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COMBINATORIAL ASPECTS OF THE IKEDA LIFT
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In loving memory of my grandparents,

Carl Albert & Lillian May Border

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On to the next adventure!

Contents

Abstract

This thesis is motivated by the problem of studying the Ikeda lift via a converse theorem due to R. Weissauer. We investigate a certain function; denoted by $\phi(a; B)$, where a is a positive integer and B is a symmetric positive-definite half-integral matrix; appearing in the Fourier coefficient formulas of a linear version of the Ikeda lift due to W. Kohnen. We develop new methods for computing $\phi(a; B)$ via the extended Gross-Keating (EGK) datum of a quadratic form and develop novel combinatorial interpretations for $\phi(a; B)$ which involve integer partitions with restrictions depending on the EGK datum attached to B at each prime.

Introduction

Based on numerical examples of Hecke eigenvalues of Siegel cuspforms of genus two, in 1978 H. Saito and N. Kurokawa [Kur78] independently conjectured a lifting of automorphic froms from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{Sp}_4(\mathbb{Z})$. The existence of this *Saito-Kurokawa* lift was quickly proven by Maass, Andrianov, and Zagier using the theory of Jacobi forms [EZ85]. In 1996, Duke and Imamoglu [DI96] proved the modularity of the Saito-Kurokawa lift using the explicit formulas for its Fourier coefficients and converse theorem due to Imai. In 2001, Ikeda [Ike01] generalized the Saito-Kurokawa lift to a lifting from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{Sp}_{4n}(\mathbb{Z})$ for all n. Like the Saito-Kurokawa lift, this *Ikeda lift* was proven using the theory of Jacobi forms. This thesis is motivated by the following two developments: (1) Kohnen [Koh02], in 2002, developed a linear version of the Ikeda lift. Kohnen's Fourier coefficient formulas for the Ikeda lift are a more direct generalization of the corresponding formulas for the Saito-Kurokawa lift; (2) Weissauer [Wei84], in an unpublished manuscript from 1984, proved a converse theorem for $\mathrm{Sp}_{2n}(\mathbb{Z})$. As far as we know, this theorem has not yet seen applications. The motivation for this thesis is to prove the modularity of the Ikeda lift using Weissauer's converse theorem and Kohnen's linear version of the Ikeda lift. However, even the first step of Duke and Imamoglu's proof – namely, writing a certain Dirichlet series attached to the lift as a Rankin-Selberg convolution – evades the author's persistent attacks. The difficulty arises due to the complicated terms – which we call *Kohnen's phi function* here – appearing in Kohnen's linear lift. This thesis investigates these terms in greater depth.

The Fourier coefficients of the Saito-Kurokawa lift are given as follows:

$$A(B) := \sum_{a \mid \operatorname{cont} B} c\left(\frac{|D_B|}{a^2}\right) a^k,$$

$$\operatorname{cont}(B) := \gcd(b_{11}, b_{12}, b_{22}), \quad D_B := -\det(2B) \qquad B = \begin{pmatrix} b_{11} & b_{12}/2 \\ b_{12}/2 & b_{22} \end{pmatrix}.$$

Above, B is a positive-definite 2×2 half-integral matrix. These matrices index the Fourier expansion of a Siegel cuspform of genus 2. The $c(n)$, $n > 0$ are the Fourier coefficients of a modular form for a congruence subgroup $\Gamma_0(4)$ of $\mathrm{SL}_2(\mathbb{Z})$. This is the form we are lifting. Kohnen's formulas for the Fourier coefficients of the Ikeda lift are given as follows:

$$A(B) = \sum_{a \mid f_B} c\left(\frac{|D_B|}{a^2}\right) a^{k-1} \phi(a; B), \qquad B \in \mathcal{S}'_{2n}(\mathbb{Z})^+,$$

where $D_B := (-1)^n \det(2B) = D_{B,0} f_B^2$ s.t. $D_{B,0}$ is a fundamental discriminant.

Above, $\mathcal{S}'_{2n}(\mathbb{Z})^+$ denotes the set of positive-definite $2n \times 2n$ half-integral matrices. These matrices index the Fourier expansion of a Siegel cuspform of degree $2n$. The term $\phi(a; B)$ is what we call Kohnen's phi function. Kohnen [Koh02] proves that for $n = 1$, we have:

$$\phi(a; B) = \begin{cases} a & \text{if } a \mid \operatorname{cont} B, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, Kohnen's formulas agree with the Saito-Kurokawa lift when $n = 1$. In this thesis, we obtain more detailed information about $\phi(a; B)$ for $n > 1$. First, we remark that $\phi(a; B)$ is multiplicative in the parameter a. Thus, it suffices to study $\phi(a; B)$ at prime powers $a = p^\mu$. Our investigations of Kohnen's phi function were driven by the following two developments:

(1) Kohnen and Choie [CK08], in 2008, showed that $\phi(p^\mu; B)$ is the coefficient of the p-local Siegel series $\tilde{F}_p(B; X)$ attached to B under a suitable basis. (2) More recently, Ikeda and Katsurada [IK18] have developed the theory of extended Gross-Keating (EGK) data attached to half-integral symmetric matrices (or, equivalently quadratic forms) over $\mathbb{Z}_p$. They also develop inductive formulas [IK22b] for computing the Siegel series $\tilde{F}_p(B; X)$ attached to B in terms of this EGK data of B. Although inductive formulas due to Katsurada [Kat99] were available much earlier, the inductive formulas using EGK data are easier to manipulate and they treat the dyadic ($p = 2$) and non-dyadic ($p \neq 2$) cases uniformly. Using these inductive formulas, conjectures about $\phi(p^\mu; B)$ obtained from numerical experiments, and calculations based on Kohnen's linear lift, we develop combinatorial interpretations of Kohnen's phi function. Namely, we prove that $\phi(p^\mu; B)$ is connected to integer partitions.

We now outline the contents of this thesis. Chapters 1, 2, and 3 are primarily expository.

In Chapter 1, we highlight essential aspects of the theory of quadratic forms and automorphic forms. This chapter mostly serves to fix notation.

In Chapter 2, we detail the historical developments of the Saito-Kurokawa and Ikeda lifts. At the end, we state Weissauer's converse theorem for $\mathrm{Sp}_{2n}(\mathbb{Z})$. Although we have not yet applied this result, it will be helpful to state it explicitly here for future reference.

In Chapter 3, we describe the extended Gross-Keating data as developed by Ikeda and Katsurada. When then connect the EGK data to classical invariants of quadratic forms.

The technical heart of this thesis lies in Chapters 4, 5, and 6.

In Chapter 4, we introduce the inductive formulas for the Siegel series. These inductive formulas are expressed in terms of a Laurent polynomial in two variables X, Y; denoted by $\mathcal{F}(H; Y, X)$; where H is an (abstract) *naive* EGK datum. This polynomial specializes to $\tilde{F}_p(B; X)$ in a sense we will describe below. An (abstract) naive EGK datum of length n is a tuple $H = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n)$; where $a_i \in \mathbb{Z}_{\geq 0}$ and $\varepsilon_i \in \mathcal{Z}_3 := \{0, 1, -1\}$; with certain restrictions enforced by Definition 3.10. The polynomial $\mathcal{F}(H; Y, X)$ for H a naive

EGK datum of length n is defined inductively in terms of the polynomial $\mathcal{F}(H';Y,X)$, where $H' := (a_1,\ldots,a_{n-1};\varepsilon_1,\ldots,\varepsilon_{n-1})$ is a naive EGK datum of length $n-1$. The inductive formulas depend only on the parity of the length n. The bulk of this chapter is then devoted to making these induction formulas as explicit as possible. After this, we give explicit formulas for $\mathcal{F}(H;Y,X)$ when H is a naive EGK datum of length 2 and 3. Finally, we strategically reorganize these inductive formulas in a way which will unveil the combinatorial aspects of $\phi(p^{\mu};B)$ in Chapter 6. The point of these calculations is that the Laurent polynomial $\mathcal{F}(H;Y,X)$ specializes to the p-local Siegel series of $B\in\mathcal{S}'_{2n}(\mathbb{Z})^{+}$ via:

$$\tilde{F}_p(B;X)=\mathcal{F}(\mathrm{EGK}(B)^{(p)};p^{\frac{1}{2}},X),$$

where $\mathrm{EGK}(B)^{(p)}$ is the EGK datum attached to B viewed as an an element of $\mathcal{S}'_n(\mathbb{Z}_p)^{+}$.

In Chapter 5, we formally introduce the Kohnen's phi function $\phi(a;B)$. We then describe the algorithm we used to calculate $\phi(p^{\mu};B)$; p an odd prime; to make conjectures. For a $B\in\mathcal{S}'_{2n}(\mathbb{Z})^{+}$; $n=1,2,3,4$; and μ satisfying $\mu<\mathrm{ord}_p(\mathrm{cont}B)$, we observed these trends:

μ	$\phi(p^{\mu};B)$; where $B\in\mathcal{S}'_2(\mathbb{Z})^{+}$
0	1
2	p^2
4	p^4
6	p^6

μ	$\phi(p^\mu; B)$; where $B \in \mathcal{S}_4'(\mathbb{Z})^+$
0	1
2	$p(p^2+p^3)$
4	$p^2(p^4+p^5+p^6)$
6	$p^3(p^6+p^7+p^8+p^9)$
8	$p^4(p^8+p^9+p^{10}+p^{11}+p^{12})$

μ	$\phi(p^\mu; B)$; where $B \in \mathcal{S}_6'(\mathbb{Z})^+$
0	1
2	$p(p^2+p^4+p^5)$
4	$p^2(p^4+p^6+p^7+p^8+p^9+p^{10})$
6	$p^3(p^6+p^8+p^9+p^{10}+p^{11}+2p^{12}+p^{13}+p^{14}+p^{15})$
8	$p^4(p^8+p^{10}+p^{11}+p^{12}+p^{13}+2p^{14}+p^{15}+2p^{16}+2p^{17}+p^{18}+p^{19}+p^{20})$

μ	$\phi(p^\mu; B)$; where $B \in \mathcal{S}_8'(\mathbb{Z})^+$
0	1
2	$p(p^2+p^4+p^6+p^7)$
4	$p^2(p^4+p^6+2p^8+p^9+p^{10}+p^{11}+p^{12}+p^{13}+p^{14})$
6	$p^3(p^6+p^8+2p^{10}+p^{11}+2p^{12}+p^{13}+2p^{14}+2p^{15}+p^{16}+p^{17}+p^{18}+p^{19}+p^{20}+p^{21})$

The formulas for $B \in \mathcal{S}_2'(\mathbb{Z})^+$ were, as stated above, already proven by Kohnen. However, the data for $n = 2, 3, 4$ is new. We were especially intrigued by the seemingly sporadic appearance of the coefficient 2 in the data for $n \geq 3$. Can we explain this phenomenon? Indeed we can. Kohnen's formula for $\phi(p^\mu; B)$ involves a sum over a set denoted by $\mathcal{D}_p(B)$. Under the assumption $\mu < \mathrm{ord}_p(\mathrm{cont}B)$, this set is:

$$\mathcal{D}_p(B) = \mathrm{GL}_{2n}(\mathbb{Z}_p)\backslash \mathrm{M}_{2n}(\mathbb{Z}_p) \cap \mathrm{GL}_{2n}(\mathbb{Q}_p).$$

Let $\mathcal{D}_p(B)_\nu$ denote the subset of $\mathcal{D}_p(B) \ni G$ with $\mathrm{ord}_p \det(G) = \nu \geq 0$. We can explicitly express the size of this set as a sum indexed over the partitions of ν. Specifically,

$$\#\mathcal{D}_p(B)_\nu := \sum_{(\lambda_i)\in[\nu]_{2n}} \prod_{i=1}^{2n} p^{(i-1)\lambda_i}.$$

where

$$[\nu]_{2n} = \left\{ \lambda := (\lambda_1, \ldots, \lambda_{2n}) \in \mathbb{Z}^{2n} \;:\; \lambda_i \geq 0,\; \sum_{i=1}^{2n} \lambda_i = \nu \right\}.$$

Using Kohnen's formulas, the formula for $\#\mathcal{D}_p(B)_\nu$, and some combinatorics, we prove:

Proposition 5.11. Let $B \in \mathcal{S}'_4(\mathbb{Z})^+$. For $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$, we have

$$\phi(p^\mu; B) = p^{\frac{\mu}{2}+\frac{\delta_2(\mu)}{2}} \sum_{(\lambda_i)\in[\lfloor\mu/2\rfloor]_2} \prod_{i=1}^{2} p^{(i+1)\lambda_i},$$

where,

$$\delta_2(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ 3 & \text{if } \mu \text{ odd.} \end{cases}$$

Proposition 5.12. Let $B \in \mathcal{S}'_6(\mathbb{Z})^+$. For $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$, we have

$$\phi(p^\mu; B) = p^{\frac{\mu}{2}+\frac{\delta_3(\mu)}{2}} \sum_{\substack{(\lambda_i)\in[\lfloor\mu/2\rfloor]_4 \\ \lambda_2=0}} \prod_{i=1}^{4} p^{(i+1)\lambda_i}.$$

where,

$$\delta_3(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ 5 & \text{if } \mu \text{ odd.} \end{cases}$$

Proposition 5.13. Let $B \in \mathcal{S}'_8(\mathbb{Z})^+$. For $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$, we have

$$\phi(p^\mu; B) = p^{\frac{\mu}{2} + \frac{\delta_4(\mu)}{2}} \sum_{\substack{(\lambda_i) \in [\lfloor \mu/2 \rfloor]_6 \\ \lambda_2 = \lambda_4 = 0}} \prod_{i=1}^{6} p^{(i+1)\lambda_i}.$$

where,

$$\delta_4(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ 7 & \text{if } \mu \text{ odd.} \end{cases}$$

These formulas agree with our numerical data. From these three results, we (correctly) conjectured and proved

Theorem 5.14. Let $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$, $n > 1$. For $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$, we have

$$\phi(p^\mu; B) = p^{\frac{\mu}{2} + \frac{\delta_n(\mu)}{2}} \sum_{\substack{(\lambda_i) \in [\lfloor \mu/2 \rfloor]_{2n-2} \\ \lambda_{2i} = 0,\ i < n-1}} \prod_{i=1}^{2n-2} p^{(i+1)\lambda_i}.$$

where,

$$\delta_n(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ (2n-1) & \text{if } \mu \text{ odd.} \end{cases}$$

We delay the proof of this result until Chapter 6. After this work, we introduce a generalization of Kohnen's phi function; denoted $\phi(\mu, H; Y)$; attached to a naive EGK datum H. This $\phi(\mu, H; Y)$ is a Laurent polynomial in the variable Y. It is related to $\mathcal{F}(H; Y, X)$ in the same way that $\phi(p^\mu; B)$ is related to $\tilde{F}_p(X)$. In particular, we have:

$$\phi(p^\mu; B) = \phi(\mu, \mathrm{EGK}(B)^{(p)}; p^{\frac{1}{2}}).$$

Continuing in Chapter 5, we develop a key lemma which allows us to compute $\phi(\mu, H; Y)$ in terms of the coefficients of $\mathcal{F}(H'; Y, X)$, where H' is the truncated version of H. By specialization, this gives us a method to compute $\phi(p^\mu; B)$. We conclude this chapter by applying this key lemma to reproduce some of the formulas we derived above for $\phi(p^\mu; B)$.

In Chapter 6, we combine all the technical results of Chapters 4 and 5 to prove and extend our conjectures. At the start of the chapter, we prove a generalization of our earlier conjecture. It is phrased in terms of $\phi(\mu, H; Y)$, i.e. the (generalized) Kohnen's phi function attached to a naive EGK datum H. Specifically, we prove:

Theorem 6.1. Let $H \in \mathcal{NEGK}_{2n}$, with $n > 1$. Then for $\mu < \mathfrak{e}_1$,

$$\phi(\mu, H; Y) = Y^{\mu+\delta_n(\mu)} \sum_{\substack{(\lambda_i)\in[\lfloor\mu/2\rfloor]_{2n-2} \\ \lambda_{2i}=0,\ i<n-1}} \prod_{i=1}^{2n-2} Y^{2(i+1)\lambda_i}, \tag{0.1}$$

where

$$\delta_n(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ 2n-1 & \text{if } \mu \text{ odd.} \end{cases}$$

The parameter $\mathfrak{e}_1$ is the EGK analogue of the invariant $\mathrm{ord}_p(\mathrm{cont}\, B)$ for $B \in \mathcal{S}'_{2n}(\mathbb{Z}_p)^+$. The proof of this conjecture requires all the technical results we developed in the previous chapters. Upon specializing this result with $H = \mathrm{EGK}(B)^{(p)}$ and $Y = p^{\frac{1}{2}}$, we recover our original conjecture for $\phi(p^\mu; B)$. The chapter ends with a technical result which shows that, under the reorganization of the inductive formulas at the end of Chapter 4, we can connect Kohnen's phi function for $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$ at $a = p^\mu$ to integer partitions with restrictions depending on the datum $\mathrm{EGK}(B)^{(p)}$. To conclude, we note that our picture of Kohnen's phi function is incomplete. However, we have constructed the appropriate framework for further developments of combinatorial descriptions of this somewhat mysterious function.

Chapter 1

Preliminaries

In this preliminary chapter, we review the basics of quadratic forms and automorphic forms. We will highlight aspects of the theory which are needed to understand subsequent chapters.

1.1 Notation

We fix some common notation. For a ring R, let $\mathrm{M}_n(R)$ denote the $n \times n$ matrices over R. For $B, U \in \mathrm{M}_n(R)$, let ${}^tU$ denote the transpose of U, let $B^{(i)}$ denote the upper lefthand $i \times i$ submatrix of B, and let $B[U] := {}^tUBU$. For a subset $S \subseteq \mathrm{M}_n(R)$, let $S^{\mathrm{nd}} \subseteq S$ be the subset of non-degenerate matrices in S. For a symmetric matrix B, we write $B \geq 0$ (resp. $B > 0$) when B is positive-semidefinite (resp. positive-definite). Let $\lfloor x \rfloor := \max\{a \in \mathbb{Z} : a \leq x\}$. Let $\mathcal{Z}_3 := \{0, 1, -1\}$.

1.2 Quadratic Forms

1.2.1 Quadratic Forms over Fields

We outline the theory of quadratic forms over fields. For a treatment of the odd characteristic, see [Cas78], [Lam05], and [O'M00]. For arbitrary characteristic, we recommend [EKM08]. Let F denote a field of any characteristic. We first recall some basic definitions:

Remark 1.1. All vector spaces in this section are finite-dimensional over the base field F.

Definition 1.2. Let V be a F-vector space. A quadratic form on V is a map $\varphi : V \longrightarrow F$ satisfying:

(1) $\varphi(av) = a^2\varphi(v)$ for all $v \in V$, $a \in F$,

(2) The map $b_\varphi : V \times V \longrightarrow F$ defined via:

$$b_\varphi(v, w) = \varphi(v + w) - \varphi(v) - \varphi(w)$$

is an F-bilinear form. We call b_φ the *polar form* of φ.

For clarity, we sometimes write V_φ for the underlying vector space associated to φ.

Definition 1.3. Let φ and ψ be quadratic forms. An *isometry* is a linear map $f : V_\varphi \longrightarrow V_\psi$ with $\varphi = \psi \circ f$. If such an isometry exists, we write $\varphi \simeq \psi$ and say φ and ψ are isometric.

Definition 1.4. For a subspace $W \subseteq V$, the restriction of φ on W is the quadratic form; denoted by $\varphi|_W$; whose polar form is given by $b_{\varphi|_W} := b_\varphi|_W$. $\varphi|_W$ is called a *subform* on W.

Definition 1.5. Let φ be a quadratic form on V. A vector $v \in V$ is called *anisotropic* if $\varphi(v) \neq 0$ and *isotropic* if $v \neq 0$ but $\varphi(v) = 0$. We call φ *anisotropic* if there are no isotropic vectors in V and *isotropic* otherwise. A $W \subseteq V$ is called *totally isotropic* whenever $\varphi|_W = 0$.

Definition 1.6. For a quadratic form φ on V, and a subspace $W \subseteq V$, let $W^{\perp}$ denote the ortogonal complement of W relative to the polar form of φ. Let ψ be a subform of φ on a subspace $W_{\psi} \subseteq V$. The restriction of φ on $(W_{\psi})^{\perp}$ is denoted by $\psi^{\perp}$ and is called the *complementary form* of ψ in φ. If $V = W \oplus U$ is a direct sum of vector spaces with $W \subseteq U^{\perp}$, we write $\varphi = \varphi|_W \perp \varphi|_U$ and call it an *internal orthogonal sum*. Thus $\varphi(w+u) = \varphi(w) + \varphi(u)$ for all $w \in W$ and $u \in U$. Note that $\varphi|_U$ is a subform of $(\varphi|_W)^{\perp}$.

Definition 1.7. For a quadratic from φ on V, define the *radical* of b_{φ} by:

$$\operatorname{rad} b_{\varphi} := \{v \in V \ : \ b_{\varphi}(v,w) = 0 \text{ for all } w \in V\};$$

and define the *quadratic radical* of φ by:

$$\operatorname{rad} \varphi = \{v \in \operatorname{rad} b_{\varphi} \ : \ \varphi(v) = 0\}.$$

Definition 1.8. Let V be an F-vector space. Define the a quadratic form on $V \oplus V^*$ by:

$$\varphi_{\mathbb{H}}(v,f) := f(v).$$

We write $\mathbb{H}(V) := \varphi_{\mathbb{H}}$ and call it the *hyperbolic form* on V. If φ is a quadratic form isometric to $\mathbb{H}(W)$ for some vector space W, φ is called a *hyperbolic form*. The form $\mathbb{H}(F)$ is called the *hyperbolic plane* and we denote it simply by $\mathbb{H}$. If $\varphi \simeq \mathbb{H}$, two vectors $e, f \in V$ satisfying $\varphi(e) = \varphi(f) = 0$ and $b_{\varphi}(e,f) = 1$ are called a *hyperbolic pair*.

A cornerstone result on quadratic forms over fields is **Witt's Decomposition Theorem**:

Theorem 1.9. ([EKM08, Theorem 8.5]) Let φ be a quadratic form on a vector space V.

Then there exist subspaces V_a and V_h of V such that

$$\varphi = \varphi|_{\mathrm{rad}\varphi} \perp \varphi|_{V_a} \perp \varphi|_{V_h}, \qquad \varphi|_{V_a} \text{ anisotropic}, \;\; \varphi|_{V_h} \text{ hyperbolic}.$$

Moreover, $\varphi|_{V_a}$ and $\varphi|_{V_h}$ are unique up to equivalence.

1.2.2 Integral Quadratic Forms

With the theory covered, we describe a way to work with quadratic forms via matrices. Specifically, we need to work with quadratic forms with coefficients lying in $R = \mathbb{Z}$ and $\mathbb{Z}_p$. Such a form can be written as a homogeneous multivariate polynomial of degree 2:

$$\varphi = \varphi(x_1, \ldots, x_n) = \sum_{1 \le i,j \le n} a_{ij} x_i x_j, \qquad a_{ij} \in R. \tag{1.1}$$

We may associate to φ a symmetric $n \times n$ matrix with coefficients in $\frac{1}{2}R$:

$$B_\varphi = (b_{ij}), \qquad b_{ij} := \frac{1}{2}(a_{ij} + a_{ji}). \tag{1.2}$$

The coefficients of B satisfy:

$$\begin{aligned} b_{ii} &\in R, \quad 1 \le i \le n \\ 2b_{ij} &\in R, \quad 1 \le i < j \le n. \end{aligned} \tag{1.3}$$

With this in mind we define:

Definition 1.10. Let $\mathcal{S}_n(R)$ denote the set of symmetric $n \times n$ matrices over R.

Definition 1.11. Let $R = \mathbb{Z}$ or $\mathbb{Z}_p$. Let $\mathcal{S}'_n(R)$ denote the set of matrices $B \in \mathcal{S}_n(\frac{1}{2}R)$ satisfying (1.3). We call $\mathcal{S}'_n(R)$ the set of $n \times n$ **half-integral** symmetric matrices over R.

Via the symmetrization trick (1.2), we have a correspondence:

$$\left\{\begin{matrix} n\text{-ary} \\ \text{quadratic forms} \\ \text{over } R \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} n \times n \text{ half-integral} \\ \text{symmetric matrices} \\ \text{over } R \end{matrix}\right\}.$$

1.2.3 Local Quantities Attached to Quadratic Forms

Here, we collect some local quantities which we will frequently need in later chapters.

Definition 1.12. For $B \in \mathcal{S}'_n(\mathbb{Z})$, we define the **signed determinant** as

$$D_B := (-4)^{\lfloor n/2 \rfloor} \det(B). \tag{1.4}$$

We define $D_{B,0}, f_B \in \mathbb{Z}$ via:

$$D_B = D_{B,0} f_B^2, \qquad D_{B,0} \text{ is a fundamental discriminant.} \tag{1.5}$$

For a prime p, we define

$$\xi_p(B) = \xi_{p,B} := \left(\frac{D_{B,0}}{p}\right) = \begin{cases} 1 & \text{if } D_B \in \mathbb{Q}_p^{\times 2}, \\ -1 & \text{if } \mathbb{Q}_p(\sqrt{D_B})/\mathbb{Q}_p \text{ is unramified quadratic,} \\ 0 & \text{if } \mathbb{Q}_p(\sqrt{D_B})/\mathbb{Q}_p \text{ is ramified quadratic.} \end{cases} \tag{1.6}$$

1.3 Automorphic Forms

In this section, we collect basic information about various types of automorphic forms: specifically, Siegel modular forms, half-integral weight modular forms, and Maass forms.

1.3.1 Siegel Modular Forms

We now review the basic theory of Siegel modular forms. For a unital commutative ring R, we define the **symplectic group of similitudes** via:

$$\mathrm{GSp}_{2n}(R) := \left\{ U \in \mathrm{GL}_{2n}(R) \ : \ {}^tUJU = \mu(U)J, \ \ \mu(U) \in R^\times, \ J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

We have the subgroup

$$\mathrm{Sp}_{2n}(R) := \{U \in \mathrm{GSp}_{2n}(R) \ : \ \mu(U) = 1\}.$$

Let

$$\Gamma^{(n)} := \mathrm{Sp}_{2n}(\mathbb{Z}).$$

For $N \geq 1$, we define the **congruence subgroups**:

$$\Gamma^{(n)}(N) := \{U \in \Gamma^{(n)} \ : \ U \equiv I_{2n} \bmod N\},$$
$$\Gamma_0^{(n)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \ : \ C \equiv 0 \bmod N \right\}.$$

Remark 1.13. For simpler notation in subsequent chapters, we denote:

$$\Gamma(N) := \Gamma^{(1)}(N), \qquad \Gamma_0(N) := \Gamma_0^{(1)}(N),$$
$$\Gamma^{(n)} := \Gamma^{(n)}(1), \qquad \text{for } n > 1.$$

The **Siegel upper half space** of genus n is:

$$\mathcal{H}_n := \{Z = X + iY \in \mathrm{M}_n(\mathbb{C}) \ : \ {}^tZ = Z, \ \ Y > 0\}.$$

In particular, we denote $\mathcal{H} := \mathcal{H}_1$. This is the Poincaré upper half plane. Let

$$\mathrm{GSp}_{2n}(\mathbb{R})^+ := \{U \in \mathrm{GSp}_{2n}(\mathbb{R}) \ : \ \mu(U) > 0\}.$$

Then $\mathrm{GSp}_{2n}(\mathbb{R})^+$ acts on $\mathcal{H}_n$ via:

$$U\langle Z\rangle := (AZ + B)(CZ + D)^{-1}, \qquad U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_{2n}(\mathbb{R})^+, \ \ Z \in \mathcal{H}_n. \tag{1.7}$$

For $k \geq 1$, the **weight** k **slash operator** on the space of complex functions on $\mathcal{H}_n$ is:

$$(f|_kU) = \det(CZ + D)^{-k} \det(U)^{\frac{k}{2}} F(U\langle Z\rangle), \qquad U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_{2n}(\mathbb{R})^+. \tag{1.8}$$

Remark 1.14. The normalizing factor $\det(U)^{\frac{k}{2}}$ in (1.8) differs from some sources; for instance Equation (15) in [Pit19] uses $\mu(U)^{nk-\frac{n(n+1)}{2}} = \det(U)^{k-\frac{(n+1)}{2}}$. The choice is a matter of convenience. In fact, [DS05] uses both normalizations in the $\mathrm{SL}_2(\mathbb{Z})$ setting (i.e. $n = 1$).

We are now ready to define a Siegel modular form:

Definition 1.15. Fix integers $k, n \geq 1$. A function $F : \mathcal{H}_n \to \mathbb{C}$ is called a **Siegel modular form** of weight k, genus n if it satisfies:

(1) F is holomorphic on $\mathcal{H}_n$,

(2) $F|_kM = F$ for $M \in \Gamma^{(n)}$,

(3) If $n = 1$, F is bounded in $Y \geq Y_0$ for any $Y_0 > 0$.

The $\mathbb{C}$-vector space of such functions is denoted by $M_k(\Gamma^{(n)})$.

A Siegel modular form F has a Fourier expansion of the form:

$$F(Z) = \sum_{\substack{B \in \mathcal{S}'_n(\mathbb{Z}) \\ B \geq 0}} A(B) e^{2\pi i \mathrm{Tr}(BZ)}, \qquad Z \in \mathcal{H}_n.$$

The subspace of **cuspforms**, denoted by $S_k(\Gamma)$, consists of forms with $A(B) = 0$ unless $B > 0$. The Fourier coefficients $A(B)$ of a Siegel modular form have certain symmetries with respect to $\Gamma^{(n)}$ and satisfy a growth condition. We summarize these properties below:

Definition 1.16. A sequence $\{A(B)\}$, $B \in \mathcal{S}_n(\mathbb{Z})^+$, is called $\mathrm{SL}_n(\mathbb{Z})$-admissible if:

(1) $|A(B)| \ll \det(B)^c$ for a fixed constant c,

(2) $A(B[U]) = A(B)$ for all $U \in \mathrm{SL}_2(\mathbb{Z})$.

A Siegel modular form of genus 1 is called an **(elliptic) modular form**. For a congruence subgroup $\Gamma \subseteq \Gamma(1)$, the set of functions $f : \mathcal{H} \to \mathbb{C}$ satisfying Definition 1.15 – with the modularity property (2) holding only for the subgroup Γ and the condition (3) replaced with holomorphy at each cusp of $\Gamma\backslash\mathcal{H}$ – is denoted $M_k(\Gamma)$. The subspace of cusps forms is denoted by $S_k(\Gamma)$. See [DS05, Chapter 1] for further details on congruence modular forms.

1.3.2 Modular Forms of Half-Integral Weight

The theory of (elliptic) modular forms can be meaningfully extended to half-integer weights. The model example is the classical theta function:

$$\theta(z) := \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}, \qquad z \in \mathcal{H}.$$

Let

$$J(\gamma, z) := \frac{\theta(\gamma(z))}{\theta(z)}, \qquad \gamma \in \Gamma_0(4).$$

Definition 1.17. A function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of half-integral weight $k + 1/2$ (where $k \in \mathbb{Z}$) with respect to the congruence subgroup $\Gamma_0(4) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \; : \; 4 \mid c \right\}$ if:

(1) f is holomorphic on $\mathcal{H}$,

(2) $f(\gamma(z)) = J(\gamma, z)^{2k+1} f(z)$,

(3) f is holomorphic at the cusps.

The space of such forms (resp. cuspforms) is denoted by $M_{k/2}(\Gamma_0(4))$ (resp. $S_{k/2}(\Gamma_0(4))$). By $M^+_{k/2}(\Gamma_0(4))$ (resp. $S^+_{k/2}(\Gamma_0(4))$) we denote the subspace of $M_{k/2}(\Gamma_0(4))$ (resp. $S_{k/2}(\Gamma_0(4))$) of forms with a Fourier expansion of the type $f(z) = \sum_{\substack{n \geq 0 \\ (-1)^k n \equiv 0,1 \bmod 4}} c(n) e^{2\pi i n z}$. This is called Kohnen's +-space.

1.3.3 Automorphic Forms on $\mathcal{H}$

In this section, we follow the excellent exposition of [Iwa02].

The central object of this section is the *hyperbolic Laplacian*:

$$\Delta := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

This is a $\Gamma(1)$-invariant differential operator on $\mathcal{H}$.

Definition 1.18. A function $f : \mathcal{H} \to \mathbb{C}$ is called **automorphic** with respect to $\Gamma(1)$ if:

$$f(\gamma(z)) = f(z), \qquad \gamma \in \Gamma(1).$$

Thus, f may be viewed as a function on the Riemann surface $\Gamma(1)\backslash\mathcal{H}$ [DS05, Chapter 2]. We denote the space of such functions by $\mathcal{A}(\Gamma(1)\backslash\mathcal{H})$

Definition 1.19. A function $f \in \mathcal{A}(\Gamma(1)\backslash\mathcal{H})$ which is a Δ-eigenfunction is called an **automorphic form** with respect to $\Gamma(1)$. Let $\mathcal{A}_s(\Gamma(1)\backslash\mathcal{H})$ denote the subspace of $\mathcal{A}(\Gamma(1)\backslash\mathcal{H})$ consisting of λ-eigenfunctions for Δ, $\lambda := s(1-s)$. Thus $\mathcal{A}_s(\Gamma(1)\backslash\mathcal{H}) = \mathcal{A}_{1-s}(\Gamma(1)\backslash\mathcal{H})$.

We outline the spectral theory of the subspace $L^2(\Gamma(1)\backslash\mathcal{H}) \subset \mathcal{A}(\Gamma(1)\backslash\mathcal{H})$ of square-integrable functions on $\Gamma(1)\backslash\mathcal{H}$. We have the orthogonal decompsition [Iwa02, Eq. (3.16)]:

$$L^2(\Gamma(1)\backslash\mathcal{H}) = \overline{C(\Gamma(1)\backslash\mathcal{H})} \oplus \overline{\mathcal{E}(\Gamma(1)\backslash\mathcal{H})}.$$

Here, the overline denotes the Hilbert space closure in $L^2(\Gamma(1)\backslash\mathcal{H})$, and

$$C(\Gamma(1)\backslash\mathcal{H}) := \text{space of cuspforms},$$
$$\mathcal{E}(\Gamma(1)\backslash\mathcal{H}) := \text{space of incomplete Eisenstein series}.$$

We have the following spectral resolutions of these two spaces:

Theorem 1.20. ([Iwa02, Theorem 4.7]) The Laplacian Δ has pure point spectrum (consisting of the so-called cuspforms) in $C(\Gamma(1)\backslash\mathcal{H})$. The eigenspaces are finite-dimensional.

Theorem 1.21. ([Iwa02, Theorem 7.2, Specialized to $\Gamma(1)$]) The space $\mathcal{E}(\Gamma(1)\backslash\mathcal{H})$ splits into Δ-invariant subspaces: $\mathcal{E}(\Gamma(1)\backslash\mathcal{H}) = \mathcal{R}(\Gamma(1)\backslash\mathcal{H}) \oplus \mathcal{E}_\infty(\Gamma(1)\backslash\mathcal{H})$. Here $\mathcal{R}(\Gamma(1)\backslash\mathcal{H})$ denotes the *residual spectrum*, which for $\Gamma(1)$ simply coincides with the constant functions;

the space $\mathcal{E}_\infty(\Gamma(1)\backslash\mathcal{H})$ is spanned by the unitary Eisenstein series $E(z, 1/2 + ir)$, $r \geq 0$; where

$$E(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} (\mathrm{Im}(\gamma(z)))^s, \qquad s \in \mathbb{C}.$$

Remark 1.22. In summary, $L^2(\Gamma(1)\backslash\mathcal{H})$ is spanned as a Hilbert space by the Δ-eigenfunctions:

(1) The constant function $\varphi_0(z) := \sqrt{3/\pi}$,

(2) An orthonormal basis of cusp forms $\varphi_1(z), \varphi_2(z), \ldots$,

(3) The unitary Eisenstein series $E(z, 1/2 + ir)$, $r \geq 0$.

1.3.4 The Symmetric Space $\mathcal{P}_n$

We now study the space $\mathcal{P}_n$ of positive-definite $n \times n$ real matrices:

$$\mathcal{P}_n = \{Y \in \mathrm{M}_n(\mathbb{R}) \;\;:\;\; {}^tY = Y, \;\; Y > 0\}$$

This space is studied in thorough detail in [Maa71], [Ter16], and [JL05]. Let $G = \mathrm{GL}_n(\mathbb{R})$.

Remark 1.23. $\mathcal{P}_n$ is a homogeneous space for G via the G-equivariant diffeomorphism:

$$O_n(\mathbb{R})\backslash\mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathcal{P}_n$$
$$O_n(\mathbb{R})U \longmapsto {}^tUU.$$

Due to longstanding tradition, the quadratic forms world thinks in terms of $K\backslash G$; whereas in the automorphic forms world, the standard is to think in terms G/K. In [JL05], Jorgenson and Lang retabulate many of the classical results about $\mathcal{P}_n$ in terms of the G/K perspective.

Since we do not make any use of the modern adelic perspective, we opt to phrase results via the classical "quadratic model" $\mathcal{P}_n \cong K\backslash G$ in spite of the "ongoing shift from right to left."

Let $Y \in \mathcal{P}_n$ and $f \in C^\infty(\mathcal{P}_n)$. Then G acts on $\mathcal{P}_n$ and on $C^\infty(\mathcal{P}_n)$ via:

$$Y[U] = {}^tUYU,$$
$$f^U(Y) = f(Y[U]), \qquad Y \in \mathcal{P}_n, U \in G.$$

The G-invariant volume element $d\mu(Y)$ on $\mathcal{P}_n$ is given by:

$$d\mu(Y) := \det(Y)^{-\frac{n+1}{2}} \prod_{1\leq i\leq j\leq n} dy_{ij}, \qquad Y = (y_{ij}) \in \mathcal{P}_n. \tag{1.9}$$

A differential operator δ on $\mathcal{P}_n$ is G*-invariant* if δ commutes with the G-action. That is,

$$(\delta f)^U = \delta f^U, \qquad U \in G, f \in C^\infty(\mathcal{P}_n).$$

Define

$$D(\mathcal{P}_n) = \text{the } \mathbb{C}\text{-algebra of } G\text{-invariant differential operators on } \mathcal{P}_n.$$

The structure of $D(\mathcal{P}_n)$ as a $\mathbb{C}$-algebra is well-known. Let

$$\frac{\partial}{\partial Y} := \left(\frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial y_{ij}}\right)_{1\leq i,j\leq n}, \qquad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.24. ([Maa71, §10],[Ter16, Theorem 1.1.2],[JL05, Theorem 4.2.3]) Let

$$\delta_i := \operatorname{Tr}\left(\left(Y\frac{\partial}{\partial Y}\right)^i\right), \qquad i = 1, \ldots, n.$$

These δ_i, $i = 1, \ldots, n$, form a basis for $D(\mathcal{P}_n)$ viewed as a $\mathbb{C}$-algebra. Therefore $D(\mathcal{P}_n)$ can be identified with the polynomial ring $\mathbb{C}[\delta_1, \ldots, \delta_n]$. In particular, $D(\mathcal{P}_n)$ is commutative.

Remark 1.25. The above is a special case of the Harish-Chandra isomorphism [HC51].

1.3.5 Automorphic Forms on $\mathcal{P}_n$

We finally define automorphic forms living on the space $\mathcal{P}_n$:

Definition 1.26. A Maass *Großencharacter* with respect to $\mathrm{SL}_n(\mathbb{Z})$ is a function u on $\mathcal{P}_n$ which satisfies:

(1) $u \in C^\infty(\mathcal{P}_n)$,

(2) u is $\mathrm{SL}_n(\mathbb{Z})$-invariant. That is, $u^U = u$ for $U \in \mathrm{SL}_n(\mathbb{Z})$,

(3) u is a simultaneous eigenfunction of $D(\mathcal{P}_n)$,

(4) u is homogenous of degree zero. That is, $\delta_1 u = 0$,

(5) For each $h \geq 0$, there exist constants c_1, c_2, depending on h, for which

$$\left|\frac{\partial^h u(Y)}{\partial Y \cdots \partial Y}\right| \leq c_1 \cdot \operatorname{Tr}(Y)^{c_2}, \qquad Y \in \mathcal{P}_n, \det(Y) = 1.$$

Remark 1.27. In modern parlance, Großencharacters are called Maass forms. Refer to [Gol06] for an account of Maass forms via the generalized upper half-plane model (N.B. this is not the same as the Siegel upper half space!). See [Ter16] for an account via the positive-definite matrix space model. Terras provides a map between these models in §1.5.4.

Chapter 2

Lifts of Siegel Modular Forms

In this chapter, we introduce the Saito-Kurokawa lift and the Ikeda lift. These provide correspondences between classical cuspforms and a certain subspace of Siegel cuspforms. The Saito-Kurokawa lift was originally conjectured based on numerical evidence of H. Saito and N. Kurokawa from their investigations of the Euler factors of the standard L-function attached to cuspidal Siegel eigenforms. For this reason, we first briefly define the standard L-function attached to a Siegel modular form, so that we can state the relation between the L-function of a classical cuspform f and the standard L-function of its lift F_f. After this, we detail the history of the Saito-Kurokawa lift: in particular, we outline the original proof which uses the theory of Jacobi forms; then we outline the later proof by Duke and Imamoglu with uses a converse theorem due to Imai. Then, we introduce the Ikeda lift in two ways: first, we introduce Ikeda's original formulation of the lift; then we introduce Kohnen's linear version of the Ikeda lift. We conclude by stating a converse theorem for $\Gamma^{(n)}$ due to Weissauer which could be used to study the Ikeda lift from a novel perspective.

2.1 Hecke Theory and L-functions

We recommend [Pit19] for further details on the Hecke theory of Siegel modular forms. For $F \in S_k(\Gamma^{(n)})$ a Hecke eigenform and prime p, there are $n+1$ so-called **Satake parameters** $\alpha_{0,p}, \alpha_{1,p}, \ldots, \alpha_{n,p} \in \mathbb{C}$ depending on F. We use these Satake parameters to define:

Definition 2.1. Let $F \in S_k(\Gamma^{(n)})$ be a Hecke eigenform. The degree $2n+1$ **standard** L-function of F is:

$$L(s, F, \mathrm{std}) := \prod_{p \text{ prime}} L_p(s, F, \mathrm{std}),$$

where

$$L_p(s, F, \mathrm{std})^{-1} := (1 - p^{-s}) \prod_{i=1}^{n} (1 - \alpha_{i,p} p^{-s})(1 - \alpha_{i,p}^{-1} p^{-s}),$$

and $\alpha_{0,p}, \alpha_{1,p}, \ldots, \alpha_{n,p}$ are the Satake p-parameters of F.

2.2 The Saito-Kurokawa Lift

The Saito-Kurokawa lift is a one-to-one correspondence between the space $S_{2k}(\Gamma(1))$ and a subspace of $S_{k+1}(\Gamma^{(2)})$ known as the Maass "Spezialschar." With a aid of the Shimura correspondence (recalled below) we express this lift concretely via Fourier expansions:

The Shimura Correspondence: Kohnen [Koh80] gave a correspondence between the space $S_{2k}(\Gamma(1))$ and the space $S^{+}_{k+1/2}(\Gamma_0(4))$. This correspondence is a sharpening of a lift due to Shimura [Shi73]; whence, it is historically also called the Shimura correspondence.

Theorem 2.2. (Saito-Kurokawa Lift) Given $f \in S_{2k}(\Gamma(1))$. Let $g \in S^{+}_{k+1/2}(\Gamma_0(4))$ be the form corresponding to f under the Shimura correspondence. Write $g(z) = \sum_{n>0} c(n)e^{2\pi inz}$. Define $F_f : \mathcal{H}_2 \to \mathbb{C}$ via $F_f(Z) := \sum_{B>0} A(B)e^{2\pi i \mathrm{Tr}(BZ)}$ with:

$$A(B) := \sum_{a \,|\, \mathrm{cont}\, B} c\left(\frac{|D_B|}{a^2}\right) a^k, \tag{2.1}$$

$$\mathrm{cont}(B) := \gcd(b_{11}, b_{12}, b_{22}), \qquad B = \begin{pmatrix} b_{11} & b_{12}/2 \\ b_{12}/2 & b_{22} \end{pmatrix}.$$

Then $F_f \in S_{k+1}(\Gamma^{(2)})$. If f is a Hecke eigenform, then so it F_f and, in this case,

$$L(s, F_f, \mathrm{std}) = \zeta(s)L(s+k, f)L(s+k-1, f).$$

Here, $L(s, f) := \sum_{n>0} a(n)n^{-s}$ where $f(z) = \sum_{n>0} a(n)e^{2\pi inz}$.

This lift was originally constructed by Maass, Andrianov, and Zagier using intermediary spaces of half-integral weight forms and so-called Jacobi forms. For a complete treatment of Jacobi forms and an account of the origins of the Saito-Kurokawa lift, refer to [EZ85]. The Saito-Kurokawa lift is a composition of three isomorphisms:

$$\begin{array}{c} \text{Maass “Spezialschar”} \subset S_{k+1}(\Gamma^{(2)}) \\ \Updownarrow \\ \text{Jacobi cuspforms of weight } k+1 \text{ and index } 1 \\ \Updownarrow \\ \text{Kohnen's +-space} \subset S_{k+1/2}(\Gamma_0(4)) \\ \Updownarrow \\ S_{2k}(\Gamma(1)). \end{array}$$

The bottom isomorphism is the Shimura correspondence. The remaining two are explicitly constructed in [EZ85].

Lifting problems have also been studied with the aid of tools called *converse theorems*.

For a survey of these applications, see [Cog07]. A converse theorem, roughly speaking, says that for suitable class of functions F (defined on an algebraic group), there is an equivalence between - on one side - the modularity of F and - on the other side - symmetries of L-functions attached to F. We will detail one such converse theorem for $\Gamma^{(2)}$, due to Imai, which Duke and Imamoglu [DI96] applied to prove the modularity of Saito-Kurokawa lift.

2.3 Duke and Imamoglu's Proof of the Saito-Kurokawa Lift

We first develop Imai's Converse Theorem:

Remark 2.3. In Definitions 2.4 and 2.8, the notation $\sum_{B>0/\mathrm{SL}_n(\mathbb{Z})}$ denotes summation over the classes of $\mathcal{S}_n(\mathbb{Z})^+$ under the equivalence relation $B_1 \equiv B_2$ if $B_2 = B_1[U]$ for a $U \in \mathrm{SL}_n(\mathbb{Z})$.

Definition 2.4. Given $\{A(B)\}$, with $B \in \mathcal{S}_2(\mathbb{Z})^+$, an $SL_2(\mathbb{Z})$-admissible sequence, and φ a Δ-eigenfunction in $L^2(\Gamma(1)\backslash\mathcal{H})$, we define the *Koecher-Maass Dirichlet series* as:

$$Z(\varphi; s) = Z(\{A(B)\}, \varphi; s) := \sum_{B>0/\mathrm{SL}_2(\mathbb{Z})} \frac{A(B)\varphi(z_B)}{e(B)\det(B)^{s+(k-1)/2}}, \qquad \mathrm{Re}(s) \text{ large},$$

where the sum runs over the $SL_2(\mathbb{Z})$-equivalence classes of $\mathcal{S}_2(\mathbb{Z})^+$ (See Remark 2.3),

$$e(B) := \#\{U \in \mathrm{SL}_2(\mathbb{Z}) \ : \ B[U] = B\} < \infty,$$

and $z_B := x + iy \in \mathcal{H}$ is defined via the Iwasawa decomposition [Gol06, §1.2]:

$$B = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \cdot K \cdot A, \qquad K \in \mathrm{SO}_2(\mathbb{R}), \ \ A \in \mathbb{R}^+.$$

Theorem 2.5. ([Ima80]) Suppose $\{A(B)\}$ is an $SL_2(\mathbb{Z})$-admissible sequence. Define $F : \mathcal{H}_2 \to \mathbb{C}$ via $F(Z) := \sum_{B>0} A(B)e^{2\pi i \mathrm{Tr}(BZ)}$. Let $k \geq 0$. The following are equivalent:

(1) $F \in S_k(\Gamma^{(2)})$,

(2) For all Δ-eigenfunctions φ in $L^2(\Gamma(1)\backslash\mathcal{H})$ with the same parity as k, the function:

$$\Lambda(\varphi; s) := (2\pi)^{-2s}\Gamma(s + k/2 - 3/4 + ir/2)\Gamma(s + k/2 - 3/4 - ir/2)Z(\varphi; s)$$

(where $\Delta\varphi = (1/4 + r^2)\varphi$) is entire, bounded in vertical strips of $\mathbb{C}$, and satisfies:

$$\Lambda(\varphi; 1 - s) = (-1)^k\Lambda(\varphi; s).$$

Duke and Imamoglu verify the modularity of the Saito-Kurokawa lift as follows:

(1) They start with a $f \in S^+_{k+1/2}(\Gamma_0(4))$ with Fourier expansion $f(z) = \sum_{n>0} c(n)e^{2\pi inz}$. They define a $\Gamma^{(2)}$-admissible sequence $\{A(B)\}$ using (2.1).

(2) For φ a Δ-eigenfunction in $L^2(\Gamma(1)\backslash\mathcal{H})$, they rewrite the Dirichlet series $Z(\varphi; s)$ as:

$$Z(\varphi; s) = 2^{2s+k}\sum_{n\geq 1}\frac{c(n)b(-n)}{n^{s+(k-1)/2}},$$

where $b(n)$ are the Fourier coefficients of a certain "modular" function; which we denote by $g(z)$; which depends only on the data of the twisting eigenfunction φ.

(3) Using the unfolding procedure, they derive a Rankin-Selberg convolution:

$$\Lambda(\varphi; s) = 2^{2s}\int_{\Gamma_0(4)\backslash\mathcal{H}} y^{k/2+1/4}f(z)g(z)\tilde{E}_\infty(z, s)\frac{dx\,dy}{y^2},$$

where $\tilde{E}_\infty(z, s)$ is a certain normalization of an Eisenstein series for $\Gamma_0(4)$,

(4) Using the functional equations of the Eisenstein series for $\Gamma_0(4)$ (with respect to the change of variable $s \mapsto 1-s$), they prove the desired functional equation for $\Lambda(\varphi; s)$:

$$\Lambda(\varphi; 1-s) = (-1)^k \Lambda(\varphi; s),$$

(5) They conclude that $F_f(Z) := \sum_{B>0} A(B) e^{2\pi i \mathrm{Tr}(BZ)}$ is in $S_{k+1}(\Gamma^{(2)})$.

2.4 The Ikeda Lift

In [Ike01], Ikeda constructed a generalization of the Saito-Kurokawa lift conjectured by Duke and Imamoglu. For $n \geq 1$, the lift is from the space $S_{2k}(\Gamma(1))$, $k \equiv n \bmod 2$ into the space $S_{k+n}(\Gamma^{(2n)})$. To write the Fourier expansion of the Ikeda lift, we recall some local quantities attached to $B \in \mathcal{S}'_{2n}(\mathbb{Z})$. Our aim here is to rapidly define the Siegel series. For further detauls, see [Ike01]. For each p, let $\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ be the unique additive character such that $\psi_p(x) = \exp((-1)^{n+1} 2\pi i x)$ for $x \in \mathbb{Z}[p^{-1}]$. The **Siegel series** for B is defined by:

$$b_p(B, s) := \sum_{R \in \mathcal{S}_{2n}(\mathbb{Q}_p)/\mathcal{S}_{2n}(\mathbb{Z}_p)} \psi_p(\mathrm{Tr}(BR)) p^{-\mathrm{ord}_p(\nu(R))s}, \qquad \mathrm{Re}(s) \gg 0$$

where

$$\nu(R) := \det(D) \cdot \mathbb{Z}_p, \qquad \text{where } C, D \in \mathrm{M}_{2n}(\mathbb{Z}_p) \text{ are coprime with } D^{-1}C = R.$$

Recall the quanties D_B, f_B, and $\xi_p(B)$ from Definition 1.12. Define $\gamma_p(B; X) \in \mathbb{Z}[X]$ by:

$$\gamma_p(B, X) := (1-X)(1 - p^n \xi_p(B) X)^{-1} \prod_{i=1}^{n} (1 - p^{2i} X^2).$$

There exists a polynomial $F_p(B;X) \in \mathbb{Z}[X]$ such that

$$b_p(B,s) = \gamma_p(B;p^{-s})F_p(B;p^{-s})$$

Put

$$\tilde{F}_p(B;X) := X^{-\mathrm{ord}_p f_B} F_p(B;p^{-n-1/2}X).$$

Then $\tilde{F}_p(B;X)$ is a symmetric Laurent polynomial. That is,

$$\tilde{F}_p(B;X) = \tilde{F}_p(B;X^{-1})$$

We are ready to state Ikeda's result:

Theorem 2.6. ([Ike01, Theorems 3.2 & 3.3]) Fix $n \geq 1$ and $k \equiv n \bmod 2$. Suppose $f \in \mathcal{S}_{2k}(\Gamma(1))$ is a normalized Hecke eigenform with Satake parameters $\alpha_p^{\pm 1}$ at p; that is,

$$(1 - p^{k-1/2}\alpha_p X)(1 - p^{k-1/2}\alpha_p^{-1}X) = 1 - a(p)X + p^{2k-1}X^2,$$
$$a(p) = p\text{th Fourier coefficient of } f.$$

Let $g \in S^+_{k+1/2}(\Gamma_0(4))$ be the form corresponding to f under the Shimura correspondence. Write $g(z) = \sum_{n>0} c(n)e^{2\pi i n z}$. Define $F_f : \mathcal{H}_{2n} \to \mathbb{C}$ via $F_f(Z) := \sum_{B>0} A(B)e^{2\pi i \mathrm{Tr}(BZ)}$, where

$$A(B) := c(|D_{B,0}|) f_B^{k-1/2} \prod_{p \mid D_B} \tilde{F}_p(B;\alpha_p), \qquad B \in \mathcal{S}_{2n}(\mathbb{Z})^+. \tag{2.2}$$

Then $F_f \in S_{k+n}(\Gamma^{(2n)})$ and F_f is a Hecke eigenform for which:

$$L(s, F_f, \mathrm{std}) = \zeta(s) \prod_{i=1}^{2n} L(s+k+n-i, f).$$

Here, $L(s,f) := \sum_{n>0} a(n)n^{-s}$ where $f(z) = \sum_{n>0} a(n)e^{2\pi inz}$.

Soon after Ikeda's work, Kohnen provides a linear version of the lift:

Theorem 2.7. ([Koh02, Theorem 1 & Corollary]) With notation as in Theorem 2.6:

$$A(B) = \sum_{a \mid f_B} c\left(\frac{|D_B|}{a^2}\right) a^{k-1}\phi(a;B).$$

Here $\phi(a;B)$ are certain integer-valued functions which we will introduce in Chapter 5. The map

$$\sum_{\substack{n>0 \\ (-1)^k n \equiv 0,1 \bmod 4}} c(n)e^{2\pi iz} \longmapsto \sum_{B \in \mathcal{S}_{2n}(\mathbb{Z})^+} \left(\sum_{a \mid f_B} c\left(\frac{|D_B|}{a^2}\right) a^{k-1}\phi(a;B) \right) e^{2\pi i \mathrm{Tr}(BZ)},$$

$$z \in \mathcal{H}, \quad Z \in \mathcal{H}_{2n}.$$

maps $S_{k+1/2}(\Gamma_0(4))$ to $S_{k+n}(\Gamma^{(n)})$ and on Hecke eigenforms agrees with Ikeda's lifting.

In the next section, we state a converse theorem due to Weissauer which could be used to prove the modularity of the Ikeda lift in the spirit of Duke and Imamoglu's proof for the Saito-Kurokawa lift. To our knowledge, this converse theorem has not yet seen applications.

2.5 Weissauer's Converse Theorem for $\Gamma^{(n)}$

We now develop Weissauer's Converse Theorem at full level:

Definition 2.8. Given $\{A(B)\}$, with $B \in \mathcal{S}_n(\mathbb{Z})^+$, an $\mathrm{SL}_n(\mathbb{Z})$-admissible sequence, and u a Großencharacter with respect to $\mathrm{SL}_n(\mathbb{Z})$, we define the *Koecher-Maass Dirichlet series*:

$$Z^{(n)}(u;s) = Z^{(n)}(\{A(B)\}, u; s) := \sum_{B>0/\mathrm{SL}_n(\mathbb{Z})} \frac{A(B)u(B)}{e(B)\det(B)^{s+(k-1)/2}}, \qquad \mathrm{Re}(s) \text{ large},$$

where the sum runs over the $\mathrm{SL}_n(\mathbb{Z})$-equivalence classes of $\mathcal{S}_n(\mathbb{Z})^+$ (See Remark 2.3), and

$$e(B) := \#\{U \in \mathrm{SL}_n(\mathbb{Z}) \ : \ B[U] = B\} < \infty.$$

Theorem 2.9. ([Wei84]) Suppose $\{A(B)\}$ is an $\mathrm{SL}_n(\mathbb{Z})$-admissible sequence, $n \geq 1$. Define $F(Z) := \sum_{B>0} A(B)e^{2\pi i \mathrm{Tr}(BZ)}$, $Z \in \mathcal{H}_n$. Let $k \geq 0$. The following are equivalent:

(1) $F \in S_k(\Gamma^{(n)})$

(2) For all Großencharacters u with respect to $\mathrm{SL}_n(\mathbb{Z})$, the function:

$$\Lambda^{(n)}(u;s) := (2\pi)^{-ns}\left(\prod_{i=1}^{n}\Gamma(s+(k-1)/2-\alpha_i)\right)Z^{(n)}(u;s),$$

is entire, bounded in vertical strips of $\mathbb{C}$, and satisfies the functional equation:

$$\Lambda^{(n)}(u;s) = i^{nk}\Lambda^{(n)}(v;1-s), \qquad v(Y) := u(Y^{-1}).$$

Note: By [Wei84, §2.1.3], $v(Y) := u(Y^{-1})$ is also a Großencharacter. The parameters $\alpha_1, \ldots, \alpha_n$ depend only on the δ_i-eigenvalues of u. See Appendix A for more details.

Remark 2.10. This is precisely Imai's Converse Theorem when $n = 2$. Indeed:

(1) Let φ be a Δ-eigenfunction and u the corresponding Großencharacter for $\mathrm{SL}_2(\mathbb{Z})$.

Suppose $\Delta\varphi = (1/4 + r^2)\varphi$. Then, by Appendix A, we have:

$$\alpha_1 = \overline{\alpha_2} = 1/4 + ir/2.$$

(2) For $Y = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in \mathcal{P}_n$ and $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma^{(1)}$, we have:

$$Y[S] = \begin{pmatrix} -d & b \\ b & -a \end{pmatrix} = (b^2 - ad) \cdot Y^{-1}.$$

Hence by (2) and (3) of Definiton 1.26, we have $v = u$.

Remark 2.11. In [Wei84, §3.1.1], the right hand side of the functional equation (for level $\Gamma_0^{(n)}(N)$) has the form $\overline{\Lambda}^{(n)}(v; 1 - \overline{s})$, where the bar denotes complex conjugation. Indeed, Weissauer's results apply to a space of modular forms denoted by $S_k(\Gamma_0^{(n)}(N), \epsilon)$, where $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is a Dirichlet character. When $N = 1$, this character is trivial; hence real; and the complex conjugation is not needed by the remark before §1.1.3 of Weissauer.

Chapter 3

The Extended Gross-Keating Data

In 1993, Gross and Keating [GK93] introduced a certain invariant attached to a n-ary quadratic form over $\mathbb{Z}_p$ with an impetus towards applications in arithmetic geometry. The invariant is easy to calculate for $p \neq 2$ via the Jordan splitting; while for $p = 2$, the Gross-Keating invariant was only well understood for $n \leq 3$ until the foundational work of Ikeda and Katsurada [IK18] on the so-called extended Gross-Keating (EGK) datum. Ikeda and Katsurada [IK22b] later demonstrate that the Siegel series $\tilde{F}_p(B, X)$ attached to a quadratic form $B \in \mathcal{S}'_n(\mathbb{Z}_p)$ is completely determined by the EGK datum of B. Moreover, in the same paper, they provide explicit inductive formulas for computing the Siegel series via the EGK datum. While explicit formulas for the Siegel series due to Katsurada [Kat99] were available much earlier, the formulas in [IK22b] treat the dyadic ($p = 2$) and non-dyadic ($p \neq 2$) settings uniformly. There has since been further work towards understanding various theoretical and computational aspects of the EGK invariants; for instance, [CIK+17], [CY20], [Cho20], and [IK22a]. In this chapter, we introduce Ikeda and Katsurada's formulation of the EGK datum. At the end of this chapter, we develop a connection (which we will need in subsequent chapters) between the Gross-Keating invariants and two basic invariants, f_B and $\mathrm{cont}B$, attached to a quadratic form $B \in \mathcal{S}'_n(\mathbb{Z}_p)$.

3.1 Extended Gross-Keating Invariant of a Quadratic Form

We now introduce the Gross-Keating invariant of a quadratic form $B \in \mathcal{S}'_n(\mathbb{Z}_p)$. Let

$$[B] := \{B[U] \ : \ U \in \mathrm{GL}_n(\mathbb{Z}_p)\}, \qquad B[U] := {}^tUBU$$

We call $[B]$ the *equivalence class* of B.

Definition 3.1. For $B \in \mathcal{S}'_n(\mathbb{Z}_p)$, define

$$S([B]) := \bigcup_{B' \in [B]} S(B'),$$

where $S(B')$, for $B' = (b'_{ij})$, is the set of non-decreasing sequences $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ which satisfy

$$\begin{aligned} \mathrm{ord}_p(b'_{ii}) &\geq a_i && (1 \leq i \leq n) \\ \mathrm{ord}_p(2b'_{ij}) &\geq (a_i + a_j)/2 && (1 \leq i, j \leq n). \end{aligned}$$

Definition 3.2. Let $B \in \mathcal{S}'_n(\mathbb{Z}_p)$. The **Gross-Keating invariant** of B, denoted by $\mathrm{GK}(B)$, is the greatest element of the set $S([B])$ with respect to the lexicographic order $\geq$ on $\mathbb{Z}^n_{\geq 0}$.

Definition 3.3. $B \in \mathcal{S}'_n(\mathbb{Z}_p)$ is an **optimal** form if $\mathrm{GK}(B) \in S(B)$.

By Definition 3.1, a matrix $B \in \mathcal{S}'_n(\mathbb{Z}_p)$ is equivalent to an optimal form.

Example 3.4. For $p \neq 2$, any $B \in \mathcal{S}'_n(\mathbb{Z}_p)^{\mathrm{nd}}$ is equivalent to a diagonal matrix of the form:

$$T = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, \qquad \mathrm{ord}_p(t_1) \leq \cdots \leq \mathrm{ord}_p(t_n).$$

Then, according to [Bou07, Proposition 2.6]:

$$\mathrm{GK}(B) = (\mathrm{ord}_p(t_1), \ldots, \mathrm{ord}_p(t_n)).$$

We introduce the technical machinery needed to define the extended Gross-Keating invariant. We need to recall some concepts from the classical theory of quadratic forms.

Definition 3.5. The **Clifford invariant** (see [Sch85, p. 333]) of $B \in \mathcal{S}'_n(\mathbb{Z}_p)$ is the Hasse invariant of the Clifford algebra (resp. even Clifford algebra) of B if n is even (resp. odd). We denote the Clifford invariant of B by $\eta(B) := \eta_B$.

Definition 3.6. Let $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ be non-decreasing. Let $a^*_1 < \cdots < a^*_r$ denote the distinct elements in $\underline{a}$. We define $n_s = \#\{i \mid a_i = a^*_s\}$ and $n^*_s = \sum_{i=1}^{s} n_i$ for $s = 1, \ldots, r$. In particular $n^*_r = n$. By convention $n^*_0 = 0$.

Definition 3.7. Let $T \in \mathcal{S}_n(\mathbb{Z}_p)$ be a diagonal matrix of the form:

$$T = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, \qquad \mathrm{ord}_p(t_1) \leq \cdots \leq \mathrm{ord}_p(t_n).$$

We define

$$\mathrm{NEGK}(T) = (a_1, \dots, a_n; \varepsilon_1, \dots, \varepsilon_n),$$

where,

$$a_i = \mathrm{ord}_p(t_i),$$

and,

$$\varepsilon_i = \begin{cases} \xi_{T^{(i)}} & \text{if } i \text{ is even,} \\ \eta_{T^{(i)}} & \text{if } i \text{ is odd,} \end{cases}$$

We call $\mathrm{NEGK}(T)$ the **naive EGK datum** attached to T.

Put $\mathcal{Z}_3 = \{0, 1, -1\}$.

Definition 3.8. For an optimal form $B \in \mathcal{S}'_n(\mathbb{Z}_p)$, we define:

$$\mathrm{EGK}(B) := (n_1, \dots, n_r; a_1^*, \dots, a_r^*; \zeta_1, \dots, \zeta_r) \in \mathbb{Z}_{\geq 0}^r \times \mathbb{Z}_{\geq 0}^r \times \mathcal{Z}_3^r.$$

Here, via Definition 3.6, the $a_1^*, \dots, a_r^*$ are computed from the Gross-Keating invariants:

$$(a_1, \dots, a_n) = \mathrm{GK}(B),$$

and

$$\zeta_s = \begin{cases} \xi_{B^{(n_s^*)}} & \text{if } n_s^* \text{ is even,} \\ \eta_{B^{(n_s^*)}} & \text{if } n_s^* \text{ is odd,} \end{cases}$$

and $n_1, \dots, n_r, n_1^*, \dots, n_r^*$ are as in Definition 3.6. For any $B \in \mathcal{S}'_n(\mathbb{Z}_p)$, we define

$$\begin{aligned} \mathrm{EGK}(B) &:= \mathrm{EGK}(B'), \\ &B' \in \mathcal{S}'_n(\mathbb{Z}_p) \text{ an optimal form such that } B' = B[U] \text{ for a } U \in \mathrm{GL}_n(\mathbb{Z}_p). \end{aligned}$$

We call $\mathrm{EGK}(B)$ the **EGK invariant** attached to B.

Remark 3.9. By [IK18, Theorem 0.4], Definition 3.8 is independent of the choice of B'.

3.2 Definition of Extended Gross-Keating Datum

In this section, we introduce an abstraction of the the extended Gross-Keating invariants.

Definition 3.10. An element $H = (a_1, \dots, a_n; \varepsilon_1, \dots, \varepsilon_n) \in \mathbb{Z}_{\geq 0}^n \times \mathcal{Z}_3^n$ is said to be a **naive EGK datum** of length n if the following conditions hold:

(N1) $a_1 \leq \cdots \leq a_n$,

(N2) Assume that i is even. Then $\varepsilon_i \neq 0$ if and only if $a_1 + \cdots + a_i$ is even,

(N3) If i is odd, then $\varepsilon_i \neq 0$,

(N4) $\varepsilon_1 = 1$,

(N5) If $i \geq 3$ and $a_1 + \cdots + a_{i-1}$ is even, then $\varepsilon_i = \varepsilon_{i-2}\varepsilon_{i-1}^{a_i + a_{i-1}}$.

We denote the set of naive EGK data of length n by $\mathcal{NEGK}_n$.

Definition 3.11. Let $H = (a_1, \dots, a_n; \varepsilon_1, \dots, \varepsilon_n)$ be a naive EGK datum of length n. Given $0 \leq m < n$, we define the m**-truncation** of H as $H^{(m)} := (a_1, \dots, a_{n-m}; \varepsilon_1, \dots, \varepsilon_{n-m})$. Observe $H^{(m)} \in \mathcal{NEGK}_{n-m}$. We denote $H^{(1)}, H^{(2)}$ as H', H'', respectively, for convenience.

With T as in Definition 3.8, we know via [IK18, Proposition 6.1] that, for $p \neq 2$, $\mathrm{NEGK}(T) \in \mathcal{NEGK}_n$; and conversely via [IK18, Remark 6.1] that for an $H \in \mathcal{NEGK}_n$ there exists a diagonal matrix of the form in Definition 3.8 for which $\mathrm{NEGK}(T) = H$. However, when $p = 2$ and $T = \mathrm{diag}(t_1, \dots, t_n)$ with $\mathrm{ord}_p(t_1) \leq \cdots \leq \cdots \mathrm{ord}_p(t_n)$, we might have $\mathrm{NEGK}(T) \notin \mathcal{NEGK}_n$. For an example, see [IK18, Remark 6.2]. Thus, the naive EGK datum attached to a diagonal T is not a robust invariant for a quadratic form. There are still uses of naive EGK datum towards the computation of the Siegel series. However, for now we introduce Ikeda and Katsurada's more refined notion of EGK datum:

Definition 3.12. Let $G = (n_1, \dots, n_r; m_1, \dots, m_r; \zeta_1, \dots, \zeta_r) \in \mathbb{Z}_{\geq 0}^r \times \mathbb{Z}_{\geq 0}^r \times \mathcal{Z}_3^r$. Put $n_s^* = \sum_{i=1}^s n_i$ for $s \leq r$. We say that G is an **EGK datum** of length n if the following conditions hold:

(E1) $n_r^* = n$ and $m_1 < \cdots < m_r$,

(E2) Assume that n_s^* is even. Then $\zeta_s \neq 0$ if and only if $m_1 n_1 + \cdots + m_s n_s$ is even,

(E3) Assume that n_s^* is odd. Then $\zeta_s \neq 0$. Moreover, we have

(a) Assume that n_i^* is even for any $i < s$. Then we have

$$\zeta_s = \zeta_1^{m_1+m_2} \zeta_2^{m_2+m_3} \cdots \zeta_{s-1}^{m_{s-1}+m_s}.$$

In particular, $\zeta_1 = 1$ if n_1 is odd.

(b) Assume that $m_1 n_1 + \cdots m_{s-1} n_{s-1} + m_s(n_s - 1)$ is even and that n_i^* is odd for some $i < s$. Let $t < s$ be the largest number such that n_t^* is odd. Then we have

$$\zeta_s = \zeta_t \zeta_{t+1}^{m_{t+1}+m_{t+2}} \zeta_{t+2}^{m_{t+2}+m_{t+3}} \cdots \zeta_{s-1}^{m_{s-1}+m_s}.$$

In particular, $\zeta_s = \zeta_t$ if $t + 1 = s$.

We denote the set of EGK data of length n by $\mathcal{EGK}_n$.

Theorem 3.13. ([IK18, Theorem 6.1]) Let $B \in \mathcal{S}'_n(\mathbb{Z})^{\mathrm{nd}}$. Then $\mathrm{EGK}(B) \in \mathcal{EGK}_n$.

Proposition 3.14. ([IK18, Proposition 6.2]) Let $H = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{NEGK}_n$. Let r, n_s, n_s^*, $1 \le s \le r$, be as in Definition 3.6. For $1 \le s \le r$, set $m_s = a_{n_s^*}$ and $\zeta_s = \varepsilon_{n_s^*}$. Then,

$$G = (n_1, \ldots, n_r; m_1, \ldots, m_r; \zeta_1, \ldots, \zeta_r) \in \mathcal{EGK}_n. \tag{3.1}$$

Via Proposition 3.14, we have a map:

$$\Upsilon = \Upsilon_n : \mathcal{NEGK}_n \longrightarrow \mathcal{EGK}_n \tag{3.2}$$

$$H \longmapsto G = \Upsilon(H), \tag{3.3}$$

where G is constructed via (3.1). By [IK18, Proposition 6.3]) we know that Υ is surjective.

3.3 Connection to the Invariants of f_B and cont B

Definition 3.15. We extend the greatest common denominator function to $\mathbb{Q}$ as follows. For $a, b \in \mathbb{Q}$, we define:

$$\gcd(a, b) := \prod_{p \text{ prime}} p^{\min\{\mathrm{ord}_p(a), \mathrm{ord}_p(b)\}}.$$

Given $B = (b_{ij}) \in \mathcal{S}_n(\mathbb{Q})$, we define a quantity:

$$\mathrm{cont}\, B := \gcd(\underbrace{b_{11}, \ldots, b_{nn}}_{n \text{ terms}}, \underbrace{2b_{12}, \ldots, 2b_{n,n-1}}_{T_{n-1} \text{ terms}}). \tag{3.4}$$

Here $T_n = \frac{n(n+1)}{2}$ denotes the nth triangular number. We call $\operatorname{cont} B$ the **content** of B.

Definition 3.16. Let $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$. For prime p, let $\mathrm{GK}(B)^{(p)} = (a_1^{(p)}, \dots, a_{2n}^{(p)})$ denote the Gross-Keating invariant of B viewed as an element of $\mathcal{S}'_{2n}(\mathbb{Z}_p)^{\mathrm{nd}}$. For $i = 1, \dots, 2n$, put

$$\mathfrak{e}_{i,B}^{(p)} := \begin{cases} a_1^{(p)} + \cdots + a_i^{(p)} & \text{if } i \text{ is odd,} \\ 2\lfloor (a_1^{(p)} + \cdots + a_i^{(p)})/2 \rfloor & \text{if } i \text{ is even.} \end{cases} \tag{3.5}$$

Let us first relate the local invariant $\mathfrak{e}_{2n,B}^{(p)}$ to the global invariant f_B:

Lemma 3.17. ([IK22a, Lemma 6.3]) Let $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$. Then, $\prod_{p \mid f_B} p^{\mathfrak{e}_{2n,B}^{(p)}} = f_B^2$.

Corollary 3.18. Let $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$. Then $\mathfrak{e}_{2n,B}^{(p)} = 2\mathrm{ord}_p f_B$.

Proof. This follows from the fundamental theorem of arithmetic. ∎

We also have a relation between the local invariant $\mathfrak{e}_{1,B}^{(p)}$ and the global invariant $\operatorname{cont} B$:

Lemma 3.19. ([IK22a, Remark 4.5]) Let $B = (b_{i,j}) \in \mathcal{S}'_{2n}(\mathbb{Z})^+$. Then $\mathfrak{e}_{1,B}^{(p)} = \min_{1 \le i,j \le n} \mathrm{ord}_p(b_{i,j}^{(1)})$, where

$$b_{i,j}^{(1)} = 2^{1-\delta_{i,j}} b_{i,j}, \qquad \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Corollary 3.20. Let $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$. Then $\mathfrak{e}_{1,B}^{(p)} = \mathrm{ord}_p(\operatorname{cont} B)$.

Proof. This follows immediately from the definition (3.4). We now provide a direct proof. By [IK18, §2], $\mathfrak{e}_{1,B}^{(p)} = 0$ for B primitive ; i.e. when $p^{-1}B \notin \mathcal{S}'_{2n}(\mathbb{Z}_p)^+ \Leftrightarrow \mathrm{ord}_p(\operatorname{cont} B) = 0$. For any $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$, write $B = p^k B'$, B' primitive. Then,

$$\mathfrak{e}_{1,B}^{(p)} = k + \mathfrak{e}_{1,B'}^{(p)} = k = k + \mathrm{ord}_p(\operatorname{cont} B') = \mathrm{ord}_p(\operatorname{cont} B).$$

■

Remark 3.21. This agrees with [CIK+17, Theorem 3.1]. The relevant cases in the statement of Theorem 3.1, in their notation, are (1) and (2.1), $\xi_{B^{[s-1]}} \neq 0$. The case (3) also seems to apply, but according to condition (PO3) in the definition of a pre-optimal form, the first block of a pre-optimal form must not be a diagonal unimodular matrix of degree 2.

Chapter 4

Laurent Polynomial of an EGK Datum

Quite recently, Ikeda and Katsurada [IK22b] have shown that p-local Siegel series $\tilde{F}_p(B, X)$ attached to a quadratic form $B \in \mathcal{S}_n'(\mathbb{Z})$ is completely determined by the EGK datum of B, viewed as an element of $\mathcal{S}_n'(\mathbb{Z}_p)$. Their proof uses deep facts about local densities of quadratic forms. In our work, we are interested in the result of their calculations: an inductive formula for a certain Laurent polynomial attached to an EGK datum H; denoted $\mathcal{F}(H; Y, X)$; in the variable $X^{\frac{1}{2}}$ with coefficients in $\mathbb{Z}[Y, Y^{-1}]$. Upon specializing to $H = \mathrm{EGK}(B)^{(p)}$ and $Y = p^{\frac{1}{2}}$, one obtains the p-local Siegel series $\tilde{F}_p(B, X)$. In this section, we develop a wealth of technical tools for computing this Laurent polynomial efficiently. A novel feature of our work is a strategic reorganization of the inductive formulas of Ikeda and Katsurada which unveils a deeper combinatorial interpretation of the coefficients of $\mathcal{F}(H; Y, X)$. At the end of this chapter, we provide explicit formulas for $\mathcal{F}(H; Y, X)$ in the case $n = 2$ and 3.

4.1 Definion of the Laurent Polynomial

We now define, à la Ikeda and Katsurada, the polynomial $\mathcal{F}(H; Y, X)$ attached to H:

Definition 4.1. For $e, \tilde{e} \in \mathbb{Z}$, $\xi \in \mathbb{R}$, we define rational functions in $X^{\frac{1}{2}}$ and $Y^{\frac{1}{2}}$ via:

$$C(e, \tilde{e}, \xi; Y, X) = \frac{Y^{\frac{\tilde{e}}{2}} X^{-\frac{e-\tilde{e}}{2}-1}(1-\xi Y^{-1}X)}{X^{-1}-X}$$

and

$$D(e, \tilde{e}, \xi; Y, X) = \frac{Y^{\frac{\tilde{e}}{2}} X^{-\frac{e-\tilde{e}}{2}}}{1-\xi X}.$$

For a positive integer i, put

$$C_i(e, \tilde{e}, \xi; Y, X) = \begin{cases} C(e, \tilde{e}, \xi; Y, X) & \text{if } i \text{ is even,} \\ D(e, \tilde{e}, \xi; Y, X) & \text{if } i \text{ is odd.} \end{cases} \tag{4.1}$$

Definition 4.2. For a sequence $\underline{a} = (a_1, \ldots, a_n)$ of integers and $1 \le i \le n$, we define

$$\mathfrak{e}_i = \mathfrak{e}_i(\underline{a}) := \begin{cases} a_1 + \cdots + a_i & \text{if } i \text{ is odd,} \\ 2\lfloor (a_1 + \cdots + a_i)/2 \rfloor & \text{if } i \text{ is even.} \end{cases}$$

We also put $\mathfrak{e}_0 = 0$.

Definition 4.3. For a naive EGK datum $H = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n)$ of length n we define a rational function $\mathcal{F}(H; Y, X)$ in $X^{\frac{1}{2}}$ and $Y^{\frac{1}{2}}$ inductively as follows. First, if $n = 1$, we set

$$\mathcal{F}(H; Y, X) = \sum_{i=0}^{a_1} X^{-\frac{a_1}{2}+i}. \tag{4.2}$$

Let $n > 1$. Then $H' = (a_1, \ldots, a_{n-1}; \varepsilon_1, \ldots, \varepsilon_{n-1})$ is a naive EGK datum of length $n - 1$.

We define, inductively,

$$\mathcal{F}(H;Y,X) = C_n(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi; Y, X)\mathcal{F}(H';Y,YX) \\ + \zeta C_n(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi; Y, X^{-1})\mathcal{F}(H';Y,YX^{-1}), \tag{4.3}$$

where

$$\xi = \begin{cases} \varepsilon_n & \text{if } n \text{ is even,} \\ \varepsilon_{n-1} & \text{if } n \text{ is odd,} \end{cases} \qquad \zeta = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \varepsilon_n & \text{if } n \text{ is odd.} \end{cases}$$

Although $\mathcal{F}(H;Y,X)$ is defined as a *rational function* in $X^{\frac{1}{2}}$ and $Y^{\frac{1}{2}}$, we know:

Proposition 4.4. ([IK22b, Proposition 4.2]) For $H = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n)$ a naive EGK datum of length n, $\mathcal{F}(H;Y,X)$ is a Laurent *polynomial* in $X^{\frac{1}{2}}$ with coefficients in $\mathbb{Z}[Y, Y^{-1}]$.

We also know that $\mathcal{F}(H;Y,X)$ has a symmetry with respect to the variable X:

Proposition 4.5. ([IK22b, Proposition 4.1]) For $H = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{NEGK}_n$,

$$\mathcal{F}(H;Y,X) = \eta_n \mathcal{F}(H;Y,X^{-1}),$$

where

$$\eta_n := \begin{cases} 1 & \text{if } n \text{ is even,} \\ \varepsilon_n & \text{if } n \text{ is odd.} \end{cases}$$

As mentioned in the previous chapter, although the naive EGK datum is not a robust invariant for a quadratic form, it finds its use in the calculation of $\mathcal{F}(H;Y,X)$. Specifically, the following result of Ikeda and Katsurada states that - although $\mathcal{F}(H;Y,X)$ is defined via an naive EGK datum - it depends only on the image of H in $\mathcal{E}GK_n$ under the Υ map (3.2). That is,

Theorem 4.6. ([IK22b, Theorem 4.1]) Let $G \in \mathcal{E}GK_n$; take $H \in \Upsilon_n^{-1}(G) \subseteq \mathcal{N}EGK_n$. Then $\mathcal{F}(H;Y,X)$ is uniquely determined by G; that is does not depend on the choice of H.

4.2 Technical Lemmas for Calculating the Laurent Polynomial

We now initiate our in-depth analysis of the Laurent polynomial attached an a naive EGK datum. The main results of this chapter are Propositions 4.8, 4.12, and 4.14. In these propositions, we work out the induction step (4.3) as explicitly as possible. From them, we obtain induction formulas for $\mathcal{F}(H;Y,X)$ involving no division in a rational function field. We use these results heavily later to study individual coefficients of the Laurent polynomial.

Our induction formulas are phrased in terms of two distinct bases for the symmetric Laurent polynomials of bounded degree in X. We now introduce these two bases formally.

Let $\mathbb{Q}(Y)$ denote the field of rational functions in the variable Y with coefficients in $\mathbb{Q}$. Let V denote the $\mathbb{Q}(Y)$-vector space of Laurent polynomials in the variable X with coefficients in $\mathbb{Q}(Y)$. For $\ell \geq 0$, let V_ℓ denote the following $(\ell+1)$-dimensional $\mathbb{Q}(Y)$-subspace of V:

$$V_\ell := \left\{ \sum_{\nu=0}^{\ell} c_\nu(Y)(X^\nu + X^{-\nu}) \ : \ c_\nu(Y) \in \mathbb{Q}(Y) \right\}.$$

That is, V_ℓ is the $\mathbb{Q}(Y)$-subspace of symmetric Laurent polynomials of degree $\leq \ell$. The set $1, X+X^{-1}, X^2+X^{-2}, \dots, X^\ell + X^{-\ell}$ is a $\mathbb{Q}(Y)$-basis of V_ℓ, which we call the **natural basis**.

We will also need to work in another basis of V_ℓ. Specifically, we define:

$$\psi_j(X) := \frac{X^j - X^{-j}}{X - X^{-1}} = X^{j-1} + X^{j-3} + \cdots + X^{-j+3} + X^{-j+1}, \qquad j \geq 0. \tag{4.4}$$

Fix an $\varepsilon \in \mathcal{Z}_3$. As in [CK08, Lemma 1], we conclude that $\psi_{j+1}(X) - \varepsilon Y^{-1}\psi_j(X)$; for $j = 0, \ldots, \ell$; is a $\mathbb{Q}(Y)$-basis of V_ℓ, which we call the **ε-Kohnen-Choie basis**. We calculate a change-of-basis formula between the ε-Kohnen-Choie basis and the natural basis of V_ℓ:

Lemma 4.7. Let $F(Y, X) \in V_\ell$. Write

$$F(Y,X) = A_0(Y) + \sum_{i=1}^{\ell} A_i(Y)(X^i + X^{-i}),$$
$$A_i(Y) \in \mathbb{Q}(Y), \quad i = 0, \ldots, \ell.$$

and

$$F(Y,X) = \sum_{j=0}^{\ell} B_j(Y)\left(\psi_{j+1}(X) - \varepsilon Y^{-1}\psi_j(X)\right), \tag{4.5}$$
$$B_j(Y) \in \mathbb{Q}(Y), \quad j = 0, \ldots, \ell.$$

Part I: To the Natural Basis:

For $i = 0, \ldots, \ell$, we have

$$A_i(Y) = A_{i,0}(Y) - \varepsilon Y^{-1} A_{i,1}(Y), \qquad A_{i,k}(Y) := \sum_{\substack{i \leq \nu \leq \ell \\ \nu - i \equiv k \bmod 2}} B_\nu(Y), \quad k = 0, 1. \tag{4.6}$$

Part II: From the Natural Basis:

We have, inductively from $j = \ell$ to 0, the relation:

$$B_j(Y) = A_j(Y) - S_{j,0}(Y) + \varepsilon Y^{-1} S_{j,1}(Y), \qquad S_{j,k}(Y) := \sum_{\substack{j<\nu\leq\ell \\ \nu-j\equiv k \bmod 2}} B_\nu(Y), \quad k = 0, 1.$$

Proof of Part I. Observe

$$\begin{aligned} & B_i(Y)(\psi_{i+1}(X) - \varepsilon Y^{-1}\psi_i(X)) \\ & \qquad = B_i(Y)(X^i - \varepsilon Y^{-1} X^{i-1} + X^{i-2} + \cdots + X^{-i+2} - \varepsilon Y^{-1} X^{-i+1} + X^{-i}). \end{aligned}$$

Using Table 4.1, we calculate:

$$A_i(Y) = \sum_{\substack{i\leq\nu\leq\ell \\ \nu\equiv i \bmod 2}} B_\nu(Y) - \varepsilon Y^{-1} \sum_{\substack{i\leq\nu\leq\ell \\ \nu\not\equiv i \bmod 2}} B_\nu(Y).$$

■

$\times$	X^ℓ	$X^{\ell-1}$	$X^{\ell-2}$	$\cdots$	X^i
$B_\ell(Y)$	1	$-\varepsilon Y^{-1}$	1	$\cdots$	$\cdots$
$B_{\ell-1}(Y)$		1	$-\varepsilon Y^{-1}$	$\cdots$	$\cdots$
$B_{\ell-2}(Y)$			1	$\cdots$	$\cdots$
$\vdots$				$\cdots$	1
$B_{i+1}(Y)$				$\cdots$	$-\varepsilon Y^{-1}$
$B_i(Y)$				$\cdots$	1
$B_{i-1}(Y)$				$\cdots$	
$\sum$	$A_\ell(Y)$	$A_{\ell-1}(Y)$	$A_{\ell-2}(Y)$	$\cdots$	$A_i(Y)$

Table 4.1: An visual aid for the proof of Part I of Lemma 4.7

Proof of Part II. From Table 4.2, we that the jth unknown coefficient $B_j(Y)$ must satisfy:

$$A_j(Y) = \sum_{\substack{j \le \nu \le \ell \\ \nu \equiv j \bmod 2}} B_\nu(Y) - \varepsilon Y^{-1} \sum_{\substack{j \le \nu \le \ell \\ \nu \not\equiv j \bmod 2}} B_\nu(Y).$$

This is a triangular system, so we assume $B_{j+1}(Y), \ldots, B_\ell(Y)$ have been computed already.

	X^ℓ	$\vdots$	X^j	X^{j-1}
	$B_\ell(\ell)$	$\vdots$	$\vdots$	$\vdots$
		$\vdots$	$\vdots$	$\vdots$
		$\vdots$	$-\varepsilon Y^{-1} B_{j+1}(Y)$	$B_{j+1}(Y)$
			$B_j(Y)$	$-\varepsilon Y^{-1} B_j(Y)$
				$B_{j-1}(Y)$
$\sum$	$A_\ell(Y)$	$\vdots$	$A_j(Y)$	$A_{j-1}(Y)$

Table 4.2: An visual aid for the proof of Part II of Lemma 4.7

We thus have the a recursive relation for computing the unknown coefficient $B_j(Y)$ given the known coefficient $A_j(Y)$ and the previously computed coefficients $B_{j+1}(Y), \ldots, B_\ell(Y)$:

$$B_j(Y) = A_j(Y) - \sum_{\substack{j < \nu \le \ell \\ \nu \equiv j \bmod 2}} B_\nu(Y) + \varepsilon Y^{-1} \sum_{\substack{j < \nu \le \ell \\ \nu \not\equiv j \bmod 2}} B_\nu(Y).$$

∎

We now develop two fundamental results for performing the induction steps in (4.3). Proposition 4.8 treats the case with n even and Lemma 4.10 treats the case with n odd.

Proposition 4.8. Fix $\ell_2 \ge \ell_1 \ge 0$ with $\ell_1 \in \frac{1}{2}\mathbb{Z}$, $\ell_2 \in \mathbb{Z}$, $\varepsilon \in \mathcal{Z}_3$. Define $m := \min\{2\ell_1, \ell_2\}$.

Let $F(Y,X)$ be a Laurent polynomial in $X^{\frac{1}{2}}$ with coefficients in $\mathbb{Q}[Y,Y^{-1}]$ of the form:

$$F(Y,X) = \sum_{i=0}^{2\ell_1} C_{-\ell_1+i}(Y) X^{-\ell_1+i}. \tag{4.7}$$

Define

$$G(Y,X) := C(2\ell_2, 2\ell_1, \varepsilon; Y, X) F(Y, YX) + C(2\ell_2, 2\ell_1, \varepsilon; Y, X^{-1}) F(Y, YX^{-1}).$$

Then,

$$G(Y,X) = G_+(X,Y) - G_-(X,Y),$$

where

$$G_+(Y,X) = \sum_{i=0}^{m} Y^i C_{-\ell_1+i}(Y) X^{-\ell_2+i} \sum_{j=0}^{\ell_2-i} X^{2j} - \varepsilon \sum_{i=0}^{m} Y^{i-1} C_{-\ell_1+i}(Y) X^{-\ell_2+i+1} \sum_{j=0}^{\ell_2-i-1} X^{2j}, \tag{4.8}$$

$$G_-(Y,X) = \sum_{i=m+1}^{2\ell_1} Y^i C_{-\ell_1+i}(Y) X^{\ell_2-i+2} \sum_{j=0}^{-\ell_2+i-2} X^{2j} - \varepsilon \sum_{i=m+1}^{2\ell_1} Y^{i-1} C_{-\ell_1+i}(Y) X^{\ell_2-i+1} \sum_{j=0}^{-\ell_2+i-1} X^{2j}. \tag{4.9}$$

Proof. We have

$$C(2\ell_2, 2\ell_1, \varepsilon; Y, X^{\pm 1}) = \frac{Y^{\ell_1} X^{\mp(\ell_2-\ell_1)\mp 1}(1 - \varepsilon Y^{-1} X^{\pm 1})}{X^{\mp 1} - X^{\pm 1}}.$$

We calculate

$$\begin{aligned}
&G(X,Y)\\
&\quad= \frac{Y^{\ell_1}X^{-(\ell_2-\ell_1)-1}(1-\varepsilon Y^{-1}X)}{X^{-1}-X}F(Y,YX)\\
&\qquad+\frac{Y^{\ell_1}X^{(\ell_2-\ell_1)+1}(1-\varepsilon Y^{-1}X^{-1})}{X-X^{-1}}F(Y,YX^{-1})\\
&\quad= Y^{\ell_1}\frac{X^{-(\ell_2-\ell_1)-1}F(Y,YX)-X^{(\ell_2-\ell_1)+1}F(Y,YX^{-1})}{X^{-1}-X}\\
&\qquad-\varepsilon Y^{\ell_1-1}\frac{X^{-(\ell_2-\ell_1)}F(Y,YX)-X^{(\ell_2-\ell_1)}F(Y,YX^{-1})}{X^{-1}-X}.
\end{aligned}$$

Note

$$F(Y,YX^{\pm 1}) = \sum_{i=0}^{2\ell_1} C_{-\ell_1+i}(Y)(YX^{\pm 1})^{-\ell_1+i} = Y^{-\ell_1}\sum_{i=0}^{2\ell_1} Y^i C_{-\ell_1+i}(Y)X^{\mp\ell_1\pm i}.$$

Thus, for $\delta \in \{0,1\}$,

$$
\begin{aligned}
&Y^{\ell_1-\delta}\frac{X^{-(\ell_2-\ell_1)-1+\delta}F(Y,YX)-X^{(\ell_2-\ell_1)+1-\delta}F(Y,YX^{-1})}{X^{-1}-X}\\
&\quad=\frac{Y^{-\delta}}{X^{-1}-X}\left\{\sum_{i=0}^{2\ell_1}Y^iC_{-\ell_1+i}(Y)X^{-\ell_2+i-1+\delta}-\sum_{i=0}^{2\ell_1}Y^iC_{-\ell_1+i}(Y)X^{\ell_2-i+1-\delta}\right\}\\
&\quad=\sum_{i=0}^{2\ell_1}Y^{i-\delta}C_{-\ell_1+i}(Y)\left\{\frac{X^{-\ell_2+i-1+\delta}-X^{\ell_2-i+1-\delta}}{X^{-1}-X}\right\}\\
&\quad=\sum_{i=0}^{2\ell_1}Y^{i-\delta}C_{-\ell_1+i}(Y)X^{-\ell_2+i+\delta}\left\{\frac{1-X^{2\ell_2-2i+2-2\delta}}{1-X^2}\right\}\\
&\quad=\sum_{i=0}^{m}Y^{i-\delta}C_{-\ell_1+i}(Y)X^{-\ell_2+i+\delta}\left\{\frac{1-X^{2\ell_2-2i+2-2\delta}}{1-X^2}\right\}\\
&\qquad\qquad-\sum_{i=m+1}^{2\ell_1}Y^{i-\delta}C_{-\ell_1+i}(Y)X^{\ell_2-i+2-\delta}\left\{\frac{1-X^{-2\ell_2+2i-2+2\delta}}{1-X^2}\right\}\\
&\quad=\sum_{i=0}^{m}Y^{i-\delta}C_{-\ell_1+i}(Y)X^{-\ell_2+i+\delta}\sum_{j=0}^{\ell_2-i-\delta}X^{2j}\\
&\qquad\qquad-\sum_{i=m+1}^{2\ell_1}Y^{i-\delta}C_{-\ell_1+i}(Y)X^{\ell_2-i+2-\delta}\sum_{j=0}^{-\ell_2+i-2+\delta}X^{2j}.
\end{aligned}
$$

■

Remark 4.9. We may easily write $G_+(Y,X)$ in terms of the ε-Kohnen-Choie basis:

$$G_+(Y,X)=\sum_{i=0}^{m}Y^iC_{-\ell_1+i}(Y)(\psi_{\ell_2-i+1}(X)-\varepsilon Y^{-1}\psi_{\ell_2-i}(X)).$$

We can also write:

$$G_-(Y,X)=\sum_{i=m+1}^{2\ell_1}Y^iC_{-\ell_1+i}(Y)(\psi_{-\ell_2+i-1}(X)-\varepsilon Y^{-1}\psi_{-\ell_2+i}(X)).$$

However, the expression $\psi_{-\ell_2+i-1}(X)-\varepsilon Y^{-1}\psi_{-\ell_2+i}(X)$ is not in the ε-Kohnen-Choie basis.

We call $G_+(Y,X)$ and $G_-(Y,X)$ the **principal part** and **reflection**, respectively, of $G(Y,X)$.

Lemma 4.10. Fix integers $k \le m \le l$; $\varepsilon \in \mathcal{Z}_3$. Given a Laurent polynomial of the form:

$$G(Y,X) = \sum_{i=k}^{m} C_i(Y) X^i \sum_{j=0}^{l-i} X^{2j} - \varepsilon \sum_{i=k}^{m} Y^{-1} C_i(Y) X^{i+1} \sum_{j=0}^{l-i-1} X^{2j}, \tag{4.10}$$

define

$$H_\pm(Y,X) := \frac{1}{1-\varepsilon X^{\pm 1}} G(Y, YX^{\pm 1}).$$

Then,

$$H_\pm(Y,X) = \sum_{i=k}^{m} C_i(Y)(YX^{\pm 1})^i \sum_{j=0}^{l-i-1} (YX^{\pm 1})^{2j} + \frac{Y^{2l} X^{\pm 2l}}{1-\varepsilon X^{\pm 1}} \sum_{i=k}^{m} Y^{-i} C_i(Y) X^{\mp i}. \tag{4.11}$$

In particular, for $\varepsilon = 0$, we have

$$H_\pm(Y,X) = \sum_{i=k}^{m} C_i(Y)(YX^{\pm 1})^i \sum_{j=0}^{l-i} (YX^{\pm 1})^{2j}. \tag{4.12}$$

Proof. We calculate

$$
\begin{aligned}
&\frac{1}{1-\varepsilon X^{\pm 1}} G(Y, YX^{\pm 1}) \\
&\quad = \frac{1}{1-\varepsilon X^{\pm 1}} \left\{ \sum_{i=k}^{m} C_i(Y)(YX^{\pm 1})^i \sum_{j=0}^{l-i} (YX^{\pm 1})^{2j} \right. \\
&\qquad\qquad\qquad \left. -\varepsilon \sum_{i=k}^{m} Y^{-1} C_i(Y)(YX^{\pm 1})^{i+1} \sum_{j=0}^{l-i-1} (YX^{\pm 1})^{2j} \right\} \\
&\quad = \frac{1}{1-\varepsilon X^{\pm 1}} \left\{ \sum_{i=k}^{m} C_i(Y)(YX^{\pm 1})^i \sum_{j=0}^{l-i} (YX^{\pm 1})^{2j} \right. \\
&\qquad\qquad\qquad \left. -\varepsilon X^{\pm 1} \sum_{i=k}^{m} C_i(Y)(YX^{\pm 1})^{i} \sum_{j=0}^{l-i-1} (YX^{\pm 1})^{2j} \right\} \\
&\quad = \sum_{i=k}^{m} C_i(Y)(YX^{\pm 1})^i \sum_{j=0}^{l-i-1} (YX^{\pm 1})^{2j} + \frac{1}{1-\varepsilon X^{\pm 1}} \sum_{i=k}^{m} C_i(Y)(YX^{\pm 1})^i (YX^{\pm 1})^{2(l-i)},
\end{aligned}
$$

and

$$
\sum_{i=k}^{m} C_i(Y)(YX^{\pm 1})^i (YX^{\pm 1})^{2(l-i)} = Y^{2l} X^{\pm 2l} \sum_{i=k}^{m} Y^{-i} C_i(Y) X^{\mp i}.
$$

■

Remark 4.11. We can easily write $G(Y, X)$ in terms of the ε-Kohnen-Choie basis:

$$
.G(Y, X) = X^l \sum_{i=k}^{m} C_i(Y)(\psi_{l-i+1}(Y) - \varepsilon Y^{-1} \psi_{l-i}(Y)).
$$

We will frequently need to switch between the natural basis and the ε-Kohnen-Choie basis.

4.3 Calculation of the Remainder Polynomial

In the next two results, we use Lemma 4.10 to finish the induction step of (4.3) for n odd.

Proposition 4.12. Suppose $H = (a_1, \ldots, a_{2n+1}; \varepsilon_1, \ldots, \varepsilon_{2n+1}) \in \mathcal{NEGK}_{2n+1}$, with $n \geq 1$. Write $H' := (a_1, \ldots, a_{2n}; \varepsilon_1, \ldots, \varepsilon_{2n}) \in \mathcal{NEGK}_{2n}$. Set $\ell = \mathfrak{e}_{2n+1}/2$, $\ell' = \mathfrak{e}_{2n}/2$. Write

$$\mathcal{F}(H'; Y, X) = \sum_{j=0}^{\ell'} \phi(\ell' - j, H'; Y) Y^{-(\ell'-j)} \left(\psi_{j+1}(X) - \varepsilon_{2n} Y^{-1} \psi_j(X) \right).$$

Above, $\phi(\mu, H'; Y)$, for $0 \leq \mu \leq \ell'$, is a rational function in $\mathbb{Q}(Y)$. Then,

$$\mathcal{F}(H; Y, X) = F(H; Y, X) + \varepsilon_{2n+1} F(H; Y, X^{-1}) + \varepsilon_{2n}^2 R(H; Y, X),$$

where

$$F(H; Y, X^{\pm 1}) = X^{\mp \frac{\mathfrak{e}_{2n+1}}{2}} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y) X^{\pm i} \sum_{j=0}^{\frac{\mathfrak{e}'_{2n}}{2} - i - 1} (Y X^{\pm 1})^{2j},$$

and

$$R(H; Y, X) = \frac{Y^{\mathfrak{e}_{2n}} X^{-\frac{\mathfrak{e}_{2n+1}}{2} + \mathfrak{e}_{2n}}}{1 - \varepsilon_{2n} X} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y) (Y^{-2} X^{-1})^i$$
$$+ \varepsilon_{2n+1} \frac{Y^{\mathfrak{e}_{2n}} X^{\frac{\mathfrak{e}_{2n+1}}{2} - \mathfrak{e}_{2n}}}{1 - \varepsilon_{2n} X^{-1}} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y) (Y^{-2} X)^i,$$

and

$$\mathfrak{e}'_{2n} := 2 \left\lfloor \frac{a_1 + \cdots + a_{2n} + 1}{2} \right\rfloor.$$

We call $R(H; Y, X)$ the **remainder** of $\mathcal{F}(H; Y, X)$.

Proof. First, via a change of variable:

$$\mathcal{F}(H';Y,X) = \sum_{i=0}^{\ell'} \phi(i,H';Y)Y^{-i}\left(\psi_{\ell'-i+1}(X) - \varepsilon_{2n}Y^{-1}\psi_{\ell'-i}(X)\right). \tag{4.13}$$

In the notation of Lemma 4.10, set

$$k = 0, \quad m = \ell', \quad l = \ell', \quad \varepsilon = \varepsilon_{2n}, \quad C_i(Y) = \phi(i,H';Y)Y^{-i}.$$

With this choice of parameters, we see via Remark 4.11, that

$$G(Y,X) = X^{\ell'}\mathcal{F}(H';Y,X).$$

Thus,

$$\begin{aligned} &C_{2n+1}(\mathfrak{e}_{2n+1},\mathfrak{e}_{2n};\varepsilon_{2n};Y,X^{\pm 1})\mathcal{F}(H;Y,YX^{\pm 1}) \\ &\quad = \frac{Y^{\ell'}X^{\mp(\ell-\ell')}}{1-\varepsilon_{2n}X^{\pm 1}}(YX^{\pm 1})^{-\ell'}G(Y,YX^{\pm 1}) \\ &\quad = \frac{X^{\mp\ell}}{1-\varepsilon_{2n}X^{\pm 1}}G(Y,YX^{\pm 1}). \end{aligned}$$

Thus, in view of (4.3), we have

$$\mathcal{F}(H;Y,X) = \frac{X^{-\ell}}{1-\varepsilon_{2n}X}G(Y,YX) + \varepsilon_{2n+1}\frac{X^{\ell}}{1-\varepsilon_{2n}X^{-1}}G(Y,YX^{-1})$$

Case I: Suppose $\varepsilon_{2n} = 0$. Then,

$$\mathcal{F}(H';Y,X) = X^{-\ell}G(Y,YX) + \varepsilon_{2n+1}X^{\ell}G(Y,YX^{-1})$$

By Condition (N2) of Definition 3.10, we have $\varepsilon_{2n} = 0$ if and only if $a_1 + \cdots + a_{2n}$ is odd.

Thus, $\mathfrak{e}_{2n}/2 = \mathfrak{e}'_{2n}/2 - 1$. Applying Lemma 4.10, we have

$$\begin{aligned} &G(Y, YX^{\pm 1}) \\ &= \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y) Y^{-i} (YX^{\pm 1})^i \sum_{j=0}^{\frac{\mathfrak{e}_{2n}}{2}-i} (YX^{\pm 1})^{2j} = \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y) X^{\pm i} \sum_{j=0}^{\frac{\mathfrak{e}'_{2n}}{2}-i-1} (YX^{\pm 1})^{2j}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{F}(H; Y, X) = X^{-\frac{\mathfrak{e}_{2n+1}}{2}} &\sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y) X^i \sum_{j=0}^{\frac{\mathfrak{e}'_{2n}}{2}-i-1} (YX)^{2j} \\ &+ \varepsilon_{2n+1} X^{\frac{\mathfrak{e}_{2n+1}}{2}} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y) X^{-i} \sum_{j=0}^{\frac{\mathfrak{e}'_{2n}}{2}-i-1} (YX^{-1})^{2j} \end{aligned}$$

Case II: Suppose $\varepsilon_{2n} \neq 0$. Then $a_1 + \cdots + a_{2n}$ is even by condition (N2) of Definition 3.10. Therefore $\mathfrak{e}_{2n}/2 = \mathfrak{e}'_{2n}/2 =: \ell'$. Using Lemma 4.10, we have

$$\begin{aligned} &\frac{X^{\mp\frac{\mathfrak{e}_{2n+1}}{2}}}{1 - \varepsilon_{2n} X^{\pm 1}} G(Y, YX^{\pm 1}) \\ &= X^{\mp\frac{\mathfrak{e}_{2n+1}}{2}} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y) Y^{-i} (YX^{\pm 1})^i \sum_{j=0}^{\frac{\mathfrak{e}'_{2n}}{2}-i-1} (YX^{\pm 1})^{2j} \\ &\qquad + \frac{Y^{\mathfrak{e}_{2n}} X^{\frac{\mp\mathfrak{e}_{2n+1}}{2} \pm \mathfrak{e}_{2n}}}{1 - \varepsilon_{2n} X^{\pm 1}} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} Y^{-i} \phi(i, H'; Y) Y^{-i} X^{\mp i} \\ &= X^{\mp\frac{\mathfrak{e}_{2n+1}}{2}} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y) X^{\pm i} \sum_{j=0}^{\frac{\mathfrak{e}'_{2n}}{2}-i-1} (YX^{\pm 1})^{2j} \\ &\qquad + \frac{Y^{\mathfrak{e}_{2n}} X^{\mp\frac{\mathfrak{e}_{2n+1}}{2} \pm \mathfrak{e}_{2n}}}{1 - \varepsilon_{2n} X^{\pm 1}} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y) (Y^{-2} X^{\mp 1})^i \end{aligned}$$

■

Remark 4.13. The symbols $\mathfrak{e}'_{2n}$ and ε^2_{2n} are notational aids to unify Cases I and II.

We now calculate the remainder $R(H;Y,X)$ in a more explicit form:

Proposition 4.14. With the setup of Proposition 4.12, let $R(H;Y,X)$ be the remainder of $\mathcal{F}(H;Y,X)$. Then $R(H;Y,X)$ is a Laurent polynomial of degree $\le \frac{\mathfrak{e}_{2n+1}}{2} - \mathfrak{e}_1$ in X satisfying:

$$R(H;Y,X) = \varepsilon_{2n+1} R(H;Y,X^{-1}).$$

Moreover,

$$\begin{aligned} &R(H;Y,X) \\ &= Y^{\mathfrak{e}_{2n}} X^{-\frac{\mathfrak{e}_{2n+1}}{2}+\mathfrak{e}_{2n}} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i,H';Y)(Y^{-2}X^{-1})^i \sum_{j=0}^{\mathfrak{e}_{2n+1}-\mathfrak{e}_{2n}-\mathfrak{e}_1+i} (\varepsilon_{2n}X)^j \\ &\qquad - \varepsilon_{2n}\varepsilon_{2n+1} Y^{\mathfrak{e}_{2n}} X^{\frac{\mathfrak{e}_{2n+1}}{2}-\mathfrak{e}_{2n}+1} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i,H';Y)(Y^{-2}X)^i \sum_{j=0}^{\mathfrak{e}_{2n}-\mathfrak{e}_1-i-1} (\varepsilon_{2n}X)^j. \end{aligned}$$

Proof. We calculate the $R(H;Y,X)$ as a formal power series in X. That is,

$$\begin{aligned}
&R(H;Y,X)\\
&\quad= \frac{Y^{\mathrm{e}_{2n}} X^{-\frac{\mathrm{e}_{2n+1}}{2}+\mathrm{e}_{2n}}}{1-\varepsilon_{2n}X} \sum_{i=0}^{\frac{\mathrm{e}_{2n}}{2}} \phi(i,H';Y)(Y^{-2}X^{-1})^i\\
&\qquad\qquad + \varepsilon_{2n+1} \frac{Y^{\mathrm{e}_{2n}} X^{\frac{\mathrm{e}_{2n+1}}{2}-\mathrm{e}_{2n}}}{1-\varepsilon_{2n}X^{-1}} \sum_{i=0}^{\frac{\mathrm{e}_{2n}}{2}} \phi(i,H';Y)(Y^{-2}X)^i\\
&\quad= \frac{Y^{\mathrm{e}_{2n}} X^{-\frac{\mathrm{e}_{2n+1}}{2}+\mathrm{e}_{2n}}}{1-\varepsilon_{2n}X^{\pm 1}} \sum_{i=0}^{\frac{\mathrm{e}_{2n}}{2}} \phi(i,H';Y)(Y^{-2}X^{-1})^i\\
&\qquad\qquad - \varepsilon_{2n}\varepsilon_{2n+1} X \frac{Y^{\mathrm{e}_{2n}} X^{\frac{\mathrm{e}_{2n+1}}{2}-\mathrm{e}_{2n}}}{1-\varepsilon_{2n}X} \sum_{i=0}^{\frac{\mathrm{e}_{2n}}{2}} \phi(i,H';Y)(Y^{-2}X)^i\\
&\quad= Y^{\mathrm{e}_{2n}} X^{-\frac{\mathrm{e}_{2n+1}}{2}+\mathrm{e}_{2n}} \sum_{j=0}^{\infty} (\varepsilon_{2n}X)^j \sum_{i=0}^{\frac{\mathrm{e}_{2n}}{2}} \phi(i,H';Y)(Y^{-2}X^{-1})^i\\
&\qquad\qquad - \varepsilon_{2n}\varepsilon_{2n+1} Y^{\mathrm{e}_{2n}} X^{\frac{\mathrm{e}_{2n+1}}{2}-\mathrm{e}_{2n}+1} \sum_{j=0}^{\infty} (\varepsilon_{2n}X)^j \sum_{i=0}^{\frac{\mathrm{e}_{2n}}{2}} \phi(i,H';Y)(Y^{-2}X)^i.
\end{aligned}$$

In the notation of Proposition 4.12, we have

$$R(H;Y,X) = \mathcal{F}(H;Y,X) - (F(H;Y,X) + \varepsilon_{2n+1}F(H;Y,X^{-1})).$$

Via Theorem 4.5, we have $\mathcal{F}(H;Y,X^{-1}) = \varepsilon_{2n+1}\mathcal{F}(H;Y,X)$. Thus,

$$R(H;Y,X^{-1}) = \varepsilon_{2n+1} R(H;Y,X). \tag{4.14}$$

We observe, for $i = 0, \ldots, \mathfrak{e}_{2n}/2$ and $j \geq 0$, we have the bounds:

$$-\frac{\mathfrak{e}_{2n+1}}{2} + \mathfrak{e}_{2n} + j - i \geq -\frac{\mathfrak{e}_{2n+1}}{2} + \frac{\mathfrak{e}_{2n}}{2} \geq -\frac{\mathfrak{e}_{2n+1}}{2} + \mathfrak{e}_1,$$
$$\frac{\mathfrak{e}_{2n+1}}{2} - \mathfrak{e}_{2n} + 1 + j + i \geq -\frac{\mathfrak{e}_{2n+1}}{2} + (\mathfrak{e}_{2n+1} - \mathfrak{e}_{2n}) + 1 \geq -\frac{\mathfrak{e}_{2n+1}}{2} + \mathfrak{e}_1.$$

Therefore, in view of (4.14), $R(H; Y, X)$ is a Laurent polynomial of degree $\leq \frac{\mathfrak{e}_{2n+1}}{2} - \mathfrak{e}_1$. Thus, in our calculation of $R(H; Y, X)$, it suffices to compute the terms up to degree $\frac{\mathfrak{e}_{2n+1}}{2} - \mathfrak{e}_1$. That is,

$$\begin{aligned} &R(H; Y, X) \\ &= Y^{\mathfrak{e}_{2n}} X^{-\frac{\mathfrak{e}_{2n+1}}{2} + \mathfrak{e}_{2n}} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y)(Y^{-2}X^{-1})^i \sum_{j=0}^{\mathfrak{e}_{2n+1} - \mathfrak{e}_{2n} - \mathfrak{e}_1 + i} (\varepsilon_{2n} X)^j \\ &\qquad - \varepsilon_{2n}\varepsilon_{2n+1} Y^{\mathfrak{e}_{2n}} X^{\frac{\mathfrak{e}_{2n+1}}{2} - \mathfrak{e}_{2n} + 1} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i, H'; Y)(Y^{-2}X)^i \sum_{j=0}^{\mathfrak{e}_{2n} - \mathfrak{e}_1 - i - 1} (\varepsilon_{2n} X)^j. \end{aligned}$$

■

Remark 4.15. This calculation $R(H, Y; X)$ via a formal series was inspired by [IK22a].

4.4 Explicit Formulas for the Laurent Polynomial for $n = 2, 3$

In this section, we will calculate the Laurent polynomial for EGK data of lengths 2 and 3.

Example 4.16. Let $H = (a_1, a_2; 1, \varepsilon_2)$ be a naive EGK datum of length 2. Let $H' = (a_1; 1)$.

Then H' is a naive EGK datum of length 1 with

$$\mathcal{F}(H';Y,X) = X^{-\frac{\mathfrak{e}_1}{2}} + X^{-\frac{\mathfrak{e}_1}{2}+1} + \cdots + X^{\frac{\mathfrak{e}_1}{2}-1} + X^{\frac{\mathfrak{e}_1}{2}}.$$

We apply Proposition 4.8 with $F(Y,X) = \mathcal{F}(H';Y,X)$; $\ell_i = \frac{\mathfrak{e}_i}{2}$, $i = 1,2$; and $C_i(Y) = 1$, $|i| \le \ell_1$. Since $\mathfrak{e}_1 \le \frac{\mathfrak{e}_2}{2}$, we have $m := \min\{2\ell_1, \ell_2\} = 2\ell_1 = \mathfrak{e}_1$. Therefore, in view of (4.3), we have

$$\mathcal{F}(H;Y,X) = \sum_{i=0}^{\mathfrak{e}_1} Y^i X^{-\frac{\mathfrak{e}_2}{2}+i} \sum_{j=0}^{\frac{\mathfrak{e}_2}{2}-i} X^{2j} - \varepsilon_2 \sum_{i=0}^{\mathfrak{e}_1} Y^{i-1} X^{-\frac{\mathfrak{e}_2}{2}+i+1} \sum_{j=0}^{\frac{\mathfrak{e}_2}{2}-i-1} X^{2j}. \tag{4.15}$$

Using (4.15), we calculate

Proposition 4.17. Let $H = (a_1, a_2; 1, \varepsilon_2)$ and $\mathcal{F}(H;Y,X)$ be an in Example 4.16. Write

$$\mathcal{F}(H;Y,X) = \sum_{i=0}^{2\ell} C^{(2)}_{-\ell+i}(Y) X^{-\ell+i}$$

Here $\ell = \mathfrak{e}_2/2 \in \mathbb{Z}$. Then,

$$C^{(2)}_{\nu}(Y) = A^{(2)}_{1,\nu}(Y) - \varepsilon_2 A^{(2)}_{2,\nu}(Y) \tag{4.16}$$

where

$$A^{(2)}_{1,\nu}(Y) = \begin{cases} \sum_{i=0}^{\lfloor \frac{\mathfrak{e}_2/2-\nu}{2} \rfloor} Y^{2i} & \mathfrak{e}_2/2 - \mathfrak{e}_1 \leq |\nu| \leq \mathfrak{e}_2/2, \nu \equiv \mathfrak{e}_2/2 \ (2), \\ Y \sum_{i=0}^{\lfloor \frac{\mathfrak{e}_2/2-\nu}{2} \rfloor} Y^{2i} & \mathfrak{e}_2/2 - \mathfrak{e}_1 \leq |\nu| \leq \mathfrak{e}_2/2, \nu \not\equiv \mathfrak{e}_2/2 \ (2), \\ \sum_{i=0}^{\lfloor \frac{\mathfrak{e}_1}{2} \rfloor} Y^{2i} & 0 \leq |\nu| \leq \mathfrak{e}_2/2 - \mathfrak{e}_1, \nu \equiv \mathfrak{e}_2/2 \ (2), \\ Y \sum_{i=0}^{\lfloor \frac{\mathfrak{e}_1}{2} \rfloor} Y^{2i} & 0 \leq |\nu| \leq \mathfrak{e}_2/2 - \mathfrak{e}_1, \nu \not\equiv \mathfrak{e}_2/2 \ (2), \end{cases}$$

$$A^{(2)}_{2,\nu}(Y) = \begin{cases} \sum_{i=0}^{\lfloor \frac{\mathfrak{e}_2/2-1-\nu}{2} \rfloor} Y^{2i} & \mathfrak{e}_2/2 - \mathfrak{e}_1 - 1 \leq |\nu| \leq \mathfrak{e}_2/2 - 1, \nu \equiv \mathfrak{e}_2/2 \ (2), \\ Y^{-1} \sum_{i=0}^{\lfloor \frac{\mathfrak{e}_2/2-1-\nu}{2} \rfloor} Y^{2i} & \mathfrak{e}_2/2 - \mathfrak{e}_1 - 1 \leq |\nu| \leq \mathfrak{e}_2/2 - 1, \nu \not\equiv \mathfrak{e}_2/2 \ (2), \\ \sum_{i=0}^{\lfloor \frac{\mathfrak{e}_1}{2} \rfloor} Y^{2i} & 0 \leq |\nu| \leq \mathfrak{e}_2/2 - \mathfrak{e}_1 - 1, \nu \equiv \mathfrak{e}_2/2 \ (2), \\ Y^{-1} \sum_{i=0}^{\lfloor \frac{\mathfrak{e}_1}{2} \rfloor} Y^{2i} & 0 \leq |\nu| \leq \mathfrak{e}_2/2 - \mathfrak{e}_1 - 1, \nu \not\equiv \mathfrak{e}_2/2 \ (2). \end{cases}$$

Proof. Let $\delta \in \{0, 1\}$. We calculate:

$$\sum_{i=0}^{\mathfrak{e}_1} Y^{i-\delta} X^{-\frac{\mathfrak{e}_2}{2}+i+\delta} \sum_{j=0}^{\frac{\mathfrak{e}_2}{2}-i-\delta} X^{2j} = \sum_{i=0}^{\mathfrak{e}_1} Y^{i-\delta} \sum_{j=0}^{\frac{\mathfrak{e}_2}{2}-i-\delta} X^{i+2j-\frac{\mathfrak{e}_2}{2}+\delta}.$$

The calculation is completed with the aid of Figures 4.1 and 4.2. ∎

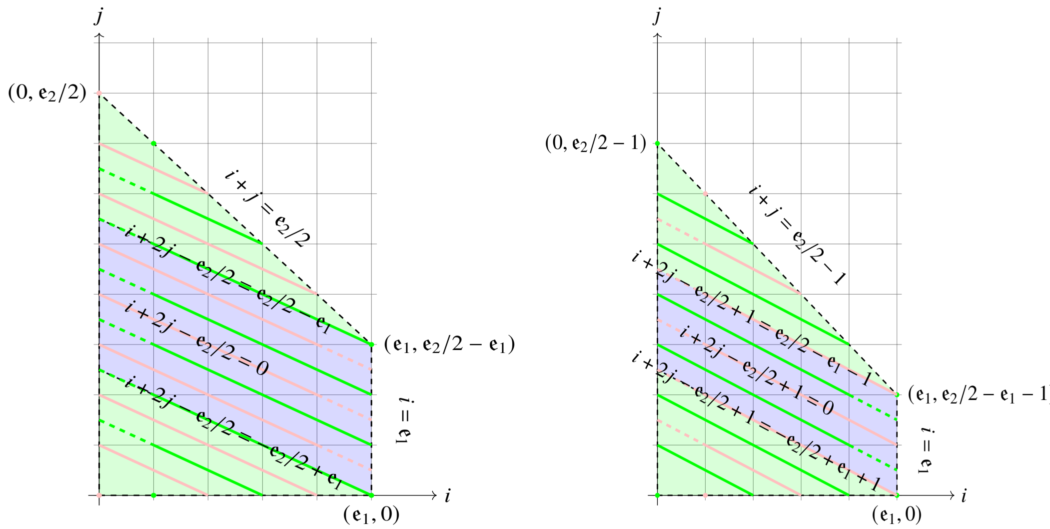

Figure 4.1: Regions of summation: $0 \le i \le e_1; 0 \le j \le \frac{e_2}{2} - i - \delta; \delta \in \{0, 1\}$.

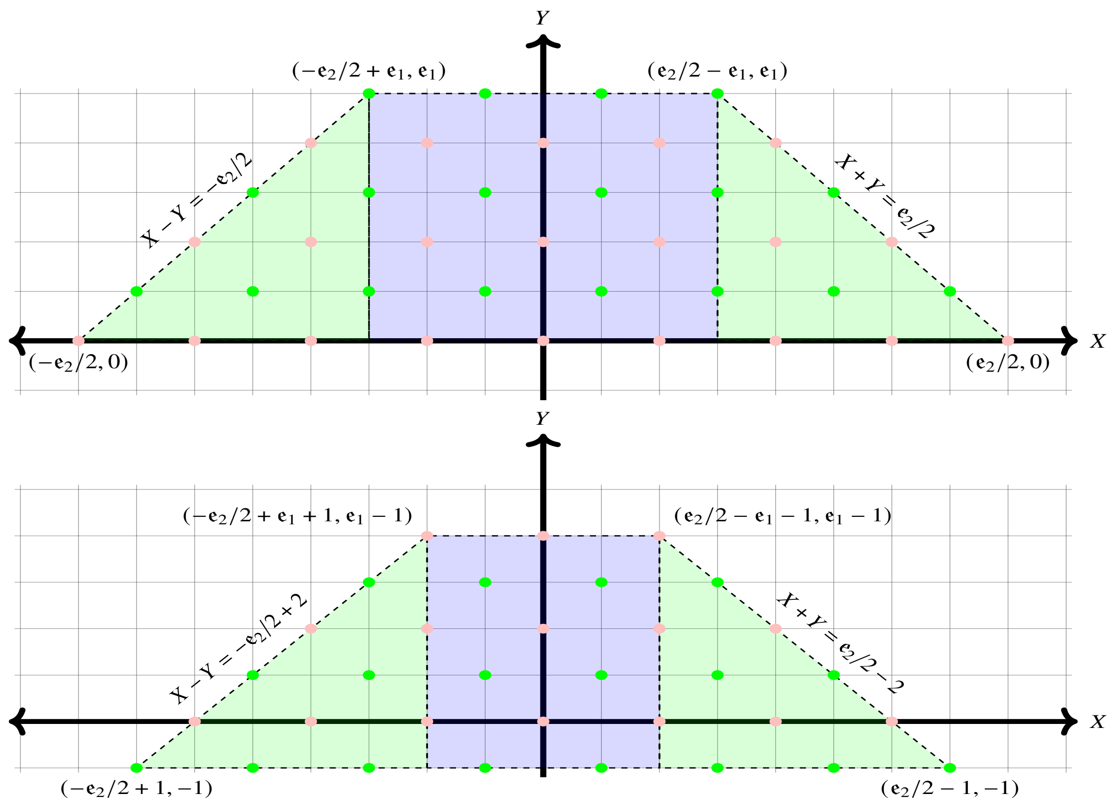

Figure 4.2: Coefficient Polygons for $\sum_{i=0}^{e_1} Y^{i-\delta} \sum_{j=0}^{\frac{e_2}{2}-i-\delta} X^{i+2j-\frac{e_2}{2}+\delta}$; $\delta \in \{0, 1\}$.

Example 4.18. Suppose that $H = (a_1, a_2, a_3; 1, \varepsilon_2, \varepsilon_3)$ is a naive EGK datum of length 3. Then $H' = (a_1, a_2; 1, \varepsilon_2) \in \mathrm{NEGK}_2$. Let $\ell = \mathfrak{e}_2/2$. By Example 4.16, we have

$$\mathcal{F}(H'; Y, X) = \sum_{i=0}^{\mathfrak{e}_1} Y^i (\psi_{\ell-i+1}(X) - \varepsilon_2 Y^{-1} \psi_{\ell-i}(X)).$$

Via a change of variable:

$$\mathcal{F}(H'; Y, X) = \sum_{j=0}^{\ell} \phi(\ell - j, H'; Y) Y^{-(\ell-j)} (\psi_{j+1}(X) - \varepsilon_2 Y^{-1} \psi_j(X)),$$

where

$$\phi(\mu, H'; Y) = \begin{cases} Y^{2\mu} & \text{if } 0 \leq \mu \leq \mathfrak{e}_1, \\ 0 & \text{if } \mathfrak{e}_1 < \mu \leq \ell. \end{cases}$$

Applying Proposition 4.12, we have

$$\mathcal{F}(H; Y, X) = F(H; Y, X) + \varepsilon_3 F(H; Y, X^{-1}) + \varepsilon_2^2 R(H; Y, X),$$

where

$$F(H; Y, X^{\pm 1}) = X^{\mp \frac{\mathfrak{e}_3}{2}} \sum_{i=0}^{\mathfrak{e}_1} (Y^2 X^{\pm 1})^i \sum_{j=0}^{\frac{\mathfrak{e}_2'}{2} - i - 1} (Y X^{\pm 1})^{2j},$$

and

$$R(H; Y, X) = \frac{Y^{\mathfrak{e}_2} X^{-\frac{\mathfrak{e}_3}{2} + \mathfrak{e}_2}}{1 - \varepsilon_{2n} X} \sum_{i=0}^{\mathfrak{e}_1} X^{-i} + \varepsilon_3 \frac{Y^{\mathfrak{e}_2} X^{\frac{\mathfrak{e}_3}{2} - \mathfrak{e}_2}}{1 - \varepsilon_2 X^{-1}} \sum_{i=0}^{\mathfrak{e}_1} X^i,$$

and

$$\mathfrak{e}_2' := 2\left\lfloor \frac{a_1+a_2+1}{2} \right\rfloor .$$

To calculate $R(H;Y,X)$, assume $\varepsilon_2 \neq 0$; for, otherwise, the term $\varepsilon_2^2 R(H;Y,X)$ vanishes. By condition (N2) of Definition 3.10, $\varepsilon_2 \neq 0$ if and only if a_1+a_2 is even. In this case, $\varepsilon_3 = \varepsilon_2^{a_2+a_3}$ by condition (N5) of Definition 3.10. We now calculate $R(H;Y,X)$ as follows:

$$\begin{aligned}
&\frac{Y^{\mathfrak{e}_2} X^{-\frac{\mathfrak{e}_3}{2}+\mathfrak{e}_2}}{1-\varepsilon_2 X} \sum_{i=0}^{\mathfrak{e}_1} X^{-i} + \varepsilon_3 \frac{Y^{\mathfrak{e}_2} X^{\frac{\mathfrak{e}_3}{2}-\mathfrak{e}_2}}{1-\varepsilon_2 X^{-1}} \sum_{i=0}^{\mathfrak{e}_1} X^{i} \\
&\quad = \frac{Y^{\mathfrak{e}_2} X^{-\frac{\mathfrak{e}_3}{2}+\mathfrak{e}_2}}{1-\varepsilon_2 X} \sum_{i=0}^{\mathfrak{e}_1} X^{-i} - \varepsilon_2\varepsilon_3 X \frac{Y^{\mathfrak{e}_2} X^{\frac{\mathfrak{e}_3}{2}-\mathfrak{e}_2}}{1-\varepsilon_2 X} \sum_{i=0}^{\mathfrak{e}_1} X^{i} \\
&\quad = Y^{\mathfrak{e}_2} X^{-\frac{\mathfrak{e}_3}{2}+\mathfrak{e}_2-\mathfrak{e}_1} \frac{1-\varepsilon_2^{a_2+a_3+1} X^{\mathfrak{e}_3-2\mathfrak{e}_2+\mathfrak{e}_1+1}}{1-\varepsilon_2 X} \sum_{i=0}^{\mathfrak{e}_1} X^{i} \\
&\quad = Y^{\mathfrak{e}_2} X^{-\frac{\mathfrak{e}_3}{2}+\mathfrak{e}_2-\mathfrak{e}_1} \frac{1-(\varepsilon_2 X)^{\mathfrak{e}_3-2\mathfrak{e}_2+\mathfrak{e}_1+1}}{1-\varepsilon_2 X} \sum_{i=0}^{\mathfrak{e}_1} X^{i} \\
&\quad = Y^{\mathfrak{e}_2} X^{-\frac{\mathfrak{e}_3}{2}+\mathfrak{e}_2-\mathfrak{e}_1} \sum_{j=0}^{\mathfrak{e}_3-2\mathfrak{e}_2+\mathfrak{e}_1} (\varepsilon_2 X)^{j} \sum_{i=0}^{\mathfrak{e}_1} X^{i}.
\end{aligned}$$

Above, we have used

$$\mathfrak{e}_3 - 2\mathfrak{e}_2 + \mathfrak{e}_1 + 1 = (a_1+a_2+a_3) - 2\mathfrak{e}_2 + a_1 + 1 \equiv a_2 + a_3 + 1 \bmod 2.$$

Summary: We have

$$
\begin{aligned}
\mathcal{F}(H;Y,X) = X^{-\frac{\mathfrak{e}_3}{2}} \sum_{i=0}^{\mathfrak{e}_1} (Y^2X)^i \sum_{j=0}^{\frac{\mathfrak{e}_2'}{2}-i-1} (YX)^{2j} \\
+ \varepsilon_3 X^{\frac{\mathfrak{e}_3}{2}} \sum_{i=0}^{\mathfrak{e}_1} (Y^2X^{-1})^i \sum_{j=0}^{\frac{\mathfrak{e}_2'}{2}-i-1} (YX^{-1})^{2j} \\
+ \varepsilon_2^2 Y^{\mathfrak{e}_2} X^{-\frac{\mathfrak{e}_3}{2}+\mathfrak{e}_2-\mathfrak{e}_1} \sum_{j=0}^{\mathfrak{e}_3-2\mathfrak{e}_2+\mathfrak{e}_1} (\varepsilon_2 X)^j \sum_{i=0}^{\mathfrak{e}_1} X^i.
\end{aligned} \tag{4.17}
$$

Using (4.17), we calculate

Proposition 4.19. Let $H = (a_1, a_2, a_3; 1, \varepsilon_2, \varepsilon_3)$ and $\mathcal{F}(H;Y,X)$ be an in Example 4.18. Write

$$
\mathcal{F}(H;Y,X) = \sum_{i=0}^{2\ell'} C^{(3)}_{-\ell'+i}(Y) X^{-\ell'+i}
$$

Here $\ell = \mathfrak{e}_3/2 \in \frac{1}{2}\mathbb{Z}$. Then,

$$
C^{(3)}_{\nu}(Y) = A^{(3)}_{1,\nu}(Y) + \varepsilon_2 A^{(3)}_{2,\nu}(Y) + \varepsilon_2^2 A^{(3)}_{3,\nu}(Y) \tag{4.18}
$$

where

$$
\begin{aligned}
A^{(3)}_{2,\nu}(Y) &= A^{(3)}_{1,-\nu}(Y), \\
A^{(3)}_{3,\nu}(Y) &= \varepsilon_2 A^{(3)}_{3,-\nu}(Y).
\end{aligned}
$$

with

$$A^{(3)}_{1,\nu}(Y) = \begin{cases} Y^{\nu+\mathfrak{e}_3/2} \sum_{i=0}^{\lfloor \frac{\nu+\mathfrak{e}_3/2}{2} \rfloor} Y^{2i} & 0 \le \nu + \mathfrak{e}_3/2 \le \mathfrak{e}_1, \nu \equiv \mathfrak{e}_3/2 \ (2), \\ Y^{\nu+\mathfrak{e}_3/2+1} \sum_{i=0}^{\lfloor \frac{\nu+\mathfrak{e}_3/2}{2} \rfloor} Y^{2i} & 0 \le \nu + \mathfrak{e}_3/2 \le \mathfrak{e}_1, \nu \not\equiv \mathfrak{e}_3/2 \ (2), \\ Y^{\nu+\mathfrak{e}_3/2} \sum_{i=0}^{\lfloor \frac{\mathfrak{e}_1}{2} \rfloor} Y^{2i} & \mathfrak{e}_1 \le \nu + \mathfrak{e}_3/2 \le \mathfrak{e}'_2 - \mathfrak{e}_1 - 2, \nu \equiv \mathfrak{e}_3/2 \ (2), \\ Y^{\nu+\mathfrak{e}_3/2+1} \sum_{i=0}^{\lfloor \frac{\mathfrak{e}_1}{2} \rfloor} Y^{2i} & \mathfrak{e}_1 \le \nu + \mathfrak{e}_3/2 \le \mathfrak{e}'_2 - \mathfrak{e}_1 - 2, \nu \not\equiv \mathfrak{e}_3/2 \ (2), \\ Y^{\nu+\mathfrak{e}_3/2} \sum_{i=0}^{\lfloor \frac{\nu+\mathfrak{e}_3/2-\mathfrak{e}'_2+2}{2} \rfloor} Y^{2i} & \mathfrak{e}'_2 - \mathfrak{e}_1 - 2 \le \nu + \mathfrak{e}_3/2 \le \mathfrak{e}'_2 - 2, \nu \equiv \mathfrak{e}_3/2 \ (2), \\ Y^{\nu+\mathfrak{e}_3/2+1} \sum_{i=0}^{\lfloor \frac{\nu+\mathfrak{e}_3/2-\mathfrak{e}'_2+2}{2} \rfloor} Y^{2i} & \mathfrak{e}'_2 - \mathfrak{e}_1 - 2 \le \nu + \mathfrak{e}_3/2 \le \mathfrak{e}'_2 - 2, \nu \not\equiv \mathfrak{e}_3/2 \ (2), \end{cases}$$

and

$$A^{(3)}_{3,\nu}(Y) = \begin{cases} \left(\sum_{i=0}^{\nu+\mathfrak{e}_3/2-\mathfrak{e}_2+\mathfrak{e}_1} \varepsilon_2^i \right) Y^{\mathfrak{e}_2} & 0 \le \nu + \mathfrak{e}_3/2 - \mathfrak{e}_2 + \mathfrak{e}_1 \le \mathfrak{e}_1, \\ \left(\varepsilon_2^{\nu+\mathfrak{e}_3/2-\mathfrak{e}_2} \sum_{i=0}^{\mathfrak{e}_1} \varepsilon_2^i \right) Y^{\mathfrak{e}_2} & 0 \le \nu + \mathfrak{e}_3/2 - \mathfrak{e}_2 \le \mathfrak{e}_3/2 - \mathfrak{e}_2. \end{cases}$$

Proof. We calculate:

$$X^{\mp\frac{\mathfrak{e}_3}{2}} \sum_{i=0}^{\mathfrak{e}_1} (Y^2 X^{\pm 1})^i \sum_{j=0}^{\frac{\mathfrak{e}'_2}{2}-i-1} (YX^{\pm 1})^{2j} = X^{\mp\frac{\mathfrak{e}_3}{2}} \sum_{i=0}^{\mathfrak{e}_1} \sum_{j=0}^{\frac{\mathfrak{e}'_2}{2}-i-1} Y^{2(i+j)} X^{\pm(i+2j)}.$$

and

$$Y^{\mathfrak{e}_2} X^{-\frac{\mathfrak{e}_3}{2}+\mathfrak{e}_2-\mathfrak{e}_1} \sum_{j=0}^{\mathfrak{e}_3-2\mathfrak{e}_2+\mathfrak{e}_1} (\varepsilon_2 X)^j \sum_{i=0}^{\mathfrak{e}_1} X^i = Y^{\mathfrak{e}_2} \sum_{i=0}^{\mathfrak{e}_1} \sum_{j=0}^{\mathfrak{e}_3-2\mathfrak{e}_2+\mathfrak{e}_1} \varepsilon_2^j X^{i+j-\frac{\mathfrak{e}_3}{2}+\mathfrak{e}_2-\mathfrak{e}_1}.$$

The calculation is completed with the aid of Figures 4.3, 4.4, and 4.5. ∎

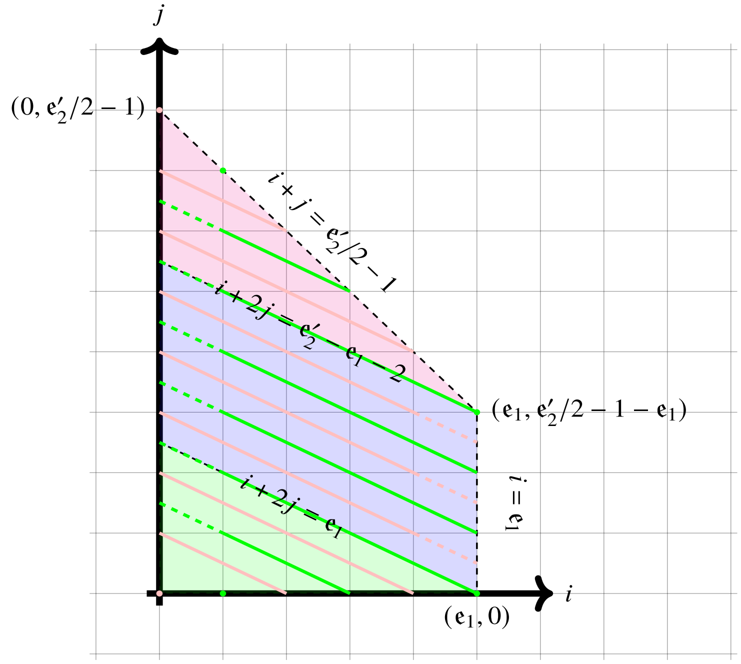

Figure 4.3: Region of summation: $0 \le i \le \mathfrak{e}_1; 0 \le j \le \frac{\mathfrak{e}_2'}{2} - i - 1$.

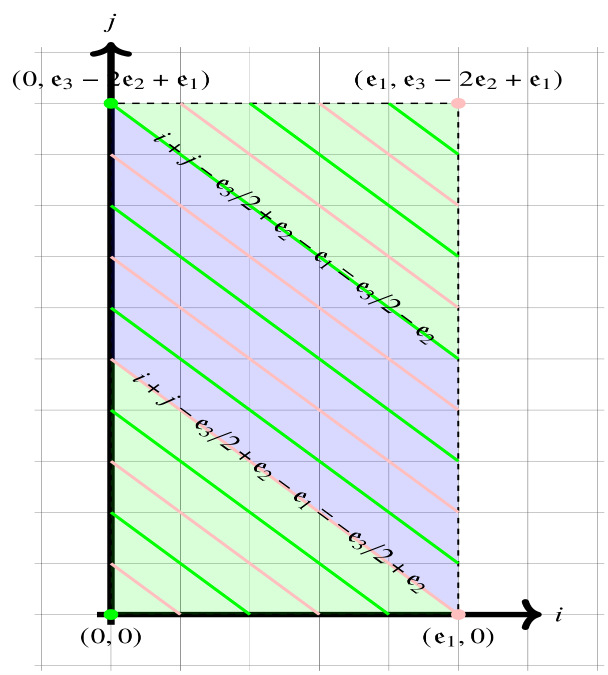

Figure 4.4: Region of summation: $0 \le i \le \mathfrak{e}_1; 0 \le j \le \mathfrak{e}_3 - 2\mathfrak{e}_2' + \mathfrak{e}_1$.

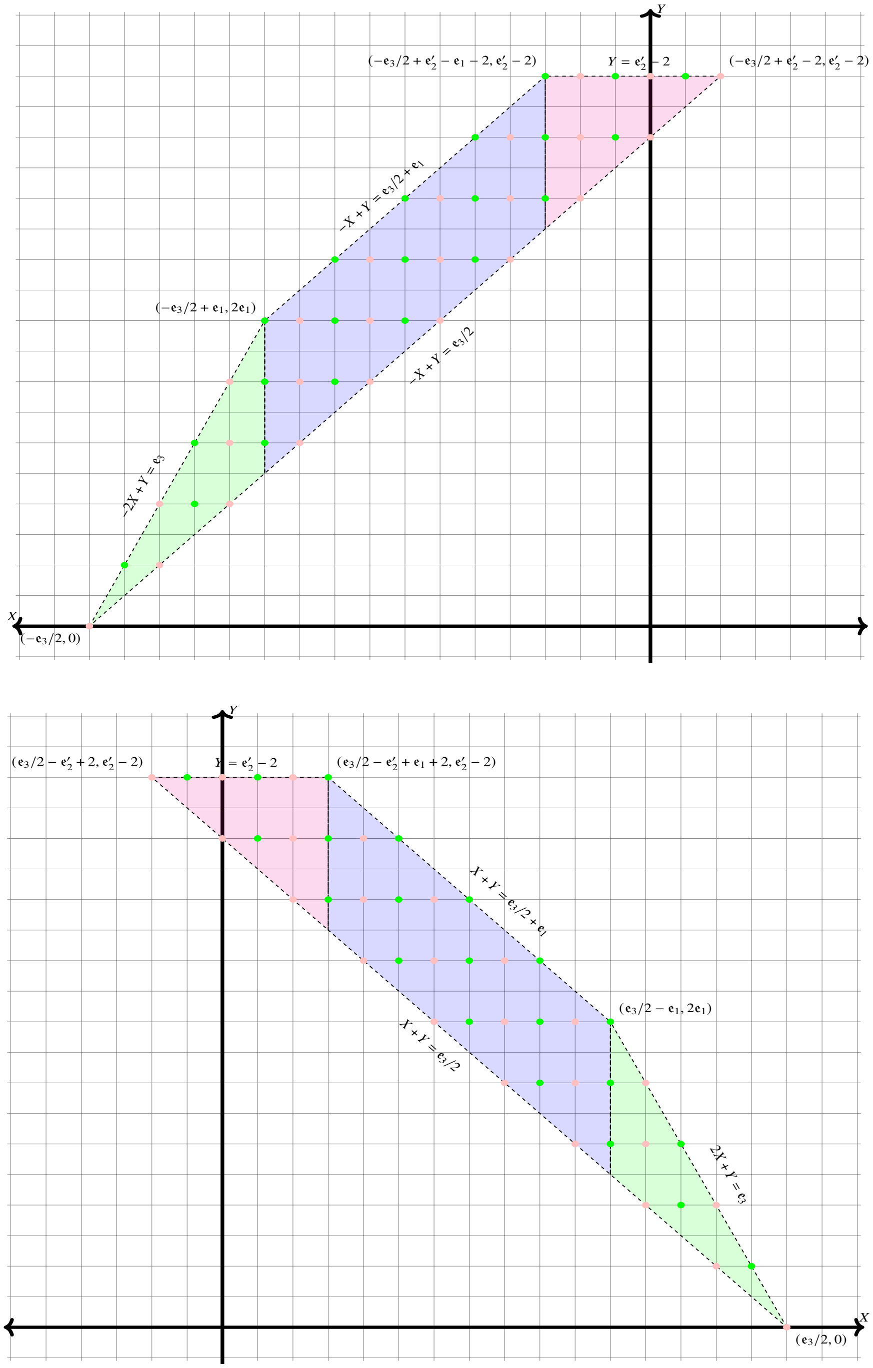

Figure 4.5: Coefficient Polygons for $X^{\mp\frac{e_3}{2}} \sum_{i=0}^{e_1} (Y^2 X^{\pm 1})^i \sum_{j=0}^{\frac{e_2'}{2}-i-1} (Y X^{\pm 1})^{2j}$.

Remark 4.20. Formulas (4.15) and (4.17) agree with Example (1) in Section 8 of [IK22b]

4.5 An Algorithm for Computing the Laurent Polynomial

We combine (4.3) with Propositions 4.8, 4.12, and 4.14 into an algorithm for computing $\mathcal{F}(H;Y,X)$. This method involves a strategic reorganization of Ikeda and Katsurada's original induction formulas. Our method allows us to profitably investigate portions of the Laurent polynomial in isolation. We will see later that our method also exposes combinatorial aspects of the coefficients of the Laurent series. We start with some definitions:

Definition 4.21. Let $H = (a_1, \ldots, a_{2n}; \varepsilon_1, \ldots, \varepsilon_{2n})$ be a naive EGK datum of length $2n$. For $n \geq 1$, define the (as Cartesian products) the sets:

$$S_1^{(n)} := \{1, 2, 3\}^{n-1}, \qquad S_2^{(n)} := \{1, -1\}^{n-1}.$$

Let

$$\zeta := (\zeta_2, \ldots, \zeta_n) \in S_1^{(n')}, \qquad \sigma := (\sigma_2, \ldots, \sigma_n) \in S_2^{(n')}.$$

For $1 \leq m \leq n$, we define Laurent polynomials; denoted $F^{(2m-1)}_{(\zeta,\sigma)}(H;Y,X)$ and $F^{(2m)}_{(\zeta,\sigma)}(H;Y,X)$; in the variable $X^{\frac{1}{2}}$ with coefficients in $\mathbb{Z}[Y, Y^{-1}]$ inductively as follows. For $m = 1$, we set:

$$F^{(1)}_{(\zeta,\sigma)}(H;Y,X) = \sum_{i=0}^{\mathfrak{e}_1} X^{-\frac{\mathfrak{e}_1}{2}+i},$$

$$F^{(2)}_{(\zeta,\sigma)}(H;Y,X) = \sum_{i=0}^{\mathfrak{e}_1} Y^i X^{-\frac{\mathfrak{e}_2}{2}+i} \sum_{j=0}^{\frac{\mathfrak{e}_2}{2}-i} X^{2j} - \varepsilon_2 \sum_{i=0}^{\mathfrak{e}_1} Y^{i-1} X^{-\frac{\mathfrak{e}_2}{2}+i+1} \sum_{j=0}^{\frac{\mathfrak{e}_2}{2}-i-1} X^{2j}.$$

Let $m \geq 2$. Write $\ell'' = \mathfrak{e}_{2m-2}/2$, $\ell' = \mathfrak{e}_{2m-1}/2$, and $\ell = \mathfrak{e}_{2m}/2$. Write

$$F^{(2m-2)}_{(\zeta,\sigma)}(H;Y,X) = \sum_{i=0}^{\frac{\mathfrak{e}_{2m-2}}{2}} C_i(Y)(\psi_{i+1}(X) - \varepsilon_{2m-2}Y^{-1}\psi_i(X)), \qquad C_i(Y) \in \mathbb{Q}(Y).$$

We define

$$F^{(2m-1)}_{(\zeta,\sigma)}(H;Y,X) = \begin{cases} F(H;Y,X) & \text{if } \zeta_m = 1, \\ \varepsilon^2_{2m-2}R(H;Y,X) & \text{if } \zeta_m = 2, \\ \varepsilon_{2m-1}F(H;Y,X^{-1}) & \text{if } \zeta_m = 3, \end{cases}$$

where

$$F(H;Y,X^{\pm 1}) = X^{\mp\frac{\mathfrak{e}_{2m-1}}{2}} \sum_{i=0}^{\frac{\mathfrak{e}_{2m-2}}{2}} C_{\ell''-i}(Y)(YX^{\pm 1})^i \sum_{j=0}^{\frac{\mathfrak{e}'_{2m-2}}{2}-i-1} (YX^{\pm 1})^{2j},$$

and

$$\begin{aligned} &R(H;Y,X) \\ &\quad = Y^{\mathfrak{e}_{2m-2}} X^{-\frac{\mathfrak{e}_{2m-1}}{2}+\mathfrak{e}_{2m-2}} \sum_{i=0}^{\frac{\mathfrak{e}_{2m-2}}{2}} C_{\ell''-i}(Y)(Y^{-1}X^{-1})^i \sum_{j=0}^{\mathfrak{e}_{2m-1}-\mathfrak{e}_{2m-2}-\mathfrak{e}_1+i} (\varepsilon_{2m-2}X)^j \\ &\qquad - \varepsilon_{2m-2}\varepsilon_{2m-1}Y^{\mathfrak{e}_{2m-2}} X^{\frac{\mathfrak{e}_{2m-1}}{2}-\mathfrak{e}_{2m-2}+1} \sum_{i=0}^{\frac{\mathfrak{e}_{2m-2}}{2}} C_{\ell''-i}(Y)(Y^{-1}X)^i \sum_{j=0}^{\mathfrak{e}_{2m-2}-\mathfrak{e}_1-i-1} (\varepsilon_{2m-2}X)^j. \end{aligned}$$

Write

$$F^{(2m-1)}_{(\zeta,\sigma)}(H;Y,X) = \sum_{i=0}^{\mathfrak{e}_{2m-1}} C_{-\ell'+i}(Y) X^{-\frac{\mathfrak{e}_{2m-1}}{2}+i}, \qquad C_i(Y) \in \mathbb{Q}(Y).$$

Let $k := \min\{2\ell', \ell\}$. We define

$$F^{(2m)}_{(\zeta,\sigma)}(H;Y,X) = \begin{cases} G_+(Y,X) & \text{if } \sigma_m = 1, \\ G_-(Y,X) & \text{if } \sigma_m = -1, \end{cases}$$

where

$$G_+(Y,X) = \sum_{i=0}^{k} Y^i C_{-\ell'+i}(Y) X^{-\ell+i} \sum_{j=0}^{\ell-i} X^{2j} - \varepsilon_{2m} \sum_{i=0}^{k} Y^{i-1} C_{-\ell'+i}(Y) X^{-\ell+i+1} \sum_{j=0}^{\ell-i-1} X^{2j},$$

and

$$G_-(Y,X) = \sum_{i=k+1}^{2\ell'} Y^i C_{-\ell'+i}(Y) X^{\ell-i+2} \sum_{j=0}^{-\ell+i-2} X^{2j} - \varepsilon_{2m} \sum_{i=k+1}^{2\ell'} Y^{i-1} C_{-\ell'+i}(Y) X^{\ell-i+1} \sum_{j=0}^{-\ell+i-1} X^{2j}.$$

The point of this formality is the following:

Theorem 4.22. Let $H = (a_1, \ldots, a_{2n}; \varepsilon_1, \ldots, \varepsilon_{2n})$ be a naive EGK datum of length $2n$. With the notation of Definition 4.21, we have, for $0 \le m < n$:

$$\mathcal{F}(H^{(2m)};Y,X) = \sum_{(\zeta,\sigma)\in S_1^{(2(n-m))} \times S_2^{(2(n-m))}} F^{(2(n-m))}_{(\zeta,\sigma)}(H;Y,X). \tag{4.19}$$

Here $H^{(2m)}$ denotes the $2m$-truncation of H from Definition 3.11.

Proof. This follows from (4.3) and Propositions 4.8, 4.12, 4.14 using an induction argument. We note especially that $R(H;Y,X)$ is defined to hold terms up to degree $\frac{\mathfrak{e}_{2(n-m)}}{2} - \mathfrak{e}_1$ at each step $m = n-1, \ldots, 0$. This might be more terms than necessary; however, we need to carry all these terms at each step since the cancellations guaranteed by the symmetry of $\mathcal{F}(H^{(2m)};Y,X)$ will not happen until the we compute the sum (4.19) at the very last step. A more careful analysis would be needed to reduce the degree bound $\frac{\mathfrak{e}_{2(n-m)}}{2} - \mathfrak{e}_1$. ■

Chapter 5

Kohnen's Phi Function

In [Koh02], Kohnen introduced a certain integer-valued function, denoted $\phi(a; B)$, where $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$ and $1 \leq a | f_B$. Notably, this function appears in the formula (21) of [Koh02] for the Fourier coefficients of the Ikeda lift. In this chapter, we first formally define $\phi(a; B)$ following Kohnen's exposition. Then, we calculate $\phi(a; B)$ explicitly under certain simplifying conditions on a using Kohnen's definition. We will see that calculating $\phi(a; B)$ for all $a | f_B$ via Kohnen's definition is likely intractable. We then introduce a novel method for computing $\phi(a; B)$ in terms of the Laurent polynomial attached to the datum $\mathrm{EGK}(B)$. We then demonstrate the application of our method to the calculation of $\phi(a; B)$ for $n = 1, 2$.

5.1 Definition of Kohnen's Phi Function

In this section, à la Kohnen, we formally define the function $\phi(a; B)$ mentioned above. First, we introduce some local quantities attached to a matrix $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$. Fix a prime p. Let $V_{B,p} := (\mathbb{F}_p^{2n}, q)$ be the quadratic space over the finite field $\mathbb{F}_p$ where q is the quadratic form obtained from the quadratic form $X \mapsto {}^tXBX$, $X \in \mathbb{Z}^{2n}$, by reducing modulo p. Write

$$V_{B,p} = V_{B,p}^{\mathrm{iso}} \oplus V'_{B,p}$$

Here, $V_{B,p}^{\mathrm{iso}}$ is a maximal isotropic subspace and $V'_{B,p}$ is a complementary subspace. Define

$$s_p = s_p(B) := \dim V_{B,p}^{\mathrm{iso}} \tag{5.1}$$

and

$$\lambda_p = \lambda_p(B) := \begin{cases} 1 & \text{if } V'_{B,p} \text{ is a hyperbolic space or } s_p = 2n, \\ -1 & \text{otherwise.} \end{cases} \tag{5.2}$$

Define a polynomial

$$H_{n,p}(B;t) := \begin{cases} 1 & \text{if } s_p = 0 \\ \prod_{j=1}^{\lfloor \frac{s_p-1}{2} \rfloor} (1 - p^{2j-1}t^2) & \text{if } s_p > 0,\ s_p \text{ odd} \\ (1 + \lambda_p(B) p^{\frac{s_p-1}{2}} t) \prod_{j=1}^{\lfloor \frac{s_p-1}{2} \rfloor} (1 - p^{2j-1}t^2) & \text{if } s_p > 0,\ s_p \text{ even} \end{cases} \tag{5.3}$$

With f_B as in Definition 1.12, we define a function $\rho_B(p^\mu)$, $\mu \geq 0$, via

$$\sum_{\mu \geq 0} \rho_B(p^\mu) t^\mu := \begin{cases} (1 - t^2) H_{n,p}(B;t) & \text{if } p | f_B, \\ 1 & \text{otherwise.} \end{cases} \tag{5.4}$$

Define

$$\mathcal{D}_p(B) := \mathrm{GL}_{2n}(\mathbb{Z}_p) \backslash \{ G \in \mathrm{M}_{2n}(\mathbb{Z}_p) \cap \mathrm{GL}_{2n}(\mathbb{Q}_p) : B[G^{-1}] \text{ is half-integral} \}. \tag{5.5}$$

Above, we recall $B[G^{-1}] := {}^tG^{-1}BG^{-1}$. We now define **Kohnen's phi function**:

Definition 5.1. For $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$, we define a function $\phi(a; B)$ for $1 \le a \mid f_B$, which we call *Kohnen's phi function* for B, as follows. For $a = p^\mu$, p prime, and $0 \le \mu \le \mathrm{ord}_p f_B$, set

$$\phi(p^\mu; B) := p^{\frac{\mu}{2}} \sum_{\nu=0}^{\lfloor \frac{\mu}{2} \rfloor} \sum_{\substack{G \in \mathcal{D}_p(B) \\ \mathrm{ord}_p \det(G) = \nu}} \rho_{B[G^{-1}]}(p^{\mu - 2\nu}). \tag{5.6}$$

We then extend $\phi(a; B)$ multiplicatively to all $1 \le a \mid f_B$.

5.2 Calculating Kohnen's Phi Function via Brute Force

We now describe the algorithm we initially used to compute $\phi(a; B)$ to develop conjectures. Let p be any prime.

Lemma 5.2. Fix $\nu \ge 0$. A set of representatives G from $\mathrm{GL}_{2n}(\mathbb{Z}_p) \backslash \mathrm{M}_{2n}(\mathbb{Z}_p) \cap \mathrm{GL}_{2n}(\mathbb{Q}_p)$ (where $\mathrm{GL}_{2n}(\mathbb{Z}_p)$ acts by left multiplication) with $\mathrm{ord}_p \det(G) = \nu$ is given by:

$$\left\{ G = \begin{pmatrix} p^{\lambda_1} & a_{1,2} & \cdots & a_{1,2n} \\ & p^{\lambda_2} & & \vdots \\ & & \ddots & a_{2n-1,2n} \\ & & & p^{\lambda_{2n}} \end{pmatrix} \ : \ \lambda_i \ge 0, \ \sum_i \lambda_i = \nu, \ 0 \le a_{ij} < p^{\lambda_j} \right\}. \tag{5.7}$$

Proof. This is seen via Gaussian elimination and a routine calculation. ∎

Remark 5.3. Observe that the representatives in (5.7) all lie in $\mathrm{M}_{2n}(\mathbb{Z}) \cap \mathrm{GL}_{2n}(\mathbb{Q})$.

If p is odd, we may assume that B takes a diagonal form:

$$B = p^{\mu} \begin{pmatrix} u_1 p^{\alpha_1} & & \\ & \ddots & \\ & & u_{2n} p^{\alpha_{2n}} \end{pmatrix}, \qquad \begin{array}{c} \alpha_1 \leq \cdots \leq \alpha_{2n}, \\ u_1, \ldots, u_{2n} \in \mathbb{Z}_p^{\times}. \end{array} \tag{5.8}$$

We record a linear algebraic lemma:

Lemma 5.4. Let $1 \leq n \leq 4$. Let

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}, \qquad G = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & a_{n,n} \end{pmatrix}.$$

Define,

$$C_{i,i} := a_{i,i} \qquad (1 \leq i \leq n).$$

Define recursively, in the lexicographic order $\preceq$ inherited from $\mathbb{Z}^2$,

$$C_{i,j} := -\left(a_{i,j} + \sum_{i<k<j} \frac{C_{i,k} a_{k,j}}{a_{k,k}} \right) \qquad (1 \leq i < j \leq n)$$

Then,

$$D[G^{-1}]_{i,j} = D'_{i,j} := \frac{1}{a_{i,i} a_{j,j}} \sum_{k=1}^{i} C_{k,i} C_{k,j} \frac{d_k}{a_{k,k}^2}, \qquad (1 \leq i \leq j \leq n) \tag{5.9}$$

Remark 5.5. The formula for $C_{i,j}$ depends only on the quantities $C_{i,k}$ where $(i,k) \prec (i,j)$.

Proof of Lemma 5.4. This was verified with the aid of SageMath. See Appendix B. ■

Remark 5.6. This is likely true for arbitrary n. The algorithm we develop in this section is prohibitively slow for $n \geq 4$, so we opt to abstain from painstakingly verifying this result.

Corollary 5.7. Let $1 \leq n \leq 4$. Let B, G be as in (5.7), (5.8), respectively. Define,

$$C_{i,i} := p^{\lambda_i} \qquad\qquad (1 \leq i \leq 2n).$$

Define recursively, in the lexicographic order $\preceq$ inherited from $\mathbb{Z}^2$,

$$C_{i,j} := -\left(a_{i,j} + \sum_{i<k<j} \frac{C_{i,k}a_{k,j}}{p^{\lambda_k}}\right) \qquad (1 \leq i < j \leq 2n)$$

Then,

$$B[G^{-1}]_{i,j} = \frac{p^{\mu}}{p^{\lambda_i}p^{\lambda_j}} \sum_{k=1}^{i} C_{k,i}C_{k,j}\frac{u_k p^{\alpha_k}}{p^{2\lambda_k}}, \qquad (1 \leq i < j \leq 2n). \tag{5.10}$$

Proof. This follows immediately by specializing Lemma 5.7. ■

5.3 Kohnen's Phi Function for $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$

Let's calculate $\phi(p^{\mu}; B)$, p any prime, $n = 1, 2, 3, 4$ in the case $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$. We begin with the following observation:

Lemma 5.8. Suppose that $B \in \mathcal{S}'_{2n}(\mathbb{Z})$ and $G \in \mathrm{M}_{2n}(\mathbb{Z}) \cap \mathrm{GL}_{2n}(\mathbb{Q})$ (recall Remark 5.3). If $2\mathrm{ord}_p \det(G) < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$, then $B[G^{-1}] \equiv 0 \bmod p$.

Proof. Indeed, we have

$$B[G^{-1}] := {}^tG^{-1}BG^{-1} = \frac{1}{\det(G)^2}B[\mathrm{adj}(G)].$$

Above, $\mathrm{adj}(G) := \det(G) \cdot G^{-1}$ denotes the adjugate of G. The adjugate lies in $M_{2n}(\mathbb{Z})$. Thus $\mathrm{ord}_p(\mathrm{cont}B[\mathrm{adj}(G)]) \geq \mathrm{ord}_p(\mathrm{cont}B)$. Therefore,

$$\begin{aligned}\mathrm{ord}_p(\mathrm{cont}B[G^{-1}]) &= \mathrm{ord}_p(\mathrm{cont}B[\mathrm{adj}(G)]) - 2\mathrm{ord}_p \det(G) \\ &\geq \mathrm{ord}_p(\mathrm{cont}B) - 2\mathrm{ord}_p \det(G) > \mathrm{ord}_p(2).\end{aligned}$$

Thus, looking at the definition (3.4), we see that $B[G^{-1}] \equiv 0 \bmod p$. ∎

We now introduce a combinatorial object related to the sets $\mathcal{D}_p(B)$. Define

$$[\nu]_d := \left\{ (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d \quad : \quad \lambda_i \geq 0, \ \sum_i \lambda_i = \nu \right\}, \tag{5.11}$$

and

$$\mathcal{D}_p(B)_\nu := \{ G \in \mathcal{D}_p(B) \quad : \quad \mathrm{ord}_p \det(G) = \nu \}. \tag{5.12}$$

We observe:

Lemma 5.9. Given $B \in \mathcal{S}'_{2n}(\mathbb{Z})$ and $2\nu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$, we have:

$$\#\mathcal{D}_p(B)_\nu = \sum_{(\lambda_i) \in [\nu]_{2n}} \prod_{i=1}^{2n} p^{(i-1)\lambda_i}. \tag{5.13}$$

Proof. Via Lemma 5.8, we know $B[G^{-1}] \equiv 0 \bmod p$ for all $G \in M_{2n}(\mathbb{Z}) \cap GL_{2n}(\mathbb{Q})$ with $\mathrm{ord}_p \det(G) = \nu$. From Remark 5.3, and definitions (5.5) and (5.12), we see $\#\mathcal{D}_p(B)_\nu$ equals the size of the set (5.7). The result follows from the parameterization given in (5.7). ∎

We now are ready to calculate:

Proposition 5.10. Let $B \in \mathcal{S}_2'(\mathbb{Z})^+$. For $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$, we have

$$\phi(p^\mu; B) = p^\mu. \tag{5.14}$$

Proof. We noted $B[G^{-1}]$ is zero modulo p for all $G \in \mathcal{D}_p(B)$ for which $2\mathrm{ord}_p \det(G) \leq \mu$. Thus $s_p(B[G^{-1}]) = 2$ and $\lambda_p(B[G^{-1}]) = 1$ for all such G. We calculate

$$\rho(p^\nu) := \rho_{B[G^{-1}]}(p^\nu) = \begin{cases} 1 & \text{if } \nu = 0 \\ p^{\frac{1}{2}} & \text{if } \nu = 1 \\ -1 & \text{if } \nu = 2 \\ -p^{\frac{1}{2}} & \text{if } \nu = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Assume μ is even. By definition,

$$\phi(p^\mu; B) := p^{\frac{\mu}{2}} \left(\#\mathcal{D}_p(B)_{\mu/2} \cdot \rho(1) + \#\mathcal{D}_p(B)_{\mu/2-1} \cdot \rho(p^2) \right).$$

Using Lemma 5.9, the quantity is parentheses equals,

$$\sum_{(\lambda_i) \in [\mu/2]_2} \prod_{i=1}^{2} p^{(i-1)\lambda_i} - \sum_{(\lambda_i) \in [\mu/2-1]_2} \prod_{i=1}^{2} p^{(i-1)\lambda_i} = \sum_{i=0}^{\mu/2} p^i - \sum_{i=0}^{\mu/2-1} p^i = p^{\frac{\mu}{2}}.$$

Thus,

$$\phi(p^\mu; B) = p^{\frac{\mu}{2}} \cdot p^{\frac{\mu}{2}} = p^\mu$$

The calculation is essentially the same when μ is odd. ∎

Proposition 5.11. Let $B \in \mathcal{S}_4'(\mathbb{Z})^+$. For $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$, we have

$$\phi(p^\mu; B) = p^{\frac{\mu}{2}+\frac{\delta_2(\mu)}{2}} \sum_{(\lambda_i)\in[\lfloor\mu/2\rfloor]_2} \prod_{i=1}^{2} p^{(i+1)\lambda_i} = p^{\mu+\left\lfloor\frac{\mu+1}{2}\right\rfloor} \sum_{i=0}^{\lfloor\mu/2\rfloor} p^i, \tag{5.15}$$

where,

$$\delta_2(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ 3 & \text{if } \mu \text{ odd.} \end{cases}$$

Proof. We've noted $B[G^{-1}]$ is zero modulo p for all $G \in \mathcal{D}(B)$ for which $2\mathrm{ord}_p \det(G) \le \mu$. Thus $s_p(B[G^{-1}]) = 4$ and $\lambda_p(B[G^{-1}]) = 1$ for all such G. We calculate

$$\rho(p^\nu) := \rho_{B[G^{-1}]}(p^\nu) = \begin{cases} 1 & \text{if } \nu = 0 \\ p^{\frac{3}{2}} & \text{if } \nu = 1 \\ -(p+1) & \text{if } \nu = 2 \\ -p^{\frac{3}{2}}(p+1) & \text{if } \nu = 3 \\ p & \text{if } \nu = 4 \\ p^{\frac{3}{2}}p & \text{if } \nu = 5 \\ 0 & \text{otherwise.} \end{cases}$$

Assume μ is even. By definition,

$$\phi(p^\mu; B) := p^{\frac{\mu}{2}} \left(\#\mathcal{D}_p(B)_{\mu/2} \cdot \rho(1) + \#\mathcal{D}_p(B)_{\mu/2-1} \cdot \rho(p^2) + \#\mathcal{D}_p(B)_{\mu/2-2} \cdot \rho(p^4) \right).$$

Using Lemma 5.9, the expression inside the parentheses equals

$$\sum_{(\lambda_i)\in[\mu/2]_4}\prod_{i=1}^{4}p^{(i-1)\lambda_i}-(p+1)\sum_{(\lambda_i)\in[\mu/2-1]_4}\prod_{i=1}^{4}p^{(i-1)\lambda_i}+p\sum_{(\lambda_i)\in[\mu/2-2]_4}\prod_{i=1}^{4}p^{(i-1)\lambda_i}.$$

Note, for $k=1,2$, we have

$$\sum_{(\lambda_i)\in[\mu/2-k]_4}\prod_{i=1}^{4}p^{(i-1)\lambda_i}=\sum_{\substack{(\lambda_i)\in[\mu/2-k+1]_4\\ \lambda_1\geq 1}}\prod_{i=1}^{4}p^{(i-1)\lambda_i}.$$

Thus

$$\begin{aligned}\phi(p^\mu;T)&=p^{\frac{\mu}{2}}\left(\sum_{\substack{(\lambda_i)\in[\mu/2]_4\\ \lambda_1=0}}\prod_{i=1}^{4}p^{(i-1)\lambda_i}-p\sum_{\substack{(\lambda_i)\in[\mu/2-1]_4\\ \lambda_1=0}}\prod_{i=1}^{4}p^{(i-1)\lambda_i}\right)\\ &=p^{\frac{\mu}{2}}\left(\sum_{(\lambda_i)\in[\mu/2]_3}\prod_{i=1}^{3}p^{i\lambda_i}-p\sum_{(\lambda_i)\in[\mu/2-1]_3}\prod_{i=1}^{3}p^{i\lambda_i}\right).\end{aligned}$$

Observe

$$p\sum_{(\lambda_i)\in[\mu/2-1]_3}\prod_{i=1}^{3}p^{i\lambda_i}=\sum_{\substack{(\lambda_i)\in[\mu/2]_3\\ \lambda_1\geq 1}}\prod_{i=1}^{3}p^{i\lambda_i}.$$

Finally,

$$\begin{aligned}\phi(p^\mu;T)&=p^{\frac{\mu}{2}}\sum_{\substack{(\lambda_i)\in[\mu/2]_3\\ \lambda_1=0}}\prod_{i=1}^{3}p^{i\lambda_i}=p^{\frac{\mu}{2}}\sum_{(\lambda_i)\in[\mu/2]_2}\prod_{i=1}^{2}p^{(i+1)\lambda_i}\\ &=p^{\frac{\mu}{2}}\sum_{k=0}^{\mu/2}p^{2\left(\frac{\mu}{2}-k\right)+3k}=p^{\frac{\mu}{2}}\sum_{k=0}^{\mu/2}p^{\mu+k}=p^{\mu+\frac{\mu}{2}}(1+p+\cdots+p^{\frac{\mu}{2}}).\end{aligned}$$

The calculation is essentially the same when μ is odd. ∎

We'll see that for $B \in \mathcal{S}_6'(\mathbb{Z})^+$, $\phi(p^\mu; B)$ no longer has a closed form series expression. The proof is quite satisfying for those who enjoy combinatorics.

Proposition 5.12. Let $B \in \mathcal{S}_6'(\mathbb{Z})^+$. For $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$, we have

$$\phi(p^\mu; B) = p^{\frac{\mu}{2}+\frac{\delta_3(\mu)}{2}} \sum_{\substack{(\lambda_i)\in[\lfloor\mu/2\rfloor]_4 \\ \lambda_2=0}} \prod_{i=1}^{4} p^{(i+1)\lambda_i}. \tag{5.16}$$

where,

$$\delta_3(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ 5 & \text{if } \mu \text{ odd.} \end{cases}$$

Proof. We've noted $B[G^{-1}]$ is zero modulo p for all $G \in \mathcal{D}(B)$ for which $2\mathrm{ord}_p \det(G) \leq \mu$. Thus $s_p(B[G^{-1}]) = 6$ and $\lambda_p(B[G^{-1}]) = 1$ for all such G. We calculate

$$\rho(p^\nu) := \rho_{B[G^{-1}]}(p^\nu) = \begin{cases} 1 & \text{if } \nu = 0 \\ p^{\frac{5}{2}} & \text{if } \nu = 1 \\ -(1+p+p^3) & \text{if } \nu = 2 \\ -p^{\frac{5}{2}}(1+p+p^3) & \text{if } \nu = 3 \\ p+p^3+p^4 & \text{if } \nu = 4 \\ p^{\frac{5}{2}}(p+p^3+p^4) & \text{if } \nu = 5 \\ -p^4 & \text{if } \nu = 6 \\ -p^{\frac{5}{2}}p^4 & \text{if } \nu = 7 \\ 0 & \text{otherwise.} \end{cases}$$

Assume μ is even. By definition,

$$\phi(p^\mu; B) := p^{\frac{\mu}{2}} \Big(\#\mathcal{D}_p(B)_{\mu/2} \cdot \rho(1) + \#\mathcal{D}_p(B)_{\mu/2-1} \cdot \rho(p^2) \\ + \#\mathcal{D}_p(B)_{\mu/2-2} \cdot \rho(p^4) + \#\mathcal{D}_p(B)_{\mu/2-3} \cdot \rho(p^6) \Big).$$

Using Lemma 5.9, the expression inside the parentheses equals

$$\sum_{(\lambda_i) \in [\mu/2]_6} \prod_{i=1}^{6} p^{(i-1)\lambda_i} - (1 + p + p^3) \sum_{(\lambda_i) \in [\mu/2-1]_6} \prod_{i=1}^{6} p^{(i-1)\lambda_i} \\ + (p + p^3 + p^4) \sum_{(\lambda_i) \in [\mu/2-2]_6} \prod_{i=1}^{6} p^{(i-1)\lambda_i} - p^4 \sum_{(\lambda_i) \in [\mu/2-3]_6} \prod_{i=1}^{6} p^{(i-1)\lambda_i}.$$

Note, for $k = 1, 2, 3$, we have

$$\sum_{(\lambda_i) \in [\mu/2-k]_6} \prod_{i=1}^{6} p^{(i-1)\lambda_i} = \sum_{\substack{(\lambda_i) \in [\mu/2-k+1]_6 \\ \lambda_1 \geq 1}} \prod_{i=1}^{6} p^{(i-1)\lambda_i}$$

Thus,

$$\begin{aligned}
&\phi(p^\mu; B)\\
&= p^{\frac{\mu}{2}} \left(\sum_{\substack{(\lambda_i)\in[\mu/2]_6 \\ \lambda_1=0}} \prod_{i=1}^{6} p^{(i-1)\lambda_i} - p \sum_{\substack{(\lambda_i)\in[\mu/2-1]_6 \\ \lambda_1=0}} \prod_{i=1}^{6} p^{(i-1)\lambda_i} \right.\\
&\qquad \left. -p^3 \sum_{\substack{(\lambda_i)\in[\mu/2-1]_6 \\ \lambda_1=0}} \prod_{i=1}^{6} p^{(i-1)\lambda_i} + p^4 \sum_{\substack{(\lambda_i)\in[\mu/2-2]_6 \\ \lambda_1=0}} \prod_{i=1}^{6} p^{(i-1)\lambda_i} \right)\\
&= p^{\frac{\mu}{2}} \left(\sum_{(\lambda_i)\in[\mu/2]_5} \prod_{i=1}^{5} p^{i\lambda_i} - p \sum_{(\lambda_i)\in[\mu/2-1]_5} \prod_{i=1}^{5} p^{i\lambda_i} \right.\\
&\qquad \left. -p^3 \sum_{(\lambda_i)\in[\mu/2-1]_5} \prod_{i=1}^{5} p^{i\lambda_i} + p^4 \sum_{(\lambda_i)\in[\mu/2-2]_5} \prod_{i=1}^{5} p^{i\lambda_i} \right)
\end{aligned}$$

Note, for $k = 1, 2$, we have

$$p \sum_{(\lambda_i)\in[\mu/2-k]_5} \prod_{i=1}^{5} p^{i\lambda_i} = \sum_{\substack{(\lambda_i)\in[\mu/2-k+1]_5 \\ \lambda_1\geq 1}} \prod_{i=1}^{5} p^{i\lambda_i}.$$

Thus,

$$\begin{aligned}
\phi(p^\mu; B) &= p^{\frac{\mu}{2}} \left(\sum_{\substack{(\lambda_i)\in[\mu/2]_5 \\ \lambda_1=0}} \prod_{i=1}^{5} p^{i\lambda_i} - p^3 \sum_{\substack{(\lambda_i)\in[\mu/2-1]_5 \\ \lambda_1=0}} \prod_{i=1}^{5} p^{i\lambda_i} \right)\\
&= p^{\frac{\mu}{2}} \left(\sum_{(\lambda_i)\in[\mu/2]_4} \prod_{i=1}^{4} p^{(i+1)\lambda_i} - p^3 \sum_{(\lambda_i)\in[\mu/2-1]_4} \prod_{i=1}^{4} p^{(i+1)\lambda_i} \right)
\end{aligned}$$

Observe

$$p^3 \sum_{(\lambda_i)\in[\mu/2-1]_4} \prod_{i=1}^{4} p^{(i+1)\lambda_i} = \sum_{\substack{(\lambda_i)\in[\mu/2]_4 \\ \lambda_2\geq 1}} \prod_{i=1}^{4} p^{(i+1)\lambda_i}.$$

Thus,

$$\phi(p^\mu; B) = p^{\frac{\mu}{2}} \sum_{\substack{(\lambda_i)\in[\mu/2]_4 \\ \lambda_2=0}} \prod_{i=1}^{4} p^{(i+1)\lambda_i}.$$

The calculation is essentially the same when μ is odd. ∎

We assume $B \in \mathcal{S}_6'(\mathbb{Z})^+$ and $0 \leq \mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$ in the table below:

μ	$\phi(p^\mu; B)$
0	1
2	$p(p^2 + p^4 + p^5)$
4	$p^2(p^4 + p^6 + p^7 + p^8 + p^9 + p^{10})$
6	$p^3(p^6 + p^8 + p^9 + p^{10} + p^{11} + 2p^{12} + p^{13} + p^{14} + p^{15})$
8	$p^4(p^8 + p^{10} + p^{11} + p^{12} + p^{13} + 2p^{14} + p^{15} + 2p^{16} + 2p^{17} + p^{18} + p^{19} + p^{20})$

We omit the proof of the next result, as the technique is the same as in Proposition 5.12.

Proposition 5.13. Let $B \in \mathcal{S}_8'(\mathbb{Z})^+$. For $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$, we have

$$\phi(p^\mu; B) = p^{\frac{\mu}{2}+\frac{\delta_4(\mu)}{2}} \sum_{\substack{(\lambda_i)\in[\lfloor\mu/2\rfloor]_6 \\ \lambda_2=\lambda_4=0}} \prod_{i=1}^{6} p^{(i+1)\lambda_i}. \tag{5.17}$$

where,

$$\delta_4(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ 7 & \text{if } \mu \text{ odd.} \end{cases}$$

We assume $B \in \mathcal{S}_8'(\mathbb{Z})^+$ and $0 \leq \mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$ in the table below:

μ	$\phi(p^\mu; B)$
0	1
2	$p(p^2 + p^4 + p^6 + p^7)$
4	$p^2(p^4 + p^6 + 2p^8 + p^9 + p^{10} + p^{11} + p^{12} + p^{13} + p^{14})$
6	$p^3(p^6 + p^8 + 2p^{10} + p^{11} + 2p^{12} + p^{13} + 2p^{14} + 2p^{15} + p^{16} + p^{17} + p^{18} + p^{19} + p^{20} + p^{21})$

With Propositions 5.11, 5.12, and 5.13 in mind, we (correctly) conjecture:

Theorem 5.14. Let $B \in \mathcal{S}_{2n}'(\mathbb{Z})^+$, $n > 1$. For $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$, we have

$$\phi(p^\mu; B) = p^{\frac{\mu}{2} + \frac{\delta_n(\mu)}{2}} \sum_{\substack{(\lambda_i) \in [\lfloor \mu/2 \rfloor]_{2n-2} \\ \lambda_{2i}=0,\ i<n-1}} \prod_{i=1}^{2n-2} p^{(i+1)\lambda_i}. \tag{5.18}$$

where,

$$\delta_n(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ (2n-1) & \text{if } \mu \text{ odd.} \end{cases}$$

Remark 5.15. We will delay the proof of Theorem 5.14 to Chapter 6.

5.4 Calculating Kohnen's Phi Function via EGK Data

In the previous section, we introduced Kohnen's phi function $\phi(a; B)$ as a sum over a rather unwieldy space of matrices denoted $\mathcal{D}_p(B)_\nu$. We saw that, under a certain simplifying condition; namely $\mu < \mathrm{ord}_p(\mathrm{cont}B) - \mathrm{ord}_p(2)$; we may compute $\phi(p^\mu; B)$ using the cardinality formula for $\mathcal{D}_p(B)_\nu$ in Lemma 5.9 (which applies for only $2\nu \leq \mu$ with the simplifying condition on μ above) and then applying some combinatorial identities. Unfortunately, calculating $\phi(a; B)$ in general via Definition 5.1 seems intractable. In this section, we will introduce an alternative way to compute $\phi(a; B)$ for all $a|f_B$. We start with the following two results which connect $\phi(a; B)$ to the Laurent polynomial for $\mathrm{EGK}(B)$. Firstly, Kohnen and Choie relate $\phi(a; B)$ to the Siegel series as follows:

Theorem 5.16. ([CK08, Main Result]) Let $B \in \mathcal{S}_{2n}(\mathbb{Z})^+$. For $p|f_B$, we have

$$\tilde{F}_p(B; X) = \sum_{j=1}^{\ell_p+1} \phi(p^{\ell_p - j+1}; B) p^{-\frac{\ell_p - j+1}{2}} \left(\psi_j(X) - \left(\frac{D_{B,0}}{p}\right) p^{-\frac{1}{2}} \psi_{j-1}(X)\right), \tag{5.19}$$

where $\ell = \ell_p := \mathrm{ord}_p f_B$ and

$$\psi_j(X) = \frac{X^j - X^{-j}}{X - X^{-1}} = X^{j-1} + X^{j-3} + \cdots + X^{-j+3} + X^{-j+1}. \tag{5.20}$$

.

Secondly, Ikeda and Katsurada connect the Siegel series to the Laurent polynomial via:

Theorem 5.17. ([IK22b, Theorem 1.1]) Let $B \in \mathcal{S}'_{2n}(\mathbb{Z}_p)^{\mathrm{nd}}$. Then,

$$\tilde{F}_p(B; X) = \mathcal{F}(\mathrm{EGK}(B); p^{\frac{1}{2}}, X).$$

We recall from (1.6) and Definition 3.8 that:

$$\varepsilon_{2n} := \xi(B) := \left(\frac{D_{B,0}}{p}\right). \tag{5.21}$$

With Theorems 5.16, 5.17, Corollary 3.18, and (5.21) in mind, we define:

Definition 5.18. Let $H = (a_1, \ldots, a_{2n}; \varepsilon_1, \ldots, \varepsilon_{2n})$ be a naive EGK datum of length $2n$. We define a function $\phi(\mu, H; Y)$, for $0 \leq \mu \leq \ell := \mathfrak{e}_{2n}/2$, implicitly from the relation:

$$\mathcal{F}(H; Y, X) = \sum_{j=1}^{\ell+1} \phi(\ell - j + 1, H; Y) Y^{-(\ell-j+1)} \left(\psi_j(X) - \varepsilon_{2n} Y^{-1} \psi_{j-1}(X)\right). \tag{5.22}$$

We call $\phi(\mu, H; Y)$ **Kohnen's phi function** attached to H.

Remark 5.19. $\phi(\mu, H; Y)$ is well-defined via Proposition 4.5 and Lemma 1 of [CK08].

Remark 5.20. Note, by comparing Definition 5.18 with (5.19), we have:

$$\phi(p^\mu; B) = \phi(\mu, \mathrm{EGK}(B)^{(p)}; p^{\frac{1}{2}}). \tag{5.23}$$

Thus $\phi(\mu, H; Y)$ captures far more data than $\phi(p^\mu; B)$ about B. In future chapters, we will study $\phi(\mu, H; Y)$ in more detail. For now, we explore applications to computing $\phi(p^\mu; B)$:

The following key result allows us to compute $\phi(\mu, H; Y)$ in terms of $\mathcal{F}(H'; Y, X)$.

Theorem 5.21. Let $H = (a_1, \ldots, a_{2n}; \varepsilon_1, \ldots, \varepsilon_{2n})$ be a naive EGK datum of length $2n$. Let $H' := (a_1, \ldots, a_{2n-1}; \varepsilon_1, \ldots, \varepsilon_{2n-1})$ and set $\ell' := \mathfrak{e}_{2n-1}/2$, $\ell := \mathfrak{e}_{2n}/2$. Write

$$\mathcal{F}(H'; Y, X) = \sum_{i=0}^{2\ell'} C_{-\ell'+i}^{(2n-1)}(Y) X^{-\ell'+i}$$

Then,

$$\phi(\mu, H; Y) = Y^{2\mu}(C^{(2n-1)}_{-\ell'+\mu}(Y) - C(\mu, G_-; Y)), \qquad 0 \leq \mu \leq \ell.$$

Here, G_- is the symmetric Laurent polynomial in X of degree $< -\ell + 2\ell' \leq \ell$ defined via:

$$G_-(Y, X) := \sum_{i=m+1}^{2\ell'} Y^i C_{-\ell'+i}(Y)(\psi_{-\ell+i-1}(X) - \varepsilon_{2n} Y^{-1}\psi_{-\ell+i}(X)), \qquad m := \min\{2\ell', \ell\}.$$

The $C(\mu, G_-; Y) \in \mathbb{Q}(Y)$ are so that, in the ε_{2n}-Kohnen-Choie basis, we have:

$$G_-(Y, X) = \sum_{i=0}^{\ell} Y^i C(i, G_-; Y)(\psi_{\ell-i+1}(X) - \varepsilon_{2n} Y^{-1}\psi_{\ell-i}(X)).$$

Proof. This follows from Proposition 4.8, Remark 4.9, and Definition 5.18. ■

Remark 5.22. The statement of Theorem 5.21 suffers from awkward indexing. The issue arises as $\phi(\mu, H; Y)$ evaluated at $\mu = 0$ corresponds to the coefficient of $X^{-\ell'}$ in $\mathcal{F}(H'; Y, X)$.

Remark 5.23. Ikeda and Katsurada have used a similar type of induction formula to obtain estimates of the Fourier coefficients of the Ikeda lift. Refer to [IK22a, Theorems 5.6 & 5.7].

As a corollary, we obtain a new tool for computing $\phi(p^\mu; B)$.

Corollary 5.24. Let $B \in \mathcal{S}'_{2n}(\mathbb{Z})^{\mathrm{nd}}$. Let $G := \mathrm{EGK}(B)^{(p)}$; take $H \in \Upsilon^{-1}(G) \subseteq \mathcal{N}EGK_n$. Let $H = (a_1, \ldots, a_{2n}; \varepsilon_1, \ldots, \varepsilon_{2n})$. Let $H' := (a_1, \ldots, a_{2n-1}; \varepsilon_1, \ldots, \varepsilon_{2n-1}) \in \mathcal{N}EGK_{n-1}$. Set $\ell' := \mathfrak{e}_{2n-1}/2$, $\ell := \mathfrak{e}_{2n}/2$. Write

$$\mathcal{F}(H'; p^{\frac{1}{2}}, X) = \sum_{i=0}^{2\ell'} C^{(2n-1)}_{-\ell'+i} X^{-\ell'+i}.$$

Then,

$$\phi(p^{\mu}; B) = p^{\mu}(C^{(2n-1)}_{-\ell'+\mu} - C(\mu, G_-)), \qquad 0 \le \mu \le \ell,$$

Here, G_- is the symmetric Laurent polynomial in X of degree $< -\ell + 2\ell' \le \ell$ defined via:

$$G_-(X) := \sum_{i=m+1}^{2\ell'} p^{\frac{i}{2}} C_{-\ell'+i} \left(\psi_{-\ell+i-1}(X) - \left(\frac{D_{B,0}}{p}\right) p^{-\frac{1}{2}} \psi_{-\ell+i}(X)\right), \qquad m := \min\{2\ell', \ell\}.$$

The $C(\mu, G_-) \in \mathbb{Q}$ are so that, in the basis $\psi_{j+1}(X) - (\frac{D_{B,0}}{p}) p^{-1/2} \psi_j(X)$, we have:

$$G_-(X) = \sum_{i=0}^{\ell} p^{\frac{i}{2}} C(i, G_-) \left(\psi_{\ell-i+1}(X) - \left(\frac{D_{B,0}}{p}\right) p^{-\frac{1}{2}} \psi_{\ell-i}(X)\right).$$

Proof. This follows from Theorems 5.16, 5.17, 5.21, Corollary 3.18, and (5.21). ∎

Remark 5.25. We recall that $\phi(p^{\mu}; B)$ is defined only for $0 \le \mu \le \mathrm{ord}_p f_B = \mathfrak{e}_{2n}/2 =: \ell$. The proofs below implicitly assume $\mu \le \ell$.

5.5 Explicit Formulas for Kohnen's Phi Function for $n = 1, 2$

We may now compute Kohnen's phi function with our new tools:

Proposition 5.26. (Confer Proposition 5.10 and [Koh02, Proposition 3]) Let $B \in \mathcal{S}_2'(\mathbb{Z}_p)^+$. Then,

$$\phi(p^{\mu}; B) = \begin{cases} p^{\mu} & \text{if } \mu \le \mathrm{ord}_p(\mathrm{cont} B), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Write $H := \mathrm{EGK}(B) = (a_1, a_2; 1, \varepsilon_2)$. We recall that $\mathrm{ord}_p(\mathrm{cont}B) = \mathfrak{e}_1 = a_1 \leq a_2$; we also recall $\mathrm{ord}_p f_B = \lfloor (a_1 + a_2)/2 \rfloor =: \mathfrak{e}_2/2 =: \ell$. Therefore $a_1 \leq \ell$. From (4.2), we have

$$\mathcal{F}(H'; p^{\frac{1}{2}}, X) = \sum_{i=0}^{2\ell'} X^{-\ell'+i}, \qquad \ell' := \frac{\mathfrak{e}_1}{2}, \quad \mathfrak{e}_1 = a_1.$$

Since $a_1 \leq a_2$, we have:

$$\ell + 1 = \frac{\mathfrak{e}_2}{2} + 1 = \left\lfloor \frac{a_1 + a_2}{2} \right\rfloor + 1 \geq \frac{a_1 + a_2 - 1}{2} + 1 > a_1 = 2\ell'.$$

In the notation of Corollary 5.24, this implies $G_-(X) = 0$. Thus,

$$\phi(\mu; B) = p^\mu C^{(1)}_{-\ell'+\mu}.$$

From the expression for $\mathcal{F}(H'; p^{\frac{1}{2}}, X)$ above, we have:

$$C^{(1)}_{-\ell'+\mu} = \begin{cases} 1 & \text{if } \mu \leq a_1, \\ 0 & \text{otherwise.} \end{cases}$$

The result follows immediately. ■

We may generalize the vanishing phenomenon of Proposition 5.26:

Proposition 5.27. Let $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$. Then $\phi(p^\mu; B) = 0$ for $\mathfrak{e}_{2n-1} < \mu \leq \mathfrak{e}_{2n}/2$.

Proof. Write $\ell := \mathfrak{e}_{2n}/2$, $\ell' := \mathfrak{e}_{2n-1}/2$. Then $\mu > 2\ell'$. If $2\ell' \geq \ell$, the statement is vacuous. Thus, we may assume $2\ell' < \ell$. In the notation of Corollary 5.24, this implies $G_-(X) = 0$. Thus,

$$\phi(\mu; B) = p^\mu C^{(1)}_{-\ell'+\mu}.$$

Moreover, since $-\ell' + \mu > -\ell' + 2\ell' = \ell' \geq 0$, we have $C^{(2n-1)}_{-\ell'+\mu} = 0$. Hence the claim. ∎

Remark 5.28. While we have $\mathfrak{e}_1 \leq \mathfrak{e}_2/2$, in general, $\mathfrak{e}_{2n-1} \nleq \mathfrak{e}_{2n}/2$.

We now record elementary estimate:

Lemma 5.29. For $(a_1, \ldots, a_{2n}) \in \mathbb{Z}^n_{\geq 0}$ non-decreasing and $\mathfrak{e}_i$ as in Definition 4.2:

$$\mathfrak{e}_{2n} - \mathfrak{e}_{2n-1} \geq \max\{a_1, a_{2n} - 1\} \geq a_1.$$

Proof. In the case $a_1 = \cdots = a_{2n}$, we have

$$\mathfrak{e}_{2n} - \mathfrak{e}_{2n-1} := 2\left\lfloor \frac{a_1 + \cdots + a_{2n}}{2} \right\rfloor - (a_1 + \cdots + a_{2n-1}) = a_{2n} = a_1.$$

Otherwise, when $a_{2n} \geq a_1 + 1$, we have

$$\begin{aligned}\mathfrak{e}_{2n} - \mathfrak{e}_{2n-1} :=& 2\left\lfloor \frac{a_1 + \cdots + a_{2n}}{2} \right\rfloor - (a_1 + \cdots + a_{2n-1}) \\ &\geq (a_1 + \cdots + a_{2n} - 1) - (a_1 + \cdots + a_{2n-1}) \\ &\geq a_{2n} - 1 \geq a_1.\end{aligned}$$

∎

Proposition 5.30. ([See Proposition 5.11]) Let $B \in \mathcal{S}'_4(\mathbb{Z}_p)^+$. For $\mu < \mathrm{ord}_p(\mathrm{cont}B)$,

$$\phi(p^\mu; B) = p^{\mu + \left\lfloor \frac{\mu+1}{2} \right\rfloor} \sum_{i=0}^{\lfloor \mu/2 \rfloor} p^i.$$

Proof sketch. We use the notation of Proposition 4.19. As in Proposition 5.26,we have:

$$C^{(3)}_{-\ell'+\mu}(Y) = A^{(3)}_{1,-\ell'+\mu} = Y^{\mu+\delta_1(\mu)} \sum_{i=0}^{\lfloor \mu/2 \rfloor} Y^{2i}.$$

Via Lemma 5.29, we have:

$$\ell - (-\ell + 2\ell') = 2\ell - 2\ell' = \mathfrak{e}_4 - \mathfrak{e}_3 \geq a_1.$$

In the notation of Corollary 5.24, this means G_- does not contribute to $\phi(p^\mu; B)$, $\mu < a_1$. That is, G_- is a polynomial of degree $< -\ell + 2\ell' \leq \ell - a_1$ in X; but $\phi(p^\mu; B)$, $\mu < a_1$ depends on the coefficients of X^ν for $\nu > \ell - a_1$. Thus, using Corollary 5.24, we calculate:

$$\phi(p^\mu; T) = p^\mu C^{(3)}_{-\ell'+\mu}(p^{\frac{1}{2}}) = p^\mu p^{\frac{\mu}{2}+\frac{\delta_1(\mu)}{2}} \sum_{i=0}^{\lfloor \mu/2 \rfloor} p^i = p^{\mu+\left\lfloor \frac{\mu+1}{2} \right\rfloor} \sum_{i=0}^{\lfloor \mu/2 \rfloor} p^i.$$

■

Remark 5.31. We have omitted details (principally, that the remaining coefficients $A^{(3)}_{2,\nu}$ and $A^{(3)}_{3,\nu}$ of Proposition 4.19 do not contribute to $\phi(p^\mu; B)$ for $\mu < a_1$) from the proof of Proposition 5.30 since we will generalize this calculation in Proposition 6.2 with full details.

Chapter 6

Combinatorial Aspects of $\mathcal{F}(H;Y,X)$

In this chapter, we fruitfully combine the results of Chapters 4 and 5 to study certain combinatorial interpretations of Kohnen's phi function (attached to an EGK datum). These combinatorial aspects; which were foreshadowed in Propositions 5.11, 5.12, 5.13 and Theorem 5.14; will be extended as far as possible with the aid of algorithm in Section 4.5.

6.1 Proof of Theorem 5.14

In this section, we prove the following generalization of Theorem 5.14:

Theorem 6.1. Let $H \in \mathcal{N}EGK_{2n}$, with $n > 1$. Then for $\mu < \mathfrak{e}_1$,

$$\phi(\mu, H; Y) = Y^{\mu+\delta_n(\mu)} \sum_{\substack{(\lambda_i)\in[\lfloor\mu/2\rfloor]_{2n-2} \\ \lambda_{2i}=0,\ i<n-1}} \prod_{i=1}^{2n-2} Y^{2(i+1)\lambda_i}, \tag{6.1}$$

where

$$\delta_n(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ 2n-1 & \text{if } \mu \text{ odd.} \end{cases}$$

Now, we prove the base case of of Theorem 6.1:

Lemma 6.2. Let $H \in \mathcal{NEGK}_4$. For $\mu < \mathfrak{e}_1$, we have

$$\phi(\mu, H; Y) = Y^{\mu+\delta_2(\mu)} \sum_{(\lambda_i)\in[\lfloor\mu/2\rfloor]_2} Y^{4\lambda_1+6\lambda_2}, \tag{6.2}$$

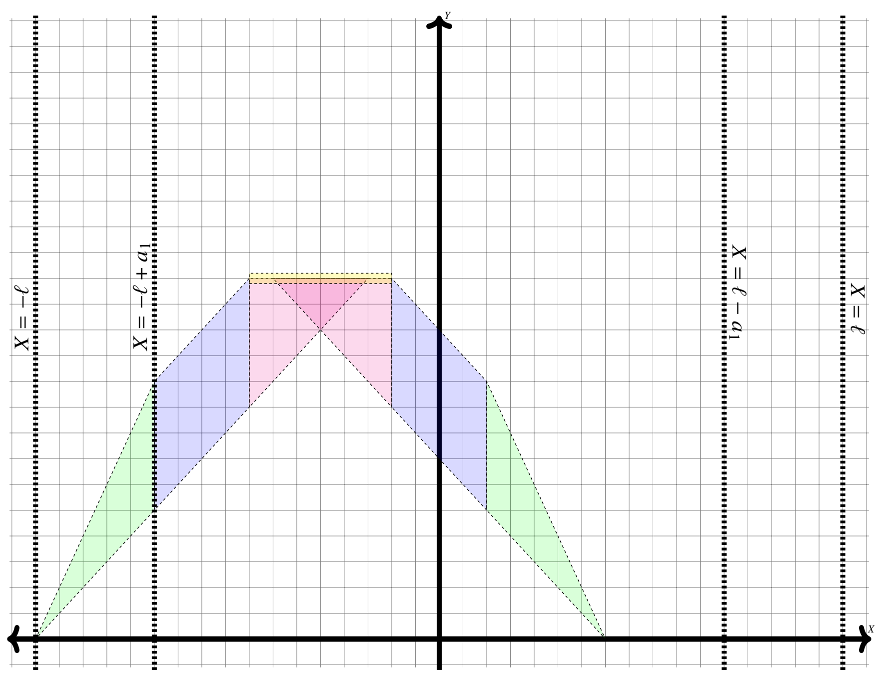

Figure 6.1: Coefficient Polygon for $\mathcal{F}(H'; Y, X)$ Shifted for Application of Theorem 5.21

Proof. Our calculation relies on Proposition 4.19, which gives an explicit formula for $\mathcal{F}(H'; Y, X)$, where $H' \in \mathcal{NEGK}_3$ is the 1-truncation of H, and Theorem 5.21, which connects the coefficients of $\mathcal{F}(H';Y,X)$ to Kohnen's phi function for H. First, we note that the terms $C(\mu, G_-;Y)$ do not contribute to $\phi(\mu, H;Y)$ for $\mu < \mathfrak{e}_1$, via the argument used in the proof of Proposition 5.30. In brief, G_- is a symmetric Laurent polynomial in X of degree $< \ell - \mathfrak{e}_1$; but $\phi(\mu, H;Y)$ for $\mu < \mathfrak{e}_1$ depends on the coefficients of X^ν for $|\nu| > \ell - \mathfrak{e}_1$. We now show that the the coefficients $A^{(3)}_{2,\nu}(Y)$ and $A^{(3)}_{3,\nu}(Y)$ in Proposition 4.19 do not contribute to $\phi(\mu, H;Y)$ for $\mu < \mathfrak{e}_1$. In view of Theorem 5.21, we must show that the coefficients $A^{(3)}_{2,\nu}(Y)$ and $A^{(3)}_{3,\nu}(Y)$ vanish when $-\ell' \leq \nu < -\ell' + \mathfrak{e}_1$. We proceed as follows:

Claim I: $A^{(3)}_{2,\nu}(Y) = 0$ for $-\ell' \leq \nu < -\ell' + \mathfrak{e}_1$.

Via Proposition 4.19, $A^{(3)}_{2,\nu}(Y) = 0$ except possibly when $\ell' - \mathfrak{e}'_2 + 2 \leq \nu \leq \ell'$. Now,

$$-\ell' + \mathfrak{e}_1 < \ell' - \mathfrak{e}'_2 + 2,$$

since

$$2\ell' - \mathfrak{e}'_2 + 2 = (a_1 + a_2 + a_3) - 2\left\lfloor \frac{a_1 + a_2 + 1}{2} \right\rfloor + 2 \geq a_3 + 1 > a_1 =: \mathfrak{e}_1.$$

Hence Claim I holds.

Claim II: $A^{(3)}_{3,\nu}(Y) = 0$ for $-\ell' \leq \nu < -\ell' + \mathfrak{e}_1$.

Via Proposition 4.19, $A^{(3)}_{3,\nu}(Y) = 0$ except possibly when $-\ell' + \mathfrak{e}_2 - \mathfrak{e}_1 \leq \nu \leq \ell' - \mathfrak{e}_2 + \mathfrak{e}_1$. We estimate

$$-\ell' + \mathfrak{e}_1 \leq -\ell' + \mathfrak{e}_2 - \mathfrak{e}_1,$$

since, via Lemma 5.29, we have

$$\mathfrak{e}_2 - \mathfrak{e}_1 \geq a_1 =: \mathfrak{e}_1.$$

Hence Claim II holds.

Therefore, as claimed, $\phi(\mu, H; Y)$ depends only on the coefficients $A^{(3)}_{1,\nu}(Y)$ for $\mu < \mathfrak{e}_1$. Thus, in view of Theorem 5.21:

$$\phi(\mu, H; Y) = Y^{2\mu} A^{(3)}_{1,-\ell'+\mu}(Y).$$

Using the formulas in Proposition 4.19, we calculate

$$\phi(\mu, H; Y) = Y^{2\mu} Y^{\mu+\delta_1(\mu)} \sum_{i=0}^{\lfloor \mu/2 \rfloor} Y^{2i} = Y^{\mu+\delta_2(\mu)} \sum_{i=0}^{\lfloor \mu/2 \rfloor} Y^{2\mu+2i+(\delta_1(\mu)-\delta_2(\mu))}.$$

On the other hand, we have:

$$\sum_{[\lfloor \mu/2 \rfloor]_2} Y^{4\lambda_1+6\lambda_2} = \sum_{i=0}^{\lfloor \mu/2 \rfloor} Y^{4\left(\left\lfloor \frac{\mu}{2} \right\rfloor - i\right)+6i} = \sum_{i=0}^{\lfloor \mu/2 \rfloor} Y^{4\left\lfloor \frac{\mu}{2} \right\rfloor+2i} = \sum_{i=0}^{\lfloor \mu/2 \rfloor} Y^{2\mu+2i+(\delta_1(\mu)-\delta_2(\mu))}.$$

Above, we have used the identity:

$$4 \left\lfloor \frac{\mu}{2} \right\rfloor = 2\mu + (\delta_1(\mu) - \delta_2(\mu)).$$

■

We will now complete the induction argument. For $\mu, \nu \geq 0$, define

$$\Pi^{(2m)}_{(\mu,\nu)} := \left\{ (\lambda_1, \ldots, \lambda_{2m}) \in [\mu]_{2m} \ : \ \lambda_{2i} = 0,\ i < m-1;\ \sum_i (i+1)\lambda_i = \nu; \right\}. \quad (6.3)$$

Our induction argument requires the following combinatorial lemma:

Lemma 6.3. We have

$$\#\Pi^{(2n)}_{(\lfloor\mu/2\rfloor,\nu/2+(2\lfloor\mu/2\rfloor-\mu))} = \sum_{i=0}^{\lfloor\mu/2\rfloor} \#\Pi^{(2n-2)}_{(i,\nu/2-\mu)}.$$

Proof. This result follows from the bijection: for $0 \le i \le \lfloor\mu/2\rfloor$,

$$\Pi^{(2n-2)}_{(i,\nu/2-\mu)} \quad \Leftrightarrow \quad \text{Subset of } \Pi^{(2n)}_{(\lfloor\mu/2\rfloor,\nu/2+(2\lfloor\mu/2\rfloor-\mu))} \ni (\lambda'_i) \text{ with } \lambda'_1 = \left\lfloor\frac{\mu}{2}\right\rfloor - i.$$

To see this, let $(\lambda_1, \cdots, \lambda_{2n-2}) \in \#\Pi^{(2n-2)}_{(i,\nu/2-\mu)}$ with $0 \le i \le \lfloor\mu/2\rfloor$. That is,

$$\sum_{j=1}^{2n-2} \lambda_j = i, \qquad \sum_{j=1}^{2n-2} (j+1)\lambda_j = \frac{\nu}{2} - \mu.$$

Define $(\lambda'_1, \ldots, \lambda'_{2n}) \in \mathbb{Z}^{2n}$ via:

$$\lambda'_1 = \left\lfloor\frac{\mu}{2}\right\rfloor - i; \qquad \lambda'_2 = 0; \qquad \lambda'_j = \lambda_{j-2}, \quad j > 2.$$

Then

$$\sum_{j=1}^{2n} \lambda'_j = \lambda'_1 + \lambda'_2 + \sum_{j=3}^{2n} \lambda'_j = \left(\left\lfloor\frac{\mu}{2}\right\rfloor - i\right) + \sum_{j=1}^{2n-2} \lambda_j = \left\lfloor\frac{\mu}{2}\right\rfloor$$

and

$$\begin{aligned}\sum_{j=1}^{2n}(j+1)\lambda'_j &= 2\lambda'_1 + 3\lambda'_2 + \sum_{j=1}^{2n-2}(j+3)\lambda_j \\ &= (2\lambda'_1 + 3\lambda'_2) + 2\sum_{j=1}^{2n-2}\lambda_j + \sum_{j=1}^{2n-2}(j+1)\lambda_j \\ &= 2\left(\left\lfloor\frac{\mu}{2}\right\rfloor - i\right) + 2i + \left(\frac{\nu}{2} - \mu\right) \\ &= \frac{\nu}{2} + \left(2\left\lfloor\frac{\mu}{2}\right\rfloor - \mu\right)\end{aligned}$$

Thus $(\lambda'_j) \in \#\Pi^{(2n)}_{(\lfloor\mu/2\rfloor, \nu/2+(2\lfloor\mu/2\rfloor-\mu))}$ with $\lambda'_1 = \left\lfloor\frac{\mu}{2}\right\rfloor - i$. The reverse direction is clear. ■

We will now prove Theorem 6.1:

Proof of Theorem 6.1. The base case is Lemma 6.2. Assume the claim holds for all $H \in \mathcal{NEGK}_{2n}$ for a fixed $n > 1$. Fix $\mu < \mathfrak{e}_1$. Let $H \in \mathcal{NEGK}_{2n+2}$. Therefore $H'' \in \mathcal{NEGK}_{2n}$. Here H'' denotes the 2-truncation of H. Via the induction hypothesis we have, for $0 \le i \le \mu$,

$$\phi(i, H''; Y) = Y^{i+\delta_n(i)} \sum_{\substack{(\lambda_k)\in[\lfloor i/2\rfloor]_{2n-2} \\ \lambda_{2k}=0,\ k<n-1}} \prod_{k=1}^{2n-2} Y^{2(k+1)\lambda_k} = Y^{i+\delta_n(i)} \sum_{j\ge 0} C^{(2n)}_{(i,j)} Y^j,$$

where,

$$C^{(2n)}_{(i,j)} := \#\Pi^{(2n-2)}_{(\lfloor i/2\rfloor, j/2)}.$$

From Proposition 4.12, we have

$$\mathcal{F}(H'; Y, X) = F(H'; Y, X) + \varepsilon_{2n+1} F(H'; Y, X^{-1}) + \varepsilon^2_{2n} R(H'; Y, X).$$

where

$$F(H';Y,X) = X^{-\frac{\mathfrak{e}_{2n+1}}{2}} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi(i,H'';Y)X^{i} \sum_{j=0}^{\frac{\mathfrak{e}'_{2n}}{2}-i-1} (YX)^{2j}. \tag{6.4}$$

As in the proof of Lemma 6.2, we will show that the coefficients of the latter two Laurent polynomials $F(H';Y,X^{-1})$ and $R(H';Y,X)$ do not contribute to $\phi(\mu,H;Y)$ for $0 \leq \mu < \mathfrak{e}_1$. Before this, we briefly note, that, as in the proofs of Proposition 5.30 and Theorem 6.2, the terms $C(\mu,G_{-};Y)$ of Theorem 5.21 do not contribute to $\phi(\mu,H;Y)$ for $0 \leq \mu < \mathfrak{e}_1$. Now,

Claim I: $F(H';Y,X^{-1})$ does not contribute to $\phi(\mu,H;Y)$ for $0 \leq \mu < \mathfrak{e}_1$.

The coefficients of X^{ν} in $F(H';Y,X^{-1})$ vanishes except when $\frac{\mathfrak{e}_{2n+1}}{2} - \mathfrak{e}'_{2n} + 2 \leq \nu \leq \frac{\mathfrak{e}_{2n+1}}{2}$. We estimate:

$$-\frac{\mathfrak{e}_{2n+1}}{2} + \mathfrak{e}_1 < \frac{\mathfrak{e}_{2n+1}}{2} - \mathfrak{e}'_{2n} + 2,$$

since

$$\mathfrak{e}_{2n+1} - \mathfrak{e}'_{2n} + 2 = (a_1 + \cdots + a_{2n} + a_{2n+1}) - 2\left\lfloor \frac{a_1 + \cdots + a_{2n} + 1}{2} \right\rfloor + 2 \geq a_{2n+1} + 1 > a_1 =: \mathfrak{e}_1.$$

Hence Claim I holds.

Claim II: $R(H';Y,X)$ does not contribute to $\phi(\mu,H;Y)$ for $0 \leq \mu < \mathfrak{e}_1$.

Indeed, by Proposition 4.14, we know $R(H';Y,X)$ is a polynomial of degree $\leq \frac{\mathfrak{e}_{2n+1}}{2} - \mathfrak{e}_1$. Hence Claim II holds.

Now, write

$$F(H';Y,X) = \sum_{i=0}^{2\ell'} C^{(2n+1)}_{-\ell'+i}(Y) X^{-\ell'+i}, \qquad C^{(2n+1)}_{-\ell'+i}(Y) \in \mathbb{Q}(Y).$$

Using (6.4), we calculate the coefficient

$$C^{(2n+1)}_{-\ell'+\mu}(Y) = \sum_{\substack{0\le i\le\mu \\ i\equiv\mu \bmod 2}} \phi(i,H'';Y)Y^{\mu-i}.$$

Above, we have used the fact that, for $0 \le i \le \mu$ with $i \equiv \mu \bmod 2$, we have:

$$i \le \frac{\mathfrak{e}_{2n}}{2}, \qquad\qquad \frac{\mu-i}{2} \le \frac{\mathfrak{e}'_{2n}}{2} - i - 1.$$

Indeed, $i \le \mu < \mathfrak{e}_1 \le \mathfrak{e}_{2n}/2$. Moreover, the latter inequality is equivalent to $i \le \mathfrak{e}'_{2n} - \mu - 2$. We estimate:

$$i \le \mu < \mathfrak{e}_1 \le \mathfrak{e}'_{2n} - \mathfrak{e}_1 < \mathfrak{e}'_{2n} - \mu.$$

As two of the inequalities are strict, we get the desired estimate. Now, from Theorem 5.21:

$$\phi(\mu,H;Y) = Y^{2\mu}C^{(2n+1)}_{-\ell'+\mu}(Y) = Y^{2\mu} \sum_{\substack{0\le i\le\mu \\ i\equiv\mu \bmod 2}} \phi(i,H'';Y)Y^{\mu-i}.$$

Plugging in the expression for $\phi(i,H'';Y)$ and noting $\delta_n(i) = \delta_n(\mu)$ for $i \equiv \mu \bmod 2$ yields:

$$\phi(\mu,H;Y) = Y^{2\mu} \sum_{\substack{0\le i\le\mu \\ i\equiv\mu \bmod 2}} \left(Y^{i+\delta_n(i)} \sum_{j\ge 0} C^{(2n)}_{(i,j)} Y^j \right) Y^{\mu-i} = Y^{\mu+\delta_n(\mu)} \sum_{\substack{0\le i\le\mu \\ i\equiv\mu \bmod 2}} \sum_{j\ge 0} C^{(2n)}_{(i,j)} Y^{2\mu+j}.$$

Swapping the order of summation and making the change of variable $\nu = 2\mu + j$:

$$\begin{aligned}\phi(\mu, H; Y) &= Y^{\mu+\delta_n(\mu)} \sum_{\nu \geq 2\mu} \left(\sum_{\substack{0 \leq i \leq \mu \\ i \equiv \mu \bmod 2}} C^{(2n)}_{(i, \nu-2\mu)} \right) Y^{\nu} \\ &= Y^{\mu+\delta_{n+1}(\mu)} \sum_{\nu \geq 2\mu} \left(\sum_{\substack{0 \leq i \leq \mu \\ i \equiv \mu \bmod 2}} C^{(2n)}_{(i, \nu-2\mu)} \right) Y^{\nu+2(2\lfloor \mu/2 \rfloor - \mu)}\end{aligned}$$

Above, we have used that $\delta_{n+1}(\mu) = \delta_n(\mu) + 2(\mu - 2\lfloor \mu/2 \rfloor)$, which follows from:

$$2(\mu - 2\lfloor \mu/2 \rfloor) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ 2 & \text{if } \mu \text{ odd.} \end{cases}$$

Now, via the definition of $C^{(2n)}_{(i,\nu-2\mu)}$ and a change of variable:

$$\sum_{\substack{0 \leq i \leq \mu \\ i \equiv \mu \bmod 2}} C^{(2n)}_{(i, \nu-2\mu)} = \sum_{\substack{0 \leq i \leq \mu \\ i \equiv \mu \bmod 2}} \#\Pi^{(2n-2)}_{(\lfloor i/2 \rfloor, \nu/2-\mu)} = \sum_{i=0}^{\lfloor \mu/2 \rfloor} \#\Pi^{(2n-2)}_{(i, \nu/2-\mu)}. \tag{6.5}$$

Thus, via Lemma 6.3, we have:

$$\phi(\mu, H; Y) = Y^{\mu+\delta_{n+1}(\mu)} \sum_{\nu \geq 2\mu} \#\Pi^{(2n)}_{(\lfloor \mu/2 \rfloor, \nu/2 + (2\lfloor \mu/2 \rfloor - \mu))} Y^{\nu+2(2\lfloor \mu/2 \rfloor - \mu)} \tag{6.6}$$

We calculate an alternate expression for this series. The definition of $\Pi^{(2n)}_{(\lfloor \mu/2 \rfloor, j/2)}$ gives:

$$\sum_{\substack{(\lambda_i) \in [\lfloor \mu/2 \rfloor]_{2n} \\ \lambda_{2i}=0,\ i<n}} \prod_{i=1}^{2n} Y^{2(i+1)\lambda_i} = \sum_{j \geq 0} \#\Pi^{(2n)}_{(\lfloor \mu/2 \rfloor, j/2)} Y^{j}.$$

By definition, we know $\Pi^{(2n)}_{(\lfloor \mu/2 \rfloor, j/2)}$ is empty unless:

$$2\left\lfloor \frac{\mu}{2} \right\rfloor = 2\sum_i \lambda_i \le \sum_i (i+1)\lambda_i = \frac{j}{2}.$$

Thus,

$$\sum_{\substack{(\lambda_i)\in[\lfloor \mu/2 \rfloor]_{2n} \\ \lambda_{2i}=0,\ i<n}} \prod_{i=1}^{2n} Y^{2(i+1)\lambda_i} = \sum_{j \ge 4\lfloor \mu/2 \rfloor} \#\Pi^{(2n)}_{(\lfloor \mu/2 \rfloor, j/2)} Y^j$$

Making the change of variable $\nu = j + 2(\mu - 2\lfloor \mu/2 \rfloor)$, we have

$$\sum_{\substack{(\lambda_i)\in[\lfloor \mu/2 \rfloor]_{2n} \\ \lambda_{2i}=0,\ i<n}} \prod_{i=1}^{2n} Y^{2(i+1)\lambda_i} = \sum_{\nu \ge 2\mu} \#\Pi^{(2n)}_{(\lfloor \mu/2 \rfloor, \nu/2+(2\lfloor \mu/2 \rfloor - \mu))} Y^{\nu+2(2\lfloor \mu/2 \rfloor - \mu)}.$$

Comparing this with (6.6), we conclude

$$\phi(\mu, H; Y) = Y^{\mu+\delta_{n+1}(\mu)} \sum_{\substack{(\lambda_i)\in[\lfloor \mu/2 \rfloor]_{2n} \\ \lambda_{2i}=0,\ i<n}} \prod_{i=1}^{2n} Y^{2(i+1)\lambda_i}.$$

This completes the induction. ■

We have thus verified a stronger version (without the $\mathrm{ord}_p(2)$ term) of Theorem 5.14:

Theorem 6.4. Let $B \in \mathcal{S}'_{2n}(\mathbb{Z}_p)^+$, $n > 1$. For $\mu < \mathrm{ord}_p(\mathrm{cont} B)$, we have

$$\phi(p^\mu; B) = p^{\frac{\mu}{2}+\frac{\delta_n(\mu)}{2}} \sum_{\substack{(\lambda_i)\in[\lfloor \mu/2 \rfloor]_{2n-2} \\ \lambda_{2i}=0,\ i<n-1}} \prod_{i=1}^{2n-2} p^{(i+1)\lambda_i}. \tag{6.7}$$

where,

$$\delta_n(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ 2n-1 & \text{if } \mu \text{ odd.} \end{cases}$$

Proof. This follows immediately from (5.23) and Theorem 6.1. ∎

We can express these results in a cleaner form (by removing the parts with $\lambda_{2i} = 0$):

Remark 6.5. Given $\lambda := (\lambda_i) \in \mathbb{Z}^n$, define the *partition weight* function:

$$w_n(\lambda) := \sum_{i=1}^{n-1} 2i\lambda_i + (2n-1)\lambda_n.$$

Then, in Theorem 6.1, we have:

$$\phi(\mu, H; Y) = Y^{\mu+\delta_n(\mu)} \sum_{\lambda \in [\lfloor \mu/2 \rfloor]_n} Y^{2w_n(\lambda)}, \tag{6.8}$$

and, in Theorem 6.4, we have:

$$\phi(p^\mu; B) = p^{\frac{\mu}{2}+\frac{\delta_n(\mu)}{2}} \sum_{\lambda \in [\lfloor \mu/2 \rfloor]_n} p^{w_n(\lambda)}.$$

These formulas cover the case $n = 1$ as well. We introduce these expressions after the proofs above as it would be awkward to keep track of the lone part which is weighted by an odd integer in $w_n(\lambda)$. This is apparent upon examining (6.11) and the proof of Lemma 6.3.

6.2 Restricted Integer Partitions

We now extend our results by studying the Laurent polynomial $F^{(2n)}_{(\mathbf{1^n},\mathbf{1^n})}$ from Section 4.5. Here $\mathbf{1}^n = \underbrace{(1,\dots,1)}_{n \text{ entries}} \in S_1^{(n)}, S_2^{(n)}$. The coefficients of $F^{(2n)}_{(\mathbf{1^n},\mathbf{1^n})}$ involve *restricted* partitions. We start by formally introducing the necessary combinatorial objects:

Definition 6.6. Let $H = (a_1,\dots,a_{2m};\varepsilon_1,\dots,\varepsilon_{2m})$ be a naive EGK datum of length $2m$. We define, for $\delta \in \{0,1\}$, the sets

$$\widetilde{[\xi,\delta]}_m^{H,\,\mathrm{res}} := \left\{ \lambda \in [\xi]_m \ : \begin{array}{c} \lambda_{m-i+1} \geq \xi'_{m-i+1} - (\mathfrak{e}'_{2n-2}/2 - 1), \quad 1 < i \leq m, \\ \xi'_{m-i+1} \leq \mathfrak{e}_{2i}/2, \quad 1 < i < m, \\ \xi'_m \leq \mathfrak{e}_1 (\leq \mathfrak{e}_2/2); \\ \text{where } \xi'_{m-i+1} := 2(\sum_{k=m-i+1}^m \lambda_k) + \delta \end{array} \right\}, \tag{6.9}$$

$$[\xi,\delta]_m^{H,\,\mathrm{res}} := \left\{ \lambda \in \widetilde{[\xi,\delta]}_m^{H,\mathrm{res}} \ : \begin{array}{c} \xi'_1 \leq \mathfrak{e}_{2m}/2; \\ \text{where } \xi'_1 := 2(\sum_{k=1}^m \lambda_k) + \delta \end{array} \right\}, \tag{6.10}$$

and

$$\widetilde{\Pi}^{H,\,\mathrm{res}}_{([\xi,\delta],\nu)} := \left\{ \lambda \in \widetilde{[\xi,\delta]}_m^{H,\,\mathrm{res}} \ : \ w_m(\lambda) = \nu \right\}, \tag{6.11}$$

$$\Pi^{H,\,\mathrm{res}}_{([\xi,\delta],\nu)} := \left\{ \lambda \in [\xi,\delta]_m^{H,\,\mathrm{res}} \ : \ w_m(\lambda) = \nu \right\}. \tag{6.12}$$

The $\mathfrak{e}_i$ are as in Definition 4.2.

Remark 6.7. In our applications, we always plug in $\xi = \lfloor \mu/2 \rfloor$ and $\delta = \delta_1(\mu)$. Thus,

$$\xi'_1 := 2\left(\sum_{i=1}^m \lambda_i\right) + \delta = 2\left\lfloor \frac{\mu}{2} \right\rfloor + \delta_1(\mu) = \mu.$$

It was essential to upgrade our notation like this to maintain consistency with the previous results. In Theorem 6.1, we were able to write $\phi(\mu, H; Y)$ as a sum over integer partitions of $\lfloor \mu/2 \rfloor$. However, for our next result, $\phi(\mu, H; Y)$ is written as a sum over integer partitions of $\lfloor \mu/2 \rfloor$ with restrictions which depend critically on the parity of μ. This information is lost in the quantity $\lfloor \mu/2 \rfloor$; hence we must include the extra parameter δ in our new notation.

Theorem 6.8. Let $H = (a_1, \ldots, a_{2n}; \varepsilon_1, \ldots, \varepsilon_{2n}) \in \mathcal{NEGK}_{2n}$. Write

$$F^{(2n)}_{(\mathbf{1^n},\mathbf{1^n})}(H; Y, X) = \sum_{j=0}^{\ell} \phi_{(\mathbf{1^n},\mathbf{1^n})}(j; H, Y) Y^{-j} (\psi_{l-j+1}(X) - \varepsilon_{2n} \psi_{l-j}(X)),$$

$$\phi_{(\mathbf{1^n},\mathbf{1^n})}(j; H, Y) \in \mathbb{Q}(Y), \quad \ell := \frac{\mathfrak{e}_{2n}}{2}.$$

Then,

$$\phi_{(\mathbf{1^n},\mathbf{1^n})}(\mu, H; Y) = Y^{\mu+\delta_n(\mu)} \sum_{\lambda \in [\lfloor \mu/2 \rfloor, \delta_1(\mu)]_n^{H,\mathrm{res}}} Y^{2w_n(\lambda)}, \qquad \mu \leq \ell, \tag{6.13}$$

where

$$\delta_n(\mu) = \begin{cases} 0 & \text{if } \mu \text{ even,} \\ 2n-1 & \text{if } \mu \text{ odd.} \end{cases}$$

We first prove the base case:

Lemma 6.9. Let $H = (a_1, a_2; \varepsilon_1, \varepsilon_2) \in \mathcal{NEGK}_2$. Write

$$F^{(2)}_{(\mathbf{1^n},\mathbf{1^n})}(H; Y, X) = \sum_{j=0}^{\ell} \phi(j; H, Y) Y^{-j} (\psi_{\ell-j+1}(X) - \varepsilon_2 Y^{-1} \psi_{\ell-j}(X)), \qquad \ell := \frac{\mathfrak{e}_{2n}}{2}.$$

Then,

$$\phi(\mu, H; Y) = Y^{\mu+\delta_1(\mu)} \sum_{\lambda \in [\lfloor \mu/2 \rfloor, \delta_1(\mu)]_1^{H,\mathrm{res}}} Y^{2w_n(\lambda)}, \qquad \mu \leq \ell.$$

Proof. By Definition 4.21, we have

$$F^{(2)}_{(\mathbf{1^n},\mathbf{1^n})} = \mathcal{F}(H; Y, X) = \sum_{j=0}^{\mathfrak{e}_1} Y^j(\psi_{\ell-i+1}(X) - \varepsilon_2 Y^{-1}\psi_{\ell-j}(X)).$$

Thus,

$$\phi(\mu; H, Y) = \begin{cases} Y^{2\mu} & \text{if } \mu \leq \mathfrak{e}_1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we have $[\lfloor \mu/2 \rfloor]_1 = \{(\lfloor \mu/2 \rfloor)\}$ and

$$2\left\lfloor \frac{\mu}{2} \right\rfloor + \delta_1(\mu) \leq \mathfrak{e}_1 \qquad \Leftrightarrow \qquad \mu \leq \mathfrak{e}_1.$$

Thus, $[\lfloor \mu/2 \rfloor, \delta_1(\mu)]_1^{H,\mathrm{res}} = \{(\lfloor \mu/2 \rfloor)\}$ if $\mu \leq \mathfrak{e}_1$. Otherwise, $[\lfloor \mu/2 \rfloor, \delta_1(\mu)]_1^{H,\mathrm{res}}$ is empty. Note that $w_1(\lambda) := \lambda_1$ for $\lambda = (\lambda_1)$. Thus,

$$Y^{\mu+\delta_1(\mu)} \sum_{\lambda \in [\lfloor \mu/2 \rfloor, \delta_1(\mu)]_m^{H,\mathrm{res}}} Y^{2w_n(\lambda)} = \begin{cases} Y^{(\mu+\delta_1(\mu))+2\lfloor \mu/2 \rfloor} & \text{if } \mu \leq \mathfrak{e}_1, \\ 0 & \text{otherwise.} \end{cases}$$

We are done since $2\lfloor \mu/2 \rfloor + \delta_1(\mu) = \mu$. ∎

We are now ready to prove Theorem 6.8:

Proof of Theorem 6.8. This proof is mostly a refinement of the proof of Theorem 6.1. The base case is Lemma 6.9. Assume the claim holds for all $H \in \mathcal{NEGK}_{2n}$ for a fixed $n \geq 1$.

Let $H \in \mathcal{NEGK}_{2n+2}$. Thus $H'' \in \mathcal{NEGK}_{2n}$, where H'' denotes the 2-truncation of H. Write

$$F^{(2n)}_{(\mathbf{1^n},\mathbf{1^n})}(H'';Y,X) = \sum_{i=0}^{\ell} \phi_{(\mathbf{1^n},\mathbf{1^n})}(i;H'',Y)Y^{-i}(\psi_{l-i+1}(X) - \varepsilon_{2n}\psi_{l-i}(X)),$$

$$\phi_{(\mathbf{1^n},\mathbf{1^n})}(i;H'',Y) \in \mathbb{Q}(Y), \quad \ell := \frac{\mathfrak{e}_{2n}}{2}.$$

By the induction hypothesis, for $0 \le i \le \mathfrak{e}_{2n}/2$, we have:

$$\phi_{(\mathbf{1^n},\mathbf{1^n})}(i,H'';Y) = Y^{i+\delta_n(i)} \sum_{\lambda \in [\lfloor i/2 \rfloor, \delta_1(i)]_n^{H'',\mathrm{res}}} Y^{2w_n(\lambda)} = Y^{i+\delta_n(i)} \sum_{j \ge 0} C^{(2n)}_{(i,j)} Y^j, \tag{6.14}$$

where

$$C^{(2n)}_{(i,j)} := \#\Pi^{H'',\,\mathrm{res}}_{([\lfloor i/2 \rfloor,\delta_1(i)],j/2)}.$$

By Definition 4.21, we have

$$F^{(2n+1)}_{(\mathbf{1^{n+1}},\mathbf{1^{n+1}})}(H;Y,X) = X^{-\frac{\mathfrak{e}_{2n+1}}{2}} \sum_{i=0}^{\frac{\mathfrak{e}_{2n}}{2}} \phi_{(\mathbf{1^n},\mathbf{1^n})}(i,H'';Y)(YX)^i \sum_{j=0}^{\frac{\mathfrak{e}'_{2n}}{2}-i-1} (YX^{\pm 1})^{2j}. \tag{6.15}$$

Now, write

$$F^{(2n+1)}_{(\mathbf{1^{n+1}},\mathbf{1^{n+1}})}(H;Y,X) = \sum_{i=0}^{2\ell'} C^{(2n+1)}_{-\ell'+i}(Y)X^{-\ell'+i}, \qquad \ell' = \frac{\mathfrak{e}_{2n+1}}{2}, \;\; C^{(2n+1)}_{-\ell'+i}(Y) \in \mathbb{Q}(Y).$$

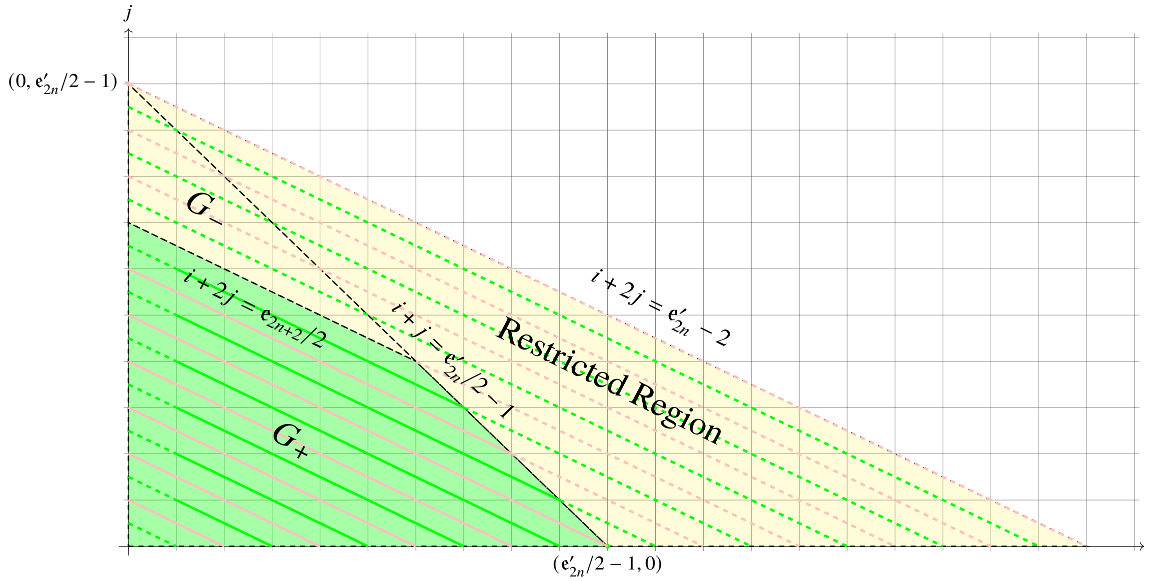

Figure 6.2: Region of summation: $0 \le i \le \frac{\mathfrak{e}_{2n}}{2}; 0 \le j \le \frac{\mathfrak{e}'_{2n}}{2} - i - 1$.

The first essential difference in the proof is here. Using (6.15) and Figure 6.2, we calculate:

$$C^{(2n+1)}_{-\ell'+\mu}(Y) = \sum_{\substack{0\le i\le B(\mu)\\ i\equiv\mu \bmod 2}} \phi_{(\mathbf{1^n},\mathbf{1^n})}(i, H''; Y)Y^{\mu-i},$$

where

$$B(\mu) := \mathfrak{e}'_{2n} - 2 - \mu.$$

This $B(\mu)$ is the i-coordinate of the intersection of the lines:

$$i + 2j = \mu$$
$$i + j = \mathfrak{e}'_{2n}/2 - 1.$$

The proof is exactly the same until we derive the identity:

$$C_{-\ell'+\mu}^{(2n+1)}(Y) = Y^{-\mu+\delta_n(\mu)} \sum_{\nu \geq 2\mu} \left(\sum_{\substack{0 \leq i \leq B(\mu) \\ i \equiv \mu \bmod 2}} C_{(i,\nu-2\mu)}^{(2n)} \right) Y^{\nu+2(2\lfloor \mu/2 \rfloor - \mu)}.$$

As before, by a change of variable:

$$\sum_{\substack{0 \leq i \leq B(\mu) \\ i \equiv \mu \bmod 2}} C_{(i,\nu-2\mu)}^{(2n)} = \sum_{\substack{0 \leq i \leq B(\mu) \\ i \equiv \mu \bmod 2}} \#\Pi_{([\lfloor i/2 \rfloor, \delta_1(i)], \nu/2-\mu)}^{H'', \mathrm{res}} = \sum_{i=0}^{\lfloor B(\mu)/2 \rfloor} \#\Pi_{([i, \delta_1(\mu)], \nu/2-\mu)}^{H'', \mathrm{res}}.$$

Above, we have used the facts that $m(\mu) \equiv \mu \bmod 2$ and that $\delta_1(i) = \delta_1(\mu)$ for $i \equiv \mu \bmod 2$.

By the same argument used in the proof of Lemma 6.3, we have:

$$\sum_{i=0}^{\lfloor B(\mu)/2 \rfloor} \#\Pi_{([i, \delta_1(\mu)], \nu/2-\mu)}^{H'', \mathrm{res}} = \#\widetilde{\Pi}_{([\lfloor \mu/2 \rfloor, \delta_1(\mu)], \nu/2+(2\lfloor \mu/2 \rfloor - \mu))}^{H, \mathrm{res}}.$$

That is, we have the following restriction on λ_1:

$$\lambda_1 \geq \left\lfloor \frac{\mu}{2} \right\rfloor - \left\lfloor \frac{B(\mu)}{2} \right\rfloor,$$

and

$$\begin{aligned} \left\lfloor \frac{\mu}{2} \right\rfloor - \left\lfloor \frac{B(\mu)}{2} \right\rfloor &= \left(\frac{\mu}{2} - \frac{\delta_1(\mu)}{2} \right) - \left(\frac{B(\mu)}{2} - \frac{\delta_1(B(\mu))}{2} \right) \\ &= \left(\frac{\mu}{2} - \frac{\delta_1(\mu)}{2} \right) - \left(\frac{\mathfrak{e}'_{2n} - 2 - \mu}{2} - \frac{\delta_1(\mu)}{2} \right) \\ &= \mu - (\mathfrak{e}'_{2n}/2 - 1) = \xi'_1 - (\mathfrak{e}'_{2n}/2 - 1), \end{aligned}$$

where

$$\xi_1' := 2(\lambda_1 + \cdots + \lambda_{n+1}) + \delta_1(\mu) = \mu.$$

Thus,

$$C^{(2n+1)}_{-\ell'+\mu}(Y) = Y^{-\mu+\delta_n(\mu)} \sum_{\nu \geq 2\mu} \#\widetilde{\Pi}^{H,\,\mathrm{res}}_{([\lfloor \mu/2 \rfloor, \delta_1(\mu)], \nu/2+(2\lfloor \mu/2 \rfloor - \mu))} Y^{\nu+2(2\lfloor \mu/2 \rfloor - \mu)}.$$

Now that we have calculated the coefficients $C^{(2n+1)}_{-\ell'+\mu}(Y)$ of $F^{(2n+1)}_{(\mathbf{1^{n+1}},\mathbf{1^{n+1}})}(H;Y,X)$, we will use Definition 4.21 to calculate $F^{(2n+2)}_{(\mathbf{1^{n+1}},\mathbf{1^{n+1}})}(H;Y,X)$. The polynomial $G_+(Y,X)$ defining $F^{(2n+2)}_{(\mathbf{1^{n+1}},\mathbf{1^{n+1}})}(H;Y,X)$ is designed to forget all the coefficients $C^{(2n+1)}_{-\ell'+\mu}(Y)$ for $\mu > \mathfrak{e}_{2n+2}/2$. This is where we pick up the second new restriction:

$$\xi_1' = \mu \leq \frac{\mathfrak{e}_{2n+2}}{2}.$$

Now, from Remark 4.9, we obtain:

$$\phi_{(\mathbf{1^{n+1}},\mathbf{1^{n+1}})}(\mu, H; Y) = Y^{\mu+\delta_n(\mu)} \sum_{\nu \geq 2\mu} \#\Pi^{H,\,\mathrm{res}}_{([\lfloor \mu/2 \rfloor, \delta_1(\mu)], \nu/2+(2\lfloor \mu/2 \rfloor - \mu))} Y^{\nu+2(2\lfloor \mu/2 \rfloor - \mu)}.$$

The remainder of this proof is, mutatis mutandis, identical to the proof of Theorem 6.1. ■

Remark 6.10. Figure 6.2 aids in understanding the two types of restrictions we obtain at each step of the proof of Theorem 6.8. The first region, labeled "Restricted Region" are restrictions obtained by due to the limits of the sum (6.15) defining $F^{(2n+1)}_{(\mathbf{1^{n+1}},\mathbf{1^{n+1}})}(H;Y,X)$. The second region, labeled "G_-" comes from the way we have constructed the polynomial $G_+(Y,X)$ in Definition 4.21 to forget all the higher order coefficients of $F^{(2n+1)}_{(\mathbf{1^{n+1}},\mathbf{1^{n+1}})}(H;Y,X)$. Recall, in this regime we no longer have an obvious ε_{2n+2}-Kohnen-Choie basis expression.

Appendix A

Maass's α_i Parameters

In this appendix, we follow the exposition of [Maa71].

We first describe the parameters $\alpha_1, \ldots, \alpha_n$ in the Γ-factors of the functional equations appearing in Weissauer's Converse Theorem. Let u be a Großencharacter on $\mathcal{P}_n$ with:

$$\delta_i u = \lambda_i u, \qquad i = 1, \ldots, n; \;\; \lambda_i \in \mathbb{C}.$$

By condition (4) of Definition 1.26, $\lambda_1 = 0$. Maass introduces a differential operator:

$$\mathrm{M}_n(Y) := |Y| \left| \frac{\partial}{\partial Y} \right|.$$

On p. 88, Maass proves the operator identity:

$$|Y|^{-s} \mathrm{M}_n(\hat{Y}) |Y|^s = (-1)^n f(s, \delta_1, \delta_2, \ldots, \delta_n), \qquad \hat{Y} := Y^{-1}.$$

Here, $f(s, x_1, \ldots, x_n)$ is a multivariate polynomial in the $n+1$ variables $s, x_1, x_2, \ldots, x_n$.

In the variable s, f is monic of degree n. We define the parameters $\alpha_1, \dots, \alpha_n$ via:

$$f(s, 0, \lambda_2, \dots, \lambda_n) = \prod_{i=1}^{n} (s - \alpha_i).$$

We now calculate α_1, α_2 for the case $n = 2$. We first remark that the hyperbolic Laplacian Δ on $\mathcal{H}$ and the differential operator $\delta_2 := \operatorname{Tr}\left(\left(Y\frac{\partial}{\partial Y}\right)^2\right)$ on $\mathcal{P}_2$ are related via $\delta_2 = -\frac{1}{2}\Delta$. For more details see [Ter16, pp. 34-37]. Let φ be a Maass form on $\mathcal{H}$ with $\Delta\varphi = (1/4+r^2)\varphi$. Let u be the corresponding Maass Großencharacter on $\mathcal{P}_2$ with $\delta_2 u = -\frac{1}{2}\Delta u = -(1/8 + r^2/2)u$. We fix two coordinate systems on $\mathcal{P}_2$:

$$Y = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad \hat{Y} = \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix}.$$

These are related via:

$$a' = \frac{c}{|Y|}, \qquad b' = \frac{-b}{|Y|}, \qquad c' = \frac{a}{|Y|}, \qquad |Y| := ac - b^2 = \frac{1}{a'c' - (b')^2}.$$

We calculate:

$$\begin{pmatrix} \frac{\partial a}{\partial a'} & \frac{\partial a}{\partial b'} & \frac{\partial a}{\partial c'} \\ \frac{\partial b}{\partial a'} & \frac{\partial b}{\partial b'} & \frac{\partial b}{\partial c'} \\ \frac{\partial c}{\partial a'} & \frac{\partial c}{\partial b'} & \frac{\partial c}{\partial c'} \end{pmatrix} = -\begin{pmatrix} a^2 & 2ab & b^2 \\ ab & ac + b^2 & bc \\ b^2 & 2bc & c^2 \end{pmatrix}.$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial a'} &= -a^2\frac{\partial}{\partial a} - ab\frac{\partial}{\partial b} - b^2\frac{\partial}{\partial c}\\ \frac{\partial}{\partial b'}, &= -2ab\frac{\partial}{\partial a} - (ac+b^2)\frac{\partial}{\partial b} - 2bc\frac{\partial}{\partial c},\\ \frac{\partial}{\partial c'} &= -b^2\frac{\partial}{\partial a} - bc\frac{\partial}{\partial b} - c^2\frac{\partial}{\partial c}.\end{aligned}$$

Remark A.1. The identities that follow are at the level of differential operators.

We calculate

$$\left|\frac{\partial}{\partial \hat{Y}}\right| := \frac{\partial}{\partial a'}\frac{\partial}{\partial c'} - \frac{1}{4}\frac{\partial}{\partial b'}\frac{\partial}{\partial b'}.$$

Via the product rule:

$$\begin{aligned}\frac{\partial}{\partial a'}\frac{\partial}{\partial c'} = a^2b^2\frac{\partial}{\partial a}\frac{\partial}{\partial a} &+ (a^2bc+ab^3)\frac{\partial}{\partial a}\frac{\partial}{\partial b} + (a^2c^2+b^4)\frac{\partial}{\partial a}\frac{\partial}{\partial c}\\ &+ ab^2c\frac{\partial}{\partial b}\frac{\partial}{\partial b} + (abc^2+b^3c)\frac{\partial}{\partial b}\frac{\partial}{\partial c} + b^2c^2\frac{\partial}{\partial c}\frac{\partial}{\partial c}\\ &+ 2ab^2\frac{\partial}{\partial a} + (abc+b^3)\frac{\partial}{\partial b} + 2b^2c\frac{\partial}{\partial c},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial b'}\frac{\partial}{\partial b'} = (2ab)^2\frac{\partial}{\partial a}\frac{\partial}{\partial a} &+ 2(2ab)(ac+b^2)\frac{\partial}{\partial a}\frac{\partial}{\partial b} + 2(2ab)(2bc)\frac{\partial}{\partial a}\frac{\partial}{\partial c}\\ &+ (ac+b^2)^2\frac{\partial}{\partial b}\frac{\partial}{\partial b} + 2(ac+b^2)(2bc)\frac{\partial}{\partial b}\frac{\partial}{\partial c} + (2bc)^2\frac{\partial}{\partial c}\frac{\partial}{\partial c}\\ &+ (6ab^2+2a^2c)\frac{\partial}{\partial a} + (6abc+2b^3)\frac{\partial}{\partial b} + (6b^2c+2ac^2)\frac{\partial}{\partial c}.\end{aligned}$$

After some routine calculation, we obtain:

$$\left|\frac{\partial}{\partial\hat{Y}}\right| = |Y|^2\left(\frac{\partial}{\partial a}\frac{\partial}{\partial c} - \frac{1}{4}\frac{\partial}{\partial b}\frac{\partial}{\partial b}\right) - \frac{1}{2}|Y|\left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}\right).$$

Via the product rule:

$$\frac{\partial}{\partial a}|Y|^s = cs|Y|^{s-1}+|Y|^s\frac{\partial}{\partial a}, \qquad \frac{\partial}{\partial c}|Y|^s = as|Y|^{s-1} + |Y|^s\frac{\partial}{\partial c},$$
$$\frac{\partial}{\partial b}|Y|^s = -2bs|Y|^{s-1} + |Y|^s\frac{\partial}{\partial b}.$$

and

$$\begin{aligned}
\frac{\partial}{\partial a}\frac{\partial}{\partial c}|Y|^s &= \frac{\partial}{\partial a}\left(as|Y|^{s-1} + |Y|^s\frac{\partial}{\partial c}\right)\\
&= \left(s|Y|^{s-1} + acs(s-1)|Y|^{s-2}\right) + as|Y|^{s-1}\frac{\partial}{\partial a} + cs|Y|^{s-1}\frac{\partial}{\partial c} + |Y|^s\frac{\partial}{\partial a}\frac{\partial}{\partial c},\\
\frac{\partial}{\partial b}\frac{\partial}{\partial b}|Y|^s &= \frac{\partial}{\partial b}\left(-2bs|Y|^{s-1} + |Y|^s\frac{\partial}{\partial b}\right)\\
&= \left(-2s|Y|^{s-1} + 4b^2s(s-1)|Y|^{s-2}\right) - 4bs|Y|^{s-1}\frac{\partial}{\partial b} + |Y|^s\frac{\partial}{\partial b}\frac{\partial}{\partial b}.
\end{aligned}$$

After some routine calculation, we obtain:

$$\mathrm{M}_n(\hat{Y})|Y|^s = |Y|^s(s^2 + As + B),$$

where

$$A = \left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}\right) - \frac{1}{2},$$
$$B = |Y|\left(\frac{\partial}{\partial a}\frac{\partial}{\partial c} - \frac{1}{4}\frac{\partial}{\partial b}\frac{\partial}{\partial b}\right) - \frac{1}{2}\left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}\right).$$

Now we calculate the G-invariant differential operators:

$$\delta_1 = \mathrm{Tr}\left(Y\frac{\partial}{\partial Y}\right), \qquad \delta_2 = \mathrm{Tr}\left(\left(Y\frac{\partial}{\partial Y}\right)^2\right).$$

Observe:

$$Y\frac{\partial}{\partial Y} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial a} & \frac{1}{2}\frac{\partial}{\partial b} \\ \frac{1}{2}\frac{\partial}{\partial b} & \frac{\partial}{\partial c} \end{pmatrix} = \begin{pmatrix} a\frac{\partial}{\partial a} + \frac{1}{2}b\frac{\partial}{\partial b} & \frac{1}{2}a\frac{\partial}{\partial b} + b\frac{\partial}{\partial c} \\ b\frac{\partial}{\partial a} + \frac{1}{2}c\frac{\partial}{\partial b} & \frac{1}{2}b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c} \end{pmatrix}.$$

We calculate, using the product rule:

$$\delta_1 = a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c},$$

$$\begin{aligned}\delta_1^2 = a^2\frac{\partial}{\partial a}\frac{\partial}{\partial a} + b^2\frac{\partial}{\partial b}\frac{\partial}{\partial b} + c^2\frac{\partial}{\partial c}\frac{\partial}{\partial c} \\ 2ab\frac{\partial}{\partial a}\frac{\partial}{\partial b} + 2ac\frac{\partial}{\partial a}\frac{\partial}{\partial c} + 2bc\frac{\partial}{\partial b}\frac{\partial}{\partial c} \\ + a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c},\end{aligned}$$

and

$$\begin{aligned}\delta_2 = a^2\frac{\partial}{\partial a}\frac{\partial}{\partial a} + \frac{1}{2}(ac + b^2)\frac{\partial}{\partial b}\frac{\partial}{\partial b} + c^2\frac{\partial}{\partial c}\frac{\partial}{\partial c} \\ 2ab\frac{\partial}{\partial a}\frac{\partial}{\partial b} + 2b^2\frac{\partial}{\partial a}\frac{\partial}{\partial c} + 2bc\frac{\partial}{\partial b}\frac{\partial}{\partial c} \\ + \frac{3}{2}\left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}\right).\end{aligned}$$

By comparing the coefficients A and B with the operators δ_1, δ_1^2, and δ_2, we see:

$$A = \delta_1 - \frac{1}{2},$$
$$B = \frac{1}{2}(\delta_1^2 - \delta_2) - \frac{1}{4}\delta_1.$$

Thus,

$$\mathrm{M}_n(\hat{Y})|Y|^s u(Y) = \left(s^2 - (1/2)s + (1/16 + r^2/4)\right)|Y|^s u(Y).$$

Now,

$$s^2 - (1/2)s + (1/16 + r^2/4) = (s - \alpha_1)(s - \alpha_2),$$

where

$$\alpha_1 = \overline{\alpha_2} = 1/4 + ir/2.$$

Appendix B

Proof of Lemma 5.4

Here, we provide the Python code we used to verify Lemma 5.4 for $1 \leq n \leq 4$.

```
# Initialize symbolic matrices D and G
var('d1,d2,d3,d4')
D = matrix(SR, 4, 4, [d1,0,0,0,\
                      0,d2,0,0,\
                      0,0,d3,0,\
                      0,0,0,d4])

var('a11,a12,a13,a14,a22,a23,a24,a33,a34,a44')
G = matrix(SR, 4, 4, [a11,a12,a13,a14,\
                      0,a22,a23,a24,\
                      0,0,a33,a34,\
                      0,0,0,a44])

# Calculate matrix product S := D[G^{-1}]
Ginv = G.inverse()
S = Ginv.transpose()*D*Ginv


```

```
    # Calculate C_{i,j} Parameters
    C = [[0,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0]]
    for i in range(0,4):
        C[i][i] = G[i][i]
        for j in range (i+1,4):
            C[i][j] = -G[i][j]
            for k in range(i+1,j):
                C[i][j] -= C[i][k]*G[k][j]/G[k][k]

    # Calculate D_{i,j}^{\prime} Coefficients
    Dp = [[0,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0]]
    for i in range(0,4):
        for j in range (i,4):
            for k in range(0,i+1):
                Dp[i][j] += C[k][i]*C[k][j]*D[k][k]/G[k][k]^2
                Dp[i][j] /= (G[i][i]*G[j][j])

    # Verify correctness of formulas
    for i in range(0,4):
        for j in range (i,4):
            assert(S[i][j] == C[i][j])
```

Bibliography

[Bou07] Irene I. Bouw. Invariants of ternary quadratic forms. *Astérisque*, (312):113–137, 2007.

[Cas78] J. W. S. Cassels. *Rational quadratic forms*, volume 13 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1978.

[Cho20] Sungmun Cho. On the local density formula and the Gross-Keating invariant with an appendix 'The local density of a binary quadratic form' by T. Ikeda and H. Katsurada. *Math. Z.*, 296(3-4):1235–1269, 2020.

[CIK+17] Sungmun Cho, Tamotsu Ikeda, Hidenori Katsurada, Chul-Hee Lee, and Takuya Yamauchi. An explicit formula for the extended gross-keating datum of a quadratic form. `https://arxiv.org/abs/1709.02772`, 2017.

[CK08] Youngju Choie and Winfried Kohnen. A remark on the Ikeda lift and local singular series polynomials. *Proc. Amer. Math. Soc.*, 136(5):1559–1563, 2008.

[Cog07] James W. Cogdell. L-functions and converse theorems for GL_n. In *Automorphic forms and applications*, volume 12 of *IAS/Park City Math. Ser.*, pages 97–177. Amer. Math. Soc., Providence, RI, 2007.

[CY20] Sungmun Cho and Takuya Yamauchi. A reformulation of the Siegel series and intersection numbers. *Math. Ann.*, 377(3-4):1757–1826, 2020.

[DI96] W. Duke and Ö. Imamoḡlu. A converse theorem and the Saito-Kurokawa lift. *Internat. Math. Res. Notices*, (7):347–355, 1996.

[DS05] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[EKM08] Richard Elman, Nikita Karpenko, and Alexander Merkurjev. *The algebraic and geometric theory of quadratic forms*, volume 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[EZ85] Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985.

[GK93] Benedict H. Gross and Kevin Keating. On the intersection of modular correspondences. *Invent. Math.*, 112(2):225–245, 1993.

[Gol06] Dorian Goldfeld. *Automorphic forms and L-functions for the group* $\mathrm{GL}(n, \mathbf{R})$, volume 99 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006. With an appendix by Kevin A. Broughan.

[HC51] Harish-Chandra. On some applications of the universal enveloping algebra of a semisimple Lie algebra. *Trans. Amer. Math. Soc.*, 70:28–96, 1951.

[IK18] Tamotsu Ikeda and Hidenori Katsurada. On the Gross-Keating invariant of a quadratic form over a non-archimedean local field. *Amer. J. Math.*, 140(6):1521–1565, 2018.

[IK22a] Tamotsu Ikeda and Hidenori Katsurada. Estimates for the Fourier coefficients of the Duke-Imamoglu-Ikeda lift. `https://arxiv.org/abs/2206.07337`, 2022.

[IK22b] Tamotsu Ikeda and Hidenori Katsurada. An explicit formula for the Siegel series of a quadratic form over a non-archimedean local field. *J. Reine Angew. Math.*, 783:1–47, 2022.

[Ike01] Tamotsu Ikeda. On the lifting of elliptic cusp forms to Siegel cusp forms of degree $2n$. *Ann. of Math. (2)*, 154(3):641–681, 2001.

[Ima80] Kaori Imai. Generalization of Hecke's correspondence to Siegel modular forms. *Amer. J. Math.*, 102(5):903–936, 1980.

[Iwa02] Henryk Iwaniec. *Spectral methods of automorphic forms*, volume 53 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, second edition, 2002.

[JL05] Jay Jorgenson and Serge Lang. *$Pos_n(R)$ and Eisenstein series*, volume 1868 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005.

[Kat99] Hidenori Katsurada. An explicit formula for Siegel series. *Amer. J. Math.*, 121(2):415–452, 1999.

[Koh80] Winfried Kohnen. Modular forms of half-integral weight on $\Gamma_0(4)$. *Math. Ann.*, 248(3):249–266, 1980.

[Koh02] Winfried Kohnen. Lifting modular forms of half-integral weight to Siegel modular forms of even genus. *Math. Ann.*, 322(4):787–809, 2002.

[Kur78] Nobushige Kurokawa. Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two. *Invent. Math.*, 49(2):149–165, 1978.

[Lam05] T. Y. Lam. *Introduction to quadratic forms over fields*, volume 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[Maa71] Hans Maass. *Siegel's modular forms and Dirichlet series*. Lecture Notes in Mathematics, Vol. 216. Springer-Verlag, Berlin-New York, 1971. Dedicated to the last great representative of a passing epoch. Carl Ludwig Siegel on the occasion of his seventy-fifth birthday.

[O'M00] O. Timothy O'Meara. *Introduction to quadratic forms*. Classics in Mathematics. Springer-Verlag, Berlin, 2000. Reprint of the 1973 edition.

[Pit19] Ameya Pitale. *Siegel modular forms*, volume 2240 of *Lecture Notes in Mathematics*. Springer, Cham, 2019. A classical and representation-theoretic approach.

[Sch85] Winfried Scharlau. *Quadratic and Hermitian forms*, volume 270 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[Shi73] Goro Shimura. On modular forms of half integral weight. *Ann. of Math. (2)*, 97:440–481, 1973.

[Ter16] Audrey Terras. *Harmonic analysis on symmetric spaces—higher rank spaces, positive definite matrix space and generalizations*. Springer, New York, second edition, 2016.

[Wei84] R. Weissauer. Siegel modular forms and Dirichlet series. *Unpublished manuscript*, 1984.