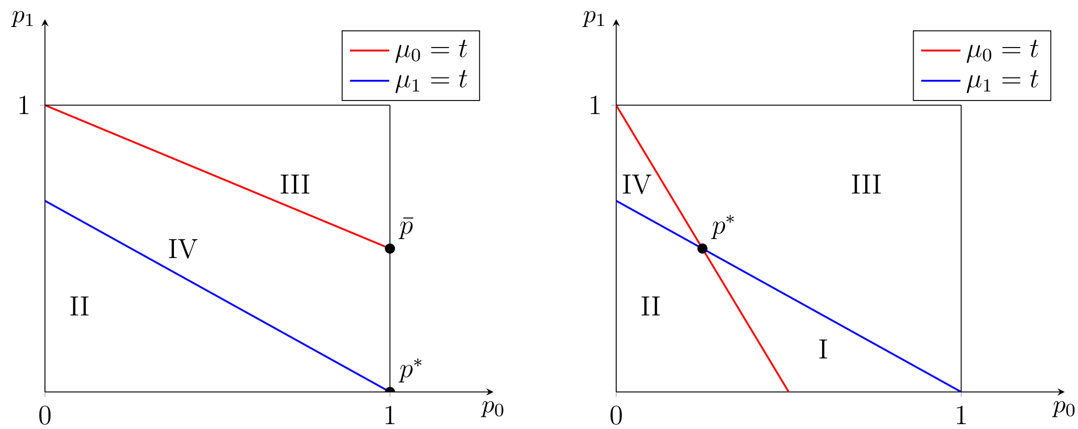
Second, combining lie detection and truth detection such that they are perfectly positively correlated is equivalent to state detection. With probability $q = r$ the Receiver learns the state w regardless of the message sent by the Sender. With such a state detection technology the analysis becomes much simpler as we just return to the Bayesian persuasion benchmark. This is because the Sender's message does not influence at all whether the Receiver learns the state, and any message m is only relevant whenever the Receiver does not learn the state. This finding contrasts with the literature on noisy cheap talk in which adding communication error or noise influences the messaging strategies and can improve welfare (Blume et al., 2007).

These observations highlight our interpretation of Bayesian persuasion under lie detection in that the Sender's messages have a literal meaning of truth and lies. Even though the Sender is committing to the strategy—or, alternatively speaking, choosing an experiment—the strategies employed by the Sender are not equivalent to just an arbitrary garbling of the state.

3.4.2 Default Action Coincides with Sender's Preferred Action

In standard Bayesian persuasion models without lie detection the Sender can always send a purely uninformative signal. Therefore, a trivial case obtains if the Receiver's default action coincides with the Sender's preferred action because the Sender can induce the Receiver to take this action with probability one by committing to an uninformative signal. However, the messages in our model have literal meanings and are subject to lie detection. Therefore, a purely uninformative signal is unavailable to the Sender. Intuitively, lie detection forces information transmission from the Sender to the Receiver which makes the Sender's optimization problem nontrivial even when the Receiver's default action coincides with the Sender's preferred action.

In this extension, we analyze the scenario in which the prior mean μ is higher than the action threshold t. The results are analogous to those in the baseline model. As before, the Sender's maximization problem is solved by considering the four sub-problems. The only change relative to the baseline model is that Region IV now exists for any $q \in [0, 1]$ as shown in Figure 3.4.



(a) $q \leq \tilde{q}$ **(b)** $q > \tilde{q}$

Figure 3.4: Equilibrium message strategies for different detection probabilities q ($\mu \geq t$).

The optimal messaging strategy p^* is always in Region IV. When $q \in \tilde{q} \equiv 1 - \frac{t(1-\mu)}{\mu(1-t)}$, the strategy $(p_0, p_1) = (1, 0)$ would induce the Receiver to take $a = 1$ with probability one and is thus optimal.[5] Under this strategy, the Sender reports $m = 0$ with probability one in both states. If it is flagged as a lie, the Receiver immediately learns that the true state is $\omega = 1$. Otherwise, her posterior mean would drop by the martingale property. Nonetheless, if q is sufficiently small, her posterior mean would be close to the prior mean which is still higher than the action threshold. Hence, the Receiver is always willing to take the favorable action.

If q is sufficiently large such a strategy is no longer sustainable and it is impossible to induce $a = 1$ with probability one. For example, in the extreme case where $q = 1$, it is as if the Receiver learns the true state. Hence, it must be that the Receiver takes

[5] It is not unique though because the Receiver actually strictly prefers to take $a = 1$ when she observes $(m = 0, \neg lie)$. Other optimal strategies include the segments from p^* to $\bar{p}$.

action $a = 1$ if and only if $\omega = 1$. In fact, the Sender's optimal messaging strategy is again characterized by the intersection of two indifference conditions: $\mu_0 = t$ and $\mu_1 = t$ as in Figure 3.4 (b).

Given this discussion above, the Sender's (Receiver's) payoff is initially constant in q when $q \leq \tilde{q}$, and then decreasing (increasing) in q when $q > \tilde{q}$. This is consistent with Proposition 3.3.

Admittedly, the fact that the Sender cannot induce the Receiver to always take the favorable action even when $\mu \geq t$, suggests some tension between our model and the standard persuasion paradigm. In the standard paradigm without lie detection this case is trivial whereas in ours it is not. However, it is easy to reconcile this tension by introducing an additional stage prior to the persuasion game, in which the Sender decides whether or not to enter the game. If he enters, the Sender and the Receiver play the persuasion game with lie detection specified in our main analysis. Otherwise, the Sender cannot send any message and the Receiver takes an action based on her prior. It is straightforward to show that the Sender enters the game if the Receiver's default action does not coincide with his preferred action. Otherwise, the Sender does not enter the game, but the Receiver always takes action $a = 1$, consistent with the standard persuasion paradigm.

3.4.3 Non-revealing Lie Detection

In a binary-state environment, the lie detection technology considered in this chapter is quite special in the sense that whenever the Receiver learns that the Sender has lied, she immediately learns the true state. However, Proposition 3.3 is not driven by this special feature.[6] Intuitively, in a binary-state environment, the lie detection technology forces the release of too much information to the Receiver. Still, we show

[6] In the three-state environment, the uniqueness of equilibrium is not necessarily guaranteed. Thus, it is hard to generalize Proposition 3.2. However, the strategic effect still appears and plays an important role.

that the Receiver does not benefit from a weak lie detection technology, even in this case. Following this reasoning, when the Receiver's posterior beliefs after a lie detection are non-degenerate, the Receiver obtains less information. Therefore, we would expect Proposition 3.3 to be strengthened instead of weakened. For example, if the state space is continuous and the prior belief has no point mass, then the lie detection technology is essentially useless because learning that a single state is impossible does not alter the Receiver's belief. In fact, even in a three-state environment, lie detection technology is useless. This section shows that both the Sender's and the Receiver's payoff are unaffected by the strength of lie detection, suggesting that fully revealing lie detection is not the driving force.[7]

Formally, let $\omega \in \{0,\, \lambda,\, 1\}$ be the state of the world and $(P_0,\, P_\lambda,\, P_1)$ be the full-support prior belief where $\lambda \in (0, 1)$. The message space is again restricted to be identical to the state space and a lie is detected with probability $q \in [0, 1]$ whenever the message is inconsistent with the true state. For simplicity, keep the player's utility functions unchanged. In particular, the Sender always prefers $a = 1$ over $a = 0$ regardless of the true state, whereas the Receiver takes an action $a = 1$ if and only if her posterior mean is higher than an action threshold t. Assume $t \in (\mu, \lambda)$, where $\mu = P_1 + \lambda P_\lambda$ is the prior mean.[8] The implication of this restriction is twofold. First, the Receiver's default action is $a = 0$. Second, if the Receiver knows the state is λ, she prefers to take an action $a = 1$. In other words, both $\omega = 1$ and $\omega = \lambda$ are favorable states for the Sender. To quantify the Receiver's utility explicitly, let her

[7] An alternative approach to model non-revealing lie detection is to modify the lie detection technology by introducing false alarms. In other words, a lie may be detected even if the message is consistent with the true state. However, this approach is more complicated. So, we adopt the approach of expanding the state space.

[8] The choice of t is not important for the extension. However, it affects two things and complicates the exposition. First, it affects the Receiver's utility function. Second, it affects the Sender's equilibrium payoff in the benchmark.

payoff function and expected payoff be

$$u_R(a,\omega) = (1-t)\cdot \mathbb{1}_{\{a=1,\omega=1\}} + (\lambda - t)\cdot \mathbb{1}_{\{a=1,\omega=\lambda\}} + t\cdot \mathbb{1}_{\{a=0,\omega=0\}},$$

and

$$U^R = (1-t)\cdot \Pr(a=1,\omega=1) + (\lambda-t)\cdot \Pr(a=1,\omega=\lambda) + t\cdot \Pr(a=0,\omega=0). \tag{3.1}$$

This utility function is analogous to the one in the main body. The Receiver would like to take the right action for each state but assigns different weights for different states. The particular choice of weights induces a decision rule that it is optimal to take $a = 1$ if and only if the posterior is higher than t.

The goal here is to show that both players' payoffs are constant in q. To this end, we first compute the Sender's payoff in the benchmark scenario ($q = 0$) and then construct a strategy that leads to the same payoff for the Sender with any detection probability. Last, we show that in any Sender's preferred equilibrium, the Receiver's payoff is constant. Moreover, this constant is independent of q.

The optimal signal/messaging strategy in the classical Bayesian Persuasion framework with a binary action has been analyzed in the literature. Ivanov (2021) shows that in a binary-action and continuous-state environment, there exists an optimal strategy with a partitional structure where the Sender sends a message if the state is above some threshold and sends another message otherwise. By applying the insight to our discrete-state model, there exists an optimal strategy with the following properties. The Sender sends one (another) message if the state is strictly higher (lower) than some threshold state. Moreover, he mixes between two messages at the threshold state. In particular, the following strategy achieves the optimum.

$$\textbf{Strategy 1:} \quad \begin{aligned} &\omega = 1 \longrightarrow m = 1 \\ &\omega = \lambda \longrightarrow m = 1 \\ &\omega = 0 \longrightarrow m = \begin{cases} 1, & w.p. \quad r \\ 0, & w.p. \quad 1 - r \end{cases} \end{aligned}$$

where r solves

$$\frac{P_1 + \lambda P_\lambda}{P_1 + P_\lambda + r P_0} = t.$$

Essentially, the mixing probability r ensures the Receiver to be indifferent after observing $m = 0$. Given Strategy 1, the Receiver takes the favorable action if and only if she receives $m = 1$. Thus,

$$U^S(0) = P_1 + P_\lambda + r P_0 = \frac{\mu}{t}.$$

Now, suppose there is a lie detection probability $q > 0$. In principle, this limits the Sender's scope to manipulate the Receiver's posterior beliefs, and thus potentially lowers the Sender's payoff. However, the following strategy yields the Sender the same payoff as in the benchmark. Moreover, this strategy is independent of q, suggesting that lie detection has not impact on the Sender's payoff at all.

$$\textbf{Strategy 2:}\quad \begin{aligned} &\omega = 1 \longrightarrow m = \lambda \\ &\omega = \lambda \longrightarrow m = 1 \\ &\omega = 0 \longrightarrow m = \begin{cases} 1, & w.p. \quad r \\ \lambda, & w.p. \quad s \\ 0, & w.p. \quad 1 - r - s \end{cases} \end{aligned}$$

where r and s respectively solve

$$\begin{cases} \frac{\lambda P_\lambda}{P_\lambda + P_0 r} = t, \\ \frac{P_1}{P_1 + P_0 s} = t. \end{cases}$$

The assumption $\mu < t < \lambda$ ensures that $s, r, s + r \in (0, 1)$. Given Strategy 2, the Receiver is indifferent after observing $(m = 1, d = \neg lie)$, $(m = 1, d = lie)$, $(m = \lambda, d = \neg lie)$, and $(m = \lambda, d = lie)$. So, she takes the favorable action if and only if she receives $m = 1$ or $m = \lambda$, regardless of the lie detection outcome. It follows that

$$U^S(q) = P_1 + P_0 r + P_\lambda + P_0 s = \frac{\lambda P_\lambda}{t} + \frac{P_1}{t} = \frac{\mu}{t} = U^S(0), \quad \forall q \in (0, 1]. \tag{3.2}$$

Lie detection is not useful here because conditional on this particular strategy, the message λ and the message 1 are always lies, whereas a message 0 is never a lie. Moreover, the probability of lie detection is constant as long as the Sender is lying. Thus, lie detection does not provide any additional information for the Receiver, no matter how strong it is.

Since we focus on the Sender's preferred equilibrium, the Receivers' payoff is potentially non-unique. Fortunately, that is not the case here. Lemma 3.1 guarantees that the Receiver's payoff is unique. In addition, it is always linear in the Sender's equilibrium payoff with a negative slope.

Lemma 3.1. *Fix a lie detection probability* $q \in [0, 1]$*. If* $\mu < t < \lambda$*, then in any Sender's preferred equilibrium,*

$$U^R(q) = (1 - t)P_1 + (\lambda - t)P_\lambda + t[1 - U^S(q)].$$

Here is the key intuition of this result. Note that both $\omega = 1$ and $\omega = \lambda$ are favorable states for the Sender. Thus, it is optimal to induce $a = 1$ under those two states, which suggests $\Pr(a = 1, \omega = 1) = \Pr(a = 1, \omega = \lambda) = 1$ in any Sender's preferred equilibrium. Then, according to Equation 3.1, the Receiver's expected payoff only depends on $\Pr(a = 0, \omega = 0)$. But this probability is completely determined by the Sender's optimality and is therefore linked to the Sender's equilibrium payoff. Roughly speaking, the Sender wishes to minimize this probability while conditional on $\Pr(a = 1, \omega = 1) = \Pr(a = 1, \omega = \lambda) = 1$.

Combining Equation 3.2 and Lemma 3.1, it is immediate that the Receiver's equilibrium payoff is also independent of q.

$$U^R(q) = (1 - t)P_1 + (\lambda - t)P_\lambda + t - \mu = tP_0.$$

Our analysis suggests that fully revealing lie detection does not drive Proposition 3.3.

3.4.4 Related Literature

Balbuzanov (2019) and Dziuda and Salas (2018) also study strategic communication in the presence of a lie detection technology but in a cheap talk setting. The largest difference between these two papers and ours therefore lies in the commitment power of the Sender. Although it is debatable whether the extreme cases of full commitment (as in Bayesian persuasion) or no commitment (as in cheap talk) constitute more plausible assumptions about real-life communication setting, we believe our model is an important step towards studying the communication games with lie detection under (partial) commitment.

This chapter also differs from Balbuzanov (2019) in the payoff functions. In Balbuzanov (2019) the Sender and the Receiver have some degree of common interest whereas there is no common interest in our model. Due to this difference the Sender's type-dependent preferences in Balbuzanov (2019) permit fully revealing equilibria in some cases as it allows the Receiver to tailor message-specific punishment actions. In particular, fully revealing equilibria exist for some intermediate degree of lie detectability if the Sender's bias is small. However, the Sender in our model never reveals the state perfectly due to the conflict in payoffs.

Dziuda and Salas (2018) do not allow for common interest and therefore, like in this chapter, and thus fully revealing equilibria are impossible in their paper. In their continuous state model, there are many off-path beliefs to be specified. To discipline these off-path beliefs, they impose two refinements. They show that in all remaining equilibria, the lowest types lie but *some* higher types tell the truth. However, the assumptions of our model allow the second refinement required by Dziuda and Salas (2018) to be violated. Therefore, irrespective of the commitment power of the Sender, our model is not nested by theirs. Furthermore, in the baseline model of Dziuda and Salas (2018), a higher lie detection probability leads to more truth-telling, which is contrary to our finding.

3.5 Conclusion

In this chapter we analyze the role of probabilistic lie detection in a model of Bayesian persuasion between a Sender and a Receiver. We show that the Sender lies more when the lie detection probability increases. As long as the lie detection probability is sufficiently small the Sender's and the Receiver's equilibrium payoff are unaffected by the lie detection technology because the Sender compensates by lying more. Once the lie detection probability is sufficiently high, the Sender can no longer maximally lie about the unfavorable state and the Sender's (Receiver's) equilibrium payoff decreases (increases) with the lie detection probability. Our model rationalizes that a sender of communication chooses to lie more frequently when it is more likely that their false statements will be flagged as lies.

This chapter explores the impact of lie detection on communication in a setting with complete commitment to a communication strategy by the Sender. It thereby establishes a useful benchmark relative to the diametrically opposed assumption of no commitment in existing cheap talk models with lie detection. However, how does lie detection influence communication behavior in intermediate settings with partial commitment? We leave this and other interesting questions to future research.

3.A Omitted Proofs

3.A.1 Proof of Proposition 3.1

We now show that type I and II strategies are suboptimal because the resulting implementation probabilities $\Pr_{\text{I}}(a=1)$ and $\Pr_{\text{II}}(a=1)$ are dominated by the probability $\Pr_{\text{III}}(a=1)$ resulting from III. To see this, note first that

$$\Pr_{\text{I}}(a=1) \leq \mu \leq \Pr_{\text{II}}(a=1).$$

The second inequality holds because $(p_0, p_1) = (0,0)$ is a type II strategy and gives value μ. In fact, for a type II strategy, it is optimal to set $p_1 = 0$ because this loosens both constraints, and improves the objective. Given this, $\mu_1 = 0 < t$ is loose. Hence the optimum requires

$$\mu_0 = \frac{\mu(1-q)}{\mu(1-q) + (1-\mu)p_0} = t,$$

and hence

$$\Pr_{\text{II}}(a=1) = \mu + \left(\frac{\mu}{t} - \mu\right)(1-q).$$

Similarly, in the maximization problem within type III strategies, it is optimal to set $p_1 = 1$. Then $\mu_0 = 0 < t$ becomes loose. The optimum requires p_0 to be as small as possible while ensuring that $\mu_1 \geq t$. Define $\overline{q} \equiv 1 - \frac{\mu(1-t)}{t(1-\mu)} \in (0,1)$, then there are two cases to consider.

- $\frac{\mu}{\mu+(1-\mu)(1-q)} \leq t$ or $q \leq \overline{q}$. In this case, there exists p_0^* s.t. $\mu_1 = t$, that is $\frac{\mu}{\mu+(1-\mu)(1-p_0^*)(1-q)} = t$. Therefore, $\Pr_{\text{III}}(a=1) = \frac{\mu}{t}$.

- $\frac{\mu}{\mu+(1-\mu)(1-q)} > t$ or $q > \overline{q}$. In this case, $\mu_1 \geq t$ can never bind. Thus, the best option is to set $p = 0$ which implies $\Pr_{\text{III}}(a=1) = \mu + (1-\mu)(1-q)$.

Clearly, in either case we have $\Pr_{\text{III}}(a=1) > \Pr_{\text{II}}(a=1)$ and therefore both type I and II strategies are suboptimal. It therefore remains to compare $\Pr_{\text{III}}(a=1)$ and $\Pr_{\text{IV}}(a=1)$.

(1) If $\frac{\mu}{\mu+(1-\mu)(1-q)} \leq t$, the type IV strategies do not exist, *i.e.*, there is no way to choose p_0, p_1 such that $\mu_1 \geq t$ and $\mu_0 \geq t$. If that were the case we would have

$$\frac{\mu p_1}{\mu p_1 + (1-\mu)(1-p_0)(1-q)} \geq t,$$

and

$$\frac{\mu(1-p_1)(1-q)}{\mu(1-p_1)(1-q) + (1-\mu)p} \geq t \iff \frac{\mu(1-p_1)}{\mu(1-p_1) + (1-\mu)\frac{p}{1-q}} \geq t,$$

which would imply

$$\frac{\mu p_1 + \mu(1-p_1)}{\mu p_1 + \mu(1-p_1) + (1-\mu)(1-p_0)(1-q) + (1-\mu)\frac{p}{1-q}} \geq t,$$

and therefore

$$t \leq \frac{\mu}{\mu + (1-\mu)(1-p_0)(1-q) + (1-\mu)\frac{p}{1-q}} \leq \frac{\mu}{\mu+(1-\mu)(1-q)},$$

where the last inequality is binding if $q = 0$ or $p = 0$. This in turn yields $t < \frac{\mu}{\mu+(1-\mu)(1-q)}$ which is a contradiction. Hence, if $\frac{\mu}{\mu+(1-\mu)(1-q)} \leq t$, the optimal strategy is

$$p_0^* = 1 - \frac{\frac{\mu(1-t)}{t(1-\mu)}}{1-q} \quad \text{and} \quad p_1^* = 1.$$

Alternatively,

$$p_0^* = \frac{\overline{q} - q}{1 - q} \quad \text{and} \quad p_1^* = 1.$$

(2) If $\frac{\mu}{\mu+(1-\mu)(1-q)} > t$, it is now possible to induce $\mu_1 \geq t, \mu_0 \geq t$. In particular, the constraints can be rewritten as two lines where the coordinates are p_0 and p_1. In particular, we have

$$\mu_1 \geq t \Leftrightarrow (1-t)\mu p_1 \geq t(1-\mu)(1-p_0)(1-q),$$

which passes through $(1,0)$ and $\left(0, \frac{t(1-\mu)(1-q)}{(1-t)\mu}\right)$ where $\frac{t(1-\mu)(1-q)}{(1-t)\mu} < 1$ by assumption. We also have

$$\mu_0 \geq t \Leftrightarrow \mu(1-t)(1-p_1) \geq t(1-\mu)\frac{p}{1-q},$$

which passes through $(0,1)$ and $\left(\frac{\mu(1-t)(1-q)}{t(1-\mu)}, 0\right)$ where $\frac{\mu(1-t)(1-q)}{t(1-\mu)} < 1$ because $t > \mu$.

Since the objective is to maximize $1-(1-\mu)(1-p_0)q$, we want to find the point in type IV strategies with the largest value of p_0. Clearly, this point is at the intersection of the two lines in Figure 3.1(b), given by

$$p_0^* = 1 - \frac{1-(1-q)\frac{\mu(1-t)}{t(1-\mu)}}{(2-q)q} \quad \text{and} \quad p_1^* = 1 - \frac{1-(1-q)\frac{t(1-\mu)}{\mu(1-t)}}{(2-q)q},$$

where $\frac{\mu(1-t)}{t(1-\mu)} \in (1-q, 1)$ by assumption. Alternatively,

$$p_0^* = \frac{1-q}{(2-q)q}(q-\overline{q}) \quad \text{and} \quad p_1^* = \frac{1-q}{(2-q)q}\left[\frac{1}{1-\overline{q}} - (1-q)\right].$$

As a result, we have $\Pr_{\mathrm{III}}(a=1) < \Pr_{\mathrm{IV}}(a=1)$ because the following inequality holds

$$\begin{aligned}\Pr_{\text{III}}(a=1) &= \mu + (1-\mu)(1-q) \\ &= 1-(1-\mu)q < 1-(1-\mu)q(1-p_0^*) = \Pr_{\text{IV}}(a=1).\end{aligned}$$

3.A.2 Proof of Proposition 3.2

- If $q \le \overline{q}$,

$$p_0^* = \frac{\overline{q}-q}{1-q} \quad \text{and} \quad p_1^* = 1.$$

Clearly, $p_0^* = 1 - \frac{1-\overline{q}}{1-q}$ decreases in q and p_1^* is constant in q.

- If $q > \overline{q}$,

$$p_0^* = \frac{1-q}{(2-q)q}(q-\overline{q}) \quad \text{and} \quad p_1^* = \frac{1-q}{(2-q)q}\left[\frac{1}{1-\overline{q}} - (1-q)\right].$$

This implies

$$\begin{aligned}\frac{\partial p_0^*}{\partial q} &= \frac{(-2q+1+\overline{q})\cdot(2-q)q - (2-2q)(1-q)(q-\overline{q})}{(2-q)^2q^2} \\ &= \frac{-q^2+(q^2-2q+2)\overline{q}}{(2-q)^2q^2}.\end{aligned}$$

Therefore,

$$\frac{\partial p_0^*}{\partial q} \ge 0 \Longleftrightarrow \frac{1}{\overline{q}} \le \frac{q^2-2q+2}{q^2} = 1 + \frac{2-2q}{q^2}.$$

RHS decreases in q, meaning the sign of the derivative at most changes one time. Since the derivative is positive at $q = \overline{q}$, but negative at $q = 1$, we conclude that p_0^* is first increasing and then decreasing in q over $(\overline{q}, 1]$.

On the other hand, p_1^* can be written as a product of $\frac{(1-q)}{(2-q)}$ and $\frac{\frac{1}{1-\overline{q}}-(1-q)}{q}$. Each term decreases in q, the it follows that p_1^* decreases in q over $(\overline{q}, 1]$.

3.A.3 Proof of Proposition 3.3

The expected payoff of the Sender is $\Pr(a = 1)$. There are two cases depending on whether $q > \overline{q}$.

- If $q \leq \overline{q}$, then the Receiver chooses $a = 1$ whenever $(m = 1,\ d = \neg lie)$ or $(m = 0,\ d = lie)$. But the latter occurs with probability 0 in the equilibrium. So,

$$U_S = \mu + (1-\mu)(1-p_0^*)(1-q) = \frac{\mu}{t},$$

  which is constant in q. Essentially, any marginal improvement in the lie detection technology (i.e., increase in q) is completely offset by less truthful reporting when $w = 0$ (i.e., decrease in p_0^*).

- If $q > \overline{q}$, then the Receiver chooses $a = 1$ always unless $(m = 1,\ d = lie)$. So,

$$U_S = 1 - (1-\mu)(1-p_0^*)q = 1 - \frac{t(1-\mu) - \mu(1-t)(1-q)}{t(2-q)},$$

  which is decreasing in q as

$$\begin{aligned}\frac{\partial U_S}{\partial q} &= \frac{-\mu(1-t)t(2-q) - t[t(1-\mu) - \mu(1-t)(1-q)]}{t^2(2-q)^2} \\ &= \frac{-\mu(1-t) - t(1-\mu)}{t(2-q)^2} \\ &< 0.\end{aligned}$$

The Receiver's expected payoff is $t \cdot \Pr(a = w = 0) + (1-t) \cdot \Pr(a = w = 1)$. Again, there are two cases.

- If $q \leq \overline{q}$, then the Receiver matches the state $w = 0$ correctly if $(w = 0, m = 0)$ or if $(w = 0, m = 1,\ d = lie)$, and matches the state $w = 1$ correctly if $w = 1$. In sum,

$$\begin{aligned} U_R &= (1-\mu)t \cdot [p_0^* + (1-p_0^*)q] + \mu(1-t) \\ &= (1-\mu)t \cdot [1 - (1-p_0^*)(1-q)] + \mu(1-t) \\ &= (1-\mu)t \cdot \left[1 - \frac{\mu(1-t)}{t(1-\mu)}\right] + \mu(1-t) \\ &= (1-\mu)t, \end{aligned}$$

  which is constant in q.

- If $q > \overline{q}$, then the Receiver matches the state $w = 0$ correctly if $(w = 0, m = 1,\ d = lie)$, and matches the state $w = 1$ correctly if $w = 1$. In sum,

$$\begin{aligned} U_R &= (1-\mu)t \cdot (1-p_0^*)q + \mu(1-t) \\ &= (1-\mu)t \cdot \frac{1-(1-q)\frac{\mu(1-t)}{t(1-\mu)}}{2-q} + \mu(1-t) \\ &= \frac{(1-\mu)t + t(1-\mu)}{2-q}, \end{aligned}$$

  which is increasing in q.

3.A.4 Proof of Lemma 3.1

In Step 1, we characterize properties of the Sender's preferred equilibria and show that in any Sender's preferred equilibrium, the Receiver always takes $a = 1$ under state 1 and λ. In Step 2, we decompose the payoff functions $U^S(q)$ and link it to $U^R(q)$.

Step 1:

Let the Sender's strategy be represented by $\boldsymbol{a} = \{a_{ij}\}_{i,j \in \{0,\lambda,1\}}$, where a_{ij} is the probability of sending message j under state i. Let X be the set of pairs $(m,\, d)$

where $(m, d) \in \{0,\, \lambda,\, 1\} \times \{lie,\, \neg lie\}$. Let $\mu_{m,d}$ denote the posterior mean after observing $(m, d) \in \{0,\, \lambda,\, 1\} \times \{lie,\, \neg lie\}$. The formulas of the posterior means are given by[9]

$$\begin{aligned}
\mu_{1,\,lie} &= \frac{q \cdot \lambda P_\lambda a_{\lambda 1}}{q \cdot (P_\lambda a_{\lambda 1} + P_0 a_{01})} \\
\mu_{1,\,\neg lie} &= \frac{P_1 a_{11} + \lambda P_\lambda a_{\lambda 1}(1-q)}{P_1 a_{11} + P_\lambda a_{\lambda 1}(1-q) + P_0 a_{01}(1-q)} \\
\mu_{\lambda,\,lie} &= \frac{q \cdot P_1 a_{1\lambda}}{q \cdot (P_1 a_{1\lambda} + P_0 a_{0\lambda})} \\
\mu_{1,\,\neg lie} &= \frac{\lambda P_\lambda a_{\lambda\lambda} + P_1 a_{1\lambda}(1-q)}{P_\lambda a_{\lambda\lambda} + P_1 a_{1\lambda}(1-q) + P_0 a_{0\lambda}(1-q)} \\
\mu_{0,\,lie} &= \frac{q \cdot (P_1 a_{10} + \lambda P_\lambda a_{\lambda 0})}{q \cdot (P_1 a_{10} + P_\lambda a_{\lambda 0})} \\
\mu_{0,\,\neg lie} &= \frac{(P_1 a_{10} + \lambda P_\lambda a_{\lambda 0})(1-q)}{P_0 a_{00} + (P_1 a_{10} + P_\lambda a_{\lambda 0})(1-q)}
\end{aligned}$$

Moreover, let $num(x) = \mu_x \cdot \Pr(x)$ to be the numerator of μ_x. Denote $X_1 = \{(m,\, d) \in X \,|\, \mu_{m,d} \geq t\}$ as the set of message-detection pairs under which the Receiver takes action $a = 1$. An observation is that the sum of six numerators equal μ and the sum of six denominators equal 1. Thus, for any strategy $\boldsymbol{a}$, the Sender's payoff is equal to

$$\sum_{x \in X_1} \Pr(x) \leq \frac{\sum_{x \in X_1} \mu_x \cdot \Pr(x)}{t} \leq \frac{\mu}{t} \tag{1.A.1}$$

We know from Equation 3.2 that $\frac{\mu}{t}$ is exactly the Sender's optimal payoff in this case. Thus, it suffices to find conditions on $\boldsymbol{a}$ such that both equalities in Equation 1.A.1 are attained. The first equality requires that $\forall x \in X_1$, the Receiver is indifferent after observing $x:\ \mu_x = t$. This immediately implies $a_{10} = a_{\lambda 0} = 0$ because otherwise $\mu_{0,\,lie} > t$ by assumption $t < \lambda$. Next, we consider two cases.

[9] In particular, $\frac{0}{0} \equiv 0$.

If $a_{1\lambda} > 0$, then the second equality requires $(\lambda,\, lie) \in X_1$. Otherwise,

$$\sum_{x \in X_1} num(x) \leq \mu - q \cdot P_1 a_{1\lambda} < \mu.$$

Analogously, $(\lambda,\, \neg lie) \in X_1$. However, if $(\lambda,\, lie), (\lambda,\, \neg lie) \in X_1$, then by the implication of the first equality, it must be that $a_{\lambda\lambda} = 0$ and thus $a_{\lambda 1} = 1$. Repeat the arguments, the second equality requires $(1,\, lie), (1,\, \neg lie) \in X_1$, and the first inequality requires $a_{11} = 0$ and thus $a_{1\lambda} = 1$. In summary, the Sender always sends message 1 under state λ and sends message λ under state 1. Moreover, $(\lambda,\, \neg lie)$, $(\lambda,\, lie)\, (1,\, \neg lie)$, $(1,\, lie)$ all induce action $a = 1$. Thus, $\Pr(a = 1,\, \omega = 1) = \Pr(a = 1,\, \omega = \lambda) = 1$.

In the second case, suppose $a_{1\lambda} = 0$, which implies $a_{11} = 1$. By the second equality, $(1,\, \neg lie) \in X_1$. Now, $a_{\lambda 1}$ must be 0. Otherwise, the second equality also requires $(1,\, lie) \in X_1$. However, then the first equality is violated as $t \leq \mu_{1,\neg lie} < \mu_{1,lie}$. In summary, the Sender is totally truthful under state 1 and λ. Moreover, $(\lambda,\, \neg lie)$, $(1,\, \neg lie)$ both induce action $a = 1$. Again, $\Pr(a = 1,\, \omega = 1) = \Pr(a = 1,\, \omega = \lambda) = 1$.

Step 2:

Note that the Sender's equilibrium payoff can be decomposed in the following way.

$$\begin{aligned} U^S(q) &= \Pr(a = 1) \\ &= P_1 \cdot \Pr(a = 1,\, \omega = 1) + P_\lambda \cdot \Pr(a = 1,\, \omega = \lambda) + P_0 \cdot \Pr(a = 1,\, \omega = 0). \end{aligned}$$

Step 1 implies $\Pr(a = 1,\, \omega = 1) = \Pr(a = 1,\, \omega = \lambda) = 1$ so that

$$U^S(q) = P_1 + P_\lambda + P_0 \cdot [1 - \Pr(a = 0,\, \omega = 0)].$$

At the same time, the Receiver's expected payoff defined in Equation 3.1 is reduced to

$$\begin{aligned} U^R(q) &= (1-t)P_1 + (\lambda - t)P_\lambda + tP_0 \cdot \Pr(a=0,\, \omega=0) \\ &= (1-t)P_1 + (\lambda - t)P_\lambda + t[1 - U^S(q)]. \end{aligned}$$

which concludes the proof.

Chapter 4

Contracts for Experimentation with Non-Common Priors

4.1 Introduction

Without experimentation, there is no innovation. Starting a profitable business or developing an invention idea typically involves considerable ex ante experimentation because the profitability of a project or the feasibility of an idea is initially uncertain. However, experimentation is often costly and thus relies on external incentives. For example, venture capitalists incentivize entrepreneurs to experiment on their start-ups. Managers in the R&D department motivate employees to experiment on designing new products. Professors induce students to experiment on different research ideas.

In such a principal-agent framework, two types of friction are likely to arise. First, if the principal cannot monitor the agent's behaviors, the experimentation is subject to moral hazard. For example, venture capitalists may not know whether entrepreneurs misuse financial funding. Similarly, managers (professors) may not know whether workers (students) are shirking. Second, the two parties may also disagree on how promising the project or the idea is. While the common-prior assumption has been accepted as a standard assumption in the literature due to its nice learning foundation, Acemoglu et al. (2016) provide a critique that shakes this foundation. They find that if two Bayesian agents have different priors and receive the same sequence of signals, then they asymptotically disagree even if there is an arbitrarily small uncertainty in interpreting the signals. Moreover, disagreement is the rule rather than the exception in real life. Entrepreneurs are well acknowledged to exhibit overconfidence (Cooper et al., 1988), probably more than venture capitalists. [1]

This chapter studies the following questions. How should the principal incentivize experimentation through contracts subject to the two aforementioned types of friction? Given the optimal contract, is there overexperimentation or underexperimentation relative to the principal's first-best criterion? If there is distortion, which friction drives it?

To answer the set of questions, I follow the model of Halac et al. (2016) (henceforth HKL (2016)) with one principal (he), one agent (she), and one project. The project can be either good or bad. Each of the two parties has a prior over the quality, and they *agree to disagree* (Aumann, 1976). In each period, the agent chooses between working and shirking, in which working incurs a fixed cost but shirking does not. If the project is good and the agent exerts effort in a period, success occurs with a fixed probability. Otherwise, a failure is certain to occur. If success occurs in a period, the principal obtains a surplus, and there is no need to experiment further.

The principal provides incentives to experiment via contracts and has full commitment power. A contract includes a working schedule and a compensation scheme. The working schedule specifies when the agent works, and the compensation scheme specifies how much compensation the agent receives in each period in the working

[1] Landier and Thesmar (2009) provides three potential reasons for entrepreneurial optimism: the "above-average" effect, the planning fallacy, and selection. In addition to these reasons, entrepreneurial overconfidence may be related to the "endowment effect". An entrepreneur owning a start-up may deem it more valuable than venture capitalists.

schedule. Once the deadline (the last period in the working schedule) is reached, the contract ends, and the experimentation ceases. The principal can incentivize the agent through both bonuses that are conditional on success and wages that are conditional on effort. Note that wages are feasible only if moral hazard is absent. Due to different priors, the two parties have joint incentives to bet with each other. Thus, limited liability is imposed on the agent's side to protect her from being exploited infinitely.

The model departs from that in HKL (2016) in two aspects. First, the types of friction in the experimentation are distinct. HKL (2016) studies the interaction of adverse selection and moral hazard. In contrast, this chapter abstracts from adverse selection and introduces heterogeneous priors. Second, in HKL (2016), the principal can achieve the first best when adverse selection is absent, i.e., the only friction comes from moral hazard. In contrast, the agent in my model is protected by limited liability. Consequently, even if there is only moral hazard, the principal cannot achieve the first best.

The optimal contract is derived for any pair of priors, both with and without moral hazard. First, assume the effort is observable so that moral hazard is absent. If, additionally, the two parties agree on the prior, then the principal simply solves an optimal stopping problem. He induces the agent to keep working until he becomes sufficiently pessimistic about the project, after which the project is abandoned. An optimal and straightforward contract is a pure wage contract that compensates for the effort cost up to the optimal stopping time. However, the optimality of this contract is not robust to the common-prior assumption. Once the two parties hold different priors, the principal can be strictly better off by offering alternative contracts.

If the agent is more optimistic than the principal, the agent essentially has strong intrinsic incentives (higher confidence in the project). She is then willing to work with weak extrinsic incentives (lower payments) because the intrinsic incentives and extrinsic incentives are substitutes. Thus, the optimal contract uses bonuses exclusively because providing incentives is cheaper through bonuses than through wages. Moreover, the principal could reuse the saved money to sustain longer experimentation. In fact, the optimal length of experimentation is exactly the optimal stopping time evaluated according to the agent's prior and is longer than the optimal stopping time evaluated according to the principal's prior. Last, since the prior enters the principal's objective function linearly, the optimal contract is independent of the principal's prior.

On the other hand, if the agent is less optimistic than the principal, providing incentives is more expensive through bonuses than through wages. Thus, bonus contracts are suboptimal relative to wage contracts. Furthermore, the two parties disagree on the probability of reaching a fixed future period (without success in the past). In particular, the agent deems success less likely at all times and thus has a higher belief in reaching that future period. The disagreement over the event's likelihood gives the principal room to exploit the agent. In particular, instead of compensating for the effort cost in every period, he can concentrate all compensation on a late period. The rearrangement of wages is appealing to the principal because, from his perspective, success is quite likely to be generated before that period. In this case, the agent always works yet does not receive any compensation. However, the period can be set arbitrarily late, which leads to the non-existence of an optimal contract. Note that this type of contract crucially depends on the principal's commitment power. If the principal lacks commitment, the agent would not initially trust the principal.

When the effort is not observable so that moral hazard plays a role, the agent can only be incentivized through bonuses. For this reason, the non-compactness issue in the previous case disappears, and an optimal contract always exists. It varies continuously in the agent's prior belief and is independent of the principal's prior belief. The optimal bonus amount decreases in the agent's prior belief (over the good state) because a highly confident agent has sufficient intrinsic incentives to work and demands lower extrinsic incentives. As a result, the optimal length of experimentation is longer when the agent is more confident since the principal can sustain the same length at a cheaper cost. In addition, the non-observability of effort significantly reduces the optimal length of experimentation due to classical arguments of moral hazard. Essentially, the principal has to give away some rents to the agent, which forces the principal to cease the experimentation much earlier. Last, the principal's payoff is strictly increasing in the agent's prior belief, which echoes Gervais et al. (2011)'s claim that overconfident managers are more attractive to firms than their rational counterparts.

In principle, the two parties may also disagree over other model components, such as the agent's ability. An experimenter may deem her marginal contribution of effort higher or lower than the principal evaluates it to be. If the agent underestimates her ability, she initially views success as less likely. However, after a failure, she updates her belief about the project quality more slowly than the principal because she attributes the failure primarily to her low ability. This new feature implies that after sufficiently long periods of failure, the agent may view success as more probable than the principal. Intuitively, the agent appears underconfident initially and overconfident after some transition. In this context, the insight that the principal employs different instruments for different agents still applies. The principal uses wages exclusively before the transition and uses bonuses exclusively after the transition. Moreover, the relative confidence of the agent with respect to the principal grows along with the experimentation, which suggests that the agent prefers the bonus in the last period over any other type of compensation more than the principal. As a result, it is optimal for the principal to concentrate the bonus on the last period.

There has been an extensive literature on experimentation in the past decades.

The first wave of papers focuses on dynamic moral hazard, either with a single agent (Bergemann and Hege, 1998, 2005; Hörner and Samuelson, 2013) or a team comprising of multiple agents (Bolton and Harris, 1999; Keller et al., 2005; Keller and Rady, 2010; Bonatti and Hörner, 2011). Some recent papers (Gomes et al., 2016; Halac et al., 2016) emphasize on an alternative friction in the experimentation—adverse selection. Specifically, Gomes et al. (2016) consider uncertainty in projects' qualities, while Halac et al. (2016) consider uncertainty in agents' abilities. However, none of the papers in the literature have studies heterogeneous beliefs in an experimentation setting, which, as argued earlier, is another plausible and pervasive friction. Therefore, this chapter provides new understandings of how explicit disagreements shape the optimal contract for experimentation.

In addition, this chapter contributes to the burgeoning literature on behavioral contract theory.[2] The central theme of this literature is that if firms are aware of consumers' potential bias, they could design contracts or set prices in a way to exploit the bias. For instance, in DellaVigna and Malmendier (2004), partially naïve hyperbolic discounters are not aware of their time inconsistency issue. In response, firms set the price of investment goods below marginal cost while pricing leisure goods above marginal cost. In Gabaix and Laibson (2006), naïve consumers ignore add-on prices, and firms respond by choosing high add-on prices. In Grubb (2009), consumers believe that they can predict their consumption more precisely than they actually can, so firms respond by setting high marginal prices for high usage. Note that, while de la Rosa (2011) also studies moral hazard models with an overconfident agent, he considers only a static problem and thus cannot capture the learning component, the heart of experimentation.

Last, this chapter is distantly related to the literature in finance concerning contract designs with heterogeneous beliefs. Landier and Thesmar (2009) presents a

[2]See Koszegi (2014) for a comprehensive survey.

simple three-period model without moral hazard in which a cohort of entrepreneurs endowed with different beliefs self-select into different types of contracts. In Gervais et al. (2011), the agent has access to a private signal about the project but overestimates the precision of the signal. In contrast, the agent has no private information in my model, and the meaning of overconfidence is different.

The chapter is organized as follows. The model is presented in Section 4.2. Sections 4.3 and 4.4, respectively, examine the optimal contract with observable and nonobservable effort. Section 4.5 explores an extension in which the disagreement is over the agent's ability. Section 4.6 concludes. All omitted proofs are included in Appendix 4.A.

4.2 Model

Environment. A principal hires an agent to work on a project. The quality of the project is unknown and can be either good or bad. As the main departure from the literature, the two parties may disagree over the expected quality. Specifically, the principal's prior on the project being good is $\mu_0^P \in (0,1)$, while the agent's prior on the project being good is $\mu_0^A \in (0,1)$. If $\mu_0^A > \mu_0^P$, the agent is referred to as being overconfident; if $\mu_0^A < \mu_0^P$, the agent is referred to as being underconfident; if $\mu_0^A = \mu_0^P$, the agent is referred to as being unbiased.[3] Moreover, the difference in the belief is common knowledge. That is, they agree to disagree. For example, the entrepreneur may be more passionate and overconfident about a start-up than the venture capitalist, but they recognize the discrepancy and maintain their own beliefs.

Time is discrete: $t = 0, 1, \ldots$. Starting from period 1, the agent chooses an effort level $a_t \in \{0, 1\}$ in each period, where $a_t = 0$ (shirking) is costless and $a_t = 1$ (working) costs the agent $c > 0$. If the project is good and the agent exerts effort in period

[3]The results of the paper do not depend on whose prior is correct. However, for expositional convenience, assume that the principal's prior is correct and that the agent is biased.

t, then the project generates an observable success in that period with probability $\lambda \in (0,1)$. Otherwise, no success can be obtained in that period. In other words, both effort and high quality are necessary for success.

Without loss of generality, a success yields the principal a payoff of 1. [4] Once success is generated, the experimentation ends, and no further effort is required. To make the problem interesting, assume that both parties deem it efficient to experiment for at least one period.

Assumption 4.1. $\mu_0^P \lambda > c,\ \mu_0^A \lambda > c.$

Last, both parties are risk neutral, have quasi-linear preferences, share a common discount factor $\delta \in (0,1)$, and are expected-utility maximizers.

Contracts. The principal has full commitment power and offers a contract in period 0. Since the class of contracts depends on the observability of effort, I introduce the class of contracts for two cases separately.

If effort is observable, then the principal can condition payment on both outcomes and effort. A contract includes a finite set of working periods $\Gamma \subset \mathbb{N}_+$, a bonus scheme $\{b_t\}_{t\in\Gamma} \geq 0$, and a wage scheme $\{w_t\}_{t\in\Gamma} \geq 0$. A period $t \notin \Gamma$ is interpreted as a locked-out period in which the agent cannot work. For any $t \in \Gamma$, b_t is a bonus given to the agent conditional on a success in period t, and w_t is a wage given to the agent whenever the agents works in period t. Last, if the agent shirks in a period $t \in \Gamma$, the contract ends immediately, and the agent is fired. This "firing" continuation contract is independent of the history and is thus dropped from the definition of a contract for brevity. Formally, denote $\mathcal{C} = \{\left(\Gamma, \{b_t\}_{t\in\Gamma}, \{w_t\}_{t\in\Gamma}\right) \mid \Gamma \subset \mathbb{N}_+,\ |\Gamma| < \infty;\ b_t, w_t \geq 0,\ \forall t \in \Gamma\}$ as the set of contracts available to the principal.

At first sight, this class of contracts seems nonstandard or special in many aspects.

[4] For simplicity, assume that the agent does not value the success directly. However, this is not essential for the main results.

First, the principal cannot penalize the agent for a failure. Second, Γ is potentially nonconnected. Third, the continuation contract after shirking is too restrictive, as the principal may wish to continue the relationship with the agent even after detecting shirking. However, it turns out that the inability to use the penalty as an instrument is almost necessary for this problem. In addition, allowing for nonconnected Γ and disallowing richer continuation contracts is without loss of generality.

It is well known that if two people hold non-common beliefs, they are willing to engage in a bet with arbitrarily large stakes. Hence, if bonus and penalty are simultaneously allowed, then whenever the priors differ, the principal could obtain an infinite payoff by betting on the outcome in period 1. [5] To limit such Dutch book arguments, assume that penalty is not available to the principal and that all payments (including both bonus and wage) are nonnegative. Essentially, the agent has limited liability.[6]

An alternative class of contracts requires Γ to be connected but allows any individually rational continuation contract after shirking. Notice that the continuation contract has a recursive structure because in every continuation contract, one must specify the plan if the agent shirks again. The recursive structure makes it cumbersome to define this alternative class. However, it turns out that this alternative class is payoff equivalent to $\mathcal{C}$ so that one can bypass the recursive structure.

To see the equivalence, first note that firing the agent imposes the lowest continuation payoff for her among all individually rational continuation contracts. If the agent decides to work because she worries that shirking leads to a low (but positive) continuation payoff, she must also decide to work if she is fired after shirking. In addition, if the agent shirks in any period given a contract in the alternative class,

[5] For example, if the agent is more optimistic, consider the following contract: $T = 1, w_1 = 0, b_1 = b, l_1 = l$, and the agent is fired if $a_1 = 0$. The principal could gain infinitely (from his point of view) by choosing an arbitrarily large b and l that satisfy $\mu_0^A \lambda b \geq (1-\mu_0^A \lambda)l + c$ and $(1-\mu_0^P \lambda)l \geq \mu_0^P \lambda(b-1)$. Such a pair (b, l) exists as $\mu_0^A > \mu_0^P$.

[6] It is, of course, feasible to allow a positive payment to the agent even when failure occurs. However, it is never optimal to use this option in an optimal contract.

she receives no compensation in that period. Thus, the principal could alternatively lock out that period. The two observations suggest that for any contract in the alternative class, there exists a contract in $\mathcal{C}$ that replicates the agent's action and the compensation in each period, thereby generating the same payoff to the principal. Conversely, any contract in $\mathcal{C}$ can also be interpreted as a contract in the alternative class. To this end, the principal simply needs to fill the holes of Γ (locked-out periods) and choose the compensation and continuation contracts carefully in those periods. Specifically, let the compensation be 0 and the continuation contract be identical to the base contract. Then, the agent deliberately chooses not to work in the locked-out periods because otherwise, she has to bear the effort cost and risk terminating the project without any additional compensation. Thus, the outcome of any contract in $\mathcal{C}$ can be perfectly replicated by a contract in the alternative class.[7]

When effort is not observable, the principal can only condition the payment on the outcome so that wages are no longer feasible. In addition, there is no need to specify the continuation contract, as shirking is not detected. As a result, the principal just needs to choose the set of working periods Γ and the bonus scheme $\{b_t\}_{t\in\Gamma}$. Formally, denote $\mathcal{C}' = \{(\Gamma, \{b_t\}_{t\in\Gamma}) \mid \Gamma \subset \mathbb{N}_+, |\Gamma| < \infty; b_t \geq 0, \forall t \in \Gamma\}$ as the set of contracts available to the principal.

Payoffs. Again, there are two cases. If effort is observable, the game ends immediately whenever the agent shirks. Given a contract $\boldsymbol{C} = (\Gamma, \{b_t\}_{t\in\Gamma}, \{w_t\}_{t\in\Gamma}) \in \mathcal{C}$ and an arbitrary action profile $\boldsymbol{a} = (a_t)_{t\in\Gamma}$, let $\bar{t} = \min\{t \in \Gamma \mid a_t = 0\}$ be the first

[7] The "firing" continuation contract is admittedly stringent. However, it is effectively equivalent to another, softer continuation contract. Instead of terminating the contract, the principal could delay the old contract by one period. For example, if the agent accepts a contract with $\Gamma = \{1, 2, 3, 4, 5\}$ but shirks in period 3, then the continuation contract requires the agent to work in periods 4, 5, and 6, where the compensation in period t of the continuation contract corresponds to the compensation in period $t-1$ of the old contract, $t \in \{4, 5, 6\}$. Intuitively, if firing deters the agent from shirking, her payoff by not shirking, denoted by U, must be weakly positive. Thus, it must be that $U \geq \delta U$, which means that the agent also prefers to work if the principal imposes the above continuation contract.

shirking period. If the agent never shirks, then $\bar{t} = \max\{t | t \in \Gamma\} + 1$. The principal's expected discounted payoff at period 0 is given by

$$\Pi_0(\boldsymbol{C}, \boldsymbol{a}) = -(1-\mu_0^P) \sum_{t\in\Gamma,\, t<\bar{t}} \delta^t w_t + \mu_0^P \sum_{t\in\Gamma,\, t<\bar{t}} \delta^t \left[\prod_{s<t,\, s\in\Gamma} (1-\lambda) \right] [\lambda(1-b_t) - w_t] \,. \tag{4.1}$$

Equation (4.1) is interpreted as follow. The project is bad with probability $1-\mu_0^P$ from the principal's view, in which case no success can be generated. The principal simply pays wages up to the last working period, i.e., $\max\{t \in \Gamma \mid t < \bar{t}\}$. With probability $\mu_0^P \left[\prod_{s<t,\, s\in\Gamma} (1-\lambda) \right] \lambda$, the project is good, and a success is generated in period t. In this case, the principal obtains a payoff 1 from success and pays the corresponding bonus b_t. Moreover, even if the project is good, he still needs to pay the wage w_t whenever no success has occurred before period t.

Analogously, the agent's expected discounted payoff at period 0 is given by

$$U_0(\boldsymbol{C}, \boldsymbol{a}) = (1-\mu_0^A) \sum_{t\in\Gamma,\, t<\bar{t}} \delta^t (w_t - c) + \mu_0^A \sum_{t\in\Gamma,\, t<\bar{t}} \delta^t \left[\prod_{s<t,\, s\in\Gamma} (1-\lambda) \right] (\lambda b_t - c + w_t). \tag{4.2}$$

The interpretation is similar except that the agent evaluates the payoffs according to her own prior μ_0^A. In her view, the project is bad with probability $1-\mu_0^A$, in which case no success is generated and she receives the wage w_t up to the last working period. With probability $\mu_0^A \left[\prod_{s<t,\, s\in\Gamma} (1-\lambda) \right] \lambda$, a success is generated in period t, and she receives the bonus b_t. Moreover, even if the project is good, she receives wage w_t whenever no success has occurred before period t.

If effort is nonobservable, the wage disappears from the expressions of payoffs, and the contract does not end when the agent shirks. Therefore, given a contract $\boldsymbol{C} = (\Gamma, \{b_t\}_{t\in\Gamma}) \in \mathcal{C}'$ and an arbitrary action profile $\boldsymbol{a} = (a_t)_{t\in\Gamma}$, the two parties' expected discounted payoffs at period 0 are modified to

$$\tilde{\Pi}_0(\boldsymbol{C}, \boldsymbol{a}) = \mu_0^P \sum_{t\in\Gamma} \delta^t \left[\prod_{s<t,\, s\in\Gamma} (1 - a_s\lambda) \right] a_t\lambda(1 - b_t). \tag{4.3}$$

$$\tilde{U}_0(\boldsymbol{C}, \boldsymbol{a}) = -(1 - \mu_0^A) \sum_{t\in\Gamma} \delta^t a_t c + \mu_0^A \sum_{t\in\Gamma} \delta^t \left[\prod_{s<t,\, s\in\Gamma} (1 - a_s\lambda) \right] a_t(\lambda b_t - c). \tag{4.4}$$

Note that the action profile explicitly appears in the cost and success probability because it is not observable. The multiplicative form is a convenient trick to express the payoffs.

4.3 Optimal Contracts with Observable Effort

4.3.1 Common-Prior Benchmark

In the simplest case, both types of friction—disagreement and moral hazard—are absent. Then, it is well known that the problem is solved by an optimal stopping rule. Essentially, the principal asks the agent to experiment until either obtaining success or becoming sufficiently pessimistic after a sequence of failures. Formally, the principal chooses a stopping time T^* to maximize the expected value of the project, given by

$$\sum_{t=1}^{T} \delta^t \left[\mu_0^P (1 - \lambda)^{t-1} (\lambda - c) - (1 - \mu_0^P) c \right].$$

The expression in the bracket is strictly decreasing in t. Hence, the optimal termination date T^* is the latest date such that the expression in the bracket is positive. Ignoring integer issues, the following equation characterizes the closed-form solution of T^*.

$$(1 - \lambda)^{T^*-1} = \frac{1 - \mu_0^P}{\mu_0^P} \cdot \frac{c}{\lambda - c}. \tag{4.5}$$

Alternatively, define μ_t^P as the principal's posterior belief at the beginning of period t given that the agent has exerted effort in all past periods.[8]

$$\mu_t^P = \frac{\mu_0^P(1-\lambda)^{t-1}}{\mu_0^P(1-\lambda)^{t-1} + (1-\mu_0^P)}.$$

Then T^* is precisely the timing at which the principal finds it worthless to experiment for one more period.

$$T^* = \max_{t\geq 0}\left\{t : \mu_t^P \lambda \geq c\right\}. \tag{4.6}$$

This optimal stopping rule is implemented by a simple wage contract such that $\Gamma = \{1, .., T^*\}$, $b_t = 0$, $w_t = c$, $\forall t \in \Gamma$. This is consistent with the insight in the classical contract theory literature that outcomes are redundant when effort is observable. However, as shown in the next subsection, this insight relies heavily on the common-prior assumption. In the absence of this assumption, there exist alternative contracts that yield the principal a strictly higher payoff.

4.3.2 Exploitative Contracts under Non-Common Priors

In this section, I first formalize and simplify the principal's problem. Then, I illustrate the nature of optimal contracts through two simple examples. Last, I generalize the results in the examples by solving the principal's problem.

Note that each contract $\boldsymbol{C} = (\Gamma, \{b_t\}_{t\in\Gamma}, \{w_t\}_{t\in\Gamma}) \in \mathcal{C}$ induces an action profile $\boldsymbol{a}$. It is without loss to restrict attention to contracts under which the agent participates and works in all periods in Γ because the principal could have locked out the periods when the agent shirks. In other words, Γ can be interpreted as the set of periods in which the principal wishes the agent to work. Let $\mathbf{1}$ denote the action profile such that $a_t = 1$, $\forall t \in \Gamma$. Then, the principal maximizes his period-0 expected discounted payoff $\Pi_0(\boldsymbol{C}, \mathbf{1})$ subject to the agent's individual rationality constraints (henceforth

[8] $\mu_1^P = \mu_0^P$ because there is no learning in period 0.

IR constraints) at each $t \in \Gamma \cup \{0\}$ and the limited liability constraints that all bonuses and wages be nonnegative.[9]

$$\begin{aligned} \sup_{\boldsymbol{C} \in \mathcal{C}} \quad & \Pi_0(\boldsymbol{C}, \mathbf{1}) \\ s.t. \quad & U_0(\boldsymbol{C}, \mathbf{1}) \geq 0 \quad (IR_0) \\ & U_\tau(\boldsymbol{C}, \mathbf{1}) \geq 0, \quad \forall \tau \in \Gamma \quad (IR_\tau) \\ & b_\tau \geq 0,\ w_\tau \geq 0, \quad \forall \tau \in \Gamma \quad (LL) \end{aligned} \tag{$\blacklozenge$}$$

Here, $U_\tau(\boldsymbol{C}, \mathbf{1})$ represents the agent's expected discounted payoff at the beginning of period τ given that she has been following and will continue to follow the action profile $\mathbf{1}$. To be precise, $U_\tau(\boldsymbol{C}, \mathbf{1})$ equals

$$(1-\mu_\tau^A) \sum_{t \in \Gamma,\, t \geq \tau} \delta^{t-\tau}(w_t - c) + \mu_\tau^A \sum_{t \in \Gamma,\, t \geq \tau} \delta^{t-\tau} \left[\prod_{\tau \leq s < t,\, s \in \Gamma} (1-\lambda) \right] (\lambda b_t + w_t - c).$$

The solution to ($\blacklozenge$) depends on the direction of the agent's bias. The following two examples illustrate the underlying ideas.

Example 4.1. *(Overconfident Agent)* $\quad \mu_0^P = \frac{1}{2},\ \lambda = \frac{4}{5},\ c = \frac{1}{5},\ \mu_0^A = \frac{4}{5},\ \delta \in (0,1).$

In this example, since $\mu_1^P = \frac{1}{2} > \frac{c}{\lambda}$ and $\mu_2^P = \frac{\mu_0^P(1-\lambda)}{\mu_0^P(1-\lambda)+(1-\mu_0^P)} = \frac{1}{6} < \frac{c}{\lambda}$, it follows by Equation (4.6) that $T^* = 1$, i.e., the principal wishes the agent to experiment for only one period if their beliefs are aligned. One naïve contract that the principal could offer is the wage contract used in the benchmark: $\Gamma = \{1\}$, $b_t = 0$, $w_t = c$. Denote this contract by $\boldsymbol{C}_1$. It can be easily computed that the principal's payoff under $\boldsymbol{C}_1$ is given by

$$\Pi(\boldsymbol{C}_1) = \delta(\mu_0^P \lambda - c) = \frac{1}{5}\delta.$$

[9] Let $\underline{t} = \min\{t \,|\, t \in \Gamma\}$; then, the IR_0 constraint is essentially equivalent to the $IR_{\underline{t}}$ constraint up to discounting because neither effort is exerted nor compensation implemented between period 0 and (the beginning of) period $\underline{t}$. However, I include both for completeness.

Now, consider an alternative contract $\boldsymbol{C}_2$: $\Gamma = \{1\}$, $b_1 = \frac{5}{16}$, $w_1 = 0$. This contract is constructed such that all of the agent's IR constraints are binding. However, this contract yields the principal a higher payoff:

$$\Pi(\boldsymbol{C}_2) = \delta[\mu_0^P \lambda(1 - b_1)] = \frac{11}{40}\delta > \frac{1}{5}\delta = \Pi(\boldsymbol{C}_1).$$

To see the reason, note that in the benchmark case, the optimal contract is not unique. The wage contract $\boldsymbol{C}_1$ is one optimal contract, while a pure bonus contract $\tilde{\boldsymbol{C}}_2$: $\Gamma = \{1\}$, $b_1 = \frac{1}{2}$, $w_1 = 0$ is another one. Since the agent is overconfident now, $\tilde{\boldsymbol{C}}_2$ leaves some fictitious revenue to the agent because she believes that success occurs with a higher probability in period 1. As a result, the principal could lower the amount of the bonus from $\frac{1}{2}$ to $\frac{5}{16}$ to increase his own payoff. Intuitively, an overconfident agent has stronger intrinsic incentives, demanding weaker extrinsic incentives. The two instruments—wages and bonuses—are equally costly when the principal faces an unbiased agent, but the former are more costly than the latter if the principal faces an overconfident agent.

However, $\boldsymbol{C}_2$ is still not optimal. The gap in their priors not only induces the principal to use bonuses as the instrument but also affects the length of experimentation. Consider a third contract $\boldsymbol{C_3}:$ $\Gamma = \{1, 2\}$, $b_1 = \frac{5}{16}$, $b_2 = \frac{9}{16}$, $w_1 = w_2 = 0$. Again, it is easy to verify that all of the agent's IR constraints are binding. Given $\boldsymbol{C}_3$, the principal obtains an even higher payoff than $\Pi(\boldsymbol{C}_2)$.

$$\Pi(\boldsymbol{C}_3) = \mu_0^P \left[\delta\lambda(1 - b_1) + \delta^2\lambda(1 - \lambda)(1 - b_2)\right] = \frac{11}{40}\delta + \frac{7}{200}\delta^2 > \Pi(\boldsymbol{C}_2).$$

Table 4.1 summarizes the payoffs under the three contracts.

Why would the principal receive a higher payoff by inducing the agent to over-experiment relative to T^*? Recall that in the benchmark case, both parties think spending more time on the project is worthless after T^*. Therefore, the principal is

	Principal's payoff	Agent's payoff	Agent's payoff evaluated by μ_0^P
$\boldsymbol{C}_1$	$\frac{1}{5}\delta$	0	0
$\boldsymbol{C}_2$	$\frac{11}{40}\delta$	0	$-\frac{3}{40}\delta$
$\boldsymbol{C}_3$	$\frac{11}{40}\delta + \frac{7}{200}\delta^2$	0	$-\frac{3}{40}\delta - \frac{3}{40}\delta^2$

Table 4.1: Players' payoffs under contracts $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, $\boldsymbol{C}_3$.

willing to give the project to the agent for free even though the agent does not value it. However, if the agent is overconfident, then after T^*, the project is still potentially valuable to her. In this numerical example, the efficient length of experimentation evaluated according to the agent's prior equals 2. Thus, the agent would love to experiment for one more period if the principal transfers the project to her. As a response, the principal could further extract the fictitious revenue from the agent by extending the deadline of experimentation and setting b_2 appropriately such that the IR constraint is binding. Essentially, it is the agent's efficient length instead of the principal's that matters.

A natural question emerges here. Can the principal repeat this trick infinitely and keep extending the experimentation? The answer is no. Intuitively, if the agent also finds it worthless to keep experimenting beyond a deadline, then there is no way to profitably incentivize her to exert effort. Indeed, $\boldsymbol{C}_3$ is an optimal contract in this example. If the principal further extends the contract to period 3, the IR constraint in period 3 requires the bonus b_3 to be higher than 1, the principal's payoff from a success. Thus, it is never in the principal's interest to offer such a contract.

$$\mu_0^A(1-\lambda)^2(\lambda b_3 - c) \geq (1-\mu_0^A)c \Longrightarrow b_3 \geq \frac{39}{16} > 1.$$

Example 4.1 suggests that providing incentives through effort is more expensive than through outcome if the agent is overconfident. The converse is true if the agent is underconfident, in which case the principal ignores the outcomes and only pays conditional on effort. However, the wage contract used in the benchmark that compensates for the effort cost in each period is still suboptimal. In Example 4.2, I explain how the principal could achieve a higher payoff by concentrating the wage on a single period.

Example 4.2. *(Underconfident Agent)* $\mu_0^P = \frac{1}{2},\ \lambda = \frac{4}{5},\ c = \frac{1}{5},\ \mu_0^A = \frac{1}{4},\ \delta \in (0,1).$

The parameters in this case are the same as before except that the agent's prior is lower than the principal's prior. Therefore, the efficient length of experimentation evaluated according to the principal's prior is still 1. If the principal adopts a wage contract $\boldsymbol{C}_4$: $\Gamma = \{1\}\,,\, b_1 = 0,\, w_1 = \frac{1}{5}$, he receives the same payoff as in Example 4.1.

$$\Pi(\boldsymbol{C}_1') = \delta(\mu_0^P \lambda - c) = \frac{1}{5}\delta.$$

However, he is strictly better off by offering $\boldsymbol{C}_5$: $\Gamma = \{1, 2\}\,,\, b_1 = b_2 = 0,\, w_1 = 0,\, w_2 = c + \frac{c}{\delta[\mu_0^A(1-\lambda)+1-\mu_0^A]} = \frac{1}{5} + \frac{1}{4\delta}$. In this contract, the agent does not receive any payment if the project succeeds in the first period. The only way to be compensated is to work in period 1, hoping for a failure, work in period 2, and receive w_2. Again, all IR constraints are satisfied. In fact, since the payment is back-loaded, IR_2 is loose, and IR_1 is binding. Under this contract, the principal receives

$$\begin{aligned}\Pi(\boldsymbol{C}_5) &= \mu_0^P \left[\delta\lambda + \delta^2 (1-\lambda)\lambda\right] - \delta^2 \left[\mu_0^P (1-\lambda) + 1 - \mu_0^P\right] w_2 \\ &= \frac{1}{4}\delta - \frac{1}{25}\delta^2 > \Pi(\boldsymbol{C}_4),\ \forall \delta \in (0,1).\end{aligned}$$

How can the principal be better off by overexperimenting relative to both parties' efficient length? This contract works because the two parties disagree on the probability of reaching period 2. The agent thinks she is likely to reach period 2 and receive significant compensation. Thus, she is willing to sacrifice the compensation in period 1. However, from the principal's perspective, the project will succeed in period 1 with a high probability, in which case he will extract all the surplus without paying a single penny to the agent. While the social surplus is maximal with only one period of experimentation, the principal still benefits from extending the experimentation because the expected compensation is reduced, and the gain may outweigh the loss.

The story of exploitation does not end here. The principal could do better by offering the following contract $\boldsymbol{C}_6$: $\Gamma = \{1, N\}$, $b_1 = b_N = 0$, $w_1 = 0, w_N = c + \frac{c}{\delta^{N-1}[\mu_0^A(1-\lambda)+1-\mu_0^A]} = \frac{1}{5} + \frac{1}{4\delta^{N-1}}$. This contract bears much resemblance to $\boldsymbol{C}_5$, except that the second period of experimentation is postponed to an arbitrary period $N > 2$ and the wage w_N is adjusted to ensure that IR_1 constraint binds. One could similarly verify that both IR constraints hold. However, under this contract, the principal obtains an even higher payoff.

$$\begin{aligned}\Pi(\boldsymbol{C}_6) &= \mu_0^P \left[\delta\lambda + \delta^N (1-\lambda)\lambda\right] - \delta^N \left[\mu_0^P (1-\lambda) + 1 - \mu_0^P\right] w_N \\ &= \frac{1}{4}\delta - \frac{1}{25}\delta^N > \Pi(\boldsymbol{C}_5), \quad \forall \delta \in (0,1).\end{aligned}$$

Table 4.2 summarizes the payoffs under the three different contracts.

	Principal's payoff	Agent's payoff	Agent's payoff evaluated by μ_0^P
$\boldsymbol{C}_4$	$\frac{1}{5}\delta$	0	0
$\boldsymbol{C}_5$	$\frac{1}{4}\delta - \frac{1}{25}\delta^2$	0	$-\frac{1}{20}\delta$
$\boldsymbol{C}_6$	$\frac{1}{4}\delta - \frac{1}{25}\delta^N$	0	$-\frac{1}{20}\delta$

Table 4.2: Players' payoffs under contracts $\boldsymbol{C}_4$, $\boldsymbol{C}_5$, $\boldsymbol{C}_6$.

The improvement of $\boldsymbol{C}_6$ over $\boldsymbol{C}_5$ may be puzzling. After all, the two parties share the same discount factor, and it seems the trick of delaying compensation should not work. This result is best understood by interpreting the discount factor as a surviving probability. In this model, there are two types of "attrition": one due to discounting and the other due to success. However, they essentially play the same role. An underconfident agent underestimates the second type of attrition, so it is as if the agent perceives a higher surviving probability. Thus, the scenario is effectively one in which the agent's discount factor is higher than the principal's. Then, it is clear that the principal wants to postpone the payment. One could also understand this result by decomposing $\Pi(\boldsymbol{C}_6)$ as follows.

$$\Pi(\boldsymbol{C}_6) = \underbrace{\delta\left[\mu_0^P\lambda - c\frac{\mu_0^P(1-\lambda)+1-\mu_0^P}{\mu_0^A(1-\lambda)+1-\mu_0^A}\right]}_{\text{Gain in Period 1}} + \underbrace{\delta^N\left[\mu_0^P(1-\lambda)(\lambda-c)-(1-\mu_0^P)c\right]}_{\text{Loss in Period } N}.$$

The first term measures the principal's "essential" payoff within period 1 because it is independent of N. This term is less than $\mu_0^P\lambda - c$, the social surplus from experimenting in period 1, thus creating an additional gain relative to the benchmark. The second term measures the principal's "essential" payoff within period N, which is negative because it is socially suboptimal to experiment for one period by construction. Postponing the second period of experimentation reduces the loss while not affecting the gain. As a result, the principal finds it profitable to postpone N. By choosing an arbitrarily large N, the principal receives a payoff converging to $\frac{1}{4}\delta$. In fact, this is still not the best that the principal could obtain. However, the point is that no optimal contract may exist due to the possibility of postponing because the principal could always include one more period of experimentation but postpone it to an arbitrarily late period.

Example 4.1 and 4.2 illustrate some important features of the "optimal" contract under non-common priors. First, the principal prefers a pure bonus contract when the agent is overconfident and prefers a pure wage contract when the agent is underconfident. Second, if the agent is underconfident, the principal is better off by concentrating the wage on the last period. Third, the principal finds it profitable to overexperiment relative to the efficient length evaluated according to μ_0^P in either case. It turns out that all these features still hold in the general setting.

The principal's maximization problem (♦) is solved in two steps. Note that, it is natural to decompose the instruments into the working schedule Γ and the compensation scheme $(\{b_t\}_{t\in\Gamma}, \{w_t\}_{t\in\Gamma})$ so that the maximization problem can be solved sequentially. Fix any arbitrary working schedule $\Gamma = \{t_1, ..., t_N\}$ where $N \in \mathbb{N}_+$ and $t_1 < ... < t_N < \infty$; the principal first chooses the associated optimal compensation scheme $\left(\{b_{t_j}\}_{j=1}^N, \{w_{t_j}\}_{j=1}^N\right)$ that satisfies the IR and LL constraints. Next, he compares across different working schedules and selects the one that delivers the highest payoff.

As hinted in the two examples, most IR constraints are loose at the optimum. Motivated by this, we consider a relaxed problem in the first step where all the $\{IR_{t_j}\}_{j=1,...,N}$ constraints are ignored. Once the relaxed problem is solved, it suffices to verify that the solution satisfies those constraints. With the payoffs explicitly written out, the principal solves the following problem.

$$\max_{\{b_{t_j}\}_{j=1}^N, \{w_{t_j}\}_{j=1}^N} -(1-\mu_0^P)\sum_{j=1}^N \delta^{t_j} w_{t_j} + \mu_0^P \sum_{j=1}^N \delta^{t_j}(1-\lambda)^{j-1}\left[\lambda(1-b_{t_j}) - w_{t_j}\right]$$

$$s.t.\ (1-\mu_0^A)\sum_{j=i}^N \delta^{t_j}(w_{t_j} - c) + \mu_0^A \sum_{j=i}^N \delta^{t_j}(1-\lambda)^{j-1}\left(\lambda b_{t_j} - c + w_{t_j}\right) \geq 0 \quad (IR_0) \quad (\blacklozenge\blacklozenge)$$

$$b_{t_j} \geq 0,\ w_{t_j} \geq 0, \quad \forall j = 1, ..., N \quad (LL)$$

There are two immediate observations. First, at the optimal solution, IR_0 must bind. Otherwise, there exists a period at which either the bonus or the wage is strictly positive. Then, however, the principal could lower it and achieve a higher payoff. Second, this is a linear programming problem since both the objective function and the constraints are linear in the choice variables. Thus, some choice variables hit the corner solutions at the optimum. Lemma 4.1 provides a characterization for the variables that are equal to 0.

Lemma 4.1. *If $\mu_0^A > \mu_0^P$, then $w_{t_j} = 0,\ \forall j = 1, ..., N$. If $\mu_0^A < \mu_0^P$, then $b_{t_j} = 0,\ \forall j = 1, ..., N$.*

Proof. See Appendix 4.A.1.

This lemma formalizes the intuition that providing incentives through bonuses (wages) is cheaper if the agent is overconfident (underconfident). Essentially, for each player, we can define the marginal rate of substitution between b_{t_j} and w_{t_j}.

$$MRS^A_{b_{t_j},w_{t_j}} = \frac{\frac{\partial U_0}{\partial b_{t_j}}}{\frac{\partial U_0}{\partial w_{t_j}}} = \frac{\mu_0^A(1-\lambda)^{j-1}\lambda}{1-\mu_0^A+\mu_0^A(1-\lambda)^{j-1}},$$
$$MRS^P_{b_{t_j},w_{t_j}} = \frac{\frac{\partial \Pi_0}{\partial b_{t_j}}}{\frac{\partial \Pi_0}{\partial w_{t_j}}} = \frac{\mu_0^P(1-\lambda)^{j-1}\lambda}{1-\mu_0^P+\mu_0^P(1-\lambda)^{j-1}}.$$

For $i \in \{A, P\}$, $MRS^i_{b_{t_j},w_{t_j}}$ measures how much wage in period t_j the player i is willing to sacrifice to have one more unit of bonus in period t_j. Due to non-common priors, $MRS^P_{b_{t_j},w_{t_j}} \neq MRS^A_{b_{t_j},w_{t_j}}$. In particular, whoever is more confident perceives a higher marginal rate of substitution. Thus, if the agent is overconfident (underconfident), the agent's relative preference for bonuses (wages) is higher than the principal's. It follows that the principal prefers to use bonuses (wages) as an instrument. In short, when the agent is more optimistic (pessimistic), her intrinsic incentives are strong (weak), lowering (raising) the cost of extrinsic incentives through bonuses. The rest of derivation depends on the direction of the agent's bias and is divided into two parts.

Overconfident Agent

If $\mu_0^A > \mu_0^P$, the wage in every period is set to 0. When we substitute the binding IR_0 into the objective function, it turns out that the objective function is reduced to a constant.

$$\Pi_0 = \mu_0^P \sum_{j=1}^{N} \delta^{t_j} \left[(1-\lambda)^{j-1}(\lambda - c) - \frac{1-\mu_0^A}{\mu_0^A} c \right]. \tag{4.7}$$

This suggests that the optimal compensation scheme is nonunique. It is nonunique since the marginal rate of substitution between bonuses in two different periods is the same for both parties. Thus, any bonus scheme that satisfies the binding IR_0 constraint and all limited liability constraints solves the relaxed problem. Furthermore, any bonus scheme that satisfies the binding IR_0 constraint, all $\{IR_{t_j}\}_{j=1,...,N}$ constraints and all limited liability constraints solves the first-step maximization problem. Thus, the most natural candidate makes all IR constraints binding so that the agent is always indifferent between working and shirking, which implies

$$\mu_0^A(1-\lambda)^{j-1}(\lambda b_{t_j} - c) - (1-\mu_0^A)c = 0. \tag{4.8}$$

Moreover, this particular solution is robust in two different senses. First, if success is also allowed in the bad state, this bonus scheme is the unique optimal solution. Second, the arrangement of bonuses across periods is time consistent, meaning that the principal does not want to revise the contract as the experimentation process goes on. This suggests that even when the principal lacks commitment, this bonus scheme remains optimal.

Given the optimal compensation scheme associated with the working schedule, the principal maximizes Π_0 in Equation (4.7) over the working schedule Γ. Denote the expression in the bracket of Equation (4.7) by C_j. Then, the principal should include all periods such that $C_j \geq 0$ and exclude all periods such that $C_j < 0$. Observe that $C_1 = \mu_0^A\lambda - c > 0$ by Assumption 4.1, $C_\infty < 0$, and C_j strictly decreasing in j. Thus, the optimal working schedule turns out to be connected, and $\Gamma^* = \{1, ..., \max\{j : C_j \geq 0\}\}$. Proposition 4.1 characterizes the optimal contract when the agent is more optimistic than the principal. Denote the principal's payoff under this contract by Π_0^*.

Proposition 4.1. *Assume that effort is observable and that $\mu_0^P < \mu_0^A$; then, the following contract is an optimal contract:* $\Gamma = \{1, ..., T^*\}$, $b_t = \frac{1}{\lambda}\left[\frac{1-\mu_0^A}{\mu_0^A} \cdot \frac{c}{(1-\lambda)^{t-1}} + c\right]$, $w_t =$

$0,\ \forall t \in \Gamma,$ *where*

$$T^* = \max_{t \geq 0} \{t : \mu_0^A (1-\lambda)^{t-1}(\lambda - c) \geq (1-\mu_0^A)c\}.$$

Ignoring integer issues, the closed-form solution of T^* *is given by*

$$(1-\lambda)^{T^*-1} = \frac{1-\mu_0^A}{\mu_0^A} \cdot \frac{c}{\lambda - c}. \tag{4.9}$$

Proof. See Appendix 4.A.2.

Proposition 4.1 echoes Example 4.1 in three aspects. First, the optimal contract is a pure bonus contract, though the observability of effort is still important because it imposes off-equilibrium threats. Second, the agent overexperiments relative to the efficient length from the principal's view because the principal could additionally extract fictitious surplus from the agent. Third, when we compare Equations (4.5) and (4.9), the optimal length of experimentation is identical to the efficient length from the agent's view. Beyond this length, it is impossible to profitably incentivize the agent.

Moreover, the bonus is strictly increasing in t. Intuitively, after many periods of failure, the agent gradually loses her confidence and intrinsic incentives. Therefore, she demands higher extrinsic incentives and higher compensation. The bonus is also strictly decreasing in μ_0^A for an apparent reason. The wider the gap between the initial priors, the higher the intrinsic incentives that the agent has, and the lower the compensation that she demands. Last, the optimal contract is independent of the principal's prior. In fact, the principal could have sold the whole project at a price Π_0^* to the agent. By the IR constraint, the agent would like to buy it. However, such a contract is excluded by limited liability.

Underconfident Agent

If $\mu_0^A < \mu_0^P$, the bonus in every period is set to 0 by Lemma 4.1. Unlike in the previous case, the objective function is no longer a constant after the binding IR_0 constraint is substituted in. Since the two parties also disagree over the marginal rate of substitution between two wages in different periods, the arrangement of wages across periods plays a role. Lemma 4.2 implies that the principal prefers to concentrate all compensation onto the last period.

Lemma 4.2. *If $\mu_0^A < \mu_0^P$, then at the optimum, $w_{t_j} = 0$, $\forall j = 1, ..., N-1$, and*

$$w_{t_N} = \frac{\sum_{j=1}^{N} \delta^{t_j} \left[\mu_0^A (1-\lambda)^{j-1} + 1 - \mu_0^A\right]}{\delta^{t_N} \left[\mu_0^A (1-\lambda)^{N-1} + 1 - \mu_0^A\right]} c.$$

Proof. See Appendix 4.A.3.

Since the compensation is concentrated on the last period, all the $\{IR_{t_j}\}_{j=1,...,N}$ constraints are looser than IR_0 because as the experimentation goes on, the agent is more likely to receive the wage w_N and, at the same time, there is less discounting. Thus, it is indeed without loss to drop these constraints in the beginning. Lemma 4.2 solves the first step of the maximization problem. Substitute the optimal compensation scheme back into the objective function; the latter is simplified to

$$\Pi_0 = \sum_{j=1}^{N} \delta^{t_j} \left\{ \mu_0^P (1-\lambda)^{j-1} \lambda - \frac{P_N^P}{P_N^A} \left[\mu_0^A (1-\lambda)^{j-1} + 1 - \mu_0^A\right] c \right\},$$

where $P_N^i = \mu_0^i (1-\lambda)^{N-1} + 1 - \mu_0^i$ is the probability that $i \in \{P, A\}$ assigns to the event that the project reaches period t_N given that the agent has exerted effort for $N-1$ periods. Since the agent is more pessimistic, she assigns higher probability to this event, i.e., $P_N^P < P_N^A$. Thus, the expression in the brace, denoted as $D(N,j)$, is strictly decreasing in j. In addition, $D_{N,1} = \mu_0^P \lambda - \frac{P_N^P}{P_N^A} c > 0$ by Assumption 4.1, and

$D_{N,\infty} < 0$. However, it is incorrect to simply include all periods such that $D(N, j) \geq 0$ since $D_{N,j}$ is also indexed by N.

There are two effects of increasing N to $N+1$. On the one hand, an additional period of experimentation means one more period of payoff, which is either positive or negative depending on the sign of $D(N+1, N+1)$. On the other hand, the payoffs in the first t_N periods unambiguously increase because $D_{N,j}$ is increasing in N.[10] Therefore, if the second effect dominates the first effect, it is beneficial to experiment for one more period even though it may not be efficient. In fact, the first effect is eventually dominated by the second effect due to discounting. As a result, an optimal working schedule and optimal contract do not exist, as shown by Proposition 4.2.

Proposition 4.2. *Assume that effort is observable and that $\mu_0^P > \mu_0^A$; then, no optimal contract exists.*

Proof. See Appendix 4.A.4.

It can be shown that the principal's payoff is bounded above by

$$\bar{\Pi}_0 = \sum_{t=1}^{\tilde{T}} \delta^t \left\{ \mu_0^P (1-\lambda)^{t-1}\lambda - \frac{1-\mu_0^P}{1-\mu_0^A} \left[\mu_0^A (1-\lambda)^{t-1} + 1 - \mu_0^A \right] c \right\},$$

where

$$\tilde{T} = \max\{t : \mu_0^P (1-\lambda)^{t-1}\lambda - \frac{1-\mu_0^P}{1-\mu_0^A} \left[\mu_0^A (1-\lambda)^{t-1} + 1 - \mu_0^A \right] c \geq 0\}.$$

This payoff corresponds to the scenario where all the positive terms in Π_0 are maximized whereas all the negative terms in Π_0 are hypothetically dropped. While being nonachievable, this bound can be approximated by a sequence of contracts $\{\boldsymbol{C}^k\}_{k=1}^{\infty}$ such that the working schedule $\Gamma^k = \{1, ..., \tilde{T}\} \cup \{k\tilde{T}+1, ..., k\tilde{T}+k\}$. As k grows

[10] As N grows larger, the disagreement over the probability of reaching period N is larger.

large, the sum of the negative payoffs in the later k periods vanishes to 0 due to discounting.

4.4 Optimal Contracts with Nonobservable Effort

Let us turn to the scenario in which the principal cannot observe the agent's effort so that a contract includes only the working schedule Γ and the bonus scheme $\{b_\tau\}_{\tau\in\Gamma}$. Again, it is without loss to consider contracts that induce the agent to work in all periods $t \in \Gamma$ because the principal has the discretion to lock out periods in which the agent shirks. However, due to the nonobservability of effort, the principal also needs to make sure that the agent indeed prefers to follow the action profile $\mathbf{1}$. The incentive-compatibility constraints (henceforth IC constraints) capture these considerations. Formally, the principal solves the following problem, where the payoff function $\tilde{\Pi}_0$ and $\tilde{U}_0$ are defined in Equations (4.3) and (4.4).

$$\begin{aligned}
\max_{\boldsymbol{C}\in\mathcal{C}'} \quad & \tilde{\Pi}_0(\boldsymbol{C},\mathbf{1}) \\
s.t. \quad & \tilde{U}_0(\boldsymbol{C},\mathbf{1}) \geq 0 \quad (IR_0) \\
& \tilde{U}_\tau(\boldsymbol{C},\mathbf{1}) \geq 0,\ \tau\in\Gamma \quad (IR_\tau) \\
& \mathbf{1} \in \underset{\boldsymbol{a}\in\{0,1\}^\Gamma}{\arg\max}\ \tilde{U}_0(\boldsymbol{C},\boldsymbol{a}) \quad (IC) \\
& b_\tau \geq 0,\ \tau\in\Gamma \quad (LL)
\end{aligned} \tag{✪}$$

Similarly, this problem could be solved in a two-step fashion. Fix any arbitrary working schedule $\Gamma = \{t_1, ..., t_N\}$, where $N \in \mathbb{N}$ and $t_1 < ... < t_N < \infty$; the principal chooses the optimal bonus scheme $\{b_{t_j}\}_{j=1}^N$ that satisfies the IR constraints, IC constraint, and LL constraints. Next, he compares across different working schedules and selects the one that delivers the highest payoff.

Note that the additional IC constraint actually embeds $2^N - 1$ constraints because there are $2^N - 1$ ways of deviating. As usual, I consider a relaxed problem in the first step. In particular, all IR constraints are initially ignored, and the (global) IC constraint is replaced by a set of LIC (local incentive-compatibility) constraints. For $j = 1, ..., N$, LIC_j states that shirking in period t_j is not as good as always working. Denote $\mathbf{1}_{-j}$ as the action profile in which the agent works in all periods in Γ but t_j. Then, consider the following relaxed problem.

$$\begin{aligned} \max_{\boldsymbol{C} \in \mathcal{C}'} \quad & \tilde{\Pi}_0(\boldsymbol{C}, \mathbf{1}) \\ s.t. \quad & \tilde{U}_0(\boldsymbol{C}, \mathbf{1}) \geq U_0(\boldsymbol{C}, \mathbf{1}_{-j}), \ \forall j = 1, ..., N \quad (LIC) \\ & b_{t_j} \geq 0, \ \forall j = 1, ..., N \quad (LL) \end{aligned} \tag{✪✪}$$

By relaxing IC to LIC, we reduce the number of incentive-compatibility constraints from $2^N - 1$ to N, making the problem more manageable. In fact, it turns out all LIC constraints bind at the optimum. The solution to the relaxed problem (✪✪) is given by Lemma 4.3.

Lemma 4.3. *Fix $\Gamma = \{t_1, ..., t_N\}$, the following bonus scheme solves the relaxed problem* (✪✪).

$$\lambda b_{t_j} c = \frac{1}{(1-\lambda)^{j-1}} \frac{1-\mu_0^A}{\mu_0^A} c + \lambda \left[\frac{\delta^{t_{j+1}-t_j}}{(1-\lambda)^j} + ... + \frac{\delta^{t_N - t_j}}{(1-\lambda)^{N-1}} \right] \frac{1-\mu_0^A}{\mu_0^A} c, \quad j = 1, ..., N. \tag{4.10}$$

Proof. See Appendix 4.A.5.

Moreover, the optimal bonus scheme given by Equation (4.10) satisfies all the dropped IR constraints and the global IC constraint. As a result, it must also solve the original problem (✪).

Lemma 4.4. *Fix $\Gamma = \{t_1, ..., t_N\}$; the bonus scheme characterized by Equation* (4.10) solves the original problem (✪).

Proof. See Appendix 4.A.6.

Observe that if the second term of the right-hand side in Equation (4.10) is ignored, then the bonus derived here is identical to the bonus derived in Equation (4.8). Hence, the agent is paid a higher bonus when effort is not observable. This is not surprising. When effort is observable, the principal knows exactly whether a failure results from shirking or purely bad luck, and shirking always leads to 0 continuation payoff for the agent. Here, the principal is unsure what caused a failure, and thus shirking always leads to a positive continuation payoff for the agent. Therefore, the principal has to pay the agent a higher bonus to incentivize the agent to work. Roughly speaking, the agent should be compensated both for potential success in the current period and for a higher chance of succeeding in future periods had she shirked in the current period. At the last period N, the bonus derived here coincides with the one under observable effort because there is no longer any future period.

After the optimal bonus scheme is substituted into the objective function, the latter is simplified to

$$\tilde{\Pi}_0(\boldsymbol{C}, \mathbf{1}) = \mu_0^P \sum_{j=1}^{N} \delta^{t_j} \left[(\lambda - c)(1-\lambda)^{j-1} - \frac{(1-\mu_0^A)}{\mu_0^A} \frac{c}{(1-\lambda)^{j-1}} \right], \tag{4.11}$$

where the derivation involves tedious algebra and can be found in Appendix 4.A.7. Denote the term in the bracket as F_j. Clearly, the optimal Γ should include only periods such that $F_j \geq 0$. Observe that F_j is decreasing in j and that $F_1 > 0$ by Assumption 4.1; it follows that $\Gamma = \{1, ..., T^{**}\}$, where, ignoring integer issues, T^{**} solves

$$(1-\lambda)^{2T^{**}-2} = \frac{1-\mu_0^A}{\mu_0^A} \cdot \frac{c}{\lambda - c}. \tag{4.12}$$

The optimal length of experimentation T^{**} increases in the agent's prior μ_0^A, which is consistent with the results in Section 4.3.2. Regardless of the observability of effort, as the agent becomes optimistic, her minimally required extrinsic incentives decreases, and thus, the principal can sustain longer experimentation.

When the agent is overconfident or unbiased, the comparison between T^* in Equations (4.5) and T^{**} in Equation (4.9) suggests that the nonobservability of effort reduces the length of experimentation significantly. This is because the agent obtains a rent when effort is nonobservable so that it costs more to incentivize the agent for a fixed working schedule Γ. Therefore, although the agent still deems the project valuable at T^{**}, the principal cannot afford to continue the experimentation. When the agent is underconfident, Proposition 4.2 implies that there is no optimal contract. Nevertheless, the principal can approximate the supreme payoff by experimenting arbitrarily long. Thus, the reduction in the length of experimentation is even more significant. Intuitively, the nonobservability of effort makes it impossible to implement a severe punishment upon shirking and, at the same time, forces the principal to use a more expensive instrument.

Now, substitute Equation (4.12) into Equation (4.10); the optimal bonus scheme is further simplified. The derivation can be found in Appendix 4.A.7.

$$\lambda b_j - c = \left[\frac{1-\delta}{(1-\lambda)^{j-1}} + \frac{\delta}{(1-\lambda)^{N-1}} \right] \frac{1-\mu_0^A}{\mu_0^A} c. \tag{4.13}$$

The bonus b_j is strictly decreasing in μ_0^A because, again, extrinsic incentives and intrinsic incentives are substitutes. Moreover, it is strictly increasing in j, meaning that the bonus is back-loaded. This also makes sense since the agent's intrinsic incentives are undermined during the experimentation process if no success occurs. Hence, the agent demands a higher bonus as the time approaches the deadline. In summary, the optimal contract with nonobserved effort is given by the following proposition.

Proposition 4.3. *The optimal contract with nonobserved effort is* $(\{1, ..., T^{**}\},\ \{0\}_{t=1}^{T^{**}},\ \{b_t\}_{t=1}^{T^{**}})$ *, where* T^{**} *and* b_t*, respectively, are solved by Equations* (4.12) and (4.13).

Proof. Omitted.

Observe that both the optimal length of experimentation and the optimal bonus scheme are independent of the principal's prior μ_0^P. This is again consistent with Proposition 4.1. However, the independence here is stronger because it does not require the agent to be overconfident. When the agent is underconfident and effort is not observable, the principal cannot exploit the agent. In the extreme case where $\mu_0^A = 0$, the agent never works because a failure is inevitable in her view.

Last, note that the optimal contract for a less optimistic agent is also acceptable for a more optimistic agent yet the principal chooses a different contract for the more optimistic agent. Revealed preference implies that he must receive a higher payoff by doing so. Thus, the principal's payoff under the optimal contract increases in the agent's prior belief. This result provides a new rationale for why overconfidence flourishes in real life. We observe many overconfident entrepreneurs not just because this bias is pervasive among all entrepreneurs but also because venture capitalists deliberately select those who exhibit overconfidence.

4.5 Heterogeneous Beliefs over the Agent's Ability

This section explores the optimal contract when the two parties agree on the quality μ_0 but disagree over λ. Roughly, λ measures the agent's ability. Given a good project, the higher the agent's ability is, the more likely success occurs. In particular, denote the principal's perceived ability by λ^P and the agent's perceived ability by λ^A. For simplicity, restrict attention to the scenario with observable effort. As implied earlier, disagreement over ability is distinct from disagreement over quality. Suppose the agent underestimates her ability, i.e., $\lambda^A < \lambda^P$; then, initially, she is less optimistic than the principal that success will occur if she works for one period.

$$\mu_0\lambda^A < \mu_0\lambda^P.$$

However, this ranking is time variant. If success does not occur despite effort, the agent updates her belief about the project downwards, albeit more slowly than the principal. As a result, if there are sufficiently many periods of failure despite effort, the agent eventually becomes more confident in the project quality and can be more optimistic than the principal that success will occur if she works for one more period. The following inequality captures this feature.

$$\mu_{t+1}^A\lambda^A = \frac{\mu_0(1-\lambda^A)^t\lambda^A}{1-\mu_0+\mu_0(1-\lambda^A)^t\lambda^A} > \frac{\mu_0(1-\lambda^P)^t\lambda^P}{1-\mu_0+\mu_0(1-\lambda^P)^t\lambda^P} = \mu_{t+1}^P\lambda^P \quad \text{for sufficiently large } t.$$

In fact, there exists a threshold period $\hat{T}$ such that $\mu_t^A\lambda^A < \mu_t^P\lambda^P$ if and only if $t \leq T$.[11] Conversely, if the agent overestimates her ability, i.e., $\lambda^A > \lambda^P$, she is initially more optimistic but later more pessimistic than the principal that success will occur if she works for one more period. Analogously, there exists a threshold period $\hat{T}$ such that $\mu_t^A\lambda^A > \mu_t^P\lambda^P$ if and only if $t \leq T$. To make the problem interesting, in either case, let $\hat{T}$ be smaller than the efficient length according to both λ^P and λ^A. Alternatively, at $\hat{T}$, both parties would like to continue experimenting.

Assumption 4.2. $(1-\lambda^i)^{\hat{T}-1} > \frac{1-\mu_0}{\mu_0}\cdot\frac{c}{\lambda^i-c}$, *for* $i \in \{P, A\}$.

To stay as close as possible to the main body, consider the same class of contracts $\mathcal{C}$, where a typical contract $\boldsymbol{C} = \{\Gamma, \{b_t\}_{t\in\Gamma}, \{w_t\}_{t\in\Gamma}\}$. Without loss of generality, restrict attention to contracts that induce an "always-working" action profile $\boldsymbol{a} = \mathbf{1}$. Analogously, the principal's problem can be solved in two steps. The principal first

[11] For simplicity, suppose that there is no integer t such that $\mu_t^A\lambda^A = \mu_t^P\lambda^P$.

solves the optimal compensation scheme associated with an arbitrary $\Gamma = \{t_1, ..., t_N\}$ and then optimizes over Γ.

Assumption 4.2 guarantees that the principal never chooses $N \leq \hat{T}$ because both parties find it worthwhile to extend by one period even after $\hat{T}$ periods of failure. Without loss, let $N > \hat{T}$. Moreover, as in the main body, all $\{IR_\tau\}_{\tau\in\Gamma}$ constraints are dropped initially and verified later on. Formally, the principal solves

$$\max_{\{b_{t_j}\}_{j=1}^N, \{w_{t_j}\}_{j=1}^N} -(1-\mu_0)\sum_{j=1}^N \delta^{t_j} w_{t_j} + \mu_0 \sum_{j=1}^N \delta^{t_j}(1-\lambda^P)^{j-1}\left[\lambda^P(1-b_{t_j}) - w_{t_j}\right]$$

$$s.t.\ (1-\mu_0)\sum_{j=i}^N \delta^{t_j}(w_{t_j} - c) + \mu_0 \sum_{j=i}^N \delta^{t_j}(1-\lambda^A)^{j-1}\left(\lambda^A b_{t_j} - c + w_{t_j}\right) \geq 0 \quad (IR_0)$$

$$b_{t_j} \geq 0,\ w_{t_j} \geq 0, \quad \forall j = 1, ..., N \quad (LL) \qquad (\blacktriangle)$$

The problem is slightly more complicated than the problems in the main body because the objective function is not linear in λ^P nor is the IR_0 constraint linear in λ^A. However, fundamentally, it is still a linear programming problem since both the objective function and the IR constraint are linear in the choice variables. Thus, it can be solved by comparing the marginal rate of substitution between instruments across players. For simplicity, assume that $\lambda^A < \lambda^P$ in what follows. The following lemma characterizes the optimal compensation scheme associated with $\Gamma = \{t_1, ..., t_N\}$.

Lemma 4.5. *Assume that $\lambda^A < \lambda^P$; then, at the optimum, $w_{t_j} = 0$ for any $j = 1, ..., N$ and $b_{t_j} = 0$ for any $j = 1, ..., N-1$. Moreover, b_{t_N} is chosen such that IR_0 binds.*

Proof. See Appendix 4.A.8.

The intuition of Lemma 4.5 is decomposed into three steps. First, since $N > \hat{T}$, the agent appears underconfident in the first $\hat{T}$ experimentation periods and appears overconfident in the later $N - \hat{T}$ experimentation periods. Thus, the principal prefers wages in the first $\hat{T}$ experimentation periods and prefers bonuses in the later $N - \hat{T}$ experimentation periods. Second, in any of the first $\hat{T}$ experimentation periods, the principal prefers to postpone the wage by one more period to exploit the agent's free labor. This suggests that w_{t_j} is a more costly instrument than $w_{t_{j+1}}$ for any $j \leq \hat{T}$. As a result of the first two observations, wages are never used at any period. Last, the ratio of the agent's confidence in success over the principal's confidence in success is increasing over time and is maximized at the last period N. Therefore, it is suboptimal to pay any bonus in the first $N - 1$ periods because the agent prefers a bonus in period N over a bonus in any other period more than the principal does. When we put the three steps together, b_{t_N} is the least costly instrument for the principal. Thus, he concentrates all compensation on b_{t_N}.

Obviously, the optimal compensation scheme derived in Lemma 4.5 satisfies all IR constraints ignored earlier because the payment is concentrated on the last period. Thus, it must solve the original problem. What is the optimal working schedule in the second step? Unfortunately, the noncompactness issue emerges again. The principal could invite the agent to work for arbitrarily many periods and promise to give an unrealistically high bonus at an arbitrarily late period.

4.6 Conclusion

Disagreement is the rule rather than the exception in many economic circumstances. However, it has received insufficient attention in the literature. As a first attempt at addressing this gap, I focus on contracting problems in a simple experimentation model. Specifically, I study how the magnitude and direction of disagreement affect the shape of the optimal contract for experimentation and how those effects interact with moral hazard. In the future, it would be interesting to study disagreements in more general settings.

As shown in Section 4.3.2, a potential issue with this approach is the nonexistence of an optimal contract in some cases, particularly when the agent is underconfident. The underlying reason for this nonexistence is the fundamental difference between an overconfident agent and an underconfident agent. The former is susceptible to bets that offer a lucrative payoff today and a low payoff in the future. However, the individual rationality constraints set the lower bound of the future payoff to 0, preventing the agent from being infinitely exploited. In contrast, an underconfident agent is susceptible to bets that offer a lucrative payoff in the future and a low payoff today. The individual rationality constraints do not protect such an agent because looking forward, there is always a chance of earning the lucrative payment. To protect the underconfident agent, her payoff needs to be nonnegative, even if we look backward. However, such conditions are hard to interpret. Thus, exploring plausible approaches to eliminating this type of bets in models featuring non-common priors is an exciting and essential direction for future work.

4.A Omitted Proofs

4.A.1 Proof of Lemma 4.1

Suppose $\mu_0^A > \mu_0^P$ and $w_{t_i} > 0$ for some $i \in \{1, .., N\}$, then the principal could lower the w_{t_i} to 0 while increase b_{t_i} by $\frac{1-\mu_0^A+\mu_0^A(1-\lambda)^{i-1}}{\mu_0^A(1-\lambda)^{i-1}} \cdot w_{t_i}$. Such modification of payments would not affect the agent's IR constraint, but improves the principal's payoff because

$$\begin{aligned}
&-\frac{1-\mu_0^A+\mu_0^A(1-\lambda)^{i-1}}{\mu_0^A(1-\lambda)^{i-1}} \cdot w_{t_i} \cdot \delta^{t_i}\mu_0^P(1-\lambda)^{i-1} + \delta^{t_i}\left[1-\mu_0^P+\mu_0^P(1-\lambda)^{i-1}\right] \cdot w_{t_i} \\
=&\delta^{t_i} w_{t_i} \cdot \frac{\mu_0^P-\mu_0^A}{\mu_0^A(1-\lambda)^{i-1}} > 0.
\end{aligned}$$

Similarly, if $\mu_0^A < \mu_0^P$, and $b_{t_i} > 0$ for some $i \in \{1, .., N\}$, then the principal could lower b_{t_i} to 0 and increase w_{t_i} appropriately.

4.A.2 Proof of Proposition 4.1

Most of the proof is contained in the main body already. Here, I only derive Equation 4.7 and verify that the optimal bonus scheme given by Equation 4.8 makes all IR constraints binding. First, after substituting the IR_0 constraint into the objective function, the latter is reduced to

$$\begin{aligned}
&\mu_0^P \sum_{j=1}^{N} \delta^{t_j}(1-\lambda)^{j-1}\left[\lambda(1-b_{t_j})-w_{t_j}\right] \\
=&\mu_0^P \sum_{j=1}^{N} \delta^{t_j}(1-\lambda)^{j-1}(\lambda-c) - \mu_0^P \sum_{j=1}^{N} \delta^{t_j}(1-\lambda)^{j-1}\left(\lambda b_{t_j}-c\right) \\
=&\mu_0^P \sum_{j=1}^{N} \delta^{t_j}(1-\lambda)^{j-1}(\lambda-c) - \mu_0^P \sum_{j=1}^{N} \delta^{t_j}(1-\lambda)^{j-1}\frac{1-\mu_0^A}{\mu_0^A} \cdot \frac{c}{(1-\lambda)^{j-1}} \\
=&\frac{\mu_0^P}{\mu_0^A} \sum_{j=1}^{N} \delta^{t_j}\left[\mu_0^A(1-\lambda)^{j-1}(\lambda-c)-(1-\mu_0^A)c\right].
\end{aligned}$$

By Bayes rule, for any $j = 1, ..., N$, the IR_{t_j} constraint is equivalent to

$$(1-\mu_0^A)\sum_{j=i}^{N}\delta^{t_j}(w_{t_j}-c)+\mu_0^A\sum_{j=i}^{N}\delta^{t_j}(1-\lambda)^{j-1}(\lambda b_{t_j}-c+w_{t_j})\geq 0.$$

By $w_{t_j} = 0$, $j = 1, ..., N$ and Equation 4.8,

$$\begin{aligned}LHS &= (1-\mu_0^A)\sum_{j=i}^{N}\delta^{t_j}(-c)+\mu_0^A\sum_{j=i}^{N}\delta^{t_j}(1-\lambda)^{j-1}(\lambda b_{t_j}-c)\\ &= \sum_{j=i}^{N}\delta^{t_j}[\mu_0^A(1-\lambda)^{j-1}(\lambda b_{t_j}-c)-(1-\mu_0^A)c]\\ &= 0\end{aligned}$$

so that IR_{t_j} binds for $j = 1, ..., N$.

4.A.3 Proof of Lemma 4.2

Suppose $w_{t_j} > 0$ for some $j \in \{1, ..., N\}$, consider the following modification.

$$(w_{t_1},...,w_{t_j},...,w_{t_N}) \longrightarrow (w_{t_1},...,0,...,w_{t_N}+\frac{\mu_0^A(1-\lambda)^{j-1}+(1-\mu_0^A)}{\mu_0^A(1-\lambda)^{N-1}+(1-\mu_0^A)}\delta^{t_j-t_N}w_{t_j}).$$

This modification does not affect the IR_0 constraint, but improves the principal's payoff because

$$\begin{aligned}&\underbrace{\frac{\mu_0^A(1-\lambda)^{j-1}+(1-\mu_0^A)}{\mu_0^A(1-\lambda)^{N-1}+(1-\mu_0^A)}\delta^{t_j-t_N}w_{t_j}\cdot\delta^{t_N}\left[\mu_0^P(1-\lambda)^{N-1}+(1-\mu_0^P)\right]}_{\text{Loss from increasing } w_{t_N}}\\ &<\underbrace{\delta^{t_j}\left[\mu_0^P(1-\lambda)^{j-1}+(1-\mu_0^P)\right]w_{t_j}}_{\text{Gain from decreasing } w_{t_j}}\\ \Longleftrightarrow&\frac{\mu_0^A(1-\lambda)^{j-1}+(1-\mu_0^A)}{\mu_0^A(1-\lambda)^{N-1}+(1-\mu_0^A)}<\frac{\mu_0^P(1-\lambda)^{j-1}+(1-\mu_0^P)}{\mu_0^P(1-\lambda)^{N-1}+(1-\mu_0^P)}\\ \Longleftrightarrow&(\mu_0^A-\mu_0^P)[(1-\lambda)^{N-1}-(1-\lambda)^{j-1}]>0.\end{aligned}$$

Apply the same argument to $\forall j = 1, ..., N-1$, it follows that all wages are concentrated on the last period. By IR_0,

$$w_{t_N} = \frac{\sum_{j=1}^{N} \delta^{t_j} \left[\mu_0^A (1-\lambda)^{j-1} + 1 - \mu_0^A\right]}{\delta^{t_N} \left[\mu_0^A (1-\lambda)^{N-1} + 1 - \mu_0^A\right]} c.$$

4.A.4 Proof of Proposition 4.2

It suffices to show that for any finite $\Gamma = \{t_1, ..., t_N\}$, there exists a finite Γ' generating a higher payoff for the principal. Recall that $T^*(\mu_0^P) = \max_{t \geq 0} \{t : \ \mu_0^P (1-\lambda)^{t-1}(\lambda - c) \geq (1-\mu_0^P)c\}$, there are two possible cases.

(1) $N < T^*(\mu_0^P)$.

By definition of $T^*(\mu_0^P)$, $D_{N,N} = \mu_0^P (1-\lambda)^{N-1}\lambda - P_N^P \cdot c = \mu_0^P (1-\lambda)^{N-1}(\lambda - c) - (1-\mu_0^P)c > 0$. Consider $\Gamma' = (t_1, ..., t_N, 1 + t_N)$, which adds one more period after the last period in Γ. By the monotonicity property of $D(N, j)$ in j, $D_{N+1,j} > D_{N,j}$, $\forall j = 1, ..., N$. In addition, $N + 1 \leq T^*(\mu_0^P)$ implies $D_{N+1,N+1} \geq 0$. So, the principal is strictly better off because the two effects of increasing N are both positive.

(2) $N \geq T^*(\mu_0^P)$.

Now, the two effects are of different directions, but the negative effect could be made negligible. Consider $\Gamma' = (t_1, ..., t_N, M + t_N)$ for M large enough. The positive effect remains unchanged and is bounded away from 0. At the same time, the negative effect, $\delta^{M+t_N} D_{N+1,N+1} = \delta^{M+t_N} \left[\mu_0^P (1-\lambda)^N (\lambda - c) - (1-\mu_0^P)c\right]$ vanishes to 0 as $M \longrightarrow \infty$. Therefore, the principal is overall strictly better off.

4.A.5 Proof of Lemma 4.3

Consider the relaxed problem.

$$\max_{\boldsymbol{C}\in\mathcal{C}'} \quad \Pi_0(\boldsymbol{C},\boldsymbol{1})$$

$$s.t. \quad U_0(\boldsymbol{C},\boldsymbol{1}) \geq U_0(\boldsymbol{C},\boldsymbol{1}_{-j}),\ \forall j=1,...,N \quad (LIC)$$

$$b_{t_j} \geq 0,\ \forall j=1,...,N \quad (LL)$$

For $i=1,...,N$, the LIC_i constraint is given below.

$$-(1-\mu_0^A)\sum_{j=1}^{N}\delta^{t_j}c+\mu_0^A\sum_{j=1}^{N}\delta^{t_j}(1-\lambda)^{j-1}(\lambda b_{t_j}-c) \geq$$
$$-(1-\mu_0^A)\sum_{j\neq i}\delta^{t_j}c+\mu_0^A\left[\sum_{j=1}^{i-1}\delta^{t_j}(1-\lambda)^{j-1}(\lambda b_{t_j}-c)+\sum_{j=i+1}^{N}\delta^{t_j}(1-\lambda)^{j-2}(\lambda b_{t_j}-c)\right].$$

After simplifying the algebra, the above inequality is equivalent to:

$$\mu_0^A\delta^{t_i}(1-\lambda)^{i-1}(\lambda b_{t_i}-c)-(1-\mu_0^A)\delta^{t_i}c-\lambda\mu_0^A\sum_{j=i+1}^{N}\delta^{t_j}(1-\lambda)^{j-2}(\lambda b_{t_j}-c)\geq 0.$$

Now, I argue that all LIC constraints must bind at the optimum using induction from the last period. First consider the LIC_N:

$$\mu_0^A(1-\lambda)^{N-1}(\lambda b_{t_N}-c)-(1-\mu_0^A)c\geq 0.$$

If LIC_N is not binding, then b_{t_N} must be positive. the principal could lower b_{t_N} marginally so that LIC_N is still satisfied. By doing so, all the preceding LIC constraints are even looser. Intuitively, the agent has an intertemporal trade-off. A lower bonus in the future gives the agent more incentives to work today because otherwise he misses a chance to win a high bonus today. However, such a change increases the principal's payoff, thereby violating the optimality. So, LIC_N must bind at the optimum, implying that

$$\lambda b_{t_N} - c = \frac{1-\mu_0^A}{\mu_0^A}\frac{1}{(1-\lambda)^{N-1}}c. \tag{1.A.1}$$

Substitute Equation 1.A.1 into LIC_{N-1}, we obtain

$$\mu_0^A(1-\lambda)^{N-2}(\lambda b_{t_{N-1}} - c) - (1-\mu_0^A)c - (1-\mu_0^A)\frac{\lambda}{1-\lambda}\delta^{t_N - t_{N-1}}c \geq 0.$$

By a similar logic, LIC_{N-1} must also bind at the optimum. Otherwise reducing $b_{t_{N-1}}$ slightly yields a higher payoff to the principal without violating any of the constraints. The binding LIC_{N-1} further implies that:

$$\lambda b_{t_{N-1}} - c = \left(1 + \lambda\frac{\delta^{t_N - t_{N-1}}}{1-\lambda}\right)\frac{1-\mu_0^A}{\mu_0^A}\frac{1}{(1-\lambda)^{N-2}}c.$$

By induction, all LIC constraints bind at the optimum, which suggest that

$$\begin{aligned}\lambda b_{t_j} - c &= \left\{1 + \lambda\left[\frac{\delta^{t_{j+1}-t_j}}{1-\lambda} + ... + \frac{\delta^{t_N - t_j}}{(1-\lambda)^{N-j}}\right]\right\}\frac{1-\mu_0^A}{\mu_0^A}\frac{1}{(1-\lambda)^{j-1}}c \\ &= \left\{\frac{1}{(1-\lambda)^{j-1}} + \lambda\left[\frac{\delta^{t_{j+1}-t_j}}{(1-\lambda)^j} + ... + \frac{\delta^{t_N - t_j}}{(1-\lambda)^{N-1}}\right]\right\}\frac{1-\mu_0^A}{\mu_0^A}c.\end{aligned}$$

4.A.6 Proof of Lemma 4.4

It suffices to verify the solution to the relaxed problem is feasible given original constraints. First, all IR constraints are satisfied because[12]

$$\begin{aligned}\tilde{U}_{t_i}(\boldsymbol{C}, \mathbf{1}) &\propto -(1-\mu_0^A)\sum_{j=i+1}^{N}\delta^{t_j}c + \mu_0^A\sum_{j=i+1}^{N}\delta^{t_j}(1-\lambda)^{j-1}(\lambda b_{t_j} - c) \\ &= (1-\mu_0^A)\sum_{j=i+1}^{N}\delta^{t_j}\Big\{(1+\lambda[\frac{\delta^{t_{j+1}-t_j}}{1-\lambda} + ... + \frac{\delta^{t_N - t_j}}{(1-\lambda)^{N-j}}])c - c\Big\} \\ &> 0.\end{aligned}$$

[12] IR_0 is equivalent to IR_{t_1}.

Next, let us verify the global IC. Note, LIC states that the agent prefers $\mathbf{1}$ over $\mathbf{1}_{-j}$ for any $j = 1, ..., N$, or compactly $\mathbf{1} \succ \mathbf{1}_{-j}$. In contrast, the global IC requires $\mathbf{1} \succ (a_1, ..., a_N), \forall a_1, ..., a_N \in \{0, 1\}$. The next claim links the two types of constraints.

Claim 4.1. *For* $\forall i = 1, ...N$, *and* $\forall a_1, ..., a_{i-1} \in \{0, 1\}$,

$$(a_1, ..., a_{i-1}, 1, ..., 1) \succ (a_1, ..., a_{i-1}, 0, 1, ..., 1).$$

Proof. Let $\bar{\boldsymbol{a}} = (a_1, ..., a_{i-1}, 1, ..., 1)$, $\underline{\boldsymbol{a}} = (a_1, ..., a_{i-1}, 0, 1, ..., 1)$, then decompose $\tilde{U}_0(\boldsymbol{C}, \bar{\boldsymbol{a}})$ and $\tilde{U}_0(\boldsymbol{C}, \underline{\boldsymbol{a}})$ into three parts respectively.

$$\begin{aligned}\tilde{U}_0(\boldsymbol{C}, \bar{\boldsymbol{a}}) = {} & \mu_0^A \sum_{j=1}^{i-1} \delta^{t_j} (1-\lambda)^{|\{s:\bar{a}_s=1, s<j\}|} (\lambda b_{t_j} - c) a_j - (1-\mu_0^A) \sum_{j=1}^{i-1} \delta^{t_j} a_j c \\ & + [\mu_0^A \delta^{t_i} (1-\lambda)^{|\{s:\bar{a}_s=1, s<i\}|} (\lambda b_{t_i} - c) - (1-\mu_0^A) \delta^{t_i} c \\ & + \mu_0^A \sum_{j=i+1}^{N} \delta^{t_j} (1-\lambda)^{|\{s:\bar{a}_s=1, s<j\}|} (\lambda b_{t_j} - c) - (1-\mu_0^A) \sum_{j=i+1}^{N} \delta^{t_j} c.\end{aligned}$$

$$\begin{aligned}\tilde{U}_0(\boldsymbol{C}, \underline{\boldsymbol{a}}) = {} & \mu_0^A \sum_{j=1}^{i-1} \delta^{t_j} (1-\lambda)^{|\{s:\underline{a}_s=1, s<j\}|} (\lambda b_{t_j} - c) a_j - (1-\mu_0^A) \sum_{j=1}^{i-1} \delta^{t_j} a_j c \\ & + 0 \\ & + \mu_0^A \sum_{j=i+1}^{N} \delta^{t_j} (1-\lambda)^{|\{s:\underline{a}_s=1, s<j\}|} (\lambda b_{t_j} - c) - (1-\mu_0^A) \sum_{j=i+1}^{N} \delta^{t_j} c.\end{aligned}$$

Since $|\{s : \bar{a}_s = 1, s < i\}| = |\{s : \underline{a}_s = 1, s < i\}|$ for $j = 1, ..., i$, the first part of $\tilde{U}_0(\boldsymbol{C}, \bar{\boldsymbol{a}})$ and $\tilde{U}_0(\boldsymbol{C}, \underline{\boldsymbol{a}})$ coincide. The second part of $\tilde{U}_0(\boldsymbol{C}, \underline{\boldsymbol{a}})$ is 0 because the agent shirks in period i under $\underline{\boldsymbol{a}}$. For $j = i+1, ..., N$, $|\{s : \bar{a}_s = 1, s < i\}| = 1 + |\{s : \underline{a}_s =$

$1, s < i\}|$ because the two action profiles only differ at the i^{th} coordinate. Therefore,

$$\begin{aligned}\tilde{U}_0(\boldsymbol{C}, \bar{\boldsymbol{a}}) - \tilde{U}_0(\boldsymbol{C}, \underline{\boldsymbol{a}}) =& \mu_0^A \delta^{t_i} (1-\lambda)^{|\{s:\bar{a}_s=1, s<i\}|}(\lambda b_{t_i} - c) - (1-\mu_0^A)\delta^{t_i} c \\ &+ \left(1 - \frac{1}{1-\lambda}\right)\mu_0^A \sum_{j=i+1}^{N} \delta^{t_j}(1-\lambda)^{|\{s:\bar{a}_s=1,s<j\}|}(\lambda b_{t_j} - c)\end{aligned}$$

Let $m = |\{s : \bar{a}_s = 1, s < i\}|$, then $|\{s : \bar{a}_s = 1, s < j\}| = m + j - i$ for $j = i+1, ..., N$. Multiply the above equation by $(1-\lambda)^{i-m-1}$ from both sides, we obtain

$$\begin{aligned}&(1-\lambda)^{i-m-1}\left[\tilde{U}_0(\boldsymbol{C}, \bar{\boldsymbol{a}}) - \tilde{U}_0(\boldsymbol{C}, \underline{\boldsymbol{a}})\right] \\ =&\mu_0^A \delta^{t_i}(1-\lambda)^{i-1}(\lambda b_{t_i} - c) - (1-\mu_0^A)\delta^{t_i} c (1-\lambda)^{i-m-1} \\ &- \frac{\lambda}{1-\lambda}\mu_0^A \sum_{j=i+1}^{N} \delta^{t_j}(1-\lambda)^{j-1}(\lambda b_{t_j} - c) \\ >&\mu_0^A \delta^{t_i}(1-\lambda)^{i-1}(\lambda b_{t_i} - c) - (1-\mu_0^A)\delta^{t_i} c - \lambda\mu_0^A \sum_{j=i+1}^{N} \delta^{t_j}(1-\lambda)^{j-2}(\lambda b_{t_j} - c) \\ =&0.\end{aligned}$$

This implies that the agent prefers $\bar{\boldsymbol{a}}$ over $\underline{\boldsymbol{a}}$. □

Now, the global IC is ensured by repeatedly applying the claim.

$$\mathbf{1} \succ (a_1, 1, ..., 1) \succ (a_1, a_2, 1, ..., 1) \succ ... \succ (a_1, ..., a_N), \forall a_1, ..., a_N \in \{0, 1\}.$$

In other words, an arbitrary action profile with some shirking periods could be improved by switching shirking to working period by period in a backward order.

4.A.7 Derivations of Equation 4.11 and 4.13

The principal's objective in the second step is given by

$$
\begin{aligned}
\Pi_0(\boldsymbol{C}, \mathbf{1}) &= \mu_0^P \sum_{j=1}^{N} \delta^{t_j} (1-\lambda)^{j-1} \lambda (1-b_t) \\
&= \mu_0^P \sum_{j=1}^{N} \delta^{t_j} (1-\lambda)^{j-1} [\lambda - c - (\lambda b_t - c)] \\
&= \mu_0^P \sum_{j=1}^{N} \delta^{t_j} (1-\lambda)^{j-1} (\lambda - c) \\
&- \frac{\mu_0^P (1-\mu_0^A)}{\mu_0^A} c \sum_{j=1}^{N} \delta^{t_j} \left\{ 1 + \lambda \left[\frac{\delta^{t_{j+1}-t_j}}{1-\lambda} + ... + \frac{\delta^{t_N - t_j}}{(1-\lambda)^{N-j}} \right] \right\} \\
&= \mu_0^P \sum_{j=1}^{N} \delta^{t_j} (1-\lambda)^{j-1} (\lambda - c) - \frac{\mu_0^P (1-\mu_0^A)}{\mu_0^A} \sum_{j=1}^{N} \frac{\delta^{t_j}}{(1-\lambda)^{j-1}} c \\
&= \mu_0^P \sum_{j=1}^{N} \delta^{t_j} \left[(\lambda - c)(1-\lambda)^{j-1} - \frac{(1-\mu_0^A)}{\mu_0^A} \frac{c}{(1-\lambda)^{j-1}} \right].
\end{aligned}
$$

Where the fourth equality involves a double summation. Equation 4.13 comes from the fact that

$$
\begin{aligned}
& \frac{1}{(1-\lambda)^{j-1}} + \lambda \left[\frac{\delta^{t_{j+1}-t_j}}{(1-\lambda)^j} + ... + \frac{\delta^{t_N - t_j}}{(1-\lambda)^{N-1}} \right] \\
=& \frac{1}{(1-\lambda)^{j-1}} \left[1 + \delta \lambda \frac{\frac{1}{(1-\lambda)^{N-j}} - 1}{\frac{1}{1-\lambda} - 1} \right] \\
=& \frac{1}{(1-\lambda)^{j-1}} \left[1 + \delta \left(\frac{1}{(1-\lambda)^{N-j}} - 1 \right) \right] \\
=& \frac{1-\delta}{(1-\lambda)^{j-1}} + \frac{\delta}{(1-\lambda)^{N-1}}.
\end{aligned}
$$

4.A.8 Proof of Lemma 4.5

It suffices to compare $MRS_{i,k}^A$ with $MRS_{i,k}^P$, where $i, k \in \{b_{t_1}, ..., b_{t_N}, w_{t_1}, ..., w_{t_N}\}$. If $MRS_{i,k}^A > MRS_{i,k}^P$ for some i, k, then the instrument k is not used in the optimal compensation scheme because the agent prefers i over k more than the principal does.

The following claim summarizes the comparison between different instruments.

Claim 4.2. *The following statements are true.*

(a) For $j \leq \hat{T}$, $MRS^A_{w_{t_{j+1}},w_{t_j}} > MRS^P_{w_{t_{j+1}},w_{t_j}}$.

(b) For $j > \hat{T}$, $MRS^A_{b_{t_N},w_{t_j}} > MRS^P_{b_{t_N},w_{t_j}}$.

(c) For $j < N$, $MRS^A_{b_{t_N},b_{t_j}} > MRS^P_{b_{t_N},b_{t_j}}$.

Part (a) and (b) jointly suggest that $w_j = 0$, $\forall j = 1, ..., N$. Part (c) suggests that $b_j = 0$, $\forall j = 1, ..., N-1$. So, it suffice to prove this claim.

Proof. I prove the three parts sequentially.

(a) If $j \leq \hat{T}$, then

$$\begin{aligned}
&\frac{\mu_0(1-\lambda^A)^{j-1}\lambda^A}{1-\mu_0+\mu_0(1-\lambda^A)^{j-1}} < \frac{\mu_0(1-\lambda^P)^{j-1}\lambda^P}{1-\mu_0+\mu_0(1-\lambda^P)^{j-1}} \\
\Longleftrightarrow &\frac{\mu_0(1-\lambda^A)^{j-1}-\mu_0(1-\lambda^A)^{j}}{1-\mu_0+\mu_0(1-\lambda^A)^{j-1}} < \frac{\mu_0(1-\lambda^P)^{j-1}+\mu_0(1-\lambda^P)^{j}}{1-\mu_0+\mu_0(1-\lambda^P)^{j-1}} \\
\Longleftrightarrow &\frac{1-\mu_0+\mu_0(1-\lambda^A)^{j}}{1-\mu_0+\mu_0(1-\lambda^A)^{j-1}} > \frac{1-\mu_0+\mu_0(1-\lambda^P)^{j}}{1-\mu_0+\mu_0(1-\lambda^P)^{j-1}} \\
\Longleftrightarrow &MRS^A_{w_{t_{j+1}},w_{t_j}} > MRS^P_{w_{t_{j+1}},w_{t_j}}.
\end{aligned}$$

(b) For $j > \hat{T}$,

$$\begin{aligned}
&\frac{\mu_0(1-\lambda^A)^{j-1}\lambda^A}{1-\mu_0+\mu_0(1-\lambda^A)^{j-1}} > \frac{\mu_0(1-\lambda^P)^{j-1}\lambda^P}{1-\mu_0+\mu_0(1-\lambda^P)^{j-1}} \\
\Longleftrightarrow &\frac{(1-\lambda^A)^{j-1}\lambda^A}{(1-\lambda^P)^{j-1}\lambda^P} > \frac{1-\mu_0+\mu_0(1-\lambda^A)^{j-1}}{1-\mu_0+\mu_0(1-\lambda^P)^{j-1}} \\
\Longrightarrow &\frac{(1-\lambda^A)^{N-1}\lambda^A}{(1-\lambda^P)^{N-1}\lambda^P} > \frac{1-\mu_0+\mu_0(1-\lambda^A)^{j-1}}{1-\mu_0+\mu_0(1-\lambda^P)^{j-1}} \\
\Longrightarrow &\frac{(1-\lambda^A)^{N-1}\lambda^A}{1-\mu_0+\mu_0(1-\lambda^A)^{j-1}} > \frac{(1-\lambda^P)^{N-1}\lambda^P}{1-\mu_0+\mu_0(1-\lambda^P)^{j-1}} \\
\Longleftrightarrow &MRS^A_{b_{t_N},w_{t_j}} > MRS^P_{b_{t_N},w_{t_j}}.
\end{aligned}$$

(c) For $j < N$,

$$
\begin{aligned}
&MRS^A_{b_{t_N},b_{t_j}}/MRS^P_{b_{t_N},b_{t_j}} \\
=&\frac{\mu_0\delta^{t_N}(1-\lambda^A)^{N-1}\lambda^A}{\mu_0\delta^{t_j}(1-\lambda^A)^{j-1}\lambda^A}\cdot\frac{\mu_0\delta^{t_j}(1-\lambda^P)^{j-1}\lambda^P}{\mu_0\delta^{t_N}(1-\lambda^P)^{N-1}\lambda^P} \\
=&\frac{(1-\lambda^A)^{N-j}}{(1-\lambda^P)^{N-j}} \\
>&1 \\
\Longrightarrow &MRS^A_{b_{t_N},b_{t_j}} > MRS^P_{b_{t_N},b_{t_j}}.
\end{aligned}
$$

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