UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

(MULTI-)PERSISTENT HOMOLOGY AND TOPOLOGICAL ROBOTICS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

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Norman, Oklahoma
2023

(MULTI-)PERSISTENT HOMOLOGY AND TOPOLOGICAL ROBOTICS

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

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Dedicated to the loving memory of my grandparents,
and
my cousin Kun Cao, who became an angel on February 3rd, 2023.

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Abstract

After developing the relevant background and proving some general results in the early chapters, the main novel content of this thesis is the computation of the i-th homology groups of the second configuration spaces of metric graphs $\mathsf{Star_k}$ and $\hat{\mathcal{H}}_{m,n}$, with two restraint parameters. These configuration spaces are filtered by the poset $(\mathbb{R}, \leq)^{\mathsf{op}} \times (\mathbb{R}, \leq)$. We study the persistence modules $PH_i((\mathsf{Star_k})^2_{-,-}; \mathbb{F})$ and $PH_i((\hat{\mathcal{H}}_{m,n})^2_{-,-}; \mathbb{F})$ where $i = 0, 1$, since higher homology vanishes for these spaces. Next, we construct a new representation over the poset given by the hyperplane arrangement of the configuration spaces of the finite graph. There is no loss of information when we restrict to the poset of chambers because the functor $PH_i(-)$ factors through the poset of chambers. Using this machinery and the homology groups we calculated, we find the direct sum decomposition of the 2-parameter persistence modules $PH_i((\mathsf{Star_k})^2_{-,-}; \mathbb{F})$ and $PH_i((\hat{\mathcal{H}}_{m,n})^2_{-,-}; \mathbb{F})$, where each summand is indecomposable. In particular, we show that $PH_0((\hat{\mathcal{H}}_{m,n})^2_{-,-}; \mathbb{F})$ and $PH_1((\hat{\mathcal{H}}_{m,n})^2_{-,-}; \mathbb{F})$ can be written as a direct sum of polytope modules.

Acknowledgements

I want to thank my advisor, Dr. Murad Özaydın, for his guidance and mentorship. Discussing math with him is always joyful, and he is always supportive and patient during my journey. In addition, he introduced me to the Category Theory and Emily Riehl's masterpiece "Category Theory in Context" in his Algebraic Topology class, which has changed my way to think about math.

I want to thank Dr. Peter Patzt. I enjoyed his lectures and learned one of the primary tools I used in this thesis from him. In addition, during the job application season, he was the first reader of my research statement and provided me many invaluable suggestions. I want to thank Dr. Michael Lesnick for letting me participate in his class virtually. He reshaped my understanding of the multi-persistence theory. I also appreciate his advice, feedback, and tremendous support during the job application season.

I want to thank my graduate committee members: Dr. Roi Docampo, Dr. Joe Havlicek, Dr. Michael Jablonski, Dr. Miroslav Kramar, and Dr. Christian Remling for their valuable time, advices and feedback on this thesis.

I want to thank my friends, Rob Merrell, Elizabeth Pacheco, Kai Sun, and Jie Zeng. At the end, I want to thank my parents for their support.

Chapter 1

Introduction

Topological data analysis (TDA) applies Topology to study data clouds' geometry and topological features. It was first inspired by the work of Marston Morse, followed by a sequence of pioneered papers, see for instance [41][27][21]. The pipeline of TDA can be (roughly) described by three main steps:

- Obtaining the point cloud with a metric;
- Constructing the geometric object (specifically, a filtered complex) from the point cloud with discrete or continuous parameters;
- Calculating the homology groups of the filtered complex and the persistence diagram /barcode. Interpreting the topological and geometric features of the data via persistence diagram /barcode.

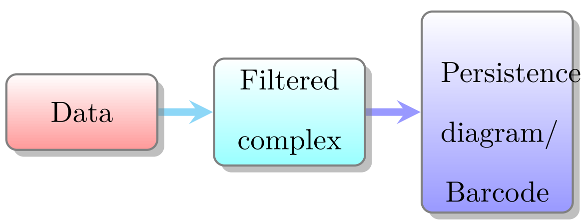


Figure 1.1: The pipeline of TDA

Let $\mathbb{F}$ be a field and $(P, \leq)$ be the set P with a partial order. A persistence module over $(P, \leq)$ is a family of $\mathbb{F}$-vector spaces $\{M_t \mid t \in P\}$ and a doubly-indexed family of linear maps $\{\rho_{st} : M_s \to M_t \mid s \leq t\}$ where $\rho_{tu}\rho_{st} = \rho_{su}$ for any $s \leq t \leq u$ in P and $\rho_{ss} = \mathrm{id}_{M_s}$ for all $s \in P$. The poset $(P, \leq)$ can be regarded as a category, and persistence modules are functors from the category $(P, \leq)$ to the category of vector spaces $\mathbf{Vect}_{\mathbb{F}}$. We use $\mathbf{Vect}_{\mathbb{F}}^{(P,\leq)}$ to denote the category of persistence modules over the poset $(P, \leq)$. For instance, when $(P, \leq) = (\mathbb{R}^n, \leq)$ with the product order where $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \in (\mathbb{R}^n, \leq)$ if and only if $a_i \leq b_i$ for all $i = 1, \ldots, n$, the objects of $\mathbf{Vect}_{\mathbb{F}}^{(\mathbb{R}^n,\leq)}$ is called n-parameter persistence modules.

One of the primary tasks in TDA is finding the proper representations (called ”barcodes”) of data clouds. The notion of barcodes is introduced in [27], and algorithms for computing barcodes of persistence modules are discovered by many researchers. Carlsson-Zomorodian (2015) studied the 1-parameter persistence modules of finite type [47], that is, $(P, \leq) = (\mathbb{N}, \leq)$ and each $M : (\mathbb{N}, \leq) \to \mathbf{Vect}_{\mathbb{F}}$ has the property that M_i is a finitely dimensional vector space for every $i \in \mathbb{N}$ and there exists $N \in \mathbb{N}$ such that morphism $M(i \leq i+1) : M_i \to M_{i+1}$ is an isomorphism for all $i \geq N$. They proved that the category of persistence modules of finite type is equivalent to the category of finitely generated non-negatively graded $\mathbb{F}[x]$-modules. Using the equivalence of the categories, one can obtain an algebraic description of the barcode. Since $\mathbb{F}[x]$ is a principal ideal domain (PID), for each persistence module $M : (\mathbb{Z}, \leq) \to \mathbf{vect}_{\mathbb{F}}$, there exists $m, n \in \mathbb{N}$ such that

$$M \cong \left(\bigoplus_{i=1}^{n} x^{a_i} \mathbb{F}[x] \right) \oplus \left(\bigoplus_{j=1}^{m} x^{b_j} \mathbb{F}[x]/(x^{d_j}) \right)$$

Therefore, the barcode of M consists of the intervals $[a_i, \infty)$ for $i = 1, \ldots, n$ and $[b_j, d_j)$ for $j = 1, \ldots, m$. The ungraded version is the structure theorem for finitely generated modules over a PID, which is a standard result in commutative algebra.

Although the theory of 1-parameter TDA provides a powerful tool for helping people understand data, the data cloud itself may naturally come with more than one parameter. Analyzing one parameter at a time is time-consuming and can be information-losing: it only contains a slice of the feature by fixing all but one parameter. For example, every RGB image has three parameters representing the three color channels. When we apply TDA theory to analyze this image, we first need to convert the RGB image into a grayscale image. In this process, we may lose information because two distinct RGB points may have the same value in grayscale. Multi-parameter topological data analysis has promising application potential in dealing with higher-dimensional data clouds. Applying a multi-parameter TDA theory, we will be able to train computers to learn RGB images without converting the images into grayscale images.

On the other hand, the data cloud may contain noise. Multi-parameter topological data analysis also has promising application potential handling data clouds with noise. By considering the density of the data as a parameter along with the radius parameter, one can remove noise from the data and analyze the shape of the data at various density levels.

One difficulty in the multi-parameter persistence theory is that there is no canonical way to define the higher-dimensional barcodes (which represent indecomposable representations) analogous to the 1-dimensional barcodes because there is no structure theorem available in $\mathbf{vect}_{\mathbb{F}}^{(P,\leq)}$ when the poset P is not a totally ordered set. For example, it is impossible to classify the indecomposable representations of 2-parameter persistence modules. Every pointwise finite-dimensional 2-parameter persistence module corresponds to a finitely generated $\mathbb{Z}^2$-graded module over $\mathbb{F}[x, y]$, up to isomorphism, but there is no classification theorem of finitely generated $\mathbb{Z}^2$-graded modules over $\mathbb{F}[x, y]$. Consequently, the indecomposable submodules of a multiparameter persistence module can be very complicated. Buchet-Escolar shows that any n-parameter persistence module can be embedded as a slice of an indecomposable $(n+1)$-parameter persistence module[13][12], that is, for every n-parameter persistence module M, there exists an indecomposable $(n+1)$-parameter persistence module N such that $M = N \circ \iota$, where $\iota : (\mathbb{Z}^n, \leq) \to (\mathbb{Z}^{n+1}, \leq)$ is a functor that is injective on objects.

In this thesis, we want to apply the multi-parameter TDA theory to analyze complicated geometric objects. In particular, we want to understand how the configuration spaces change along with the changes in the proximity conditions. In 2013, Dover-Özaydın studied the restricted configuration space of metric graphs and gave an upper bound for the number of homeomorphism types of its n-th configuration space over proximity conditions.[26] As a continuation of their pioneering work, we study the multi-parameter persistence modules given by the filtrations of the n-th configuration spaces (with two restraint parameters) of finite metric graphs. In addition to the parameters representing the minimum distance allowed between each pair of robots, we also consider the parameters representing the length of edges (denoted by $\mathbf{L} = (L_e)_{e \in E}$, where E is the set of edges of the underlying graph).

Let $\Gamma = (V, E)$ be a graph and X be a geometric realization of Γ. Given an edge length vector $\mathbf{L} = (L_e)_{e \in E} \in \mathbb{R}_{>0}^{|E|}$, the metric graph $X_{\mathbf{L}}$ is the space X endowed with the metric satisfying

1. each edge e is isometric to the interval $[0, L_{e_1}]$;
2. $X_{\mathbf{L}}$ has the path metric (denoted by δ): for any $x, y \in X_{\mathbf{L}}$, $\delta(x, y)$ is the length of a shortest path from x to y.

The n-th configuration space of X with restraint parameter $\mathbf{r} \in \mathbb{R}_{>0}^{\binom{n}{2}}$ and edge length vector $\mathbf{L}$ is

$$X^n_{\mathbf{r},\mathbf{L}} = \{(x_1, \dots, x_n) \in (X_{\mathbf{L}})^n \mid \delta(x_i, x_j) \geq r_{ij}, i, j = 1, \dots, n\}$$

The object that we are investigating is a filtration of the spaces $\{X^n_{\mathbf{r},\mathbf{L}}\}$. This filtration naturally comes with $\binom{n}{2} + |E|$ parameters [26]. In particular, when all robots have the same size and all edges have fixed lengths, the problem reduces to a single-parameter problem, and one can apply the (single-parameter) TDA theory to analyze the filtration of configuration spaces.

In Section 1.1, we discuss an example which motivates this thesis.

1.1 First Example

Let $Y_{L_{e_1}}$ be a metric graph with the shape of Y with the following metric: two edges of $Y_{L_{e_1}}$ have length 1, and the length of the other edge (labeled by e_1) varies. Let e_2 and e_3 denote the other two edges of $Y_{L_{e_1}}$. In this example, we interpret the metric graph $Y_{L_{e_1}}$ as a Y-shaped rail and consider two distinct thick robots (denoted by red robot and green robot) moving on $Y_{L_{e_1}}$, see figure 1.2. Let r denote the minimum distance required between the two robots and let L_{e_1} denote the length of the edge e_1. We denote its configuration space by $Y^2_{r,L_{e_1}}$.

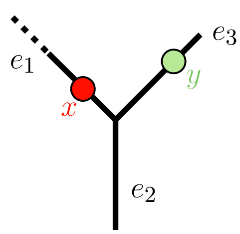


Figure 1.2

One cell of $Y^2_{r,L_{e_1}}$ is given by (1.1).

$$\begin{aligned} &0 \le x \le 1 = L_{e_2} \\ &0 \le y \le 1 = L_{e_3} \\ &x + y \ge r \end{aligned} \tag{1.1}$$

$$\begin{aligned} &0 \le x \le 1 = L_{e_3} \\ &0 \le y \le 1 = L_{e_2} \\ &x + y \ge r \end{aligned} \tag{1.2}$$

The system of inequalities (1.1) corresponds to the case when the red robot is on edge e_2 and the green robot is on edge e_3, while the system of inequalities (1.2) corresponds to the case when the red robot is on edge e_3 and the green robot is on edge e_2.

Similarly, the system of inequalities (1.3) (1.4) (1.5) gives 2-dimensional cells of $Y^2_{r,L_{e_1}}$: for $i = 2, 3$,

$$\begin{aligned} &0 \le x_1 \le L_{e_1} \\ &0 \le x_2 \le 1 = L_{e_i} \\ &x_1 + x_2 \ge r \end{aligned} \tag{1.3}$$

$$\begin{aligned} &0 \le x_1 \le 1 = L_{e_i} \\ &0 \le x_2 \le L_{e_1} \\ &x_1 + x_2 \ge r \end{aligned} \tag{1.4}$$

$$\begin{aligned} &0 \le x_1 \le L_{e_1} \\ &0 \le x_2 \le L_{e_1} \\ &x_1 + x_2 \ge r \end{aligned} \tag{1.5}$$

There is another type of 2-dimensional cells in $Y^2_{r,L_{e_1}}$ and those cells correspond to the case when the red robot and green robot are on the same edge of Y: for $i = 1, 2, 3$,

$$\begin{aligned} &0 \le x \le 1 = L_{e_i} \\ &0 \le y \le 1 = L_{e_i} \\ &x - y \ge r \end{aligned} \tag{1.6}$$

$$\begin{aligned} &0 \le x \le 1 = L_{e_i} \\ &0 \le y \le 1 = L_{e_i} \\ &y - x \ge r \end{aligned} \tag{1.7}$$

The system of inequalities (1.6) corresponds to the case when the distance between the red robot and the center of $Y_{L_{e_1}}$ is greater than the distance between the green robot and the center of $Y_{L_{e_1}}$, while the system of inequalities (1.7) corresponds to the opposite scenario.

Note that the critical hyperplanes are

$$\begin{aligned} r &= 1 \\ r &= 2 \\ r &= L_{e_1} \\ r &= L_{e_1} + 1 \end{aligned} \tag{1.8}$$

We then obtain the hyperplane arrangement in the parameter space where each point (r, L_{e_1}) in the hyperplane corresponds to a restricted configuration space $Y^2_{r,L_{e_1}}$, as shown in figure 1.3. Note that $Y^2_{r_1,l_1}$ and $Y^2_{r_2,l_2}$ are homotopy equivalent (or even stronger, isotopy equivalent) when the points (r_1, l_1) and (r_2, l_2) are contained in the same chamber.

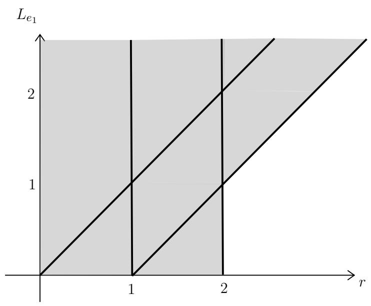


Figure 1.3: The hyperplane arrangement in $\mathbb{R}^2$ associated to $Y^2_{r,L_{e_1}}$

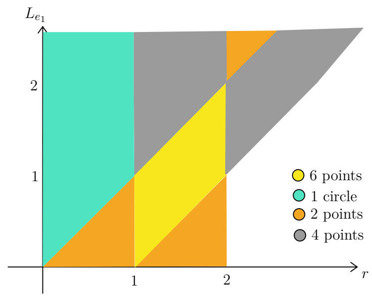


Figure 1.4: Homotopy type of $Y^2_{r,L_{e_1}}$

We want to understand how much the space will change if we perturb the parameters. For example, when $L_{e_1} = 1$, the space $Y^2_{r,1}$ changes as r goes from 0.2 to 2:

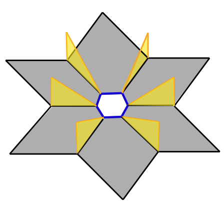


Figure 1.5: $Y^2_{0.2,1}$

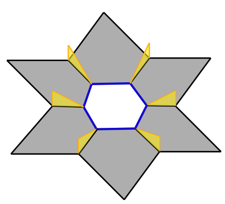


Figure 1.6: $Y^2_{0.4,1}$

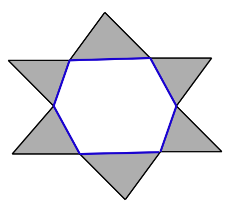


Figure 1.7: $Y^2_{1,1}$

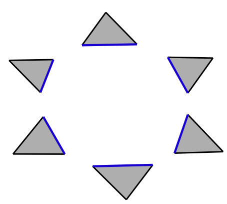


Figure 1.8: $Y^2_{1.3,1}$

Figure 1.9: $Y^2_{2,1}$

Note that the space $Y^2_{0.4,1}$ is a subspace of $Y^2_{0.2,1}$, and $Y^2_{1,1}$ is a subspace of $Y^2_{0.4,1}$. In general, for $a \leq b \in \mathbb{R}_{>0}$,

$$Y^2_{b,1} \subseteq Y^2_{a,1}$$

We hence obtain a filtration with one parameter r with a given edge length $L_{e_1} = 1$. Apply the 0-th homology functor $H_0(-;\mathbb{F})$ to each step of the filtration then we obtain a persistence module $PH_0(Y^2_{-,1};\mathbb{F})$ over the poset $(\mathbb{R}_{>0}, \leq)^{\mathsf{op}}$. The barcode of $H_0(Y^2_{-,1};\mathbb{F})$ is shown in figure 1.10.

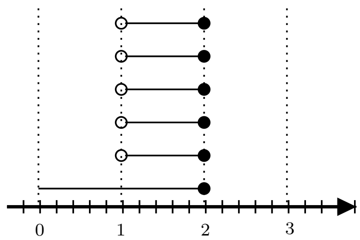


Figure 1.10: The barcode of $PH_0(Y^2_{-,1};\mathbb{F})$

Now we consider $Y^2_{r,2}$ with r varies. In other words, we consider the restricted configuration spaces of Y when the length of edge e_1 is 2. Figure 1.11- 1.15 illustrate the changes of $Y^2_{r,1}$ as r goes from 0.2 to 2:

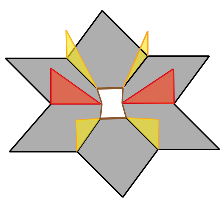


Figure 1.11: $Y^2_{0.2,2}$

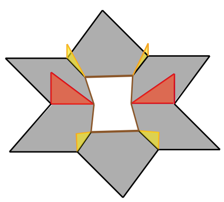


Figure 1.12: $Y^2_{0.4,2}$

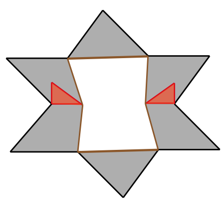


Figure 1.13: $Y^2_{1,2}$

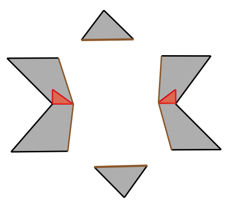


Figure 1.14: $Y^2_{1.3,2}$

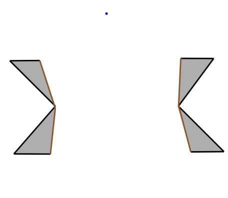


Figure 1.15: $Y^2_{2,2}$

Note that for $a \leq b \in \mathbb{R}_{>0}$,

$$Y^2_{b,2} \subseteq Y^2_{a,2}$$

We hence obtain a filtration with one parameter r with a given edge length $L_{e_1} = 2$. Note that $Y^2_{r,2}$ is homotopy equivalent to S^1 when $0 < r \leq 1$. When $1 < r \leq 3$, $Y^2_{r,2}$ is homotopy equivalent to the space of 4 points. Applying the 0-th homology functor $H_0(-;\mathbb{F})$ to each step of the filtration then we obtain a persistence module $PH_0(Y^2_{-,2};\mathbb{F})$ over the poset $(\mathbb{R}_{>0}, \leq)^{\mathsf{op}}$. The barcode of $H_0(Y^2_{-,2}; \mathbb{F})$ is shown in figure 1.16.

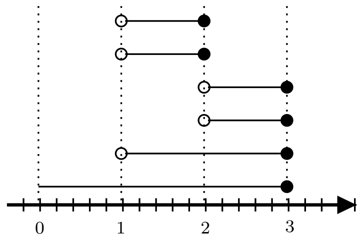


Figure 1.16: The barcode of $PH_0(Y^2_{-,2}; \mathbb{F})$

The homotopy type of $Y^2_{r,L_{e_1}}$ for each r and L_{e_1} is given in Figure 1.4. We want to understand the 2-parameter persistence module $PH_i(Y^2_{-,-}; \mathbb{F})$. In particular, we want to answer the following questions:

- Is $PH_i(Y^2_{-,-}; \mathbb{F})$ interval indecomposable?

- What are the indecomposable direct summands of $PH_i(Y^2_{-,-}; \mathbb{F})$?

Chapter 2

Preliminaries

In this chapter, we revisit some basic notions and results which serve as background. We will also introduce the notations that we are going to use in this thesis.

2.1 Some Category Theory

2.1.1 Basic Notions

Definition 2.1. *A* ***category*** C *consists of the following data:*

- *a collection of objects, denoted by* $\mathsf{ob}\,\mathsf{C}$*;*
- *a collection of morphisms between objects, denoted by* $\mathsf{Hom}_{\mathsf{C}}(*, *)$*, such that*
  - *for each object* c *of* C, $\mathsf{id}_c \in \mathsf{Hom}_{\mathsf{C}}(c, c)$*;*
  - $f \in \mathsf{Hom}_{\mathsf{C}}(c, c')$ *and* $g \in \mathsf{Hom}_{\mathsf{C}}(c', c'')$ *implies* $g \circ f \in \mathsf{Hom}_{\mathsf{C}}(c, c'')$.

  satisfying the following axioms:

  - *for any* $f \in \mathsf{Hom}_{\mathsf{C}}(c, c')$, $f \circ \mathsf{id}_c = f = \mathsf{id}_{c'} \circ f$*;*

– *for any* $f \in \mathsf{Hom}_{\mathbf{C}}(c, c')$*,* $g \in \mathsf{Hom}_{\mathbf{C}}(c', c'')$*, and* $h \in \mathsf{Hom}_{\mathbf{C}}(c'', c''')$*,*

$$h \circ (g \circ f) = (h \circ g) \circ f$$

When the collection of morphisms of $\mathbf{C}$ *is a set,* $\mathbf{C}$ *is called a* ***small category****.* $\mathbf{C}$ *is called a* ***locally small category*** *when* $\mathsf{Hom}_{\mathbf{C}}(c, c')$ *is a set for all* $c, c' \in \mathsf{ob}\,\mathbf{C}$*.*

Definition 2.2. *A category* $\mathbf{C}$ *is* ***thin*** *if* $\mathsf{Hom}_{\mathbf{C}}(c, c')$ *contains at most one element for all* $c, c' \in \mathsf{ob}\,\mathbf{C}$*.*

Definition 2.3. *A category* $\mathbf{C}$ *is* ***connected*** *if* $\mathbf{C}$ *is not empty and for all* $c, c' \in \mathsf{ob}\,\mathbf{C}$*,* c *and* c' *is connected by a finite zigzag of morphisms.*

Here are some categories that we are going to use in this thesis:

Example 1 (Poset as a category)**.** Let $(P, \leq)$ be a partially ordered set (poset, in short). We can view $(P, \leq)$ as a category where its objects are elements of P and in which $c \to c' \in \mathsf{mor}\,(P, \leq)$ if and only if $c \leq c'$. △

Example 2 (Path category of a graph)**.** Let $\Gamma = (V, E)$ be a graph. The path category of Γ, denoted by $\mathbf{Path}(\Gamma)$, is a category where its objects are vertices of Γ and morphisms are paths in Γ. △

Example 3 (Category of groups)**.** The category of groups, denoted by $\mathbf{Group}$, has groups as its objects and group homomorphisms as its morphisms. The category of abelian groups, denoted by $\mathbf{Ab}$, has abelian groups as its objects and group homomorphisms as its morphisms. $\mathbf{Ab}$ is a full subcategory of $\mathbf{Group}$. △

Example 4 (Category of R-modules)**.** Let R be a commutative ring with 1. The category of R-modules, denoted by $\mathbf{Mod}_R$, has R-modules as its objects and R-homomorphisms as its morphisms. A full subcategory of $\mathbf{Mod}_R$ consists of finitely generated R-modules is denoted by $\mathbf{Mod}_R^{\mathsf{f.g.}}$. △

Example 5 (Category of vector spaces)**.** The category of vector spaces over the field $\mathbb{F}$, denoted by $\mathsf{\mathbf{Vect}}_{\mathbb{F}}$, has vector spaces as its objects and linear transformations as its morphisms. A full subcategory of $\mathsf{\mathbf{Vect}}_{\mathbb{F}}$ consists of finite dimensional vector spaces is denoted by $\mathsf{\mathbf{vect}}_{\mathbb{F}}$.

△

Example 6 (Category of topological spaces)**.** The category of topological spaces, denoted by $\mathsf{\mathbf{Top}}$, has topological spaces as its objects and continuous maps as its morphisms. △

Definition 2.4. *A category $\mathsf{\mathbf{D}}$ is a* ***subcategory*** *of $\mathsf{\mathbf{C}}$ if $\mathsf{ob}\,\mathsf{\mathbf{D}}$ is a subcollection of $\mathsf{ob}\,\mathsf{\mathbf{C}}$ and $\mathsf{Hom}_{\mathsf{\mathbf{D}}}(d, d')$ is a subcollection of $\mathsf{Hom}_{\mathsf{\mathbf{C}}}(d, d')$ for all $d, d' \in \mathsf{ob}\,\mathsf{\mathbf{D}}$. Moreover, when $\mathsf{Hom}_{\mathsf{\mathbf{D}}}(d, d') = \mathsf{Hom}_{\mathsf{\mathbf{C}}}(d, d')$, $\mathsf{\mathbf{D}}$ is called a full subcategory of $\mathsf{\mathbf{C}}$.*

Definition 2.5. *Given categories $\mathsf{\mathbf{C}}$ and $\mathsf{\mathbf{D}}$, a* ***(covariant) functor*** *$\mathcal{F} : \mathsf{\mathbf{C}} \to \mathsf{\mathbf{D}}$ consists of the following data:*

- *for all $c \in \mathsf{ob}\,\mathsf{\mathbf{C}}$, $\mathcal{F}c \in \mathsf{ob}\,\mathsf{\mathbf{D}}$;*
- *for all $f \in \mathsf{Hom}_{\mathsf{\mathbf{C}}}(c, c')$, $\mathcal{F}c \in \mathsf{Hom}_{\mathsf{\mathbf{D}}}(\mathcal{F}c, \mathcal{F}c')$*

satisfying the following axioms:

- *$\mathcal{F}\,\mathsf{id}_c = \mathsf{id}_{\mathcal{F}c}$ for all $c \in \mathsf{ob}\,\mathsf{\mathbf{C}}$;*
- *for any $f \in \mathsf{Hom}_{\mathsf{\mathbf{C}}}(c, c')$ and $g \in \mathsf{Hom}_{\mathsf{\mathbf{C}}}(c', c'')$,*

$$\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$$

Here are some functors that we are going to use in this thesis:

Example 7 (Forgetful functors)**.** Forgetful functors (denoted by $\mathcal{U}$) are functors that forget some structures in the source category. Here are some forgetful functors:

1. $\mathcal{U} : \mathsf{\mathbf{Group}} \to \mathsf{\mathbf{Set}}$, forgetting the group structure;

2. $\mathcal{U}$: $\mathsf{Group} \to \mathsf{Set}_*$, forgetting the group structure but keeping the information of identity element for each group;

3. $\mathcal{U} : \mathsf{Top} \to \mathsf{Set}$, forgetting the topological structure;

4. $\mathcal{U} : \mathsf{Ring} \to \mathsf{Ab}$, forgetting the multiplicative operation;

5. $\mathcal{U} : \mathsf{Mod}_R \to \mathsf{Ab}$, forgetting the scalar multiplication.

$\triangle$

Example 8 (Homology)**.** Let i be a natural number. The i-th homology group defines a functor $H_i(-) : \mathsf{Top} \to \mathsf{Ab}$ where H_i sends each X to $H_i(X)$ and each continuous map $f : X \to Y$ to a group homomorphism $H_i(f) : H_i(X) \to H_i(Y)$. $\triangle$

Example 9 (Persistence module)**.** Consider the poset $(\mathbb{R}, \leq)$ as a category. A persistence module is a functor $M : (\mathbb{R}, \leq) \to \mathsf{Vect}_{\mathbb{F}}$ where M sends each $a \in \mathbb{R}$ to a vector space M_a and each arrow $a \leq b$ to a linear transformation $M(a \leq b) : M_a \to M_b$. Since $(\mathbb{R}, \leq)$ is thin, the functoriality of M implies $M(a \leq a) = \mathsf{id}_{M_a}$ and $M(a \leq c) = M(b \leq c) \circ M(a \leq b)$. $\triangle$

Example 10 (Quiver representation)**.** Let $\mathsf{Path}(\Gamma)$ be the path category of a direct graph Γ. A quiver representation over Γ is a functor $\mathcal{F} : \mathsf{Path}(\Gamma) \to \mathsf{Vect}_{\mathbb{F}}$ where $\mathcal{F}$ sends each vertex v to a vector space $\mathcal{F}v$ and each path $f : v \to w$ to a linear transformation $\mathcal{F}f : \mathcal{F}v \to \mathcal{F}w$. $\triangle$

Definition 2.6. *Given two functors $\mathcal{F}, \mathcal{G} : \mathsf{C} \to \mathsf{D}$, a **natural transformation** $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$ consists of the following data:*

- *for all $c \in \mathsf{ob}\,\mathsf{C}$, $\alpha_c : \mathcal{F}c \to \mathcal{G}c \in \mathsf{mor}\,\mathsf{D}$;*

- *for all $f : c \to c' \in \mathsf{Hom}_{\mathsf{C}}(c, c')$,the following diagram commutes in D:*

$$\begin{tikzcd} \mathcal{F}c \arrow[r, "\mathcal{F}f"] \arrow[d, "\alpha_c"'] & \mathcal{F}c' \arrow[d, "\alpha_{c'}"] \\ \mathcal{G}c \arrow[r, "\mathcal{G}f"'] & \mathcal{G}c' \end{tikzcd}$$

Here are some functor categories that we are going to use in this thesis:

Example 11 (Category of persistence modules)**.** Let $(\mathbb{R}, \leq)$ be a poset (as a category). We use $\mathbf{Vect}_{\mathbb{F}}^{(\mathbb{R},\leq)}$ to denote the category of persistence modules, where its objects are functors from $(\mathbb{R}, \leq)$ to $\mathbf{Vect}_{\mathbb{F}}$ and morphisms are natural transformations. △

Example 12 (Category of quiver representations)**.** Let $\mathbf{Path}(\Gamma)$ be the path category on a direct graph Γ. We use $\mathbf{Vect}_{\mathbb{F}}^{\mathbf{Path}(\Gamma)}$ to denote the category of quiver representations over Γ, where its objects are functors $\mathbf{Path}(\Gamma) \to \mathbf{Vect}_{\mathbb{F}}$ and morphisms are natural transformations.

△

The category of persistence modules and the category of quiver representations are examples of the category of functors.

Example 13 (Category of functors)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. Then the category of functors from $\mathbf{C}$ to $\mathbf{D}$ has functors $\mathbf{C} \to \mathbf{D}$ as its objects and natural transforms as its morphisms. We use $\mathbf{D}^{\mathbf{C}}$ to denote this category. △

The category of functors may not be locally small. In particular, when $\mathbf{C}$ and $\mathbf{D}$ are locally small, $\mathbf{D}^{\mathbf{C}}$ is not guaranteed to be locally small. The next proposition is useful.

Proposition 2.7. Let $\mathbf{C}$ be a locally small category and $\mathbf{D}$ be a small category, then $\mathbf{D}^{\mathbf{C}}$ is locally small.

Definition 2.8. *Let $\mathcal{F}, \mathcal{G} : \mathbf{C} \to \mathbf{D}$ be functors. We say $\mathcal{F}$ is* ***naturally isomorphic*** *to $\mathcal{G}$ (denoted by $\mathcal{F} \cong \mathcal{G}$) if there exists a natural transformation $\lambda : \mathcal{F} \Rightarrow \mathcal{G}$ such that λ_c is an isomorphism in $\mathbf{D}$ for all $c \in \mathsf{ob}\,\mathbf{C}$.*

Definition 2.9. *A functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ is said to be* ***full*** *if the map $\mathsf{Hom}_C(c, c') \to \mathsf{Hom}_D(\mathcal{F}c, \mathcal{F}c')$ is surjective, and is said to be* ***faithful*** *if the map $\mathsf{Hom}_C(c, c') \to \mathsf{Hom}_D(\mathcal{F}c, \mathcal{F}c')$ is injective. $\mathcal{F}$ is said to be* ***essentially surjective*** *if for all $d \in \mathsf{ob}\,\mathbf{D}$ there exists $c \in \mathsf{ob}\,\mathbf{C}$ such that $\mathcal{F}c \cong d$.*

Definition 2.10. *Let C and D be two categories. We say C is **equivalent** to D if there exists $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ and $\mathcal{G} : \mathsf{D} \to \mathsf{C}$ such that $\mathcal{F} \circ \mathcal{G} \cong \mathsf{id}_{\mathsf{D}}$ and $\mathcal{G} \circ \mathcal{F} \cong \mathsf{id}_{\mathsf{C}}$.*

Theorem 2.11. *C and D are equivalent if there exists a functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ which is fully-faithful and essentially surjective.*

2.1.2 Limits and Colimits

In this section, a diagram in C (over a category J, called a shape) is a functor $\mathcal{F} : \mathsf{J} \to \mathsf{C}$, where J is the path category of a directed graph.

Example 14 (Trivial diagram)**.** Let $c \in \mathsf{ob}\,\mathsf{C}$. A trivial diagram in C over a given shape J is a functor $c : \mathsf{J} \to \mathsf{C}$ where c sends each vertex of J to c and each arrow of J to id_c. △

Definition 2.12. *Let $\mathcal{F} : \mathsf{J} \to \mathsf{C}$ be a functor. A **cone over** $\mathcal{F}$ with summit c is a natural transformation $\lambda : c \Rightarrow \mathcal{F}$ and a **cone under** $\mathcal{F}$ with nadir c is a natural transformation $\lambda : \mathcal{F} \Rightarrow c$.*

Definition 2.13. *Let $\mathcal{F} : \mathsf{J} \to \mathsf{C}$ be a functor. The **limit** of $\mathcal{F}$, denoted by $\mathsf{lim}_{\mathsf{J}}\mathcal{F}$, is an object of C along with a universal cone $\lambda : \mathsf{lim}_{\mathsf{J}}\mathcal{F} \Rightarrow \mathcal{F}$ satisfying the following condition: for all $\mu : c \Rightarrow \mathcal{F}$, there exists a unique natural transformation $\alpha : c \Rightarrow \mathsf{lim}_{\mathsf{J}}\mathcal{F}$ such that $\mu_i = \lambda_i \circ \alpha_i$ for all $i \in \mathsf{ob}\,\mathsf{J}$.*

Equivalently, we can define $\mathsf{lim}_{\mathsf{J}}\mathcal{F}$ to be the representative of the functor $\mathsf{Cone}(-, \mathcal{F}) : \mathsf{C} \to \mathbf{Set}$, in other words,

$$\mathsf{Hom}_{\mathsf{C}}(-, \mathsf{lim}_{\mathsf{J}}\mathcal{F}) \cong \mathsf{Cone}(-, \mathcal{F})$$

Dually, we can define colimits.

Definition 2.14. *Let $\mathcal{F} : \mathsf{J} \to \mathsf{C}$ be a functor. The **colimit** of $\mathcal{F}$, denoted by $\mathsf{colim}_{\mathsf{J}}\mathcal{F}$, is an object of C along with a universal cone $\lambda : \mathcal{F} \Rightarrow \mathsf{lim}_{\mathsf{J}}\mathcal{F}$ satisfying the following condition:*

for all $\mu : \mathcal{F} \Rightarrow c$*, there exists a unique natural transformation* $\alpha : \mathsf{colim}_{\mathbf{J}}\mathcal{F} \Rightarrow c$ *such that* $\mu_i = \alpha_i \circ \lambda_i$ *for all* $i \in \mathsf{ob}\,\mathbf{J}$.

Example 15 (Product and Coproduct)**.** Let $\mathbf{J}$ be a discrete category and $\mathcal{F} : \mathbf{J} \to \mathbf{C}$. Assume $\mathbf{C}$ has limits and colimits over $\mathbf{J}$. The product of $\mathcal{F}_j$, denoted by $\prod\limits_{j\in\mathbf{J}} \mathcal{F}_j$, is the limit of $\mathcal{F}$. In other words, $\prod\limits_{j\in\mathbf{J}} \mathcal{F}_j = \mathsf{lim}_{\mathbf{J}}\mathcal{F}$. Dually, the coproduct of $\mathcal{F}_j$, denoted by $\coprod\limits_{j\in\mathbf{J}} \mathcal{F}_j$, is the colimit of $\mathcal{F}$. In other words, $\coprod\limits_{j\in\mathbf{J}} \mathcal{F}_j = \mathsf{colim}_{\mathbf{J}}\mathcal{F}$. △

2.1.3 Direct limit and its Properties

A set $\mathcal{I}$ is called a **directed set** if it has a preorder with an additional property that every pair of elements of $\mathcal{I}$ has an upper bound. A **directed system** in $\mathbf{C}$ over the directed set $\mathcal{I}$ (denoted by $\{X_i\}_{i\in\mathcal{I}}$) is a functor $X : \mathcal{I} \to \mathbf{C}$. Since $\mathcal{I}$ is a preorder, X has the following properties:

1. for all $i, j \in \mathcal{I}$, $|\mathsf{Hom}_{\mathbf{C}}(X_i, X_j)| \le 1$;
2. $X(i \le i) = \mathsf{id}_{X_i}$ for all $i \in \mathcal{I}$;
3. $X(j \le k) \circ X(i \le j) = X(i \le k)$, for all $i, j, k \in \mathcal{I}$ where $i \le j \le k$.

Definition 2.15. *Let* $\{X_i\}_{i\in\mathcal{I}}$ *be a directed system. The* ***direct limit*** *of* $\{X_i\}_{i\in\mathcal{I}}$ *is the colimit of* X*, in other words,*

$$\varinjlim X_i = \mathsf{colim}_{\mathcal{I}} X$$

Notation: We denote the (unique) map $X(i \le j) : X_i \to X_j$ $(i \le j)$ by f_{ij}; we use p_i represents the map $X_i \to \varinjlim X_i$, which is a leg map of the colimit cone.

Note that, by the universal property of the direct sum, there exists a unique morphism $\bigoplus\limits_{i\in\mathcal{I}} X_i \to \varinjlim X_i$. In particular, the following proposition provides an explicit construction of $\varinjlim X_i$. Let $\lambda_i : X_i \to \bigoplus\limits_{i} X_i$ be the canonical morphism for all $i \in \mathcal{I}$.

Proposition 2.16. Let $\mathsf{C} = \mathsf{Mod}_R$. Then $\varinjlim X_i = \mathsf{colim}_{\mathcal{I}} X \cong \bigoplus X_i/S$, where S is the submodule of M generated by $\lambda_j \circ X(i \leq j) - \lambda_i$ for all $i \leq j \in \mathcal{I}$.

Proof. See [42] Proposition 5.23. □

Lemma 2.1. Let $\{X_i\}_{i\in\mathcal{I}}$ be a directed system where $X_i \in \mathsf{Mod}_R$ for all $i \in \mathcal{I}$. Then L is the direct limit of $\{X_i\}_{i\in\mathcal{I}}$ iff the following conditions hold:

1. Given $x \in L$, there exists $i \in \mathcal{I}$ and $x_i \in X_i$ such that $p_i(x_i) = x$.
2. If there exists $x_i \in X_i$ for some $i \in \mathcal{I}$ such that $p_i(x_i) = 0$, there exists $j \in \mathcal{I}$ with $j \geq i$ such that $f_{ij}(x_i) = 0$.

Proof of lemma. ($\Rightarrow$) is given in [46] Lemma 2.6.14.

($\Leftarrow$) We want to show for any cocone under $\{X_i\}_{i\in I}$ with nadir M with leg maps $(m_i)_{i\in I}$, there exists a unique map

$$\phi : L \to M$$

such that $m_i = \phi \circ p_i$ for all $i \in I$. By the condition (1), we are forced to define $\phi(x) = m_i(x_i)$ where $p_i(x_i) = x$ for some $i \in I$. Suppose there is another x_j in X_j such that $p_j(x_j) = x$, without loss of generality, we assume $i \leq j$. Notice that $p_j(f_{ij}(x_i)) = p_i(x_i)$ hence $p_j(x_j - f_{ij}(x_i)) = p_j(x_j) - p_i(x_i) = x - x = 0$. Therefore, by condition (2), there exists $k \in I$ with $k \geq j$ such that $f_{jk}(x_j - f_{ij}(x_i)) = 0$. Hence $m_k f_{jk}(x_j - f_{ij}(x_i)) = 0$, i.e., $m_j(x_j) - m_i(x_i) = 0$, which implies ϕ is well-defined.

Theorem 2.17. *Let $\{X_i\}_{i\in\mathcal{I}}$ be a directed system where X_i are open subset of an ambient topological space and $X_i \to X_j$ is an inclusion map if $i \leq j$. Then*

$$\varinjlim \pi_n(X_i) \cong \pi_n(\varinjlim X_i)$$

Proof. Given $[\sigma] \in \pi_n(\varinjlim X_i)$, then σ is a map from S^n to $\varinjlim X_i = \bigcup X_i$. Since S^n is compact, there exists X_k such that $\mathsf{Im}\,\sigma \subseteq X_k$. That means $[\sigma] \in \pi_n(X_k)$.

On the other hand, if there exists $[\sigma] \in \pi_n(X_i)$ such that $\sigma \sim 0$ in $\pi_n(\varinjlim X_i)$. Let $H : S^n \times I \to \varinjlim X_i$ be the homotopy. Note that $S^n \times I$ is compact, there must exist X_k such that $\mathsf{Im}\,H \subseteq X_k$. Hence $\sigma \sim 0$ in $\pi_n(X_k)$.

By the Lemma 2.1, we conclude that $\varinjlim \pi_n(X_i) \cong \pi_n(\varinjlim X_i)$.

$\square$

Theorem 2.18. $\varinjlim -$ *is exact.*

Proof. Let $\{X_i\}_{i\in\mathcal{I}}, \{Y_i\}_{i\in\mathcal{I}}, \{Z_i\}_{i\in\mathcal{I}}$ be directed systems where for each $i \in \mathcal{I}$, $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ is exact at Y_i. Denote $\alpha_{ij} : Y_i \to Y_j$ and $\beta_{ij} : Z_i \to Z_j$. Denote $p_i : X_i \to \varinjlim X_i$, $q_i : Y_i \to \varinjlim Y_i$ and $r_i : Z_i \to \varinjlim Z_i$ We want to show the following sequence is exact:

$$\varinjlim X_i \xrightarrow{F} \varinjlim Y_i \xrightarrow{G} \varinjlim Z_i$$

First, the existence of F and G is clear by the universal property of direct limits.

Second, notice that $g_i \circ f_i = 0$ for all $i \in \mathcal{I}$ (which means the map $g_i \circ f_i : X_i \to Z_i$ factors through 0), hence we have $G \circ F = 0$.

On the other hand, given $y \in \mathsf{ker}G$, we have $G(y) = 0$. By the previous lemma, there exists $y_i \in Y_i$ for some $i \in \mathcal{I}$ such that $q_i(y_i) = y$. Hence we have $(G \circ q_i)(y_i) = G(y) = 0$. Note that $G \circ q_i = r_i g_i$ we have $r_i g_i(y_i) = 0$. By the previous lemma, there exists Z_j such that $\beta_{ij}(g_i(y_i)) = 0$. Note that $\beta_{ij} g_i = g_j \alpha_{ij}$ hence $g_j \alpha_{ij}(y_i) = 0$. Note that $X_j \xrightarrow{f_j} Y_j \xrightarrow{g_j} Z_j$ is exact, there exists $x_j \in X_j$ such that $f_j(x_j) = \alpha_{ij}(y_i)$. Hence we have

$$F \circ p_j(x_j) = q_j f_j(x_j) = q_j \alpha_{ij}(y_i) = q_i(y_i) = y$$

Hence $\mathsf{ker}G \subseteq \mathsf{Im}\,F$. $\square$

2.1.4 Generators of Categories and Projective Objects

Definition 2.19. *Let C be a category. A collection of objects $\{U_i\}_{i\in I}$ is called a* ***collection of generators*** *of C if for any pair of distinct morphisms $f, g \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, there exists $u \in \mathrm{Hom}_{\mathsf{C}}(U_i, A)$ for some $i \in I$ such that $fu \neq gu$.*

Note that if $P \in \mathsf{C}$ is projective and $h : P \to U_i$ is an epimorphism where U_i is a generator of C, then P is a generator of C because: for distinct functions $f, g : A \to B$ such that $fu \neq gu$, where $u : U_i \to A$, h is an epimorphism implies $(fu)h \neq (gu)h$

Proposition 2.20. Assume C has coproduct. Then an object U generates C if and only if there exists a epimorphism $\gamma : \mathsf{colim}_J U \to A$ for every $A \in \mathsf{C}$.

Proof. Assume U generates C. Then for any pair of distinct morphisms $f, g \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, there exists $u : U \to A$ in C such that $fu \neq gu$. Let $\mathsf{colim}_J U$ denote that coproduct of U over the shape $J = \mathrm{Hom}_{\mathsf{C}}(U, A)$. Note that $\gamma : \mathsf{colim}_J U \to A$ is an epimorphism: let $h \neq k$ where $h, k \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $u : U \to A$ in C such that $hu \neq ku$. Let $\sigma_u : U \to \mathsf{colim}_J U$ be the u-th leg map of the colimit cone. Then $u = \gamma\sigma_u$. Since

$$h\gamma\sigma_u = hu \neq ku = k\gamma\sigma_u$$

we conclude that $h\gamma \neq k\gamma$. Therefore, γ is an epimorphism.

Conversely, let $f, g \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ to be a pair of distinct morphisms. Because there exists a epimorphism $\gamma : \mathsf{colim}_J U \to A$ for every $A \in \mathsf{ob}\,\mathsf{C}$, $f\gamma \neq g\gamma$. Then there exists $\sigma_u : U \to \mathsf{colim}_J U$ (which is the u-th leg map of the colimit cone) such that

$$f\gamma\sigma_u \neq g\gamma\sigma_u$$

where $\gamma\sigma_u : U \to A$. $\square$

Definition 2.21. *Assume* $\mathbf{C}$ *has coproduct and* $\{U_i\}_{i\in I}$ *be a collection of generators of* $\mathbf{C}$. *We call an object* $A \in \mathbf{C}$ *is* ***finitely generated*** *if there exists an epimorphism* $\bigoplus_{k=1}^{n} U_{i_k} \to A$ *where* $i_1, \ldots, i_n \in I$ *for some natural number* n. A *is free if there exists* $J \subseteq I$ *such that* $A \cong \bigoplus_{i\in J} U_i$.

Definition 2.22. *Let* $\mathbf{A}$ *be a small abelian category. Let* $\mathcal{E}$ *be a collection of epimorphisms of* $\mathbf{A}$. *An object* A *of* $\mathbf{A}$ *is* ***$\mathcal{E}$-projective*** *if* $\mathrm{Hom}(A, \alpha)$ *is an epimorphism for all* $\alpha \in \mathcal{E}$. *The collection of* $\mathcal{E}$*-projectives is denoted by* $\mathfrak{p}\mathcal{E}$. *Let* $\mathcal{P}$ *be a collection of objects of* $\mathbf{A}$ *and* $\mathfrak{e}\mathcal{P}$ *denote the collection*

$$\mathfrak{e}\mathcal{P} = \{\alpha \in \mathrm{Mor}\mathbf{A} \mid \alpha \textit{ is an epimorphism and } \mathrm{Hom}(A, \alpha) \textit{ is an epimorphism for all } A \in \mathcal{P}.\ \}$$

Note that $\mathcal{E} \subseteq \mathcal{E}'$ implies $\mathfrak{p}\mathcal{E} \supseteq \mathfrak{p}\mathcal{E}'$ and $\mathcal{P} \subseteq \mathcal{P}'$ implies $\mathfrak{e}\mathcal{P} \supseteq \mathfrak{e}\mathcal{P}'$. Moreover, $\mathfrak{e}\mathfrak{p}\mathcal{E} \supseteq \mathcal{E}$ and $\mathfrak{p}\mathfrak{e}\mathcal{P} \supseteq \mathcal{P}$. We can view $\mathfrak{e}$ as a functor from the power sets of $\mathbf{A}$ to the power sets of the collection of the epimorphisms of $\mathbf{A}$ and view $\mathfrak{p}$ as a functor from the power sets of the collection of the epimorphisms of $\mathbf{A}$ to the power sets of $\mathbf{A}$. Then $\mathfrak{e} \dashv \mathfrak{p}$ and we obtain the Galois connection between those two categories:

$$\mathfrak{e}\mathfrak{p}\mathfrak{e}\mathcal{P} = \mathfrak{e}\mathcal{P} \quad \text{and} \quad \mathfrak{p}\mathfrak{e}\mathfrak{p}\mathcal{E} = \mathfrak{p}\mathcal{E}$$

Definition 2.23. *Let* $\mathbf{A}$ *be an abelian category and* $\mathcal{E}$ *be a collection of epimorphisms of* $\mathbf{A}$. $\mathcal{E}$ *is called* ***closed*** *if there exists a collection of objects of* $\mathbf{A}$*, denoted by* $\mathcal{P}$*, such that* $\mathcal{E} = \mathfrak{e}\mathcal{P}$. *Moreover, if for every* $A \in \mathrm{Obj}\mathbf{A}$ *there exists* $\alpha : P \to A$ *where* $P \in \mathfrak{p}\mathcal{E}$*, then* $\mathcal{E}$ *is called a* ***projective class*** *of* $\mathbf{A}$.

Lemma 2.2. Let $\mathbf{A}$ be an abelian category and $\mathcal{E}$ be a projective class of $\mathbf{A}$. If there is a morphism $\alpha : A \to B$ in $\mathbf{A}$ such that $\mathrm{Hom}_{\mathbf{A}}(P, \alpha)$ is an epimorphism for all $P \in \mathfrak{p}\mathcal{E}$, then

$\alpha \in \mathcal{E}$.

Proof of lemma. Since $\mathcal{E}$ be a projective class of $\mathbf{A}$, $\mathcal{E} = \mathfrak{e}\mathcal{P}$ for some $\mathcal{P} \subseteq \mathrm{Mor}\mathbf{A}$ and there exists $\beta \in \mathcal{E}$ where $\beta : P \to B$ and $P \in \mathfrak{p}\mathcal{E}$. Note that $\mathrm{Hom}_{\mathbf{A}}(P, \alpha)$ is an epimorphism, there exists $\gamma : P \to A$ in $\mathbf{A}$ such that $\alpha \circ \gamma = \beta$. Since β is an epimorphism, so is α. Since $\mathrm{Hom}_{\mathbf{A}}(P, \alpha)$ is an epimorphism for all $P \in \mathfrak{p}\mathcal{E}$, we obtain $\alpha \in \mathfrak{e}\mathfrak{p}\mathcal{E}$. Note that $\mathcal{E} = \mathfrak{e}\mathcal{P}$, we have

$$\mathfrak{e}\mathfrak{p}\mathcal{E} = \mathfrak{e}\mathfrak{p}\mathfrak{e}\mathcal{P} = \mathfrak{e}\mathcal{P} = \mathcal{E}$$

Therefore, $\alpha \in \mathcal{E}$. ■

Lemma 2.3. Let $T : \mathbf{A} \to \mathbf{B}$ be a faithful functor and $\alpha : A \to A'$ be a morphism in $\mathbf{A}$. If $T\alpha$ is an epimorphism, so is α.

Proof of lemma. Assume there exists $f, g \in \mathrm{Mor}\mathbf{A}$ such that $f \circ \alpha = g \circ \alpha$. Applying the functor T to this equation, by the functoriality of T, we get $Tf \circ T\alpha = Tg \circ T\alpha$. Note that $T\alpha$ is an epimorphism, therefore, $Tf = Tg$. Hence $f = g$ because T is faithful. ■

Theorem 2.24. *Consider $S : \mathbf{B} \to \mathbf{A}$ and $T : \mathbf{A} \to \mathbf{B}$ such that $S \dashv T$. Assume T is faithful. Let $\mathcal{E}$ be a projective class of $\mathbf{B}$. Then $T^{-1}\mathcal{E}$ is a projective class of $\mathbf{A}$ and $\mathfrak{p}(T^{-1}\mathcal{E})$ consists of objects of the form SP or retractions of SP, where $P \in \mathfrak{p}\mathcal{E}$.*

Proof. Given $\alpha \in T^{-1}\mathcal{E}$, $T\alpha \in \mathcal{E}$. Hence for all $P \in \mathfrak{p}\mathcal{E}$, $\mathrm{Hom}_{\mathbf{B}}(P, T\alpha)$ is an epimorphism. Since $S \dashv T$, $\mathrm{Hom}_{\mathbf{B}}(P, T\alpha)$ is an epimorphism implies $\mathrm{Hom}_{\mathbf{A}}(SP, \alpha)$ is an epimorphism. Therefore, $SP \in \mathfrak{p}(T^{-1}\mathcal{E})$.

Given $\alpha \in \mathfrak{e}\mathfrak{p}(T^{-1}\mathcal{E})$, $\mathrm{Hom}_{\mathbf{A}}(SP, \alpha)$ is an epimorphism for all $P \in \mathfrak{p}\mathcal{E}$ because $SP \in \mathfrak{p}(T^{-1}\mathcal{E})$. Since $S \dashv T$, this implies $\mathrm{Hom}_{\mathbf{B}}(P, T\alpha)$ is an epimorphism for all $P \in \mathfrak{p}\mathcal{E}$. Hence

$T\alpha \in \mathfrak{ep}\mathcal{E}$. Since $\mathcal{E}$ is a projective class, hence is closed, there exists $\mathcal{P} \subseteq \mathrm{Obj}\mathbf{B}$ such that $\mathcal{E} = \mathfrak{e}\mathcal{P}$. Hence $\mathfrak{ep}\mathcal{E} = \mathcal{E}$ and $T\alpha \in \mathcal{E}$. Thus $\alpha \in T^{-1}\mathcal{E}$, in other words, $T^{-1}\mathcal{E}$ is closed.

Given $A \in \mathrm{Obj}\mathbf{A}$, note that $TA \in \mathrm{Obj}\mathbf{B}$ and $\mathcal{E}$ is a projective class, there exists $\alpha : P \to TA$ in $\mathcal{E}$ for some $P \in \mathfrak{p}\mathcal{E}$. Since $S \dashv T$, $\alpha^\flat := \epsilon_A \circ S\alpha \in \mathrm{Hom}_{\mathbf{A}}(SP, A)$. Note that $\epsilon_A \circ S\alpha \in T^{-1}\mathcal{E}$, that is, $T\alpha^\flat \in \mathcal{E}$: for all $Q \in \mathfrak{p}\mathcal{E}$, the map

$$\mathrm{Hom}_{\mathbf{B}}(Q, T\alpha^\flat) : \mathrm{Hom}_{\mathbf{B}}(Q, TSP) \to \mathrm{Hom}_{\mathbf{B}}(Q, TA)$$

is an epimorphism because $\mathrm{Hom}_{\mathbf{B}}(Q, \alpha)$ is an epimorphism, and we have the following commutative diagram.

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathbf{B}}(Q,P) \arrow[dl, "\mathrm{Hom}_{\mathbf{B}}(Q,\eta_P)"'] \arrow[dr, two heads, "\mathrm{Hom}_{\mathbf{B}}(Q,\alpha)"] & \\
\mathrm{Hom}_{\mathbf{B}}(Q,TSP) \arrow[rr, "\mathrm{Hom}_{\mathbf{B}}(Q,T\alpha^\flat)"'] & & \mathrm{Hom}_{\mathbf{B}}(Q,TA)
\end{tikzcd}$$

Note that $SP \in \mathfrak{p}(T^{-1}\mathcal{E})$, we conclude that $T^{-1}\mathcal{E}$ is a projective class.

Given $A \in \mathfrak{p}(T^{-1}\mathcal{E})$, note that $TA \in \mathrm{Obj}\mathbf{B}$ and $\mathcal{E}$ is a projective class, there exists $\alpha : P \to TA$ in $\mathcal{E}$ for some $P \in \mathfrak{p}\mathcal{E}$. Note that $\alpha^\flat \in T^{-1}\mathcal{E}$ and $A \in \mathfrak{p}(T^{-1}\mathcal{E})$, $\mathrm{Hom}_{\mathbf{A}}(A, \alpha^\flat)$ is an epimorphism, in particular, there exists $\gamma : A \to SP$ such that $\alpha^\flat \circ \gamma = \mathsf{id}_A$. Hence A is a retract of SP for some $P \in \mathfrak{p}\mathcal{E}$. $\square$

Proposition 2.25. Consider $S : \mathbf{B} \to \mathbf{A}$ and $T : \mathbf{A} \to \mathbf{B}$ such that $S \dashv T$. Assume T is faithful. Let $\mathcal{E}$ be a projective class of $\mathbf{B}$. If there exists a functor $R : \mathbf{A} \to \mathbf{B}$ such that

1. there exists $\mu : RS \Rightarrow \mathsf{id}_{\mathbf{B}}$;

2. for all $\alpha : SP \to A$ where $P \in \mathfrak{p}\mathcal{E}$ and $A \in \mathfrak{p}(T^{-1}\mathcal{E})$, $R(\alpha)$ is an isomorphism implies

α is an isomorphism.

Then $A \in \mathfrak{p}(T^{-1}\mathcal{E})$ implies $A \cong SP$ for some $P \in \mathfrak{p}\mathcal{E}$.

Proof. Let $A \in \mathfrak{p}(T^{-1}\mathcal{E})$. $T^{-1}\mathcal{E}$ is a projective class of A, there exists $P \in \mathfrak{p}\mathcal{E}$ and $\alpha : SP \to A$ in $T^{-1}\mathcal{E}$ such that A is a retract of SP, i.e.,

$$A \to SP \to A$$

Apply R to the above diagram, we obtain

$$RA \to RSP \to RA$$

Note that $\mu : RS \Rightarrow \mathsf{id}$ is an equivalence, in particular, $RSP \cong P$, we get

$$RA \to P \to RA$$

Hence RA is a retract of P. Since $P \in \mathfrak{p}\mathcal{E}$, so is RA. Note that SRA is a retract of SP because

$$SRA \to SP \to SRA$$

We claim that $SRA \cong A$.

Lemma 2.4. The co-unit morphism $\epsilon_A : STA \to A$ is an element of $T^{-1}\mathcal{E}$.

Proof of lemma. Apply T to $\epsilon_A : STA \to A$, we obtain $T\epsilon_A : TSTA \to TA$. Note that for every $P \in \mathfrak{p}\mathcal{E}$,

$$T\epsilon_{A*} : \mathrm{Hom}(P, TSTA) \to \mathrm{Hom}(P, TA)$$

is an epimorphism: for every $f \in \mathrm{Hom}(P, TA)$, note that $\eta_{TA} \circ f \in \mathrm{Hom}(P, TSTA)$ and

$$T\epsilon_{A*}(\eta_{TA} \circ f) = T\epsilon_A \circ \eta_{TA} \circ f = \mathsf{id}_{TA} \circ f = f$$

By Lemma 2.2, $T\epsilon_A \in \mathcal{E}$. Therefore, $\epsilon_A \in T^{-1}\mathcal{E}$. ■

Since A is $T^{-1}\mathcal{E}$-projective, there exists $\gamma : A \to STA$ such that $\epsilon_A \circ \gamma = \mathsf{id}_A$. Consider

$$SRA \xrightarrow{SR\gamma} SRSTA \xrightarrow{S\mu_{TA}} STA \xrightarrow{\epsilon_A} A$$

By applying R to the above diagram, we obtain

$$RSRA \xrightarrow{RSR\gamma} RSRSTA \xrightarrow{RS\mu_{TA}} RSTA \xrightarrow{R\epsilon_A} RA$$

Note that the diagram

$$\begin{tikzcd} RSRSTA \arrow[r, "RS\mu_{TA}"] \arrow[d, "\mu_{RSTA}"'] & RSTA \arrow[d, "\mu_{TA}"] \\ RSTA \arrow[r, "\mu_{TA}"'] & TA \end{tikzcd}$$

is a commutative because μ is a natural transformation. In other words, $\mu_{TA} \circ \mu_{RSTA} = \mu_{TA} \circ RS\mu_{TA}$. Because μ_{TA} is an isomorphism, we obtain $\mu_{RSTA} = RS\mu_{TA}$. Consider the following diagram

$$\begin{tikzcd} RSRA \arrow[r, "RSR\gamma"] \arrow[d, "\mu_{RA}"'] & RSRSTA \arrow[d, "RS\mu_{TA}"] \\ RA \arrow[r, "R\gamma"'] & RSTA \end{tikzcd}$$

This diagram is commutative because μ is a natural transformation and $RS\mu_{TA} = \mu_{RSTA}$.

Therefore,

$$\begin{aligned} R\epsilon_A \circ RS\mu_{TA} \circ RSR\gamma &= R\epsilon_A \circ R\gamma \circ \mu_{RA} \\ &= R(\epsilon_A \circ \gamma) \circ \mu_{RA} \\ &= R(\mathsf{id}_A) \circ \mu_{RA} \\ &= \mu_{RA} \end{aligned} \tag{2.1}$$

Therefore, $R\epsilon_A \circ RS\mu_{TA} \circ RSR\gamma$ is an isomorphism because μ_{RA} is an isomorphism. □

Theorem 2.26. *Let* I *be a poset and* $\mathcal{E}$ *be a projective class of* vect. *Then the projective class of* $\mathsf{vect}_{\mathbb{F}}^{\mathsf{I}}$, *denoted by* $\mathcal{E}^{(\mathsf{I})}$, *consist of morphisms in* $\mathsf{vect}_{\mathbb{F}}^{\mathsf{I}}$ *which are pointwise* $\mathcal{E}$*-projective and* $\mathfrak{p}\mathcal{E}^{(\mathsf{I})}$ *consist of objects of the form* $\bigoplus_{i \in \mathsf{I}} S_i(P_i)$ *where* $P_i \in \mathfrak{p}\mathcal{E}$ *for all* $i \in \mathsf{I}$.

2.2 Graphs and Posets

A **directed graph** (also called **digraph** or **quiver**) $Q = (V, E, s, t)$ consists of two sets V (called the set of vertices) and E (called the set of arrows), and two maps $s, t : E \to V$ where s sends each edge to its source, and t sends each edge to its target. $Q = (V, E, s, t)$ is said to be **finite** if V and E are finite sets. We can view a quiver $Q = (V, E, s, t)$ as a category where V is the set of objects and the set of morphisms consists of all finite (directed) paths in Q. We say the quiver Q is **connected** if Q is a connected as a category.

A **graph** $\Gamma = (V, E, *)$ is a digraph equipped with an involution * on the set of arrows A such that $a^{**} = a$, $sa^* = ta$ and $sa = ta^*$ for all $a \in A$. The edge E of Γ are pairs $\{a, a^*\}$ for all $a \in A$, i.e., E is a set of unoriented arrow of Γ. When V and E are finite sets, we call Γ a **finite graph**. We can view $\Gamma = (V, E, *)$ as a category where V is the set of objects and the set of morphisms consists of all finite paths in Γ. We say the graph Γ is **connected** if Γ is a connected as a category.

A **tree** is a connected graph with no loops or cycles. A **star graph** with k leaves is a tree with $k+1$ vertices and k edges such that there is exactly one vertex has degree k and the degree of the other vertices is 1.

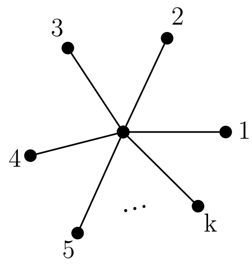


Figure 2.1: Star_k

A **partially ordered set** (or **poset**) is a set P with a partial order $\leq$:

1. $a \leq a$ for all $a \in P$;

2. $a \leq b$ and $b \leq a$ implies $a = b$, for all $a, b \in P$;

3. $a \leq b$ and $b \leq c$ implies $a \leq c$, $a, b, c \in P$.

We use $(P, \leq)$ to denote the set P with the partial order $\leq$.

A subposet I of $(P, \leq)$ is **connected** if I is connected as a subcategory of $(P, \leq)$. I is **convex** if for every $x \leq z \leq y \in P$, $x, y \in \mathsf{I}$ implies $z \in \mathsf{I}$.

Definition 2.27 (Interval of a poset)**.** $\mathsf{I} \subseteq (P, \leq)$. *$\mathsf{I}$ is called an **interval** if it is convex and connected.*

$(P, \leq)$ is **locally finite** if every interval of P is finite.

A poset $(P, \leq)$ is **bounded** if P contains an initial element and a terminal element as a category. A subset U of $(P, \leq)$ is called an upset if for any $x \in U$, $x \leq y$ implies $y \in U$ for any $y \in (P, \leq)$. Dually, a subset D of $(P, \leq)$ is called an downset if for any $x \in D$, $w \leq x$ implies $w \in D$ for any $w \in (P, \leq)$. Given an upset U (as a subposet of $(P, \leq)$), we call U a principal upset if there exists $a \in P$ such that $a \leq x$ implies $x \in U$ for any $x \in (P, \leq)$. Dually, given an upset D (as a subposet of $(P, \leq)$), we call D a principal downset if there exists $b \in P$ such that $y \leq b$ implies $y \in D$ for any $y \in (P, \leq)$.

Let $a, b \in P$. a is said to **cover** b if $b < a$ and there is no element $c \in P$ such that $b < c < z$. We write $b \prec a$ when a covers b.

When $(P, \leq)$ is finite, we can construct a graph on the plane as follows:

1. each element of $(P, \leq)$ gives a vertex of the digraph;
2. For all $a, b \in P$, place the vertex a above[1] the vertex b if $b < a$;
3. For all $a, b \in P$, place a line segment between a and b if $b \prec a$.

This digraph is called the **Hasse diagram** of $(P, \leq)$

2.3 Category of Graded Modules

Let J be a monoid. A **J-graded ring** R is a ring such that $R \cong \bigoplus_{i \in J} R_i$ where R_i is a subgroup of R for all $i \in J$, such that $R_i \cdot R_j \subseteq R_{i+j}$. In addition, when R is an algebra over a field $\mathbb{F}$ and R_i is a vector space over $\mathbb{F}$ for all $i \in J$, R is called a **graded $\mathbb{F}$-algebra**. Let R and $\hat{R}$ be two J-graded rings. $f : R \to \hat{R}$ is a **graded ring homomorphism** if f is a ring homomorphism and $f(R_i) \subseteq \hat{R}_i$ for all $i \in J$. The category of J-graded rings, denoted by $\mathsf{\textbf{Gr}^J\textbf{Ring}}$, has graded rings as objects and graded ring homomorphisms as its morphisms.

[1] a is above b means that the y-coordinate of a is greater than the y-coordinate of b on the xy-plane.

Example 16. $\mathbb{F}[x]$ is a $\mathbb{N}$-graded ring via $\mathbb{F}[x]_i := \mathbb{F}x^i$ for all $i \in \mathbb{N}$. $\triangle$

Recall that a partially ordered set is a pair $(S, \leq)$ consisting of a set S and a partial order $\leq$ on S. A group G is called a partially ordered group if its underlying set $\mathcal{U}G$ is a partially ordered set, where $\mathcal{U}$ is the forgetful functor $\mathcal{U} : \textsf{Group} \to \textsf{Set}$, satisfying the following condition

$$\text{if } a \leq b \in G, \text{ then } ag \leq bg, \text{ for all } g \in G$$

A ring can have multiple gradings.

Example 17. $R := \mathbb{F}[x_1, \ldots, x_d]$ is a $\mathbb{N}$-graded ring via $\mathbb{F}[x_1, \ldots, x_d]_i = R_i$, where R_i is the abelian group generated by the monomials of degree i. $\triangle$

Example 18. $R := \mathbb{F}[x_1, \ldots, x_d]$ is a $\mathbb{N}^d$-graded ring via $\mathbb{F}[x_1, \ldots, x_d]_{n_1,\ldots,n_d} = \mathbb{F}x_1^{n_1} \cdots x_d^{n_d}$. $\triangle$

Proposition 2.28. Let J be a monoid and R be a J-graded ring. Then

1. R_0 is a subring of a J-graded ring R, where $0 \in J$ is the identity element;

2. $1_R \in R_0$;

3. $R_i \in \textsf{Mod}_{R_0}$ for all $i \in J$.

Let J be a monoid and R be an J-graded ring. An R-module M is called an **(J-graded) R-module** if there exists a collection of subgroups of M (denoted by $\{M_i\}$) such that $M \cong \bigoplus_{i \in J} M_i$ and $R_i \cdot M_j \subseteq M_{i+j}$. Let M, N be (J-graded) R-modules. $f : M \to N$ is a **graded R-module homomorphism** if f is an R-module homomorphism and $f(M_i) \subseteq N_i$ for all $i \in J$. The category of (J-)graded R-modules, denoted by $\textsf{Gr}^{\textsf{J}}\textsf{Mod}_R$, has graded R-modules as objects and graded R-module homomorphisms as its morphisms.

When $J = \mathbb{Z} \times \mathbb{Z}$, we call M a **bigraded module**. Given two bigraded modules M and N, a morphism $f : M \to N$ has degree (a, b) if $f = (f_{ij})_{ij \in J}$ where $f_{ij} : M_{ij} \to N_{i+a,j+b}$.

2.4 Mayer-Vietoris Spectral Sequences

In this section, we give an introduction to the Mayer-Vietoris spectral sequences. The set up for the general spectral sequences can be found in any standard textbooks on homological algebra, for example, [46] and [42]. We are going to use $\mathbf{A}$ to denote an abelian category.

A **(homology) spectral sequence** in $\mathbf{A}$ is a collection of objects $\{E^r_{pq}\}$ of $\mathbf{A}$, for all $p, q \in \mathbb{Z}$ and $r \geq a$ (for a fixed a), together with morphisms $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$ such that $d^r_{pq} \circ d^r_{p-r,q+r-1} = 0$, satisfying

$$E^{r+1}_{pq} \cong \frac{\mathsf{ker} d^r_{pq}}{\mathsf{Im}\, d^r_{p+r,q-r+1}}$$

For each r, the collection of E^r_{pq} is called the r**-th page** of the spectral sequence. $\{E^r_{pq}\}$ is said to be **bounded** if for each $n := p + q$ (called the total degree of E^r_{pq}), there exists only finitely many non-zero term E^a_{pq}.

When $E^r_{pq} = 0$ unless $p \geq 0$ and $q \geq 0$, $\{E^r_{pq}\}$ is called a **first quadrant spectral sequence**. When r is sufficiently large, $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$ is the zero map since E^r_{pq}=0 or $E^r_{p-r,q+r-1} = 0$. Hence $E^r_{pq} = E^{r+1}_{pq}$ for large r. Note that every first quadrant spectral sequence is bounded.

Definition 2.29. *Let $\{E^r_{pq}\}$ be a bounded spectral sequence. Then $\{E^r_{pq}\}$* ***converges*** *to H (denoted by $E^r_{pq} \Rightarrow H$) if there exists a collection of objects $H_n \in \mathsf{ob}\,\mathbf{A}$ and a filtration*

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1} H_n \subseteq F_p H_n \subseteq \cdots \subseteq F_t H_n = H_n$$

for each n and

$$E^{\infty}_{pq} \cong \frac{F_pH_n}{F_{p-1}H_n} = \frac{F_pH_{p+q}}{F_{p-1}H_{p+q}}$$

for each $p, q \in \mathbb{Z}$ where $p + q = n$.

We will heavily use the next proposition in chapter 4.

Proposition 2.30. Let $\{E^r_{pq}\}$ be a bounded spectral sequence where $E^2_{pq} = 0$ unless $p = 0$ or $p = 1$. If $E^r_{pq} \Rightarrow H$, then there exists a short exact sequence

$$0 \to E^2_{0n} \to H_n \to E^2_{1,n-1} \to 0$$

for each n.

Proof. Note that $E^r_{pq} \Rightarrow H$, there exists a filtration of H such that for each n,

$$0 = F_sH_n \subseteq \cdots \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq \cdots \subseteq F_tH_n = H_n$$

such that $E^{\infty}_{pq} \cong \dfrac{F_pH_{p+q}}{F_{p-1}H_{p+q}}$. For each p, q, $d^2_{pq} : E^2_{pq} \to E^r_{p-2,q+1}$ is a zero morphism because $E^2_{pq} = 0$ unless $p = 0$ or $p = 1$. Therefore, the E^2 page has no no-trivial arrows. Hence $E^2 = E^{\infty}$, and

$$E^2_{pq} \cong \frac{F_pH_{p+q}}{F_{p-1}H_{p+q}}$$

Note that $n = p + q$, hence

- when $p \neq 0$ and $p \neq 1$, $E^2_{pq} = 0 = \dfrac{F_pH_n}{F_{p-1}H_n}$. Hence $F_pH_n = F_{p-1}H_n$ for all $p \neq 0$ and $p \neq 1$, and the filtration becomes to

$$0 = F_sH_n = \cdots = F_{-2}H_n = F_{-1}H_n \subseteq F_0H_n \subseteq F_1H_n = F_2H_n = \cdots = F_tH_n = H_n$$

- when $p = 0$, $E^2_{pq} = E^2_{0q} \cong \dfrac{F_0H_n}{F_{-1}H_n}$. Since $F_{-1}H_n = 0$, we obtain $E^2_{0q} \cong F_0H_{0+q}$;

- when $p = 1$, $E^2_{pq} = E^2_{1q} \cong \dfrac{F_1 H_n}{F_{1-1} H_n} = \dfrac{H_n}{F_0 H_n}$.

Note that there is a short exact sequence

$$0 \to F_0 H_n \hookrightarrow H_n \to H_n / F_0 H_n \to 0$$

Therefore,

$$0 \to E^2_{0,n} \to H_n \to E^2_{1,n-1} \to 0$$

$\square$

Definition 2.31. *A **double complex** is an ordered triple (M, d', d''), where M is a bigraded module and d', d'' are differential maps of bidegree $(-1, 0)$ and $(0, -1)$ such that $d' \circ d'' + d'' \circ d' = 0$.*

In other words, for all $p, q \in \mathbb{Z}$,

$$d'_{pq} : M_{p,q} \to M_{p-1,q}$$

and

$$d''_{pq} : M_{p,q} \to M_{p,q-1}$$

Definition 2.32. *A **total complex** of a double complex (M, d', d'') is a complex where its n-th term is $\bigoplus_{p+q=n} M_{p,q}$. We use $\mathsf{Tot}(M)_n$ to denote the term $\bigoplus_{p+q=n} M_{p,q}$.*

Lemma 2.5. $\{\mathsf{Tot}(M)_n\}$ is a chain complex, where $d = \sum_{p+q=n} d'_{pq} + d''_{pq}$.

Let X be a CW-complex. Consider a cover $U := \{X_i\}_{i \in I}$ of X where I is a totally ordered set and X_i is a subcomplex of X for all $i \in I$. Note that for $i_1 < i_2 < \cdots < i_n$, there is an inclusion map

$$C(X_{i_1} \oplus \cdots \oplus X_{i_n}) \to C(X_{i_1} \oplus \cdots \oplus \widehat{X_{i_k}} \oplus \cdots \oplus X_{i_n})$$

where $\widehat{X_{i_k}}$ means we remove the k-th direct summand from the given expression. Therefore, it induces a unique map

$$\bigoplus_{i_1<\cdots<i_n} C(X_{i_1} \oplus \cdots \oplus X_{i_n}) \to \bigoplus_{j_1<\cdots<j_{n-1}} C(X_{j_1} \oplus \cdots \oplus X_{j_{n-1}})$$

Therefore, we obtain a sequence

$$\begin{aligned} \cdots \to \bigoplus_{i_1<\cdots<i_n} C(X_{i_1} \oplus \cdots \oplus X_{i_n}) \to \bigoplus_{j_1<\cdots<j_{n-1}} C(X_{j_1} \oplus \cdots \oplus X_{j_{n-1}}) \to \cdots \\ \to \bigoplus_i C(X_i) \to C(X) \to 0 \end{aligned} \tag{2.2}$$

Lemma 2.6. The sequence given in (2.2) is exact, and $E_{pq} \Rightarrow H_{p+q}(X)$ where

$$E^2_{pq} = H_p(E^1_{pq}) = H_p(\bigoplus_{\sigma \in \mathsf{Nerve}(U)} H_q(X_\sigma))$$

Proof. See [10] pp 166-167. □

2.5 Quiver Representations

Let Q be a quiver (viewed as a category). A **quiver representation** over the quiver Q is a functor $M : \mathbf{Path}(Q) \to \mathbf{Vect}_{\mathbb{F}}$. M is **thin** if $\dim Mv \leq 1$ for all $v \in \mathsf{ob}\,\mathbf{Path}(Q)$.

Let $\mathbb{A}_n$ denote the quivers whose underlying graph is

where the graph has n vertices.

Theorem 2.33. *Any representation of a type $\mathbb{A}_n$ quiver is a direct sum of thin indecomposable representations.*

The converse of Theorem 2.33 is true as well.

Theorem 2.34. *Let Q be a finite connected quiver. If every indecomposable representation of the quiver Q is thin, then Q is a type $\mathbb{A}_n$ quiver.*

Let $\Gamma = (V, E, s, t)$ be a finite quiver and $k\Gamma$ be the path algebra of Γ. The **arrow ideal** R_Γ of $k\Gamma$ is the two-sided ideal generated by the arrows (i.e., path of length 1) of Γ. R_Γ has a natural $\mathbb{N}$-grading:

$$R_\Gamma \cong \bigoplus_{i=1}^{\infty} k\Gamma_i$$

where Γ_i is the set of paths of Γ which have length i. We use R_Γ^m to denote the subalgebra of R_Γ generated by all paths of length greater or equal to m:

$$R_\Gamma^m \cong \bigoplus_{i=m}^{\infty} k\Gamma_i$$

It is clear that $R_\Gamma \supseteq R_\Gamma^2 \supseteq R_\Gamma^3 \supseteq R_\Gamma^4 \supseteq \cdots \supseteq R_\Gamma^m \supseteq \cdots$

Definition 2.35. *A two-sided ideal I of $k\Gamma$ is an* ***admissible ideal*** *if there exists an integer $m \geq 1$ such that*

$$R_\Gamma^m \supseteq I \supseteq R_\Gamma^2$$

Example 19. Let $\Gamma = (V, E, s, t)$ be a finite quiver and $a, b, c, d \in E$. Let I_1 be the ideal of $k\Gamma$ generated by $ab - cd$ and I_2 be the ideal of $k\Gamma$ generated by $ab - c$. Then I_1 is an admissible ideal while I_2 is not. $\triangle$

Theorem 2.36 (Fitting's lemma)**.** *Let R be a ring and $M \in \mathsf{Mod}_R$ with finite length. Then for any $\phi \in \mathsf{End}(M)$, there exists a positive integer n such that*

$$M \cong \mathsf{ker}\phi^n \oplus \mathsf{Im}\,\phi^n$$

2.6 Configuration Spaces

Configuration space is one of the key notions used in robot motion planning. Given a motion planning problem, each point of the configuration space corresponds to a possible position where robots can appear without colliding with other robots. For a topological space X, the n**-th configuration space** of X is

$$X^{\underline{n}} = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\} = \{\mathbf{x} : \underline{n} \to X \mid \mathbf{x} \text{ is injective}, \underline{n} := \{1, \ldots, n\}\}$$

We interpret the n-th configuration space of a topological space X as n distinguished robots (treated as points) moving on X. When X is path-connected, the n-th configuration space $X^{\underline{n}}$ is path-connected if any pairs of robots can interchange positions on X.

Example 20. Let $X = [0,1]$ and $n = 2$. The second configuration space of X is:

$$[0,1]^{\underline{2}} = \{(x_1, x_2) \in [0,1] \times [0,1] \mid x_1 \neq x_2\}$$

It is clear that $[0,1]^{\underline{2}}$ is not path-connected and not compact. △

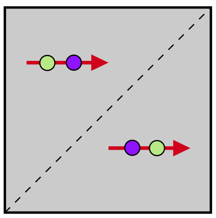


Figure 2.2: $[0,1]^{\underline{2}}$.

Example 21 ($Y^{\underline{2}}$)**.** Let Y denote the topological space of shape $Y = [0,1] \sqcup [0,1] \sqcup [0,1]/0 \sim 0 \sim 0$ (as a subspace of $\mathbb{R}^2$). The second configuration space of Y is:

We interpret $Y^{\underline{2}}$ as the configuration space of two robots moving on the shape Y and different regions of $Y^{\underline{2}}$ correspond to different positions of the robots on Y. For any $i \in$

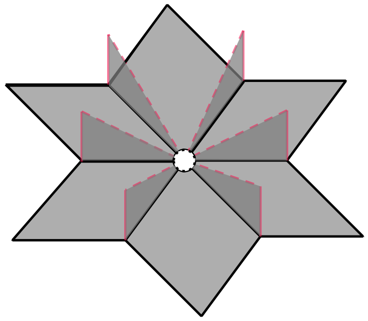


Figure 2.3: $Y^{\underline{2}}$: Second configuration space of Y (all edges have length 1)[1]

$\{A, B\}$, let x_i denote the position of robot i on edge 1 where the distance between the robot i and the center of Y is x_i. Similarly, for any $i \in \{A, B\}$, let y_i denote the position of robot i on edge 2 where the distance between the robot i and the center of Y is y_i, and let z_i denote the position of robot i on edge 3 where the distance between the robot i and the center of Y is z_i. With these notation, every point of $Y^{\underline{2}}$ can be written uniquely in terms of x_i, y_i and z_i (where $i \in \{A, B\}$):

$$(x_A, x_B, y_A, y_B, z_A, z_B)$$

Some symmetries exist on $Y^{\underline{2}}$. For example, in figure 2.4, any point in the yellow rectangle has a mirror image in the green rectangle about the common edge of the two rectangles. For simplicity, we assume edge 2 is the common edge of the yellow rectangle and the green rectangle, and robot B is moving on edge 2. Then any point in the yellow rectangle represents the case in which robot A is on edge 1 at position x_A and robot B is on edge 2, and the distance between robot B and the center of Y is y_B. In other words, every point in the yellow rectangle can be represented by $(x_A, 0, 0, y_B, 0, 0)$ and its mirror image is $(0, 0, 0, y_B, x_A, 0)$. Note that one obtains $(0, 0, 0, y_B, x_A, 0)$ from $(x_A, 0, 0, y_B, 0, 0)$ by moving the robot A from edge 1 to the same position on edge 3 while the position of the robot B stays the same.

In figure 2.5, any point in the yellow rectangle has a symmetrical point in the green rectangle from the center of Y. Any point $(x_A, 0, 0, y_B, 0, 0)$ in the yellow rectangle represents the case in which robot A is on edge 1 at position x_A and robot B is on edge 2 at position y_B. Note that $(0, y_B, x_A, 0, 0, 0)$ is the symmetrical point of $(x_A, 0, 0, y_B, 0, 0)$ and one get $(0, y_B, x_A, 0, 0, 0)$ from $(x_A, 0, 0, y_B, 0, 0)$ by exchanging the positions of robot A and robot B.

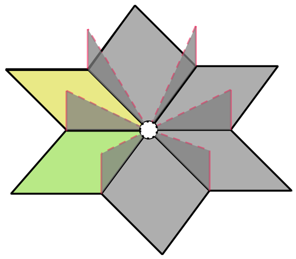


Figure 2.4

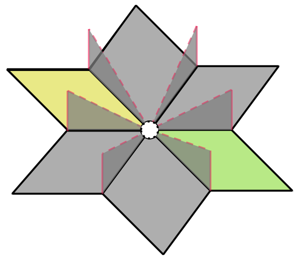


Figure 2.5

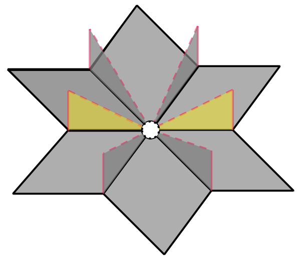


Figure 2.6

In figure 2.6, any point in the yellow triangle on the left has a symmetrical point in the yellow rectangle on the right from the center of Y. Because any point in the yellow triangle on the left represents the case in which robot A is on edge 2 at position y_A and robot B is on edge 2 at position y_B. Hence one obtains $(0, 0, y_B, y_A, 0, 0)$ from $(0, 0, y_A, y_B, 0, 0)$ in the configuration space $Y^{\underline{2}}$ by exchanging the positions of robot A and robot B.

It is clear that $Y^{\underline{2}}$ is path-connected, hence $[0,1]^{\underline{2}}$ is not homotopy equivalent to $Y^{\underline{2}}$: if $[0,1]^{\underline{2}}$ is homotopy equivalent to $Y^{\underline{2}}$, then $\pi_n([0,1]^{\underline{2}}) \cong \pi_n(Y^{\underline{2}})$ for any $n \in \mathbb{N}$. In particular, $\pi_0([0,1]^{\underline{2}}) \cong \pi_0(Y^{\underline{2}})$, which is a contradiction because $\pi_0([0,1]^{\underline{2}}) \cong \mathbb{Z} \oplus \mathbb{Z}$ while $\pi_0(Y^{\underline{2}}) \cong \mathbb{Z}$. Therefore, homotopy equivalent spaces do not always have homotopy equivalent configuration spaces. △

Here are some well-known facts about the n-th configuration spaces.

Proposition 2.37. If X is connected, Hausdorff and compact topological space, $X^{\underline{n}}$ (when $n \geq 2$) is not compact.

Proof. Since X is compact, Tychonoff's theorem implies X^n is compact. Note that X is Hausdorff, hence the diagonal of X is closed, i.e., $\Delta_{i<j} := \{(x_1, \ldots, x_n) \in X^n : x_i = x_j\}$ is

closed. Hence the finite union $\bigcup_{1\leq i<j\leq n} \Delta_{i\leq j}$ is closed. Because $X^{\underline{n}} = X^n - \Delta$, $X^{\underline{n}}$ is open. Assume $X^{\underline{n}}$ is compact. Since X is Hausdorff, so is X^n. Therefore, $X^{\underline{n}}$ is closed in X^n because $X^{\underline{n}}$ is a compact subspace of a Hausdorff space. Hence $X^{\underline{n}}$ is both open and closed. Note that X is connected, so is X^n. Therefore, $X^{\underline{n}} = X^n$. Contradiction. □

Proposition 2.38. If X is Hausdorff and not discrete, $X^{\underline{n}}$ (when $n \geq 2$) is not compact.

Proposition 2.39. Let X be a connected compact simplicial complex, $X^{\underline{n}}$ is connected unless

- $X \cong D^1$ and $n \geq 2$;
- $X \cong S^1$ and $n \geq 3$.

For a topological space X, there is a free group action on the n-th configuration spaces of X:

$$S_n \times X^{\underline{n}} \to X^{\underline{n}}$$

$$(\sigma, \mathbf{x}) \mapsto \mathbf{x} \circ \sigma$$

Since $\mathbf{x}$ is injective, S_n acts freely on $X^{\underline{n}}$. Denote $\binom{X}{n} := X^{\underline{n}}/S_n$ to the quotient space of $X^{\underline{n}}$ under the group action S_n.

Definition 2.40. $\binom{X}{n} := X^{\underline{n}}/S_n$ *is called the* n***-th unlabeled configuration space*** *of* X*.*

Similar to the labeled configuration space $X^{\underline{n}}$, we can interpret the n-th unlabeled configuration space of X as n robots (treat as points) moving on X as follows: given $2 \leq k \leq n$, let S_k denote the subgroup of S_n generated by $\{(1,2),(1,\ldots,k)\}$. Then $X^{\underline{n}}/S_k$ is the configuration space of n robots where the first k robots are indistinguishable. If X is Hausdorff, the canonical projection map $X^{\underline{n}} \to X^{\underline{n}}/S_n$ is a covering space, and S_n is the group of deck transformations of this covering space.

In the real-world scenario, however, there is a minimal distance allowed between each pair of robots because the robots are thick objects, and we cannot treat them as points. To measure the size of the robots, we need to define a metric on the graph. Deeley [25] introduced the notion of configuration spaces of thick particles on a graph X in which the graph is simple, and each edge of the graph gives the shortest distance between its endpoints. Given a graph X with a metric δ, the n-th thick particle configuration space with parameter r is

$$X_r^n = \{(x_1, \ldots, x_n) \in X^n \mid \delta(x_i, x_j) \geq r\}$$

He studied the second thick particle configuration space X_r^2 and gave an estimate on the homotopy types of X_r^2:

Theorem 2.41 (Deeley, 2011[25])**.** $\{X_r^2\}$ *has finitely many homotopy types, the number of which is bounded above by an exponential function in the number of edges.*

Dover and Özaydın[26] introduced the notion of configuration spaces with restraint parameters of finite metric graphs as a generalization of the configuration spaces of thick particles on a metric graph:

Definition 2.42. *Let $\Gamma = (V, E)$ be a finite connected graph and $X = |\Gamma|$ be the geometric realization of Γ. X is a* ***metric graph*** *if for each edge e, there is a positive number L_e such that each geometric edge is isometric to $[0, L_e]$. X has the path metric δ, i.e., for any $x, y \in X$,*

$$\delta(x, y) = \text{the length of a shortest path from } x \text{ to } y.$$

Note that in this definition, Γ is allowed to have multiple edges. Moreover, for the sake of simplicity, we subdivide each loop of Γ into two edges if Γ has loops. We are slightly abusing the notation by denoting the new graph by Γ and denoting the realization of such Γ by X. As a consequence, X becomes a regular CW complex.

Definition 2.43. *Let* (X, δ) *be a metric space and let* $\mathbf{r} = (r_{ij})_{i<j} \in \mathbb{R}_{\geq 0}^{\binom{n}{2}}$. *The* n***-th configuration space with restraint parameter*** $\mathbf{r} = (r_{ij})_{i<j}$ *is:*

$$X_{\mathbf{r}}^n = \{(x_1, \ldots, x_n) \in X^n \mid \delta(x_i, x_j) \geq r_{ij}\}$$

Proposition 2.44. Let X be a regular CW-complex, then X^n is a regular CW-complex.

Proof. Since the finite product of CW-complexes is a CW-complex ([29], Theorem A6) and X is a CW-complex, X^n is a CW-complex. □

Let (X, δ) be a metric graph. Maximal cells of a regular cell structure for $X_{\mathbf{r}}^n$ are obtained by intersecting $X_{\mathbf{r}}^n$ with the maximal cells of X^n. In other words, $X_{\mathbf{r}}^n$ is a regular CW-complex.

2.6.1 Parametric Polytopes and their Parameter Spaces

A parametric polytope $c_{\mathbf{b}}$ in $\mathbb{R}^n$ is determined by $Ax \leq \mathbf{b}$ where A is an $m \times n$ matrix and $\mathbf{b}$ is a $m \times 1$ column vector representing the constraints. Each feasible solution of $Ax \leq \mathbf{b}$ corresponds to a point of $c_{\mathbf{b}}$.

$c_{\mathbf{b}}$ is a CW-complex, and the cell structure is also encoded in $Ax \leq \mathbf{b}$. Let $\boldsymbol{\beta} = \{\lambda_1, \ldots, \lambda_n\}$. Then $A_{\boldsymbol{\beta}}$ is the submatrix of A with rows $\lambda \in \boldsymbol{\beta}$ and $\mathbf{b}_{\boldsymbol{\beta}} = (b_\lambda)_{\lambda \in \boldsymbol{\beta}}$. A potential 0-dimensional cell of $c_{\mathbf{b}}$ is given by $A_{\boldsymbol{\beta}}^{-1}\mathbf{b}_{\boldsymbol{\beta}}$ for some $\boldsymbol{\beta}$ (where $A_{\boldsymbol{\beta}}^{-1}$ exists) and such potential 0-dimensional cell is indeed an 0-dimensional cell of $c_{\mathbf{b}}$ if $A_{\boldsymbol{\beta}}^{-1}\mathbf{b}_{\boldsymbol{\beta}}$ is a solution of $Ax \leq \mathbf{b}$. We label each 0-cell by $S_{\mathbf{b}} := \{\beta \mid A_\beta \text{ is a solution of } Ax \leq \mathbf{b}\}$.

We assign a label $v_{\mathbf{b}}$ to each vertex of the parametric polytope where $v_{\mathbf{b}} = \{\lambda \mid A_\lambda A^{-1}\mathbf{b} = b_\lambda\}$. Let $T_{\mathbf{b}}$ denote the type of $c_{\mathbf{b}}$: $T_{\mathbf{b}} = \{v_{\mathbf{b}}(\beta) \mid \beta \in S_{\mathbf{b}}\}$ Dover and Özaydın [26] showed

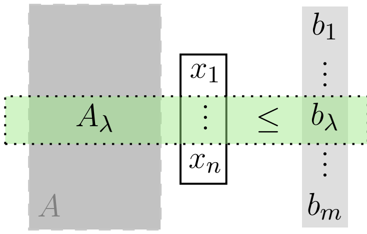


Figure 2.7: The λ-th constraint of the parametric polytope

the following lemmas:

Lemma 2.7. The family of polytopes $\{c_{\mathbf{b}}\}_{\mathbf{b}\in\mathbb{R}}$ has only finitely many combinatorial types.

Lemma 2.8. Let $\mathbf{b}' \in \mathbb{R}^n$. If $T_{\mathbf{b}'} \neq \emptyset$, then the set of all $\mathbf{b} \in \mathbb{R}^n$ with $T_{\mathbf{b}} = T_{\mathbf{b}'}$ is convex.

Using parametric polytopes, we characterize the cellular structure of $X^n_{\mathbf{r}}$. Each maximal cell of $X^n_{\mathbf{r}}$ can be characterized by a system of inequalities:

For $1 \le i \le n$:

$$0 \le x^i \le L_{e_i} \tag{2.3}$$

For $1 \le i < j \le n$:

$$\begin{aligned} x_i + x_j + \delta(a_i, a_j) &\ge r_{ij};\\ L_{e_i} - x_i + x_j + \delta(b_i, a_j) &\ge r_{ij};\\ x_i + L_{e_j} - x_j + \delta(a_i, b_j) &\ge r_{ij};\\ L_{e_i} - x_i + L_{e_j} - x_j + \delta b_i, b_j &\ge r_{ij} \end{aligned} \tag{2.4}$$

If there exists $1 \le i, j \le n$ such that $e_i = e_j$, with further assumption that $x_i \le x_j$,

$$\begin{aligned} x_i - x_j &\le 0;\\ x_i - x_j &\le -r_{ij};\\ L_{e_i} - x_i + x_j + \delta(a_i, b_i) &\ge r_{ij} \end{aligned} \tag{2.5}$$

For a metric graph (X,δ) and given $n,\mathbf{r}$, there is a natural correspondence between the dimensions of the cells of $X^n_{\mathbf{r}}$ and the positions of the robots on the graph X:

$$\begin{aligned}
0\text{-cell of } X^n_{\mathbf{r}} &\leftrightarrow \text{all } n \text{ robots are on different vertices}\\
1\text{-cell of } X^n_{\mathbf{r}} &\leftrightarrow n-1 \text{ robots are on different vertices}\\
&\ \text{and 1 robot on the interior of an edge of } X\\
2\text{-cell of } X^n_{\mathbf{r}} &\leftrightarrow n-2 \text{ robots are on different vertices}\\
&\ \text{and 2 robots on the interior of edges of } X\\
&\vdots\\
n\text{-cell of } X^n_{\mathbf{r}} &\leftrightarrow \text{all } n \text{ robots are on the interior of edges of } X
\end{aligned}$$

As a consequence, $X^n_{\mathbf{r}}$ is compact.

Dover and Özaydın studied the homotopy, homeomorphism, and isotopy types of $X^n_{\mathbf{r}}$ over the space of parameters r. They showed [26] that if $X^n_{\mathbf{r}}$ and $X^n_{\mathbf{s}}$ have the same combinatorial type (same face poset of cells), then $X^n_{\mathbf{r}}$ is isotopic to $X^n_{\mathbf{s}}$. In addition, Dover and Özaydın provided a polynomial upper bound (which depends on the number of the edges of X) for the number of isotopy types:

Theorem 2.45 (Dover-Özaydın, 2013[26])**.** *For a metric graph X with E edges, the number of homotopy types for the family of spaces $\{X^2_r\}_{r\geq 0}$ is bounded above by*

$$\frac{9}{2}E^2 - \frac{5}{2}E + 1$$

The number of isotopy types is bounded above by

$$9E^2 - 5E$$

Theorem 2.46 (Dover-Özaydın, 2013[26])**.** *Let n be a fixed positive integer and Ω be a d-dimensional affine subspace of $\mathbb{R}^{\binom{n}{2}}$. The number of isotopy types of $\{X^n_{\mathbf{r}}\}_{\mathbf{r}\in\Omega}$ is bounded above by a polynomial of degree nd in the number of edges of X.*

On the other hand, when the norm of parameter $\mathbf{r}$ is sufficiently small, $X^n_{\mathbf{r}}$ is homotopy equivalent to the n-th configuration space $X^{\underline{n}}$:

Proposition 2.47. Let $X = (V, E)$ be a connected regular metric graph and let L_e denote the edge length of $e \in E$. Define $a := \min_{e\in E} L_e$. If $||\mathbf{r}||_\infty < \dfrac{a}{n}$, then the inclusion $X^n_{\mathbf{r}} \hookrightarrow X^{\underline{n}}$ is a homotopy equivalence.

Proof. If $\mathbf{r} \leq \mathbf{s}$ and $||\mathbf{r}||_\infty \leq ||\mathbf{s}||_\infty < \dfrac{a}{n}$, then $\mathbf{r}$ and $\mathbf{s}$ belong to the same chamber in the parameter space. Note that each chamber is convex, the maps $X^n_{\mathbf{r}} \to X^n_{\mathbf{s}} \hookrightarrow X^n_{\mathbf{r}}$ and $X^n_{\mathbf{s}} \hookrightarrow X^n_{\mathbf{r}} \to X^n_{\mathbf{s}}$ are isotopic to the identity maps. Thus $\pi_k(X^n_{\mathbf{s}}) \to \pi_k(X^n_{\mathbf{r}})$ is an isomorphism for all $k \in \mathbb{N}$. Note that $\varinjlim X^n_{\mathbf{r}} \cong X^{\underline{n}}$, we update the colimit cone with the leg maps $i_{\mathbf{r}} : X^n_{\mathbf{r}} \hookrightarrow X^{\underline{n}}$, for all $\mathbf{r}$. Note that, by Theorem 2.18, $\varinjlim$ is exact, therefore,

$$\varinjlim \pi_k(X^n_{\mathbf{r}}) \cong \pi_k(\varinjlim X^n_{\mathbf{r}}) = \pi_k(X^{\underline{n}})$$

By Lemma 2.1, there exists $\mathbf{r}$ such that the induced map $\pi_k i_{\mathbf{r}} : \pi_k(X^n_{\mathbf{r}}) \to \pi_k(X^{\underline{n}})$ is an isomorphism. By Whitehead's theorem, we conclude that the inclusion $X^n_{\mathbf{r}} \hookrightarrow X^{\underline{n}}$ is a homotopy equivalence. $\square$

Chapter 3

Multiparameter Persistence Modules

In Section 3.1, we introduce the notions of P-indexed objects and persistent homology functors. In Section 3.2, we state the stability of persistence modules. In Section 3.3, we briefly discuss the Zigzag persistence modules. In Section 3.4, we show that $\mathbf{Vect}_{\mathbb{F}}^{\mathbf{C}}$ is equivalent to a full subcategory of $\mathbf{Mod}_{\mathbb{F}\mathbf{C}}$ when $\mathbf{C}$ be a small category and $\mathbb{F}$ be a field. Next, in Section 3.5, we discuss some important properties of thin polytope modules. Finally, we give a reduction algorithm for computing limits and colimits of diagrams in $\mathbb{R}^2$ with connected boundaries in Section 3.6.

3.1 Persistence Modules

Let $\mathbb{F}$ be a field and $(P, \leq)$ be a poset.

Let $\mathbf{C}$ be a category. $M : (P, \leq) \to \mathbf{C}$ is called a **(P-indexed) persistence object**. M is called a **(P-indexed) filtration** when $\mathbf{C} = \mathbf{Top}$. M is called a **(P-indexed) persistence module** when $\mathbf{C} = \mathbf{Vect}_{\mathbb{F}}$. In other words, the ($P$-indexed) persistence module M consists of a family of $\mathbb{F}$-vector spaces $\{M_t \mid t \in P\}$ and a doubly-indexed family of linear maps $\{\rho_{st} : M_s \to M_t \mid s \leq t\}$ where $\rho_{tu}\rho_{st} = \rho_{su}$ for any $s \leq t \leq u$ in P and $\rho_{ss} = \mathsf{id}_{M_s}$

for all $s \in P$. We use $\textsf{Vect}_{\mathbb{F}}^{(P,\leq)}$ to denote the category of persistence modules over the poset $(P, \leq)$. In this case, M can be viewed as a poset representation over the poset $(P, \leq)$. When $(P, \leq) = (\mathbb{R}^n, \leq)$ with the product order, where $(a_1, \dots, a_n) \leq (b_1, \dots, b_n) \in (\mathbb{R}^n, \leq)$ if and only if $a_i \leq b_i$ for all $i = 1, \dots, n$, the objects of $\textsf{Vect}_{\mathbb{F}}^{(\mathbb{R}^n,\leq)}$ is called **n-parameter persistence modules**. When $M(i)$ is a finite dimensional vector space for all $i \in (\mathbb{R}^n, \leq)$, M is an object of $\textsf{vect}_{\mathbb{F}}^{(\mathbb{R}^n,\leq)}$.

We can define a functor $PH_i(-) : \textsf{Top}^{(P,\leq)} \to \textsf{Ab}^{(P,\leq)}$ as follows:

- At the object level, $PH_i(-)$ sends each P-indexed filtration M to a P-indexed persistence abelian group $H_i(M)$, where $H_i(M) : (P, \leq) \to \textsf{Vect}_{\mathbb{F}}$ sends each $p \in P$ to $H_i(M_p)$ and each arrow $p \leq q$ to the group homomorphism $H_i(M_p) \to H_i(M_q)$ induced by the inclusion map $M_p \hookrightarrow M_q$ in $\textsf{Top}$;

- An the morphism level, $PH_i(-)$ sends each natural transformation $\alpha : M \Rightarrow N$ to $H_i(\alpha) : H_i(M) \Rightarrow H_i(N)$. The naturality of $H_i(\alpha)$ is clear because $H_i(-)$ is a functor hence it preserves commutative diagrams.

In other words, for each $M \in \textsf{ob}\,\textsf{Top}^{(P,\leq)}$, $PH_i(M)$ can be obtained by post-composing $H_i(-)$ with M. The functoriality of $PH_i(-)$ is clear.

Definition 3.1. *$PH_i(-) : \textsf{Top}^{(P,\leq)} \to \textsf{Ab}^{(P,\leq)}$ is called the **i-th persistent homology functor**.*

Definition 3.2. *Let $(P, \leq)$ be a poset and $\textsf{I} \subseteq (P, \leq)$ is an interval. Define the **interval module** $\mathbb{F}\textsf{I}$ as*

$$\mathbb{F}\textsf{I}_t = \begin{cases} \mathbb{F}, & \textit{if } t \in \textsf{I}; \\ 0, & \textit{if } t \notin \textsf{I} \end{cases} \quad \textit{and} \quad \textsf{Hom}(\mathbb{F}\textsf{I}_t, \mathbb{F}\textsf{I}_s) = \begin{cases} \{\textsf{id}_{\mathbb{F}}\}, & \textit{if } t \leq s \in \textsf{I}; \\ 0, & \textit{else} \end{cases}$$

When $(P, \leq) = (\mathbb{R}^n, \leq)$ *and* $n \geq 2$*, we call* $\mathbb{F}\mathbf{I}$ *the* ***polytope module*** *over* $\mathbf{I}$*.*

Definition 3.3. *Let* $M \in \mathbf{Vect}_{\mathbb{F}}^{(P,\leq)}$*.* M *is* ***decomposable*** *if there exists non-trivial subrepresentation* N *and* N' *such that* $M_t \cong N_t \oplus N'_t$ *for all* $t \in P$*. We say* M *is* ***indecomposable*** *if it is not decomposable.*

Lemma 3.1. Interval modules are thin and indecomposable.

Proof. Let M be an interval module. The support of M is an interval of $(P, \leq)$. Consider the endomorphism ring of M. We want to show $\mathsf{End}\,(M) \cong \mathbb{F}$. Let $f \in \mathsf{End}(M)$. Note that for every $p \in (P, \leq)$, $f_p : \mathbb{F} \to \mathbb{F}$ is a linear transformation, hence $f_p(x) = c_f x$ for some $c_f \in \mathbb{F}$. Let p' be a point in the support of M. Note that there exists a zigzag path from p to p' because the support of M is an interval. Since f is a morphism between two representations and M is a polytope module, we are forced to have $f_q(x) = c_f x$. Define $\Phi : \mathsf{End}(M) \to \mathbb{F}$ by $\Phi(f) = c_f$ for each $f \in \mathsf{End}(M)$. It is clear that Φ is bijective, so we only need to show Φ is a ring homomorphism. Note that $\Phi(f+g) = c_f + c_g = \Phi(f) + \Phi(g)$ and $\Phi(f \circ g) = c_f \cdot g = \Phi(f)\Phi(g)$, hence Φ is a ring homomorphism. $\square$

3.2 Stability Theorems of Persistence Modules

In Section 3.2.1 and Section 3.2.2, we review the definition and properties of the interleaving distances and Bottleneck distance between functors from a (locally finite) poset $(P, \leq)$ to a category $\mathbf{C}$ with nice properties (for example, $\mathbf{C}$ is a Krull-Remak-Schmidt category.) Note that the notions of interleaving distances and Bottleneck distance can be automatically applied on the category $\mathbf{vect}_{\mathbb{F}}{}^{(P,\leq)}$ because $\mathbf{vect}_{\mathbb{F}}$ is an abelian Krull-Remak-Schmidt category. In Section 3.2.3, we state the stability theorems of the 1-parameter persistence modules.

3.2.1 Interleaving distance

Let $(P, \leq)$ be a poset. A **translation functor** is an endofunctor

$$T : (P, \leq) \to (P, \leq)$$

consists of the following data:

- for each object $i \in (P, \leq)$, $i \mapsto T(i)$;
- for each morphism $i \leq j \in (P, \leq)$, $T(i) \leq T(j)$.

Example 22 (Translation functor on $(\mathbb{Z}, \leq)$)**.** The n-translation functor on $(\mathbb{Z}, \leq)$ is an endofunctor

$$[n] : (\mathbb{Z}, \leq) \to (\mathbb{Z}, \leq)$$

consists of the following data:

- for each object $i \in (\mathbb{Z}, \leq)$, $[n](i) = i + n$;
- for each morphism $i \leq j \in (\mathbb{Z}, \leq)$, $[n](i) = i + n \leq j + n = [n](j)$.

It is straightforward to verify $[n]$ is a functor. △

Remark. The collection of translation functors on a given poset $(P, \leq)$ has a monoidal structure with respect to compositions.

Let $(P, \leq, d)$ be a poset with a metric d and $T : (P, \leq) \to (P, \leq)$ is an n-translation functor. The **height** of T is

$$h(T) = \sup_{i \in (P, \leq)} \{d(i, T(i))\}$$

Definition 3.4 (T-Interleaved persistence modules)**.** *Let $M, N \in \mathbf{C}^{(P,\leq)}$ where $\mathbf{C}$ and $(P, \leq)$ are defined as above. Define $\eta : \mathsf{id} \Rightarrow T^2$. M, N are T-**Interleaved** if there exists natural transformations $\alpha : M \Rightarrow N \circ T$ and $\beta : N \Rightarrow M \circ T$ such that $\beta_{T(i)} \circ \alpha_i = M(\eta_i)$ and $\alpha_{T(i)} \circ \beta_i = M(\eta_i)$. The **interleaving distance** between M and N is*

$$d_I(M, N) = \inf\{\epsilon \mid M \textit{ and } N \textit{ are } T\textit{-interleaved and } h(T) = \epsilon.\}$$

A real-valued function $d : X \times X \to \mathbb{R}_{\geq 0}$ is a pseudometric if it satisfies the following conditions:

1. $d(x, x) \geq 0$ for all $x \in X$;

2. $d(x, y) = d(y, x)$ for all $x, y \in X$;

3. $d(x, y) + d(y, z) \geq d(x, z)$ $x, y, z \in X$.

In addition, if for all $x, y \in X$, $d(x, y) = 0$ implies $x = y$, then d is a metric.

Proposition 3.5. d_I is a pseudometric.

Proof. It is clear d_I is symmetric. For the triangle inequality, given $L, M, N \in \mathbf{C}^{(P,\leq)}$, define

$$A = \{\epsilon_1 \mid L \text{ and } M \text{ are } T\text{-interleaved and } h(T) = \epsilon_1.\}$$

$$B = \{\epsilon_2 \mid M \text{ and } N \text{ are } T\text{-interleaved and } h(T) = \epsilon_2.\}$$

Then

$$d_I(L, M) = \inf A \quad \text{and} \quad d_I(M, N) = \inf B$$

For all $\epsilon > 0$, $d_I(L, M) + \dfrac{\epsilon}{2}$ is not a lower bound of A. Hence there exists $T_1 : (P, \leq) \to (P, \leq)$ such that L and M are T_1-interleaved and $h(T_1) \leq d_I(L, M) + \dfrac{\epsilon}{2}$. Similarly,

$d_I(M,N) + \frac{\epsilon}{2}$ is not a lower bound of B implies there exists $T_2 : (P, \leq) \to (P, \leq)$ such that M and N are T_2-interleaved and $h(T_2) \leq d_I(M,N) + \frac{\epsilon}{2}$. Note that L and N are $T_2 \circ T_1$-interleaved, therefore,

$$h(T_2 \circ T_1) \leq h(T_1) + h(T_2) \leq d_I(L,M) + \frac{\epsilon}{2} + d_I(M,N) + \frac{\epsilon}{2}$$

Thus

$$d_I(L,N) \leq d_I(L,M) + d_I(M,N) + \epsilon$$

Becase ϵ is arbitrary, we conclude that

$$d_I(L,N) \leq d_I(L,M) + d_I(M,N)$$

$\square$

Example 23 (Interleaving between objects of $\mathbf{C}^{(\mathbb{R}^n,\leq)}$)**.** Let $(P, \leq)$ in the definition 3.4 be $(\mathbb{R}^n, \leq)$ and $\mathbf{C}$ be a category. For $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$, define $\vec{u} : (\mathbb{R}^n, \leq) \to (\mathbb{R}^n, \leq)$ where $\vec{u}(x_1, \ldots, x_n) = (x_1 + u_1, \ldots, x_n + u_n)$ for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then $M, N \in \mathbf{C}^{(\mathbb{R}^n,\leq)}$ are $\vec{u}$-interleaved iff $\alpha : M \Rightarrow N \circ \vec{u}$ and $\beta : N \Rightarrow M \circ \vec{u}$ such that $\beta_{\vec{u}(i)} \circ \alpha_i = M(\eta_i)$ and $\alpha_{\vec{u}(i)} \circ \beta_i = M(\eta_i)$. For $\epsilon > 0$ and $\vec{u} = (\epsilon, \ldots, \epsilon)$, in particular, we say M and N are ϵ-interleaved if they are $\vec{u}$-interleaved. $\triangle$

Let $I \in (\mathbb{R}^n, \leq)$ be an interval. The interval module $\mathbb{F}I$ is called **ϵ-trivial** if for any $(a_1, \ldots, a_n) \in I$, $(a_1 + \epsilon, \ldots, a_n + \epsilon)$ is not in I. Given $M \in \mathbf{vect}_{\mathbb{F}}^{(\mathbb{R}^n,\leq)}$, let $B(M)$ be the multiset of barcodes corresponds to M. Define $B(M)_\epsilon$ to be the multiset of barcodes that are not ϵ-trivial.

There is a natural partially order on the collection of translation functors:

Definition 3.6. *Let S, T be translation functors on $(P, \leq)$. $S \leq T$ if and only if $S(i) \leq T(i)$ for every $i \in (P, \leq)$.*

Remark. Assume M, N are T-interleaved. Then for any endofunctor $S \geq T$ on $(P, \leq)$, M, N are also S-interleaved.

Here is a useful fact about interleaving distance:

Proposition 3.7 (Proposition 3.6,[11])**.** Let $\mathbf{D}$ and $\mathbf{E}$ be two categories. Let $M, N \in \mathbf{D}^{(\mathbb{R},\leq)}$ and $H : \mathbf{D} \to \mathbf{E}$. If M and N are ϵ-interleaved, then so are HM and HN. Therefore,

$$d_I(HM, HN) \leq d_I(M, N)$$

More generally,

Proposition 3.8. Let $\mathbf{D}$ and $\mathbf{E}$ be two categories. Let $M, N \in \mathbf{D}^{(P,\leq)}$ and $H : \mathbf{D} \to \mathbf{E}$. If M and N are T-interleaved, then so are HM and HN. Therefore,

$$d_I(HM, HN) \leq d_I(M, N)$$

Proof. Since M and N are T-interleaved, there exists $\phi : M \Rightarrow N \circ T$, $\psi : N \Rightarrow M \circ T$ and $\eta_T : \mathsf{id}_{(P,\leq)} \Rightarrow T$ such that

$$M \circ \eta_{T^2} = (\psi T) \circ \phi \tag{3.1}$$

$$N \circ \eta_{T^2} = (\phi T) \circ \psi \tag{3.2}$$

By the functoriality of H, $H\phi : H \circ M \Rightarrow H \circ N \circ T$ and $H\psi : H \circ N \Rightarrow H \circ M \circ T$ are natural transformations. Now we are going to show H preserves equation (3.1) and equation

(3.2). Note that for each $i \in (P, \leq)$,

$$(H \circ M \circ \eta_{T^2})_i = H \circ M((\eta_{T^2})_i) = H \circ ((\psi T)_i \circ \phi_i) = ((H\psi T) \circ \phi)_i \tag{3.3}$$

Therefore, HM and HN are T-interleaved. $\square$

3.2.2 Bottleneck distance

For 1-parameter persistence modules, the bottleneck distance is a metric on the collection of persistence diagrams. Alternatively, the bottleneck distance evaluates the difference between the barcodes of two persistence modules. In this section we will recall the definition of bottleneck distance in different categories.

Let A, B be two multisets. A **matching** between A and B is a bijection σ from a subset A' of A to a subset B' of B. A' is called the **coimage** of σ and B' is called the **image** of σ. We denote such matching by $\sigma : A \rightleftharpoons B$.

Take $(P, \leq)$ to be $(\mathbb{R}^n, \leq)$, there is one special kind of matching that plays a significant role in the isometry theorem (see section 3.2.3).

Definition 3.9 (ϵ-matchings)**.** *An* ϵ***-matching*** *between mutilsets* A *and* B *is a matching* σ *such that:*

- $A_{2\epsilon} \subseteq coimage(\sigma)$;
- $B_{2\epsilon} \subseteq image(\sigma)$;
- *for any* $I \in A$ *and* $J \in B$ *such that* $J = \sigma(I)$*, the interval modules* $\mathbb{F}I$ *and* $\mathbb{F}J$ *are* ϵ*-interleaved.*

Definition 3.10 (Bottleneck distance[11])**.** *Let* A, B *be two multisets of barcodes. The* *bottleneck distance between A, B is defined by*

$$d_b(A,B) = \inf_{\sigma: A \rightleftharpoons B} \sup_{I \in \mathsf{coimage}(\sigma)} d_I(\mathbb{F}I, \mathbb{F}\sigma(I))$$

This definition is equivalent to the classic definition of bottleneck distance. For details, see proposition 4.12 and 4.13 of [11].

Let A, B be two multisets and $\sigma : A \rightleftharpoons B$ a matching. Let d be a metric on a multiset Σ containing A and B. Let $W : \Sigma \to [0, \infty)$ (treat the output as the 'width' of the input) such that

$$||W(I) - W(J)|| \leq d(I, J)$$

The **height** of σ is

$$h(\sigma) = \max\{\max_{I \in \mathsf{coimage}(\sigma)}\{d(I, \sigma(I))\}, \max_{I \notin \mathsf{coimage}(\sigma)}\{W(I)\}, \max_{J \notin \mathsf{image}(\sigma)}\{W(J)\}\}$$

Definition 3.11 (Generalized bottleneck distance)**.** *For Σ and W defined as above, the bottleneck distance d_b between two mutiset is*

$$d_b(A,B) = \min\{h(\sigma) \mid \sigma \textit{ is a matching between } A \textit{ and } B.\}$$

3.2.3 Stability theorems

Cohen-Steiner-Edelsbrunner-Harer proved the Bottleneck stability theorem for persistence diagrams [23]:

Theorem 3.12. *Let X be a triangulable space with continuous tame functions $f, g : X \to \mathbb{R}$. Let $D(f)$ ($D(g)$, resp.) be the set of intervals where the endpoints of each interval are critical* *point of f (g, resp.). Then the persistence diagrams satisfy*

$$||d_b(D(f), D(g))|| \leq ||f - g||_\infty$$

Bubenik-Scott generalized the above theorem[11]:

Theorem 3.13. *Let X be a topological space with two functions $f, g : X \to \mathbb{R}$. Let $F \in \mathbf{Top}^{(\mathbb{R},\leq)}$ be defined by $F(a) = f^{-1}(\infty, a]$ for $a \in \mathbb{R}$ and $F(a \leq b)$ is given by inclusion. Define G similarly. Let $H : \mathbf{Top} \to \mathbf{D}$. Then*

$$d_b(HF, HG) \leq ||f - g||_\infty$$

There is some connection between the bottleneck distance of the given barcodes and the interleaving distance between the corresponding persistence modules. Bauer-Lesnick showed the following theorem:

Theorem 3.14 (Algebraic stability theorem)**.** *[7] Let $M, N \in \mathbf{vect}_{\mathbb{F}}^{(\mathbb{N},\leq)}$ with finite support, then*

$$d_b(B(M), B(N)) \leq d_I(M, N)$$

Lesnick proved that d_b and d_I are 'equal'[35] (Theorem 3.4):

Theorem 3.15 (Isometry theorem)**.** *For any $\epsilon \geq 0$, pointwise-finite-dimensional persistence modules M and N are ϵ-interleaved iff there exists an ϵ-matching between the mutisets of barcodes $B(M)$ and $B(N)$. In particular,*

$$d_b(B(M), B(N)) = d_I(M, N)$$

3.3 First Example of Multipersistence Modules: Zigzag Persistence Modules

A **zigzag** is quiver $\mathfrak{Q}$ with alternated arrows

$$\bullet \leftrightarrow \bullet \leftrightarrow \cdots \leftrightarrow \bullet \leftrightarrow \bullet$$

where $\leftrightarrow$ means $\rightarrow$ or $\leftarrow$.

Definition 3.16. *Let $\mathbb{F}$ be a field. Let M denote a sequence (with length n) of finite-dimensional vector spaces over $\mathbb{F}$:*

$$M_1 \xleftrightarrow{f_{12}} M_2 \leftrightarrow \cdots \leftrightarrow M_{n-1} \xleftrightarrow{f_{n-1,n}} M_n$$

where

- $f_{i,i+1} : V_i \leftrightarrow V_{i+1}$ *means either* $V_i \rightarrow V_{i+1}$ *or* $V_i \leftarrow V_{i+1}$*;*
- $f_{i,i+1} V_i \leftrightarrow V_{i+1}$ *is a linear map for all* $i = 1, \ldots, n-1$*;*
- *if* $f_{i-1,i}$ *and* $f_{i,i+1}$ *is composable, then* $f_{i-1,i+1} = f_{i,i+1} \circ f_{i-1,i}$

Such M is called a ***zigzag persistence module****.*

Zigzag modules are 2-parameter persistence modules[1] because we can embed a zigzag $a_1 \leftrightarrow a_2 \leftrightarrow \cdots \leftrightarrow a_m$ into $\mathbb{Z}^2$ where $a_i \in (\mathbb{Z}^{\mathsf{op}} \times \mathbb{Z}, \leq)$ and $m \in \mathbb{N}$. Unlike the general 2-parameter persistence modules which don't have a classification of indecomposables, the indecomposable representations of a zigzag module are fairly simple to classify: Any Zigzag persistence module can be decomposed into intervals because its underlying quiver $\mathfrak{Q}$ is of type $\mathbb{A}_n$.

[1] they are objects of $\mathbf{vect}_{\mathbb{F}}^{(\mathbb{Z}^2, \leq)}$.

Theorem 3.17 (Gabriel)**.** *A connected quiver is of finite type if and only if its underlying graph is one of the Dykin diagrams:*

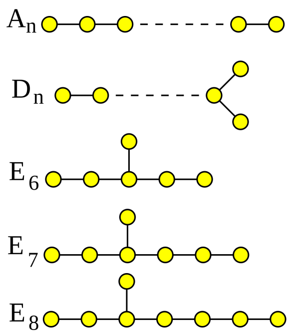


Figure 3.1: Simply laced Dynkin diagrams (Figure credit to https://en.wikipedia.org/wiki/ADE_classification)

By Gabriel's theorem, any $\mathbb{A}_n$ quiver is of finite type, i.e., there are only finitely many indecomposable representations of $\mathfrak{Q}$ up to isomorphism. Therefore, every finite-dimensional zigzag module is a direct sum of interval modules. Botman shows that the statement is also true for pointwise finite-dimensional zigzag modules over an infinite zigzag.

Theorem 3.18. *[8] Let V be a pointwise finite-dimensional zigzag module of type $\mathbb{A}_\infty$. There exists a multiset of intervals $\mathcal{F}$ such that $V \cong \bigoplus_{\mathbf{I} \in \mathcal{F}} \mathbb{F}\mathbf{I}$.*

3.3.1 Zigzag Persistence Modules vs Persistence Modules

Definition 3.19. *Let M be a zigzag module and let $M\big|_{[p,q]}$ be the restriction of M to an interval $[p,q]$. A feature of M over $[p,q]$ is a direct summand of $M\big|_{[p,q]}$ which is isomorphic to the interval module $\mathbb{F}[p,q]$.*

Carlsson-de Silva pointed out [15] that the features of a zigzag persistence module M are submodules of M, while for the 1-parameter persistence module, the features of M is also equivalent to the direct summands of M.

Proposition 3.20 ([15], proposition 2.8)**.** Let M be a persistence module of length n, and let $1 \leq p \leq q \leq n$. The following are equivalent:

1. The linear map $M_p \to M_q$ is nonzero;

2. There exist nonzero elements $x_i \in M_i$ for $p \leq i \leq q$ such that $x_{i+1} = M_{i \leq i+1}(x_i)$;

3. There exists a submodule of $M\big|_{[p,q]}$ which is isomorphic to $\mathbb{F}[p,q]$;

4. There exists a direct summand of $M\big|_{[p,q]}$ which is isomorphic to $\mathbb{F}[p,q]$.

Example 24. Consider a zigzag persistence module M:

$$\mathbb{F} \longleftarrow \mathbb{F}^2 \longrightarrow \mathbb{F}$$

$$x \hookleftarrow (x,y) \mapsto y$$

Note that $M \cong N \oplus N'$ where N is

$$\mathbb{F} \longleftarrow \mathbb{F} \longrightarrow 0$$

$$x \hookleftarrow x \mapsto 0$$

and N' is

$$0 \longleftarrow \mathbb{F} \longrightarrow \mathbb{F}$$

$$0 \hookleftarrow y \mapsto y$$

Both N and N' are interval modules hence they are indecomposable. However, there exists another submodule of M which is isomorphic to an interval module:

$$\mathbb{F} \longleftarrow \Delta \longrightarrow \mathbb{F}$$

$$x \leftarrow\!\shortmid (x, x) \mapsto x$$

where Δ is the diagonal of $\mathbb{F}^2$. Note that this submodule is not a direct summand of M. $\triangle$

3.4 Equivalences of Categories

It is known that the category of 1-parameter persistence modules $\mathbf{Vect}_{\mathbb{F}}^{(\mathbb{R},\leq)}$ is equivalent to the category of modules over the monoid ring generated by x^α, where $\alpha \in \mathbb{R}$. In particular, $\mathbf{Vect}_{\mathbb{F}}^{(\mathbb{Z},\leq)}$ is equivalent to the category of the modules over the polynomial ring $\mathbb{F}[x]$.

This fact can be generalized as follows: Let $\mathbf{C}$ be a small category and R be a commutative ring. Define

$$R\mathbf{C} = \{\sum a_i f_i \mid a_i \in R, a_i \neq 0 \text{ for finitely many } i,\ f_i \in \mathsf{mor}\,\mathbf{C}\}$$

In other words, $R\mathbf{C}$, as a set, consists of all formal linear combinations of the form $\sum_i a_i f_i$. Equip $R\mathbf{C}$ the binary operation: for $f, g \in \mathsf{mor}\,\mathbf{C}$,

$$f \cdot g = \begin{cases} f \circ g, & \text{if the range of } g \text{ is the domain of } f; \\ 0, & \text{else} \end{cases}$$

Extend it linearly, we obtain

$$\sum_i a_i f_i \cdot \sum_j b_j g_j = \sum_{i,j} (a_i b_j) f_i \cdot g_j$$

On the other hand, for any $c \in R$ and $\sum_i a_i f_i \in R\mathbf{C}$, define

$$c \cdot \sum_i a_i f_i = \sum_i (c a_i) f_i$$

$R\mathbf{C}$ with the above binary operation and scalar multiplication is called the **category algebra** of $\mathbf{C}$.

Proposition 3.21. Let $\mathbf{C}$ be a small category and $\mathbb{F}$ be a field. Then there exists a fully faithful functor

$$\mathbf{Vect}_{\mathbb{F}}^{\mathbf{C}} \to \mathbf{Mod}_{\mathbb{F}\mathbf{C}}$$

Proof. Define $\Phi : \mathbf{Vect}_{\mathbb{F}}^{\mathbf{C}} \to \mathbf{Mod}_{\mathbb{F}\mathbf{C}}$ as follows:

At the object level Given $M \in \mathbf{Vect}_{\mathbb{F}}^{\mathbf{C}}$, define $\Phi(M) = \bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Mc$. Note that $\Phi(M)$ is well-defined as a vector space[2]. Define an $\mathbb{F}\mathbf{C}$-action on $\bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Mc$: it suffices to define the $\mathbb{F}\mathbf{C}$-action coordinate-wise, then extend it linearly.

Given $c, d \in \mathsf{ob}\,\mathbf{C}$ and $f \in \mathsf{Hom}_{\mathbf{C}}(c,d)$, define $f_c = \mathsf{inc}_d \circ Mf \circ \mathsf{proj}_c$, i.e., f_c is the composition of the morphisms in the diagram:

$$\bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Mc \xrightarrow{\mathsf{proj}_c} Mc \xrightarrow{M(c \xrightarrow{f} d)} M_d \xrightarrow{inc_d} \bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Mc$$

For any $x \in \bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Mc$ and $f \in \mathsf{Hom}_{\mathbf{C}}(c,d)$, define

$$f \cdot x = f_c(x)$$

This construction gives a ring action on $\Phi(M)$: given $f \in \mathsf{Hom}_{\mathbf{C}}(c,d)$, $g \in \mathsf{Hom}_{\mathbf{C}}(d,e)$,

[2]Because $\mathbf{Vect}_{\mathbb{F}}$ is complete and cocomplete.

and $x \in \Phi(M)$,

$$\begin{aligned} g(f(x)) &= g(f_c(x)) = g_d(f_c(x)) \\ &= \mathsf{inc}_e \circ Mg \circ \mathsf{proj}_d \circ \mathsf{inc}_d \circ Mf \circ \mathsf{proj}_c \\ &= \mathsf{inc}_e \circ Mg \circ (\mathsf{proj}_d \circ \mathsf{inc}_d) \circ Mf \circ \mathsf{proj}_c \\ &= \mathsf{inc}_e \circ Mg \circ Mf \circ \mathsf{proj}_c \\ &= \mathsf{inc}_e \circ M(g \circ f) \circ \mathsf{proj}_c \\ &= (g \circ f) \cdot x \end{aligned} \tag{3.4}$$

At the morphism level Let $\alpha : M \Rightarrow N$ be a morphism in $\mathbf{Vect}_{\mathbb{F}}^{\mathbf{C}}$. Let $\Phi(\alpha)$ denote the yet-to-be-defined morphism in $\mathbf{Mod}_{\mathbb{F}\mathbf{C}}$ corresponding to α. As a morphism in $\mathbf{Vect}_{\mathbb{F}}$, we are forced to define $\Phi(\alpha) = (\alpha_c)_{c \in \mathsf{ob}\,\mathbf{C}}$. Note that $\Phi(\alpha)$ is an $\mathbb{F}\mathbf{C}$-morphism: it suffices to show that for every $f \in \mathsf{mor}\,\mathbf{C}$, $\Phi(\alpha)(f \cdot x) = f \cdot \Phi(\alpha)(x)$ for all $x \in \bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Mc$. By the naturality of α, we have

$$\mathcal{N}f \circ \alpha_c = \alpha_d \circ Mf, \text{ if the source of } f \text{ is } c \text{ and the target of } f \text{ is } d$$

Hence

$$\begin{aligned} f \cdot \Phi(\alpha)(x) &= \mathsf{inc}_d \circ Nf \circ \mathsf{proj}_c \circ \Phi(\alpha)(x) \\ &= \mathsf{inc}_d \circ Nf \circ \alpha_c \circ \mathsf{proj}_c(x) \\ &= \mathsf{inc}_d \circ \alpha_d \circ Mf \circ \mathsf{proj}_c(x) \\ &= \Phi(\alpha) \circ \mathsf{inc}_d \circ Mf \circ \mathsf{proj}_c(x) \\ &= \Phi(\alpha)(f \cdot x) \end{aligned} \tag{3.5}$$

Now we are going to show

$$\phi : \mathsf{Hom}(M, N) \to \mathsf{Hom}(\bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Mc, \bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Nc)$$
$$\alpha \mapsto \Phi(\alpha) = (\alpha_c)_{c \in \mathsf{ob}\,\mathbf{C}}$$

is a bijection for every $M, N \in \mathbf{Vect}_{\mathbb{F}}^{\mathbf{C}}$. The well-definedness and injectivity of ϕ is clear by the universal property of the coproduct. Now we are going to show that ϕ is surjective. Given $(f_c)_{c \in \mathsf{ob}\,\mathbf{C}} : \bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Mc \to \bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Nc$, define $\alpha : M \Rightarrow N$ by $\alpha_c = f_c$ for all $c \in \mathsf{ob}\,\mathbf{C}$. Consider the following diagram:

$$\begin{tikzcd}
Mc \arrow[r, "\mathsf{inc}_c"] \arrow[d, "f_c"] & \bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Mc \arrow[r, "f\cdot"] \arrow[d, "(f_c)_{c \in \mathsf{ob}\,\mathbf{C}}"] & \bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Mc \arrow[r, "\mathsf{proj}_d"] \arrow[d, "(f_c)_{c \in \mathsf{ob}\,\mathbf{C}}"] & Md \arrow[d, "f_d"] \\
Nc \arrow[r, "\mathsf{inc}_c"] & \bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Nc \arrow[r, "f\cdot"] & \bigoplus_{c \in \mathsf{ob}\,\mathbf{C}} Nc \arrow[r, "\mathsf{proj}_d"] & Nd
\end{tikzcd}$$

Note that each small square of the above diagram is commutative:

- the square on the left is commutative by the universal property of coproducts,
- the square on the right is commutative since the horizontal arrows are projection maps,
- the square in the middle is commutative since $(f_c)_{c \in \mathsf{ob}\,\mathbf{C}}$ is an $\mathbb{F}\mathbf{C}$-homomorphism.

Therefore, the outer square is commutative. Note that the composition of the morphisms in the top row is Mf because

$$\begin{aligned}
\mathsf{proj}_d \circ f \circ \mathsf{inc}_c(y) &= \mathsf{proj}_d \circ f_c(\mathsf{inc}_c(y)) \\
&= \mathsf{proj}_d \circ \mathsf{inc}_d \circ Mf \circ \mathsf{proj}_c \circ \mathsf{inc}_c(y) \qquad (3.6) \\
&= Mf(y)
\end{aligned}$$

Similarly, the composition of the morphisms in the bottom row is Nf. Hence we obtain the following commutative diagram for all $g \in G$ and $s \in S$:

$$\begin{array}{ccc} Mc & \xrightarrow{Mf} & Md \\ \Big\downarrow{\scriptstyle f_c} & & \Big\downarrow{\scriptstyle f_d} \\ Nc & \xrightarrow{Nf} & Nd \end{array}$$

Therefore, α is a natural transformation. $\square$

As a consequence, $\mathbf{Vect}_{\mathbb{F}}^{\mathbf{C}}$ is equivalent to a full subcategory of $\mathbf{Mod}_{\mathbb{F}\mathbf{C}}$.

3.5 Thin Polycode Modules

Let $(P, \leq)$ be a connected poset. Recall that a representation $M \in \mathbf{vect}_{\mathbb{F}}^{(P,\leq)}$ is called a thin representation, if M_i is either $\mathbb{F}$ or 0 for each $i \in (P, \leq)$. We have seen in Lemma 3.1 that Polytope modules are thin and indecomposable. The next theorem show that the converse of Lemma 3.1 is also true (up to isomorphism) when $(P, \leq) = (\mathbb{R}^2, \leq)$.

Lemma 3.2. Every indecomposable thin persistence module $M \in \mathbf{vect}_{\mathbb{F}}^{(\mathbb{R}^2,\leq)}$ has connected support.

Proof. Let $(P, \leq)$ be the support of M (in other words, $M_i \neq 0$ for all $i \in P$ and $M(i \leq j) \neq 0$ for $i \leq j \in P$). We first claim that P is connected: If P is not connected, then there exists $i, j \in P$ where there is no zigzag path between i and j. Define

$$S = \{x \in P : x \leq i \text{ or } x \geq i\}$$

and

$$T = S^c$$

Note that $S \cap T = \emptyset$ we have $M \cong N \oplus N'$ where

$$N_i = \begin{cases} M_i, & \text{if } i \in S \\ 0, & \text{else} \end{cases} \qquad N(i \leq j) = \begin{cases} M(i \leq j), & \text{if } i \leq j \in S \\ 0, & \text{else} \end{cases}$$

and

$$N'_i = \begin{cases} M_i, & \text{if } i \in T \\ 0, & \text{else} \end{cases} \qquad N'(i \leq j) = \begin{cases} M(i \leq j), & \text{if } i \leq j \in T \\ 0, & \text{else} \end{cases}$$

Contradiction. Therefore, $(P, \leq)$ is connect. □

Lemma 3.3. Let $M \in \mathbf{Vect}_{\mathbb{F}}^{(\mathbb{R}^2, \leq)}$ be a thin persistence module and let P be the support of M. Then for any $a, b \in P$ such that $a < b$ in $\mathbb{R}^2$, if there exists a zigzag path from a to b in P then $M(a < b) : M_a \to M_b$ is not 0.

Proof. Since $(\mathbb{R}^2, \leq)$ is a thin category, we can assume such zigzag path in P consists of horizontal and vertical arrows. We say an arrow on zigzag path is a good arrow if the orientation of the arrow is coincide with the orientation of the zigzag path; otherwise we say the arrow is a bad arrow. WLOG assuming such zigzag path is reduced: the zigzag path doesn't have two (or more) consecutive horizontal or vertical arrows, i.e., we either

- combine the two (or more) consecutive horizontal or vertical arrows if they are all good arrows or all bad arrows;
- get a new (shorter) arrow from combining a good horizontal (vertical, resp) arrow with a bad horizontal (vertical, resp) arrow.

  - The new arrow is good if the length of the original good arrow is strictly greater than the length of the original bad arrow;
  - The new arrow is bad if the length of the original good arrow is strictly less than the length of the original bad arrow;
  - The new arrow is a vertex if the length of the original good arrow is equal to the length of the original bad arrow.

Induction on the length of the zigzag path.

- length$= 1$. There is nothing to show.
- length$= N \to N+1$.
  - If the zigzag path has at least one self-intersection (denote a self-intersection by c) then we can write the zigzag path as follows

$$a - x_1 - \cdots - x_m - c - y_1 - \cdots - y_n - c - z_1 - \cdots z_{N-m-n-2} - b$$

    where $n \geq 1$. (Note that if $c = a$ or $c = b$, the induction hypothesis strikes.) Therefore,

$$a - x_1 - \cdots - x_m - c - z_1 - \cdots z_{N-m-n-2} - b$$

    is a zigzag path from a to b with length at most N. By induction hypothesis, $M(a < b) \neq 0$.
  - Now we assume the zigzag path has no self-intersection.
    * If the zigzag path consists of good arrows, then $M(a < b) \neq 0$;
    * If we have a bad (B) arrow on the zigzag path, then
      · there exists two consecutive good (G) arrows adjacent to the bad (B) arrow, i.e., BGG or GGB;

or

- there exists two consecutive bad (B) arrows adjacent to the good (G) arrow, i.e., GBB or BBG

Otherwise,

- the zigzag path consists of bad arrows;

  or

- good arrow and bad arrow alternate on the zigzag path.

contradicting to the assumption that $a < b$.

Because $(\mathbb{R}^2, \leq)$ is thin, we can substitute BBG/GBB/GGB/BGG with two new arrows (may be degenerate). Therefore, the length of the new zigzag path is at most N. By the induction hypothesis, $M(a < b) \neq 0$.

In conclusion, if there exists a zigzag path from $a < b$ in P then $M(a < b) \neq 0$. □

Definition 3.22 (weight of a zigzag path)**.** *Let p be a zigzag path in $(\mathbb{R}^2)$ and $M \in \mathbf{Vect}_{\mathbb{F}}^{(\mathbb{R}^2,\leq)}$ be a thin persistence module. The weight of p is*

$$\mathsf{weight}(p) = \prod_{i \to j \textit{ is a good arrow of } p} M(i \leq j) \cdot \prod_{k \to l \textit{ is a bad arrow of } p} M(k \leq l)^{-1}$$

Corollary 3.23. *Let $M \in \mathbf{Vect}_{\mathbb{F}}^{(\mathbb{R}^2,\leq)}$ be a thin persistence module and let P be the support of M. Then for any $a, b \in P$ such that $a < b$ in $\mathbb{R}^2$, if there exists a zigzag path from a to b in P with weight weight ω, then $M(a < b) : M_a \to M_b$ is the scalar multiplication by ω.*

Proof. Let p be a (reduced) zigzag path from a to b with weight ω. Say $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Define

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 \mid a_1 \leq x \leq b_1\}$$

$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 \mid a_2 \leq y \leq b_2\}$$

$$\Omega = \Omega_1 \cup \Omega_2$$

Refining p as follows:

- if an arrow across the boundary of Ω, we add a vertex at the intersection, subdividing the arrow into two subarrows. If the original arrow is good (bad, resp), then the subarrows are good (bad, resp).

Denote the refined zigzag path of p by $\widetilde{p}$. Note that by the above construction, $\mathsf{weight}(\widetilde{p}) = \mathsf{weight}(p)$.

Induction on the length of $\widetilde{p}$.

- length$= 1, 2$. There is nothing to show.
- length$= N \to N + 1$
  - If there exists an arrow $c \to d$ on $\widetilde{p}$ which is in the convex hull of $a \to b$ (i.e., $\Omega_1 \cap \Omega_2$). Note that $a \leq c \leq d \leq b$. By the induction hypothesis, $\omega = S_{ac} \cdot M(c < d) \cdot S_{db}$, where S_{ac} is the weight of a staircase path from a to c and S_{db} is the weight of a staircase path from d to b.
  - Assume there is no arrow of $\widetilde{p}$ that is in the convex hull of $a \to b$. WLOG we assume $a \to b$ is not vertical. Then there exists a good (G) horizontal arrow $c \to d$ (say $c = (c_1, c_2)$ and $d = (d_1, d_2)$) on $\widetilde{p}$ such that $c_1 = a_1$.
    * If $c_2 = d_2 > b_2$: note that the length of the zigzag path from a to c (on $\widetilde{p}$) is at most N, by the induction hypothesis, the weight of such path equals to the weight of S_{ac} (the staircase from a to c). Note that $a \leq c$ and $a_1 = c_1$, therefore, S_{ac} is the vertical arrow $a \to c$. In addition, $a \to c$ is a good (G)

arrow. Note that there exists a unique square in $\mathbb{R}^2$ such that $a \to c$ and $c \to d$ are two sides of the square. Let e denote the vertex of this square along with a, c, d. Because this square is commutative, the weight of $a \to c \to d$ equals to the weight of $a \to e \to d$. Note that $e \leq b$ and $[e, d) - d \rightsquigarrow b$ (where $d \rightsquigarrow b$ is in $\widetilde{p}$) is a zigzag path from e to b with length at most N. By the induction hypothesis, the weight of such path equals to the weight of S_{eb} (the staircase from e to b). Note that $a \to e$ is a good (G) arrow, therefore, $[a, e) - S_{eb}$ is a staircase from a to b. Denote this staircase by S_{ab}. Hence $\omega = \mathsf{weight}(S_{ab})$.

∗ If $c_2 = d_2 < a_2$: note that the length of the zigzag path from a to c (on $\widetilde{p}$) is at most N, by the induction hypothesis, the weight of such path equals to the inverse of the weight of S_{ca} (S_{ca} denotes the staircase from c to a). Note that $a \leq c$ and $a_1 = c_1$, therefore, S_{ca} is the vertical arrow $c \to a$. Let e be the intersection of the line $x = d_2$ and the arrow $a \to b$. Note that there exists a unique square in $\mathbb{R}^2$ such that $c \to d$ and $d \to e$ are two sides of the square. Let f denote the vertex of this square along with a, d, e. Because this square is commutative, the weight of $c \to d \to e$ equals to the weight of $c \to f \to e$. In addition,

$$\begin{aligned} \mathsf{weight}(a \to f \to e) &= \mathsf{weight}(c \to a)^{-1}\mathsf{weight}(c \to f \to e) \\ &= \mathsf{weight}(c \to a)^{-1}\mathsf{weight}(c \to d \to e) \end{aligned} \tag{3.7}$$

Note that $e \leq b$ and $[e, d) - d \rightsquigarrow b$ (where $d \rightsquigarrow b$ is in $\widetilde{p}$) is a zigzag path from e to b with length at most N. By the induction hypothesis, the weight of such path equals to the weight of S_{eb} (the staircase from e to b). Hence

$$\begin{aligned} \omega &= \mathsf{weight}(c \to a)^{-1} \cdot \mathsf{weight}(c \to d \to e) \cdot \mathsf{weight}((d \to e)^{-1} \rightsquigarrow b) \\ &= \mathsf{weight}(a \to f \to e) \cdot \mathsf{weight}(S_{eb}) \end{aligned} \tag{3.8}$$

Note that $S_{a,b} := a \to f \to e - S_{eb}$ is a staircase from a to b. Hence $\omega = \mathsf{weight}(S_{ab}) = M(a < b)$.

In conclusion, for any $a, b \in P$ such that $a < b$ in $\mathbb{R}^2$, if there exists a zigzag path p from a to b in P with weight weight ω, then $M(a < b) : M_a \to M_b$ is the scalar multiplication by ω. □

Theorem 3.24. *Every indecomposable thin persistence module $M \in \mathbf{vect}_{\mathbb{F}}^{(\mathbb{R}^2,\leq)}$ is isomorphic to a polytope module.*

Proof. Let $(P, \leq)$ be the support of M. Lemma 3.2 implies that $(P, \leq)$ is connected. Now we assume $(P, \leq)$ is not convex. Hence there exists $a \leq c \leq b \in \mathbb{Z}^2$ such that $a, b \in P$ and $c \notin P$. Therefore, $M_c = 0$. Corollary 3.23 implies that $M(a \leq b) = \omega \neq 0$ for some $\omega \in \mathbb{R}$. Since M is a persistence module, we have

$$M(a \leq b) = M(c \leq b) \circ M(a \leq c) \tag{3.9}$$

The righthand side of Equation (3.9) inicates $M(a \leq b)$ factor through $M_c = 0$, therefore, $\omega = 0$, contradiction. Hence $(P, \leq)$ is convex.

Now we construct the morphism between M and $\mathbb{F}P$. Fix $a_0 \in P$, define $\alpha : M \Rightarrow \mathbb{F}P$:

for all $b \in P$ and $x \in Mb$,

$$\alpha_b(x) = \begin{cases} \mathsf{weight}(b \to a)x, & \text{if } b \in P \\ 0, & \text{else} \end{cases}$$

Corollary 3.23 ensures α is well-defined. It is clear that α is a natural transformation, and it is a natural isomorphism because α_b is invertible for all $b \in P$, where $\alpha_b^{-1}(x)\mathsf{weight}(a \to b)x$. The weight of the trivial path has to be 1 because $M(c \leq c) = \mathsf{id}$ for all $c \in P$. □

3.6 Reduction for Computing Rank Invariants

Let $(P, \leq)$ be a poset. Recall that a subset U of $(P, \leq)$ is called an upset if for any $x \in U$, $x \leq y$ implies $y \in U$ for any $y \in (P, \leq)$. Dually, a subset D of $(P, \leq)$ is called an downset if for any $x \in D$, $w \leq x$ implies $w \in D$ for any $w \in (P, \leq)$. Given an upset U (as a subposet of $(P, \leq)$), we call U a principal upset if there exists $a \in P$ such that $a \leq x$ implies $x \in U$ for any $x \in (P, \leq)$. Dually, given an upset D (as a subposet of $(P, \leq)$), we call D a principal downset if there exists $b \in P$ such that $y \leq b$ implies $y \in D$ for any $y \in (P, \leq)$.

When $(P, \leq) = (\mathbb{R}^2, \leq)$, a subposet Q of $(\mathbb{R}^2, \leq)$ is called bounded if there exists a principal upset U and a principal downset D such that $Q \subseteq U \cap D$.

Proposition 3.25. Let C be a complete category and $P \subseteq (\mathbb{R}^2, \leq)$ be a bounded poset and the boundary of the upset (denoted by ∂P^+) is connected. Then for any $M \in \mathsf{C}^P$, $\mathsf{lim} M = \mathsf{lim} M\big|_{\partial P^+}$.

Proof. It is clear that there exists a unique morphism $\phi : \mathsf{lim} M \to \mathsf{lim} M\big|_{\partial P^+}$ because the restriction of the limit cone $\mathsf{lim} M \Rightarrow M$ is a cone over ∂P^+. Now we show there exists a unique morphism $\psi : \mathsf{lim} M\Big|_{\partial P^+} \to \mathsf{lim} M$. Let $\sigma : \mathsf{lim} M\Big|_{\partial P^+} \Rightarrow M\Big|_{\partial P^+}$ denote the limit cone and σ_i denote the leg of σ for each $i \in \partial P^+$. Given $p \in P$, there is $i \in \min \partial P^+$ ($\min \partial P^+$ is a subset of ∂P^+ consists of all minimal elements of ∂P^+) such that $i \leq p$. Define $\sigma_p = M(i \leq p) \circ \sigma_i$. We are going to show σ_p is well-defined, i.e., if there exists another $j \in \min \partial P^+$ such that $j \leq p$, then $M(i \leq p) \circ \sigma_i = M(j \leq p) \circ \sigma_j$.

If $i \vee j \in \partial P^+$: it suffices to show $M(i \leq p) \circ \sigma_i = M(i \vee j \leq p) \circ \sigma_{i \vee j} = M(j \leq p) \circ \sigma_j$. Note that

$$M(i \leq p) \circ \sigma_i = M(i \vee j \leq p) \circ M(i \leq i \vee j) \circ \sigma_i = M(i \vee j \leq p) \circ \sigma_{i \vee j}$$

and

$$M(j \leq p) \circ \sigma_j = M(i \vee j \leq p) \circ M(j \leq i \vee j) \circ \sigma_j = M(i \vee j \leq p) \circ \sigma_{i \vee j}$$

therefore, $M(i \leq p) \circ \sigma_i = M(j \leq p) \circ \sigma_j$.

If $i \vee j \notin \partial P^+$: WLOG we define $\sigma_{i \vee j} = M(i \leq i \vee j) \circ \sigma_i$ and we assume the x-value of i is less than j. . It suffices to show $\sigma_{i \vee j}$ is well-defined. Induction on the number of the minimum elements of intermediate steps between i and j. $n = 0$ is clear by the previous case. Assume the statement is true when $n = N$. When $n = N + 1$, note that there exists $a \geq a \wedge b \leq b$ on the intermediate steps between i and j such that a and b are maximal elements of ∂P^+ and $a \wedge b \in \partial P^+$. Note that for any cone σ over ∂P^+, there is a unique $\sigma_{a \vee b}$ such that... Define $\sigma_{a \vee b} = M(a \vee b \leq a \wedge b) \circ \sigma_{a \wedge b}$. $\sigma_{a \vee b}$ is well-defined: for any $a \wedge b \leq x \leq a \vee b$,

$$M(x \leq a \wedge b) \circ \sigma_x = M(x \leq a \wedge b) \circ M(a \wedge b \leq x) \circ \sigma_{a \wedge b} = M(a \vee b \leq a \wedge b) \circ \sigma_{a \wedge b}$$

Now we obtain a new poset $\widehat{\partial P^+}$ from ∂P^+ by deleting the vertex $a \wedge b$ and replacing the arrows $a \wedge b \leq a$ and $a \wedge b \leq b$ by $a \leq a \vee b$ and $b \leq a \vee b$. Note that there is a unique morphism from $\gamma : \mathsf{lim} M\big|_{\partial P^+} \to \mathsf{lim} M\big|_{\widehat{\partial P^+}}$ and $\widehat{\partial P^+}$ has one less intermediate steps than ∂P^+ (after combining arrows with the same direction). By induction hypothesis, $\tau_{i \vee j}$ is well-defined where $\tau : \mathsf{lim} M\big|_{\widehat{\partial P^+}} \Rightarrow M\big|_{\widehat{\partial P^+}}$.

Let y be an element of ∂P^+ such that $y \leq i \vee j$ and $y \neq a \wedge b$. Note that

$$M(y \leq i \vee j) \circ \sigma_y = M(y \leq i \vee j) \circ \tau_y \circ \gamma = M(i \leq i \vee j) \circ \tau_i \circ \gamma = M(i \leq i \vee j) \circ \sigma_i$$

When $y = a \wedge b$, note that

$$\begin{aligned} M(a \wedge b \leq i \vee j) \circ \sigma_{a \wedge b} &= M(a \vee b \leq i \vee j) \circ M(a \wedge b \leq a \vee b) \circ \sigma_{a \wedge b} \\ &= M(a \vee b \leq i \vee j) \circ \sigma_{a \vee b} \\ &= M(a \vee b \leq i \vee j) \circ \tau_{a \vee b} \circ \gamma \\ &= M(i \leq i \vee j) \circ \tau_i \circ \gamma \\ &= M(i \leq i \vee j) \circ \sigma_i \end{aligned}$$

Therefore, we conclude that $\sigma_{i \vee j}$ is well-defined. $\square$

A dual argument of proposition 3.25 shows that the colimit of a persistence module $M \in C^P$ is determined by the colimit of its boundary downset:

Proposition 3.26 (Dual of Proposition 3.25)**.** Let $\mathbf{C}$ be a complete category and $P \subseteq (\mathbb{R}^2, \leq)$ be a bounded poset and the boundary of the downset (denoted by ∂P^-) is connected. Then for any $M \in \mathbf{C}^P$, $\mathsf{colim} M = \mathsf{colim} M\big|_{\partial P^-}$.

Chapter 4

Decomposing $PH_\bullet(X^2_{-,-};\mathbb{F})$

Let (X,δ) be a metric graph. Recall that Star_k denotes the geometric realization of the star graph with k leaves, where the length of each edge is 1. We use $\hat{\mathsf{Star}}_k$ denote the metric star graph with k leaves where the length of an edge (denoted by e_1) is L_{e_1}, while the lengths of the other edges are 1. Note that the vector $(L_{e_1},1,\ldots,1)\in\mathbb{R}^{|E|}_{>0}$ is the edge length vertor of $\hat{\mathsf{Star}}_k$. By a slight abuse of notation, we use L_{e_1} to represent the vector $(L_{e_1},1,\ldots,1)$.

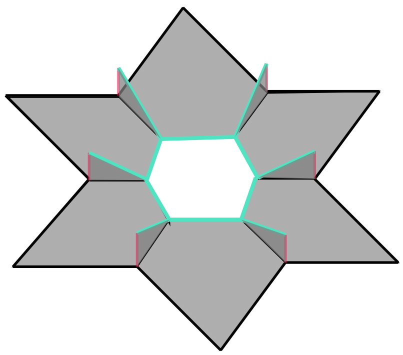


Figure 4.1: $(\mathsf{Star}_3)^2_{0.5}$

In this chapter, we are going to compute the homology groups of the second configuration spaces with restraint parameter r and edge length parameter L_{e_1} of some special graphs. We first introduce a new poset representation for $PH_i(X^2_{-,-};\mathbb{F})$ in Section 4.1. In Section, 4.2, we give the decomposition of $PH_i(Y^2_{-,-};\mathbb{F})$. In Section 4.3, we compute the homology groups of the second configuration spaces of $\hat{\mathsf{Star}}_k$, and we give the decomposition of $PH_i((\hat{\mathsf{Star}}_k)^2_{-,-};\mathbb{F})$ in Section 4.4.

Next, in section 4.5, we compute the homology groups of the second configuration spaces of H-shaped graphs, and we give the decomposition of $PH_i(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$ in Section 4.6. In section 4.7, we compute the homology groups of the second configuration spaces of the generalized H-shaped graphs, and we give the decomposition of $PH_i((\hat{\mathcal{H}}_{m,n})^2_{-,-};\mathbb{F})$ in Section 4.8. In Section 4.9, we give a strategy for calculating $PH_i(\mathbf{T}^2_{r,L_{e_1}})$ for any metric tree, where the length of all but one edge (denoted by e_1) of $\mathbf{T}$ is 1. Finally, in Section 4.10, we discuss some properties of $H_i(\mathsf{Tree}^2_{r,L_{e_1}};\mathbb{F})$, where Tree is a metric tree with arbitrary edge lengths.

4.1 Discretize the poset

Let (X,δ) be a finite metric graph and e_1 is an edge of X. When there are finitely many hyperplanes in the parameter space of $X^2_{r,L_{e_1}}$, the number of chambers in the hyperplane arrangement of $X^2_{r,L_{e_1}}$ is also finite. We may associate the hyperplane arrangement with a poset (denoted by $(P,\leq)$), and the Hasse diagram of $(P,\leq)$ can be constructed as follows:

- each chamber of the hyperplane arrangement is an element of $(P,\leq)$;
- each arrow corresponds to a wall between two chambers, and the orientation of the arrow is given by the filtration of the spaces $(X)^2_{r,L_{e_1}}$, satisfying the following condition:

the arrow is not a composition of two or more consecutive arrows

For example, the poset $(P,\leq)$ associated to the hyperplane arrangement of $Y^2_{r,L_{e_1}}$ is shown in Figure 4.2.

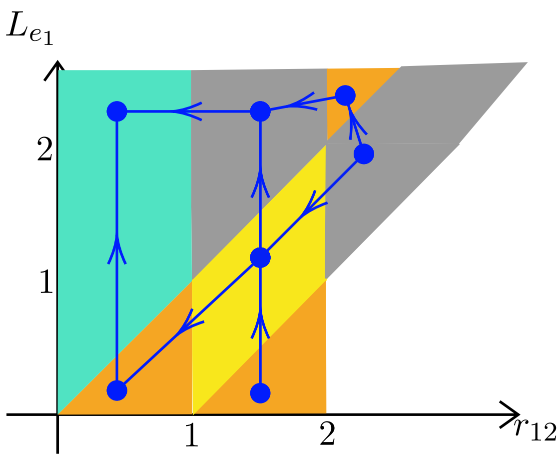


Figure 4.2: $(P, \leq)$ associated to the hyperplane arrangement of $PH_0(Y^2_{r,L_{e_1}}; \mathbb{F})$

Note that the construction of $(P, \leq)$ from a hyperplane arrangement of a finite metric graph is functorial, i.e., the construction above gives a functor (denoted by $\mathcal{F}$)

$$\mathcal{F} : (\mathbb{R}_{>0}, \leq)^{\mathsf{op}} \times (\mathbb{R}_{>0}, \leq) \to (P, \leq)$$

At the object level, $\mathcal{F}$ sends $(r, L_{e_1}) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ to $p \in P$, where p represents the chamber that contains (r, L_{e_1}).

At the morphism level, $\mathcal{F}$ sends $(r, L_{e_1}) \to (r', L'_{e_1})$ to the unique morphism $p \to p'$, where p represents the chamber that contains (r, L_{e_1}) and p' represents the chamber that contains (r', L'_{e_1}). Note that when $p = p'$, (r, L_{e_1}) and (r', L'_{e_1}) are in the same chamber, and $\mathcal{F}$ sends $(r, L_{e_1}) \to (r', L'_{e_1})$ to id_p. Hence, in particular,

$$\mathcal{F}\,\mathsf{id}_{(r,L_{e_1})} = \mathsf{id}_p$$

Note that $\mathcal{F}$ is well-defined because there is no hyperplane with a negative slope. Since $(P, \leq)$ is thin, $\mathcal{F}$ is a functor.

When (r, L_{e_1}) and (r', L'_{e_1}) lie in the same chamber of the parameter space, $X^2_{r,L_{e_1}}$ and $X^2_{r',L'_{e_1}}$ have the same homotopy type (in fact, homeomorphism type). Therefore, each persistence module $PH_i(X^2_{-,-};\mathbb{F})$ has a well-defined poset representation $M : (P, \leq) \to$ $\textbf{Vect}_{\mathbb{F}}$ for a poset $(P, \leq)$. Conversely, with the information of the hyperplane arrangement of $X^2_{r,L_{e_1}}$, the poset $(P, \leq)$ which is constructed as above, and a poset representation $M :$ $(P, \leq) \to \textbf{Vect}_{\mathbb{F}}$, we can recover $PH_i(X^2_{-,-};\mathbb{F})$ from M by defining $PH_i(X^2_{r,L_{e_1}};\mathbb{F}) = M(p)$ for all (r, L_{e_1}) that lies in the chamber represented by p, and $PH_i(X^2_{(r,L_{e_1})\leq(r',L'_{e_1})};\mathbb{F}) =$ $M(p \leq p')$ where (r, L_{e_1}) lies in the chamber represented by p and (r', L'_{e_1}) lies in the chamber represented by p'.

In other words, when X is a finite metric graph, the functor $PH_i(X^2_{-,-};\mathbb{F})$ factors through the category $(P, \leq)$.

$$\begin{tikzcd}
(\mathbb{R}_{>0}, \leq)^{\mathsf{op}} \times (\mathbb{R}_{>0}, \leq) \arrow[r, "\mathcal{F}"] \arrow[dr, "PH_i(X^2_{-,-};\mathbb{F})"'] & (P, \leq) \arrow[d, "M"] \\
& \textbf{Vect}_{\mathbb{F}}
\end{tikzcd}$$

4.2 Decomposition of $PH_i(Y^2_{-,-};\mathbb{F})$

In this section, we decompose the poset representation $PH_i(Y^2_{-,-};\mathbb{F})$ (for $i = 0, 1$) into a direct sum of indecomposable representations.

Note that the hyperplane arrangement of $Y^2_{r,L_{e_1}}$ can be interpreted as a functor

$$Y^2_{-,-} : (\mathbb{R}_{>0}, \leq)^{\mathsf{op}} \times (\mathbb{R}_{>0}, \leq) \to \textbf{Top}$$

where $Y^2_{-,-}$ sends $(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ to $Y^2_{a,b}$ and sends the unique arrow $(a,b) \to (a',b')$ to the inclusion map $\iota : Y^2_{a,b} \to Y^2_{a',b'}$, for all $a' \leq a$ and $b \leq b'$. Post-composing the i-th homology functor $H_i(-)$ with $Y^2_{-,-}$, we obtain

$$PH_i(Y^2_{-,-}) : (\mathbb{R}_{>0}, \leq)^{\mathsf{op}} \times (\mathbb{R}_{>0}, \leq) \to \mathbf{Ab}$$

In other words, at the object level, for each $(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$,

$$PH_i(Y^2_{a,b}) = H_i(Y^2_{a,b})$$

At the morphism level, $PH_i(Y^2_{-,-})$ sends each morphism $(a,b) \to (a',b')$ to a group homomorphism

$$\iota_* : H_i(Y^2_{a,b}) \to H_i(Y^2_{a',b'})$$

where ι_* is induced by the inclusion map $\iota : Y^2_{a,b} \to Y^2_{a',b'}$ in $\mathbf{Top}$.

Note that $PH_1(Y^2_{-,-};\mathbb{F})$ is an interval module because the support of $PH_1(Y^2_{-,-};\mathbb{F})$ is an interval, we immediately have

Theorem 4.1. $PH_1(Y^2_{-,-};\mathbb{F})$ *is interval decomposable.*

Now we are going to decompose $PH_0(Y^2_{-,-};\mathbb{F})$. Note that there are finitely many chambers in the hyperplane arrangement of $Y^2_{r,L_{e_1}}$, we may associate the hyperplane arrangement with the Hasse diagram of a poset (denoted by $(P,\leq)$) as follows:

- each chamber of the hyperplane arrangement is an element of $(P,\leq)$;
- each arrow corresponds to a wall between two chambers, and the orientation of the arrow is given by the filtration of the spaces $Y^2_{r,L_{e_1}}$.

We associate $PH_0(Y^2_{-,-};\mathbb{F})$ with a representation over $(P,\leq)$:

$$
\begin{tikzcd}
\mathbb{F} & \mathbb{F}^4 \arrow[l] & \mathbb{F}^2 \arrow[l] \\
\mathbb{F}^2 \arrow[u] & \mathbb{F}^6 \arrow[l] \arrow[u] & \mathbb{F}^4 \arrow[l] \arrow[u] \\
& \mathbb{F}^2 \arrow[u] &
\end{tikzcd}
$$

$$
= \begin{tikzcd}
< c_1 > & < c_1, c_3, c_4, c_6 > \arrow[l, "\alpha"'] & < c_1, c_4 > \arrow[l, hook'] \\
< c_3, c_6 > \arrow[u, "\lambda"] & < c_1, c_2, c_3, c_4, c_5, c_6 > \arrow[l, "\gamma"'] \arrow[u, "\beta"] & < c_1, c_2, c_4, c_5 > \arrow[l, hook'] \arrow[u, "\epsilon"'] \\
& < c_3, c_6 > \arrow[u, hook] &
\end{tikzcd}
\tag{4.1}
$$

where

- α maps every basis element of $< c_1, c_3, c_4, c_6 >$ to c_1;
- γ maps c_1, c_5, c_6 to c_1 and maps c_2, c_3, c_4 to c_2;
- λ maps c_3, c_6 to c_1;
- β maps c_1, c_2 to c_1, maps c_4, c_5 to c_4, maps c_3 to c_3, and maps c_6 to c_6;
- ϵ maps c_1, c_2 to c_1 and maps c_4, c_5 to c_4;
- unlabeled maps are inclusion maps.

By an abuse of notation, we use $PH_0(Y^2_{-,-};\mathbb{F})$ to denote the P-indexed persistence module given by (4.1).

Theorem 4.2. *$PH_0(Y^2_{-,-};\mathbb{F}) \cong M_1 \oplus M_2 \oplus E_1 \oplus E_2 \oplus F$ where*

$$M_1 = \begin{tikzcd}
\mathbb{F} & \mathbb{F}^2 \arrow[l, "\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]"'] & \mathbb{F} \arrow[l, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"'] \\
\mathbb{F} \arrow[u] & \mathbb{F}^2 \arrow[l, "\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]"] \arrow[u, "\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]"] & \mathbb{F} \arrow[l, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"'] \arrow[u] \\
 & \mathbb{F} \arrow[u, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] &
\end{tikzcd}$$

$$M_2 = \begin{tikzcd}
0 & \mathbb{F} \arrow[l] & \mathbb{F} \arrow[l] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] \\
 & 0 \arrow[u] &
\end{tikzcd}$$

$$E_1 = \begin{tikzcd}
0 & 0 \arrow[l] & 0 \arrow[l] \\
\mathbb{F} \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] \\
 & 0 \arrow[u] &
\end{tikzcd}$$

$$E_2 = \begin{tikzcd}
0 & 0 \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] \\
 & 0 \arrow[u] &
\end{tikzcd}$$

$$F = \begin{tikzcd}
0 & \mathbb{F} \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & \mathbb{F} \arrow[u] &
\end{tikzcd}$$

Proof. We choose a basis for each vector space $PH_0(\hat{\mathcal{H}}^2_{r,L_{e_1}};\mathbb{F})$:

$$\begin{tikzcd}
< c_1 > & < c_1, c_3 - c_6, c_4 - c_1, c_6 > \arrow[l, "\alpha"'] & < c_1, c_4 - c_1 > \arrow[l, hook] \\
< c_3 - c_6, c_6 > \arrow[u, "\lambda"] & \left\langle \begin{matrix} c_1, c_2 - c_1 + c_5 - \\ c_4, c_3 - c_6, c_4 - \\ c_1, c_5 - c_4, c_6 \end{matrix} \right\rangle \arrow[l, "\gamma"'] \arrow[u, "\beta"] & \left\langle \begin{matrix} c_1, c_2 - c_1 + c_5 - \\ c_4, c_4 - c_1, c_5 - c_4 \end{matrix} \right\rangle \arrow[l, hook] \arrow[u, "\epsilon"'] \\
 & < c_3 - c_6, c_6 > \arrow[u, hook] &
\end{tikzcd} \tag{4.2}$$

Define

$$M_1 = \begin{tikzcd}
< c_1 > & < c_1, c_6 > \arrow[l, "\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]"'] & < c_1 > \arrow[l, hook] \\
< c_6 > \arrow[u, "\mathsf{id}"] & < c_1, c_6 > \arrow[l, "\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]"] \arrow[u, "\left[\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right]"] & < c_1 > \arrow[l, hook] \arrow[u, "\mathsf{id}"'] \\
 & < c_6 > \arrow[u, hook] &
\end{tikzcd} \cong \begin{tikzcd}
\mathbb{F} & \mathbb{F}^2 \arrow[l, "\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]"'] & \mathbb{F} \arrow[l] \\
\mathbb{F} \arrow[u] & \mathbb{F}^2 \arrow[l, "\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]"] \arrow[u, "\left[\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right]"] & \mathbb{F} \arrow[l] \arrow[u] \\
 & \mathbb{F} \arrow[u] &
\end{tikzcd} \tag{4.3}$$

$$M_2 = \begin{tikzcd} 0 & < c_4 - c_1 > \arrow[l] & < c_4 - c_1 > \arrow[l, hook'] \\ 0 \arrow[u] & < c_4 - c_1 > \arrow[l] \arrow[u, "\mathsf{id}"] & < c_4 - c_1 > \arrow[l, hook'] \arrow[u, "\mathsf{id}"'] \\ & 0 \arrow[u, hook] & \end{tikzcd} \cong \begin{tikzcd} 0 & \mathbb{F} \arrow[l] & \mathbb{F} \arrow[l] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] \\ & 0 \arrow[u] & \end{tikzcd} \tag{4.4}$$

$$E_1 = \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l, hook'] \\ 0 \arrow[u] & < c_2 - c_1 + c_5 - c_4 > \arrow[l] \arrow[u] & < c_2 - c_1 + c_5 - c_4 > \arrow[l, hook'] \arrow[u] \\ & 0 \arrow[u, hook] & \end{tikzcd} \cong \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] \\ & 0 \arrow[u] & \end{tikzcd} \tag{4.5}$$

$$E_2 = \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l, hook'] \\ 0 \arrow[u] & < c_5 - c_4 > \arrow[l] \arrow[u] & < c_5 - c_4 > \arrow[l, hook'] \arrow[u] \\ & 0 \arrow[u, hook] & \end{tikzcd} \cong \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] \\ & 0 \arrow[u] & \end{tikzcd} \tag{4.6}$$

$$F = \begin{tikzcd} 0 & < c_6 - c_3 > \arrow[l] & 0 \arrow[l, hook'] \\ < c_6 - c_3 > \arrow[u] & < c_6 - c_3 > \arrow[l, "\mathsf{id}"'] \arrow[u, "\mathsf{id}"] & 0 \arrow[l, hook'] \arrow[u] \\ & < c_6 - c_3 > \arrow[u, hook, "\mathsf{id}"] & \end{tikzcd} \cong \begin{tikzcd} 0 & \mathbb{F} \arrow[l] & 0 \arrow[l] \\ \mathbb{F} \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ & \mathbb{F} \arrow[u] & \end{tikzcd} \tag{4.7}$$

Note that M_2, E_1, E_2, and F are interval modules, hence by Lemma 3.1, they are indecomposable. Now we are going to show that M_1 is indecomposable. By the Fitting's lemma (Theorem 2.36), it suffices to show that $\mathsf{End}(M_1)$ does not contain any idempotents except 0 and id.

Note that any linear transformation $\mathbb{F} \to \mathbb{F}$ is a scalar multiplication. We use $x : \mathbb{F} \to \mathbb{F}$ to denote the linear transformation that sends f to xf for all $f \in \mathbb{F}$. Let $\phi \in \mathsf{End}(M_1)$. It consists of the following data:

- $x : \mathbb{F} \to \mathbb{F}$;
- $y : \mathbb{F} \to \mathbb{F}$;
- $z : \mathbb{F} \to \mathbb{F}$;
- $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the morphism $\mathbb{F}^2 \to \mathbb{F}^2$ under the given basis,

such that

$$x \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.8}$$

$$y \circ [1 \quad 1] = A \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.9}$$

$$z \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.10}$$

Equation 4.8 implies

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \tag{4.11}$$

Hence $a = x$ and $c = 0$. Update $A = \begin{bmatrix} x & b \\ 0 & d \end{bmatrix}$.

Equation 4.9 implies

$$[x \quad b+d] = [y \quad y] \tag{4.12}$$

Hence $x = y$ and $b + d = y$. Update $A = \begin{bmatrix} x & b \\ 0 & x-b \end{bmatrix}$.

Equation 4.10 implies

$$\begin{bmatrix} b \\ x-b \end{bmatrix} = \begin{bmatrix} 0 \\ z \end{bmatrix} \tag{4.13}$$

Hence $b = 0$ and $x = z$. Update $A = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$. Therefore, $\mathsf{End}(M_1) \cong \mathbb{F}$. Since $\mathbb{F}$ is a field, it does not contain any idempotents except 0 and id. Thus M_1 is indecomposable.

□

The indecomposables of $PH_0(Y^2_{-,-};\mathbb{F})$ is given in Figure 4.3. The number on the colored block indicates the multiplicity of $\mathbb{F}$ in the representation.

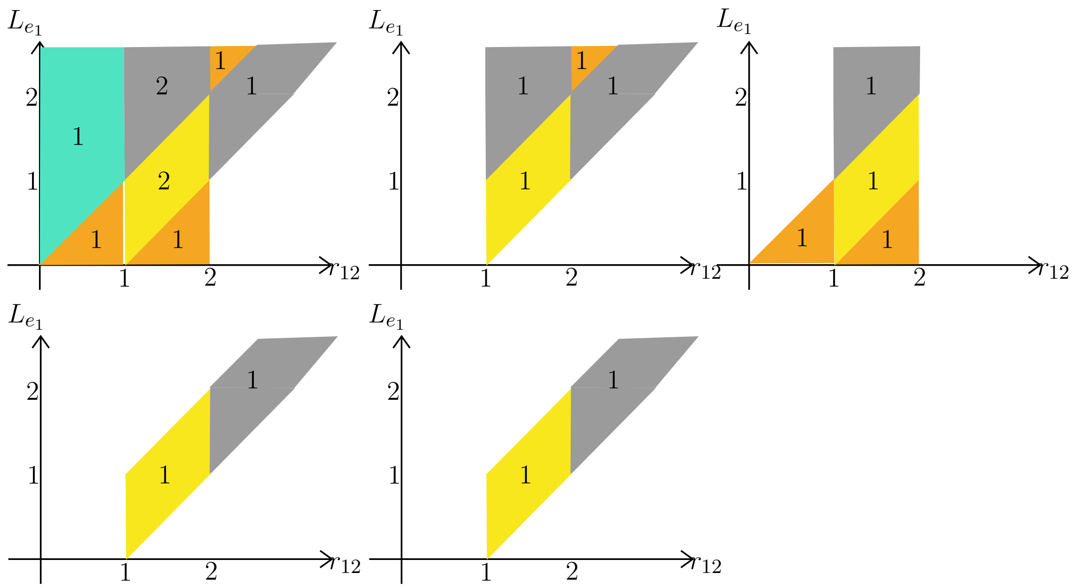


Figure 4.3: The indecomposables of $PH_0(Y^2_{-,-};\mathbb{F})$

4.3 Configuration Spaces of Star Graphs

We begin this section with an observation.

Proposition 4.3. Let $k \geq 4$ and $k \in \mathbb{Z}$. Let $\hat{\mathsf{Star}}_k$ be the metric graph where every edge but e_1 has length 1. Then $(\hat{\mathsf{Star}}_k)^2_{r,L_{e_1}}$ and $Y^2_{r,L_{e_1}} := (\hat{\mathsf{Star}}_3)^2_{r,L_{e_1}}$ have the same critical hyperplanes.

Proof. The following inequalities give a description of the parametric polytopes of $(\hat{\mathsf{Star}}_k)^2_{r,L_{e_1}}$:

For $i, j \in \{2, \ldots, k\}$ and $i \neq j$,

$$\begin{aligned} &0 \leq x \leq 1 = L_{e_i} \\ &0 \leq y \leq 1 = L_{e_j} \\ &x + y \geq r \end{aligned} \tag{4.14}$$

For $i \in \{2, \ldots, k\}$ and $j = 1$,

$$\begin{aligned} &0 \leq x \leq 1 = L_{e_i} \\ &0 \leq y \leq L_{e_1} \\ &x + y \geq r \end{aligned} \tag{4.15}$$

For $j \in \{2, \ldots, k\}$ and $i = 1$,

$$\begin{aligned} &0 \leq x \leq L_{e_1} \\ &0 \leq y \leq 1 = L_{e_j} \\ &x + y \geq r \end{aligned} \tag{4.16}$$

For $i, j \in \{2, \ldots, k\}$, $i = j$, and $y \geq x$,

$$\begin{aligned}
&0 \leq x \leq 1 \\
&0 \leq y \leq 1 \\
&y - x \geq r \\
&y \geq x
\end{aligned} \tag{4.17}$$

For $i, j \in \{2, \ldots, k\}$, $i = j$, and $y \leq x$,

$$\begin{aligned}
&0 \leq x \leq 1 \\
&0 \leq y \leq 1 \\
&x - y \geq r \\
&y \leq x
\end{aligned} \tag{4.18}$$

For $i = j = 1$, and $y \geq x$,

$$\begin{aligned}
&0 \leq x \leq L_{e_1} \\
&0 \leq y \leq L_{e_1} \\
&y - x \geq r \\
&y \geq x
\end{aligned} \tag{4.19}$$

For $i = j = 1$, and $y \leq x$,

$$\begin{aligned}
&0 \leq x \leq L_{e_1} \\
&0 \leq y \leq L_{e_1} \\
&x - y \geq r \\
&y \leq x
\end{aligned} \tag{4.20}$$

Compare the inequality systems (4.14), (4.15), (4.16), (4.17), (4.18), (4.19), and (4.20) with inequality systems (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), and (1.8), we conclude that

$(\hat{\mathsf{Star}}_k)^2_{r,L_{e_1}}$ and $Y^2_{r,L_{e_1}}$ have the same type of parametric polytopes. Therefore, $(\hat{\mathsf{Star}}_k)^2_{r,L_{e_1}}$ and $Y^2_{r,L_{e_1}}$ have the same critical hyperplanes. □

The hyperplane arrangement of $(\hat{\mathsf{Star}}_k)^2_{r,L_{e_1}}$ is given in Figure 1.3.

Proposition 4.4. Let r be a positive number and $k \geq 3$. Then

$$H_0((\mathsf{Star}_k)^2_r) \cong \begin{cases} \mathbb{Z}, & \text{if } 0 < r \leq 1 \\ \mathbb{Z}^{k^2-k}, & \text{if } 1 < r \leq 2 \\ 0, & \text{if } 2 < r \end{cases} \quad \text{and} \quad H_1((\mathsf{Star}_k)^2_r) \cong \begin{cases} \mathbb{Z}^{k(k-3)+1}, & \text{if } 0 < r \leq 1 \\ 0, & \text{if } 1 < r \end{cases}$$

Proof. We run induction on k.

When $k = 3$ Consider the following cover of $(\mathsf{Star}_3)^2_r$:

$$\begin{aligned} U_{11} &= (e_1)^2_r \\ U_{12} &= e_1 \times \mathsf{Star}_2 - \{(x,y) \in e_1 \times \mathsf{Star}_2 \mid \delta(x,y) < r\} \\ U_{21} &= \mathsf{Star}_2 \times e_1 - \{(x,y) \in \mathsf{Star}_2 \times e_1 \mid \delta(x,y) < r\} \\ U_{22} &= (\mathsf{Star}_2)^2_r \end{aligned} \tag{4.21}$$

Note that $(e_1)^2_r \simeq \begin{cases} \{*_1, *_2\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases}$ where $\{*_1, *_2\}$ is a subspace of $(e_1)^2_r$ consists of two points. On the other hand,

$$e_1 \times \mathsf{Star}_2 - \{(x,y) \in e_1 \times \mathsf{Star}_2 \mid \delta(x,y) < r\} \simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq 1 \\ \{*_1, *_2\}, & \text{if } 1 < r \leq 2 \\ \emptyset, & \text{if } 2 < r \end{cases}$$

Similarly,

$$\mathsf{Star}_2 \times e_1 - \{(x,y) \in \mathsf{Star}_2 \times e_1 \mid \delta(x,y) < r\} \simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq 1 \\ \{*_1, *_2\}, & \text{if } 1 < r \leq 2 \\ \emptyset, & \text{if } 2 < r \end{cases}$$

In addition, note that

$$H_0((\mathsf{Star}_2)^2_r) \cong \begin{cases} \mathbb{Z}^2, & \text{if } 0 < r \leq 2 \\ 0, & \text{if } 2 < r \end{cases}$$

Now let's consider the intersections of U_{ij}. Note that

$$\begin{aligned} U_{11} \cap U_{12} &\simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases} \\ U_{11} \cap U_{21} &\simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases} \\ U_{22} \cap U_{12} &\simeq \begin{cases} \{*_1, *_2\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases} \\ U_{22} \cap U_{21} &\simeq \begin{cases} \{*_1, *_2\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases} \\ U_{11} \cap U_{22} &= \emptyset \\ U_{12} \cap U_{21} &= \emptyset \end{aligned} \tag{4.22}$$

Note that there are natural inclusions

$$\begin{aligned}
U_{11} \cap U_{12} &\hookrightarrow U_{11} \\
U_{11} \cap U_{12} &\hookrightarrow U_{12} \\
U_{11} \cap U_{21} &\hookrightarrow U_{11} \\
U_{11} \cap U_{21} &\hookrightarrow U_{21} \\
U_{22} \cap U_{12} &\hookrightarrow U_{22} \\
U_{22} \cap U_{12} &\hookrightarrow U_{12} \\
U_{22} \cap U_{21} &\hookrightarrow U_{22} \\
U_{22} \cap U_{21} &\hookrightarrow U_{21}
\end{aligned} \tag{4.23}$$

When $0 < r \leq 1$: The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
2 & \vdots & 0 & 0 \\
1 & H_1((\mathsf{Star}_2)^2_r) & 0 & 0 \\
0 & \mathbb{Z}^4 \oplus H_0((\mathsf{Star}_2)^2_r) \xleftarrow{d^1} & \mathbb{Z}^6 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.23). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let b_1, b_2 be the generators of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_1, \hat{b}_2$ be the generators of $H_0(U_{22} \cap U_{21})$. Note that U_{11} has two path components, while $U_{11} \cap U_{12}$ and $U_{11} \cap U_{21}$ lie in different path components. Choose a generator (denoted by c_1) from the path component of U_{11} containing $U_{11} \cap U_{12}$, and choose a generator (denoted by c_2) from the path component of U_{11} containing $U_{11} \cap U_{21}$. Let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$d^1(a) = c_1 + e$$

and

$$d^1(\hat{a}) = c_2 + \hat{e}$$

On the other hand, let f_1, f_2 be the generators of $H_0(U_{22}) \cong \mathbb{Z}^2$, where f_1 and f_2 lie in different path components of U_{22}, then

$$d^1(b_1) = e + f_1, \quad d^1(b_2) = e + f_2,$$

and

$$d^1(\hat{b}_1) = \hat{e} + f_1, \quad d^1(\hat{b}_2) = \hat{e} + f_2$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 5, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^5$ and $\mathsf{ker} d^1 \cong \mathbb{Z}^1$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|cccc} q & & & & \\ 2 & \vdots & 0 & 0 & \\ 1 & H_1((\mathsf{Star}_2)^2_r) & 0 & 0 & \\ 0 & \mathbb{Z} & \mathbb{Z} & 0 & \\ \hline & 0 & 1 & 2 & p \end{array}$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\mathsf{Star}_3)^2_r) \cong \mathbb{Z}$$

and

$$H_1((\mathsf{Star}_3)^2_r) \cong H_1((\mathsf{Star}_2)^2_r) \oplus \mathbb{Z} \cong \mathbb{Z}$$

When $1 < r \leq 2$**:** The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
2 & \vdots & 0 & 0 \\
1 & H_1((\mathsf{Star}_2)^2_r) & 0 & 0 \\
0 & \mathbb{Z}^6 & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_0((\mathsf{Star}_3)^2_r) \cong \mathbb{Z}^6$$

and

$$H_1((\mathsf{Star}_3)^2_r) = 0$$

In conclusion,

$$H_0((\mathsf{Star}_3)^2_r) \cong \begin{cases} \mathbb{Z}, & \text{if } 0 < r \leq 1 \\ \mathbb{Z}^6, & \text{if } 1 < r \leq 2 \\ 0, & \text{if } 2 < r \end{cases} \tag{4.24}$$

and

$$H_1((\mathsf{Star}_3)^2_r) \cong \begin{cases} \mathbb{Z}, & \text{if } 0 < r \leq 1 \\ 0, & \text{if } 1 < r \end{cases} \tag{4.25}$$

When $k \geq 4$ Consider the following cover of $(\mathsf{Star}_k)^2_r$:

$$\begin{aligned} U_{11} &= (e_1)^2_r \\ U_{12} &= e_1 \times \mathsf{Star}_{k-1} - \{(x,y) \in e_1 \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \\ U_{21} &= \mathsf{Star}_{k-1} \times e_1 - \{(x,y) \in \mathsf{Star}_{k-1} \times e_1 \mid \delta(x,y) < r\} \\ U_{22} &= (\mathsf{Star}_{k-1})^2_r \end{aligned} \tag{4.26}$$

Note that $(e_1)^2_r \simeq \begin{cases} \{*_1, *_2\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases}$ where $\{*_1, *_2\}$ is a subspace of $(e_1)^2_r$ consists of two points. On the other hand,

$$e_1 \times \mathsf{Star}_{k-1} - \{(x,y) \in e_1 \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq 1 \\ \{*_1, *_2, \dots, *_{k-1}\}, & \text{if } 1 < r \leq 2 \\ \emptyset, & \text{if } 2 < r \end{cases}$$

where $\{*_1, *_2, \dots, *_{k-1}\}$ is a subspace of $(e_1)^2_r$ consists of $(k-1)$ points.

Now let's consider the intersections of U_{ij}. Note that

$$\begin{aligned}
U_{11} \cap U_{12} &\simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases} \\
U_{11} \cap U_{21} &\simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases} \\
U_{22} \cap U_{12} &\simeq \begin{cases} \{*_1, \ldots, *_{k-1}\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases} \\
U_{22} \cap U_{21} &\simeq \begin{cases} \{*_1, \ldots, *_{k-1}\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases} \\
U_{11} \cap U_{22} &= \emptyset \\
U_{12} \cap U_{21} &= \emptyset
\end{aligned} \tag{4.27}$$

Note that there are natural inclusions

$$\begin{aligned}
U_{11} \cap U_{12} &\hookrightarrow U_{11} \\
U_{11} \cap U_{12} &\hookrightarrow U_{12} \\
U_{11} \cap U_{21} &\hookrightarrow U_{11} \\
U_{11} \cap U_{21} &\hookrightarrow U_{21} \\
U_{22} \cap U_{12} &\hookrightarrow U_{22} \\
U_{22} \cap U_{12} &\hookrightarrow U_{12} \\
U_{22} \cap U_{21} &\hookrightarrow U_{22} \\
U_{22} \cap U_{21} &\hookrightarrow U_{21}
\end{aligned} \tag{4.28}$$

When $0 < r \leq 1$: The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|cccc}
q & & & & \\
2 & \vdots & 0 & 0 & \\
1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 & \\
0 & \mathbb{Z}^4 \oplus H_0((\mathsf{Star}_{k-1})^2_r) & \xleftarrow{d^1} \mathbb{Z}^{2k} & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.28). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let $b_1, \ldots, b_{k-1}$ be the generators of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_1, \ldots, \hat{b}_{k-1}$ be the generators of $H_0(U_{22} \cap U_{21})$. Note that U_{11} has two path components, while $U_{11} \cap U_{12}$ and $U_{11} \cap U_{21}$ lie in different path components. Choose a generator (denoted by c_1) from the path component of U_{11} containing $U_{11} \cap U_{12}$, and choose a generator (denoted by c_2) from the path component of U_{11} containing $U_{11} \cap U_{21}$. Let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$d^1(a) = c_1 + e$$

and

$$d^1(\hat{a}) = c_2 + \hat{e}$$

Let f be the generator of $H_0(U_{22}) \cong \mathbb{Z}$, then for all $i = 1, \ldots, k-1$,

$$d^1(b_i) = e + f$$

and

$$d^1(\hat{b}_i) = \hat{e} + f$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

where the third and the fourth columns are repeated for $(k-1)$ times. Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

The rank of this matrix is 4, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^4$ and $\mathsf{ker} d^1 \cong \mathbb{Z}^{2k-4}$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|ccc}
q & & & \\
2 & \vdots & 0 & 0 \\
1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}^{2k-4} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\mathsf{Star}_k)^2_r) \cong \mathbb{Z}$$

and the following sequence is a short exact sequence:

$$0 \to H_1((\mathsf{Star}_{k-1})^2_r) \to H_0((\mathsf{Star}_k)^2_r) \to \mathbb{Z}^{2k-4} \to 0 \tag{4.29}$$

Note that $\mathbb{Z}^{2k-4}$ is free, the short exact sequence (4.29) splits, hence

$$\begin{aligned} H_0((\mathsf{Star}_k)^2_r) &\cong H_1((\mathsf{Star}_{k-1})^2_r) \oplus \mathbb{Z}^{2k-4} \\ &\cong \mathbb{Z}^{2k-4} \oplus \mathbb{Z}^{2(k-1)-4} \oplus \cdots \oplus \mathbb{Z}^{2(4)-4} \oplus \mathbb{Z} \\ &\cong \mathbb{Z}^{k(k-3)+1} \end{aligned} \tag{4.30}$$

When $1 < r \leq 2$**:** The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|cccc}
q & & & & \\
2 & \vdots & 0 & 0 & \\
1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 & \\
0 & \mathbb{Z}^{2k-2} \oplus H_0((\mathsf{Star}_{k-1})^2_r) \xleftarrow{d^1} & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$\begin{aligned} H_0((\mathsf{Star}_k)^2_r) &\cong H_0((\mathsf{Star}_{k-1})^2_r) \oplus \mathbb{Z}^{2k-2} \\ &\cong \mathbb{Z}^{2k-2} \oplus \mathbb{Z}^{2(k-1)-2} \oplus \cdots \mathbb{Z}^{2(4)-2} \oplus \mathbb{Z}^{6} \\ &\cong \mathbb{Z}^{k^2-k} \end{aligned} \tag{4.31}$$

and

$$H_1((\mathsf{Star}_k)^2_r) \cong H_1((\mathsf{Star}_{k-1})^2_r)$$

In conclusion,

$$H_0((\mathsf{Star}_k)^2_r) \cong \begin{cases} \mathbb{Z}, & \text{if } 0 < r \leq 1 \\ \mathbb{Z}^{k^2-k}, & \text{if } 1 < r \leq 2 \\ 0, & \text{if } 2 < r \end{cases} \tag{4.32}$$

and

$$H_1((\mathsf{Star}_k)^2_r) \cong \begin{cases} \mathbb{Z}^{k(k-3)+1}, & \text{if } 0 < r \leq 1 \\ 0, & \text{if } 1 < r \end{cases} \tag{4.33}$$

$\square$

Now we are ready to compute $H_i((\hat{\mathsf{Star}}_k)^2_r)$. Consider the following cover of $(\hat{\mathsf{Star}}_k)^2_r$:

$$\begin{aligned} U_{11} &= (e_1)^2_r \\ U_{12} &= e_1 \times \mathsf{Star}_{k-1} - \{(x,y) \in e_1 \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \\ U_{21} &= \mathsf{Star}_{k-1} \times e_1 - \{(x,y) \in \mathsf{Star}_{k-1} \times e_1 \mid \delta(x,y) < r\} \\ U_{22} &= (\mathsf{Star}_{k-1})^2_r \end{aligned} \tag{4.34}$$

Note that there are natural inclusions

$$\begin{aligned}
U_{11} \cap U_{12} &\hookrightarrow U_{11} \\
U_{11} \cap U_{12} &\hookrightarrow U_{12} \\
U_{11} \cap U_{21} &\hookrightarrow U_{11} \\
U_{11} \cap U_{21} &\hookrightarrow U_{21} \\
U_{22} \cap U_{12} &\hookrightarrow U_{22} \\
U_{22} \cap U_{12} &\hookrightarrow U_{12} \\
U_{22} \cap U_{21} &\hookrightarrow U_{22} \\
U_{22} \cap U_{21} &\hookrightarrow U_{21}
\end{aligned} \tag{4.35}$$

Proposition 4.5. Let L_{e_1} be a positive number. If $r \leq L_{e_1}$ and $r \leq 1$. Then

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^{k(k-3)+1}$$

Proof. Consider the cover of $(\hat{\mathsf{Star}}_k)^2_r$ given in Equation (4.34). Note that $U_{11} = (e_1)^2_r \simeq \{*_1, *_2\}$. Moreover,

$$U_{12} = e_1 \times \mathsf{Star}_{k-1} - \{(x,y) \in e_1 \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \simeq \{*\}$$

and

$$U_{21} = \mathsf{Star}_{k-1} \times e_1 - \{(x,y) \in \mathsf{Star}_{k-1} \times e_1 \mid \delta(x,y) < r\} \simeq \{*\}$$

Now let's consider the intersections of U_{ij}. Note that

$$\begin{aligned} & U_{11} \cap U_{12} \simeq \{*\} \simeq U_{11} \cap U_{21} \\ & U_{22} \cap U_{12} \simeq \{*_1, \ldots, *_{k-1}\} \simeq U_{22} \cap U_{21} \\ & U_{11} \cap U_{22} = \emptyset = U_{12} \cap U_{21} \end{aligned} \tag{4.36}$$

The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|cccc} q & & & & \\ 2 & \vdots & 0 & 0 & \\ 1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 & \\ 0 & \mathbb{Z}^4 \oplus H_0((\mathsf{Star}_{k-1})^2_r) \xleftarrow{d^1} & \mathbb{Z}^{2k} & 0 & \\ \hline & 0 & 1 & 2 & p \end{array}$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.35). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let $b_1, \ldots, b_{k-1}$ be the generators of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_1, \ldots, \hat{b}_{k-1}$ be the generators of $H_0(U_{22} \cap U_{21})$. Note that U_{11} has two path components, while $U_{11} \cap U_{12}$ and $U_{11} \cap U_{21}$ lie in different path components. Choose a generator (denoted by c_1) from the path component of U_{11} containing $U_{11} \cap U_{12}$, and choose a generator (denoted by c_2) from the path component of U_{11} containing $U_{11} \cap U_{21}$. Let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$d^1(a) = c_1 + e$$

and

$$d^1(\hat{a}) = c_2 + \hat{e}$$

Let f be the generator of $H_0(U_{22}) \cong \mathbb{Z}$, then for all $i = 1, \ldots, k-1$,

$$d^1(b_i) = e + f$$

and

$$d^1(\hat{b}_i) = \hat{e} + f$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

where the third and the fourth columns are repeated for $(k-1)$ times. Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

The rank of this matrix is 4, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore,

$$\mathsf{Im}\, d^1 \cong \mathbb{Z}^4$$

and

$$\mathsf{ker} d^1 \cong \mathbb{Z}^{2k-4}$$

Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|ccc}
q & & & \\
2 & \vdots & 0 & 0 \\
1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}^{2k-4} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$
\begin{aligned}
H_1((\hat{\mathsf{Star}}_k)^2_r) &\cong H_1((\mathsf{Star}_{k-1})^2_r) \oplus \mathbb{Z}^{2k-4} \\
&\cong \mathbb{Z}^{k(k-3)+1}
\end{aligned} \tag{4.37}
$$

and

$$
H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z} \tag{4.38}
$$

□

Proposition 4.6. Let L_{e_1} be a positive number. If $r > L_{e_1}$ and $r \leq 1$. Then

$$
H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^{(k-1)(k-4)+1}
$$

Proof. Consider the cover of $(\hat{\mathsf{Star}}_k)^2_r$ given in Equation (4.34). Note that $U_{11} = (e_1)^2_r = \emptyset$. On the other hand,

$$
U_{12} = e_1 \times \mathsf{Star}_{k-1} - \{(x,y) \in e_1 \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \simeq \{*_1, \ldots, *_{k-1}\}
$$

and

$$
U_{21} = \mathsf{Star}_{k-1} \times e_1 - \{(x,y) \in \mathsf{Star}_{k-1} \times e_1 \mid \delta(x,y) < r\} \simeq \{*_1, \ldots, *_{k-1}\}
$$

Now let's consider the intersections of U_{ij}. Note that

$$
\begin{aligned}
&U_{11} \cap U_{12} = \emptyset = U_{11} \cap U_{21} \\
&U_{22} \cap U_{12} \simeq \{*_1, \dots, *_{k-1}\} \simeq U_{22} \cap U_{21} \\
&U_{11} \cap U_{22} = \emptyset = U_{12} \cap U_{21}
\end{aligned}
\tag{4.39}
$$

The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|cccc}
q & & & & \\
2 & \vdots & & 0 & 0 \\
1 & H_1((\mathsf{Star}_{k-1})^2_r) & & 0 & 0 \\
0 & \mathbb{Z}^{2k-2} \oplus H_0((\mathsf{Star}_{k-1})^2_r) & \xleftarrow{d^1} & \mathbb{Z}^{2k-2} & 0 \\
\hline
 & 0 & & 1 & 2 \quad p
\end{array}
$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.35). Let $b_1, \dots, b_{k-1}$ be the generators of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_1, \dots, \hat{b}_{k-1}$ be the generators of $H_0(U_{22} \cap U_{21})$. Let $e, \dots, e_{k-1}$ be the generators of $H_0(U_{12})$ and $\hat{e}_1, \dots, \hat{e}_{k-1}$ be the generators of $H_0(U_{21})$. Let f be the generator of $H_0(U_{22}) \cong \mathbb{Z}$, then for all $i = 1, \dots, k-1$,

$$
d^1(b_i) = e_i + f
$$

and

$$
d^1(\hat{b}_i) = \hat{e}_i + f
$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The rank of this matrix is $2k-2$, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^{2k-2}$ and $\mathsf{ker} d^1 = 0$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|cccc} q & & & & \\ 2 & \vdots & 0 & 0 & \\ 1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 & \\ 0 & \mathbb{Z} & 0 & 0 & \\ \hline & 0 & 1 & 2 & p \end{array}$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_1((\hat{\mathsf{Star}}_k)^2_r) \cong H_1((\mathsf{Star}_{k-1})^2_r) \cong \mathbb{Z}^{(k-1)(k-4)+1} \tag{4.40}$$

and

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z} \tag{4.41}$$

□

Proposition 4.7. Let L_{e_1} be a positive number. If $r \leq L_{e_1}$ and $1 < r \leq 2$. Then

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^{k^2-k+2} \quad \text{and} \quad H_1((\hat{\mathsf{Star}}_k)^2_r) = 0$$

Proof. Consider the cover of $(\hat{\mathsf{Star}}_k)^2_r$ given in Equation (4.34). Note that $U_{11} = (e_1)^2_r = \{*_1, *_2\}$. On the other hand,

$$U_{12} = e_1 \times \mathsf{Star}_{k-1} - \{(x,y) \in e_1 \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \simeq \{*\}$$

and

$$U_{21} = \mathsf{Star}_{k-1} \times e_1 - \{(x,y) \in \mathsf{Star}_{k-1} \times e_1 \mid \delta(x,y) < r\} \simeq \{*\}$$

Now let's consider the intersections of U_{ij}. Note that

$$\begin{aligned} U_{11} \cap U_{12} &\simeq \{*\} \simeq U_{11} \cap U_{21} \\ U_{22} \cap U_{12} &= \emptyset = U_{22} \cap U_{21} \\ U_{11} \cap U_{22} &= \emptyset = U_{12} \cap U_{21} \end{aligned} \tag{4.42}$$

The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|cccc} q & & & & \\ 2 & \vdots & 0 & 0 & \\ 1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 & \\ 0 & \mathbb{Z}^4 \oplus H_0((\mathsf{Star}_{k-1})^2_r) \xleftarrow{d^1} & \mathbb{Z}^2 & 0 & \\ \hline & 0 & 1 & 2 & p \end{array}$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.35). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$. On the other hand, let $f_1, \ldots, f_{k^2-k}$ be the generators of $H_0(U_{22}) \cong \mathbb{Z}^{k^2-k}$, then for all $i = 1, \ldots, k-1$,

$$d^1(a) = c_1 + e$$

and

$$d^1(\hat{b}_i) = c_2 + \hat{e}$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

The rank of this matrix is 2, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^2$ and $\mathsf{ker} d^1 = 0$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|ccc} q & & & \\ 2 & \vdots & 0 & 0 \\ 1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 \\ 0 & \mathbb{Z}^{k^2-k+2} & 0 & 0 \\ \hline & 0 & 1 & 2 \quad p \end{array}$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_1((\hat{\mathsf{Star}}_k)^2_r) \cong H_1((\mathsf{Star}_{k-1})^2_r) = 0 \tag{4.43}$$

and

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^{k^2-k+2} \tag{4.44}$$

□

Proposition 4.8. Let L_{e_1} be a positive number. If $L_{e_1} < r \leq L_{e_1} + 1$ and $1 < r \leq 2$. Then

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^{k^2+k-2} \quad \text{and} \quad H_1((\hat{\mathsf{Star}}_k)^2_r) = 0$$

Proof. Consider the cover of $(\hat{\mathsf{Star}}_k)^2_r$ given in Equation (4.34). Note that $U_{11} = (e_1)^2_r = \emptyset$. On the other hand, $U_{12} = e_1 \times \mathsf{Star}_{k-1} - \{(x,y) \in e_1 \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \simeq \{*_1, \ldots, *_{k-1}\}$ and $U_{21} = \mathsf{Star}_{k-1} \times e_1 - \{(x,y) \in \mathsf{Star}_{k-1} \times e_1 \mid \delta(x,y) < r\} \simeq \{*_1, \ldots, *_{k-1}\}$.

Now let's consider the intersections of U_{ij}. Note that

$$\begin{aligned} U_{11} \cap U_{12} &= \emptyset = U_{11} \cap U_{21} \\ U_{22} \cap U_{12} &= \emptyset = U_{22} \cap U_{21} \\ U_{11} \cap U_{22} &= \emptyset = U_{12} \cap U_{21} \end{aligned} \tag{4.45}$$

The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc} q & & & \\ 2 & \vdots & 0 & 0 \\ 1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 \\ 0 & \mathbb{Z}^{2k-2} \oplus H_0((\mathsf{Star}_{k-1})^2_r) & 0 & 0 \\ \hline & 0 & 1 & 2 \quad p \end{array}$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_1((\hat{\mathsf{Star}}_k)^2_r) \cong H_1((\mathsf{Star}_{k-1})^2_r) = 0 \tag{4.46}$$

and

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^{k^2+k-2} \tag{4.47}$$

□

Proposition 4.9. Let L_{e_1} be a positive number. If $r > L_{e_1} + 1$ and $1 < r \leq 2$. Then

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^{k^2-k} \quad \text{and} \quad H_1((\hat{\mathsf{Star}}_k)^2_r) = 0$$

Proof. Consider the cover of $(\hat{\mathsf{Star}}_k)^2_r$ given in Equation (4.34). Note that $U_{11} = (e_1)^2_r = \emptyset$. On the other hand, $U_{12} = e_1 \times \mathsf{Star}_{k-1} - \{(x,y) \in e_1 \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} = \emptyset$ and $U_{21} = \mathsf{Star}_{k-1} \times e_1 - \{(x,y) \in \mathsf{Star}_{k-1} \times e_1 \mid \delta(x,y) < r\} = \emptyset$.

Now let's consider the intersections of U_{ij}. Note that

$$\begin{aligned} U_{11} \cap U_{12} = \emptyset = U_{11} \cap U_{21} \\ U_{22} \cap U_{12} = \emptyset = U_{22} \cap U_{21} \\ U_{11} \cap U_{22} = \emptyset = U_{12} \cap U_{21} \end{aligned} \tag{4.48}$$

The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc} q & & & \\ 2 & \vdots & 0 & 0 \\ 1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 \\ 0 & H_0((\mathsf{Star}_{k-1})^2_r) & 0 & 0 \\ \hline & 0 & 1 & 2 \quad p \end{array}$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_1((\hat{\mathsf{Star}}_k)^2_r) \cong H_1((\mathsf{Star}_{k-1})^2_r) = 0 \tag{4.49}$$

and

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^{k^2-k} \tag{4.50}$$

□

Proposition 4.10. Let L_{e_1} be a positive number. If $r \leq L_{e_1}$ and $r > 2$. Then

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^2 \quad \text{and} \quad H_1((\hat{\mathsf{Star}}_k)^2_r) = 0$$

Proof. Consider the cover of $(\hat{\mathsf{Star}}_k)^2_r$ given in Equation (4.34). Note that $U_{11} = (e_1)^2_r \simeq \{*_1, *_2\}$. On the other hand,

$$U_{12} = e_1 \times \mathsf{Star}_{k-1} - \{(x,y) \in e_1 \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \simeq \{*\}$$

and

$$U_{21} = \mathsf{Star}_{k-1} \times e_1 - \{(x,y) \in \mathsf{Star}_{k-1} \times e_1 \mid \delta(x,y) < r\} \simeq \{*\}$$

Now let's consider the intersections of U_{ij}. Note that

$$\begin{aligned} &U_{11} \cap U_{12} \simeq \{*\} \simeq U_{11} \cap U_{21} \\ &U_{22} \cap U_{12} = \emptyset = U_{22} \cap U_{21} \\ &U_{11} \cap U_{22} = \emptyset = U_{12} \cap U_{21} \end{aligned} \tag{4.51}$$

The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|cccc}
q & & & & \\
2 & \vdots & 0 & 0 & \\
1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 & \\
0 & \mathbb{Z}^4 \oplus H_0((\mathsf{Star}_{k-1})^2_r) \xleftarrow{d^1} & \mathbb{Z}^2 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.35). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let c_1, c_2 be the generators of $H_0(U_{11})$, where c_1 and c_2 lie in different path components of U_{11}. Let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$. Note that

$$d^1(a) = c_1 + e$$

$$d^1(\hat{a}) = c_2 + \hat{e}$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Run the row reduction, we obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The rank of this matrix is 2, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^2$ and $\mathsf{ker} d^1 = 0$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|cccc}
q & & & & \\
2 & \vdots & 0 & 0 & \\
1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 & \\
0 & \mathbb{Z}^2 & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_1((\hat{\mathsf{Star}}_k)^2_r) \cong H_1((\mathsf{Star}_{k-1})^2_r) = 0 \tag{4.52}$$

and

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^2 \tag{4.53}$$

$\square$

Proposition 4.11. Let L_{e_1} be a positive number. If $L_{e_1} < r \leq L_{e_1} + 1$ and $r > 2$. Then

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^{2k-2} \quad \text{and} \quad H_1((\hat{\mathsf{Star}}_k)^2_r) = 0$$

Proof. Consider the cover of $(\hat{\mathsf{Star}}_k)^2_r$ given in Equation (4.34). Note that $U_{11} = (e_1)^2_r = \emptyset$. On the other hand, $U_{12} = e_1 \times \mathsf{Star}_{k-1} - \{(x,y) \in e_1 \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \simeq \{*_1, \ldots, *_{k-1}\}$ and $U_{21} = \mathsf{Star}_{k-1} \times e_1 - \{(x,y) \in \mathsf{Star}_{k-1} \times e_1 \mid \delta(x,y) < r\} \simeq \{*_1, \ldots, *_{k-1}\}$.

Now let's consider the intersections of U_{ij}. Note that

$$\begin{aligned} U_{11} \cap U_{12} = \emptyset = U_{11} \cap U_{21} \\ U_{22} \cap U_{12} = \emptyset = U_{22} \cap U_{21} \\ U_{11} \cap U_{22} = \emptyset = U_{12} \cap U_{21} \end{aligned} \tag{4.54}$$

The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc} q & & & \\ 2 & \vdots & 0 & 0 \\ 1 & H_1((\mathsf{Star}_{k-1})^2_r) & 0 & 0 \\ 0 & \mathbb{Z}^{2k-2} \oplus H_0((\mathsf{Star}_{k-1})^2_r) & 0 & 0 \\ \hline & 0 & 1 & 2 \quad p \end{array}$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_1((\hat{\mathsf{Star}}_k)^2_r) \cong H_1((\mathsf{Star}_{k-1})^2_r) = 0 \tag{4.55}$$

and

$$H_0((\hat{\mathsf{Star}}_k)^2_r) \cong \mathbb{Z}^{2k-2} \tag{4.56}$$

□

In summary, when $k \geq 4$, the rank of $H_0((\hat{\mathsf{Star}}_k)^2_r)$ for all $r > 0$ and $L_{e_1} > 0$ is shown in Figure 4.4. When $k \geq 4$, the rank of $H_1((\hat{\mathsf{Star}}_k)^2_r)$ for all $r > 0$ and $L_{e_1} > 0$ is shown in Figure 4.5.

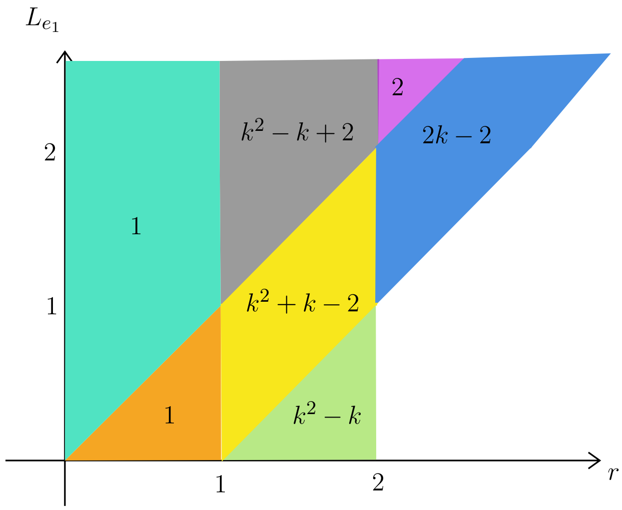


Figure 4.4: The rank of $H_0((\hat{\mathsf{Star}}_k)^2_r)$

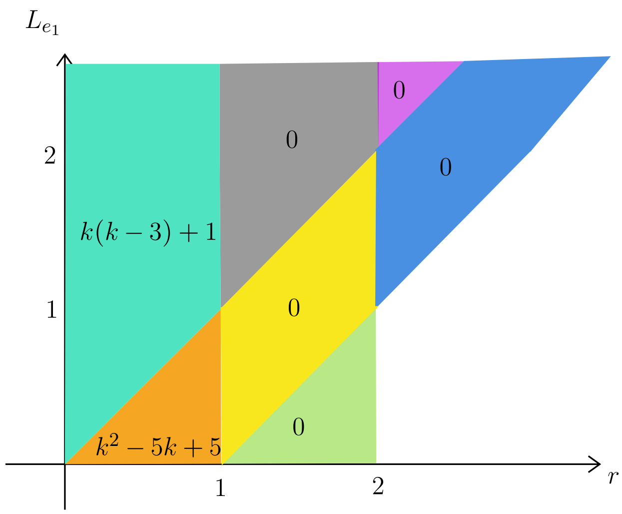


Figure 4.5: The rank of $H_1((\hat{\mathsf{Star}}_k)^2_r)$

4.4 Decomposition of $PH_i((\mathsf{Star_k})^2_{-,-};\mathbb{F})$

In the previous section, we computed $H_i((\hat{\mathsf{Star_k}})^2_r)$ for all $r, L_{e_1} \in \mathbb{R}_{>0}$. Recall that $\hat{\mathsf{Star_k}}$ is a metric star graph where the length of all but one edge (denoted by e_1) is 1, and the length of edge e_1 is L_{e_1}. In this section, we define

$$(\mathsf{Star}_k)^2_{r,L_{e_1}} := (\hat{\mathsf{Star}}_k)^2_r$$

to emphasize that $(\hat{\mathsf{Star_k}})^2_r$ is determined by two parameters r and L_{e_1}.

Note that the hyperplane arrangement of $(\mathsf{Star}_k)^2_{r,L_{e_1}}$ can be interpreted as a functor

$$(\mathsf{Star}_k)^2_{-,-} : (\mathbb{R}_{>0}, \leq)^{\mathsf{op}} \times (\mathbb{R}_{>0}, \leq) \to \textbf{Top}$$

where $(\mathsf{Star}_k)^2_{-,-}$ sends $(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ to $(\mathsf{Star}_k)^2_{a,b}$ and sends the unique arrow $(a,b) \to (a',b')$ to the inclusion map $\iota : (\mathsf{Star}_k)^2_{a,b} \to (\mathsf{Star}_k)^2_{a',b'}$, for all $a' \leq a$ and $b \leq b'$. Post-composing the i-th homology functor $H_i(-)$ with $(\mathsf{Star}_k)^2_{-,-}$, we obtain

$$PH_i((\mathsf{Star}_k)^2_{-,-}) : (\mathbb{R}_{>0}, \leq)^{\mathsf{op}} \times (\mathbb{R}_{>0}, \leq) \to \textbf{Ab}$$

In other words, at the object level, for each $(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$,

$$PH_i((\mathsf{Star}_k)^2_{a,b}) = H_i((\mathsf{Star}_k)^2_{a,b})$$

At the morphism level, $PH_i((\mathsf{Star}_k)^2_{-,-})$ sends each morphism $(a,b) \to (a',b')$ to a group homomorphism

$$\iota_* : H_i((\mathsf{Star}_k)^2_{a,b}) \to H_i((\mathsf{Star}_k)^2_{a',b'})$$

where ι_* is induced by the inclusion map $\iota : (\mathsf{Star}_k)^2_{a,b} \to (\mathsf{Star}_k)^2_{a',b'}$ in **Top**.

One natural question is whether or not it can be written as a direct sum of polycodes. If it is not a direct sum of polycodes, what are the indecomposable direct summands of $PH_i((\mathsf{Star_k})^2_{-,-};\mathbb{F})$? In this section, we give the decompositions of $PH_0((\mathsf{Star_k})^2_{-,-};\mathbb{F})$ and $PH_1((\mathsf{Star_k})^2_{-,-};\mathbb{F})$.

Note that there are finitely many chambers in the hyperplane arrangement of $(\mathsf{Star_k})^2_{r,L_{e_1}}$, we may associate the hyperplane arrangement with the Hasse diagram of a poset (denoted by $(P,\leq)$) as follows:

- each chamber of the hyperplane arrangement is an element of $(P,\leq)$;
- each arrow corresponds to a wall between two chambers, and the orientation of the arrow is given by the filtration of the spaces $(\mathsf{Star}_k)^2_{r,L_{e_1}}$.

We associate $PH_0((\mathsf{Star_k})^2_{-,-};\mathbb{F})$ with a representation over $(P,\leq)$:

$$
\begin{tikzcd}
\mathbb{F} & \mathbb{F}^{k^2-k+2} \arrow[l] & \mathbb{F}^2 \arrow[l] \\
\mathbb{F} \arrow[u] & \mathbb{F}^{k^2+k-2} \arrow[l] \arrow[u] & \mathbb{F}^{2k-2} \arrow[l] \arrow[u] \\
 & \mathbb{F}^{k^2-k} \arrow[u] &
\end{tikzcd}
$$

$$
= \begin{tikzcd}
< f > & < e_1, \hat{e}_1, f_1, \ldots, f_{k^2-k} > \arrow[l, "\alpha"'] & < e_1, \hat{e}_1 > \arrow[l, hook'] \\
< f > \arrow[u] & \left\langle \begin{matrix} e_1, \ldots, e_{k-1}, \hat{e}_1, \ldots, \hat{e}_{k-1}, \\ f_1, \ldots, f_{k^2-k} \end{matrix} \right\rangle \arrow[l, "\gamma"'] \arrow[u, "\beta"] & < e_1, \ldots, e_{k-1}, \hat{e}_1, \ldots, \hat{e}_{k-1} > \arrow[l, hook'] \arrow[u, "\epsilon"'] \\
 & < f_1, \ldots, f_{k^2-k} > \arrow[u, hook] &
\end{tikzcd}
\tag{4.57}
$$

where α and γ maps every basis element to f, β maps e_i to e_1, maps $\hat{e}_i$ to $\hat{e}_1$ for all $i = 1, \ldots, k-1$ and maps f_j to f_j itself for all $j = 1, \ldots, k^2-k$, and ϵ maps e_i to e_1 and maps $\hat{e}_i$ to $\hat{e}_1$ for all $i = 1, \ldots, k-1$. All other maps are inclusions. By an abuse of notation, we use $PH_0((\mathsf{Star_k})^2_{-,-};\mathbb{F})$ to denote the P-indexed persistence module given by (4.57).

Similarly, we associate $PH_1((\mathsf{Star_k})^2_{-,-};\mathbb{F})$ with a representation over $(P, \leq)$:

$$\begin{tikzcd}
\mathbb{F}^{k(k-3)+1} & 0 \arrow[l] & 0 \arrow[l] \\
\mathbb{F}^{k^2-5k+5} \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & 0 \arrow[u] &
\end{tikzcd} \tag{4.58}$$

By an abuse of notation, we use $PH_1((\mathsf{Star_k})^2_{-,-};\mathbb{F})$ to denote the P-indexed persistence module given by (4.58).

Theorem 4.12. $PH_1((\mathsf{Star_k})^2_{-,-};\mathbb{F})$ *is interval decomposable.*

Proof. The support of $PH_1((\mathsf{Star_k})^2_{-,-};\mathbb{F})$ is a A_2 quiver. Note that A_2 quivers are interval decomposable (Theorem 2.33), and there are only 3 types of thin indecomposable representations of a A_2 type quiver, up to isomorphism:

$$\mathbb{F} \to 0, \quad \mathbb{F} \xrightarrow{id} \mathbb{F}, \quad \text{and } 0 \to \mathbb{F}$$

Note that $\begin{tikzcd}
\mathbb{F} & 0 \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & 0 \arrow[u] &
\end{tikzcd}$ and $\begin{tikzcd}
0 & 0 \arrow[l] & 0 \arrow[l] \\
\mathbb{F} \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & 0 \arrow[u] &
\end{tikzcd}$ are indecomposable because

they are simple representations. On the other hand, note that

$$\mathsf{End}\left(\begin{tikzcd} \mathbb{F} & 0 \arrow[l] & 0 \arrow[l] \\ \mathbb{F} \arrow[u, "\mathsf{id}"] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ & 0 \arrow[u] & \end{tikzcd}\right) \cong \mathbb{F}$$

By Fitting's Lemma (Theorem 2.36), $\begin{tikzcd} \mathbb{F} & 0 \arrow[l] & 0 \arrow[l] \\ \mathbb{F} \arrow[u, "\mathsf{id}"] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ & 0 \arrow[u] & \end{tikzcd}$ is indecomposable.

Assume there exists $m, n > 0$ such that $M := \begin{tikzcd} \mathbb{F}^m & 0 \arrow[l] & 0 \arrow[l] \\ \mathbb{F}^n \arrow[u, "\alpha"] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ & 0 \arrow[u] & \end{tikzcd}$ is indecomposable.

Note that we can decompose $\alpha : \mathbb{F}^n \to \mathbb{F}^m$ into a direct sum of the thin indecomposable representations

$$\mathbb{F} \to 0, \quad \mathbb{F} \xrightarrow{id} \mathbb{F}, \quad \text{and } 0 \to \mathbb{F}$$

Extend these representation as

$$\begin{tikzcd}
\mathbb{F} & 0 \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
& 0 \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
0 & 0 \arrow[l] & 0 \arrow[l] \\
\mathbb{F} \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
& 0 \arrow[u] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{F} & 0 \arrow[l] & 0 \arrow[l] \\
\mathbb{F} \arrow[u, "\mathsf{id}"] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
& 0 \arrow[u] &
\end{tikzcd}$$

Since M is indecomposable and $m, n > 0$, we must have $m = n = 1$. Therefore, every indecomposable subrepresentation of $PH_1((\mathsf{Star_k})^2_{-,-};\mathbb{F})$ is isomorphic to one of the following thin indecomposable representations:

$$
\begin{tikzcd}
\mathbb{F} & 0 \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & 0 \arrow[u] &
\end{tikzcd},
\begin{tikzcd}
0 & 0 \arrow[l] & 0 \arrow[l] \\
\mathbb{F} \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & 0 \arrow[u] &
\end{tikzcd},
\begin{tikzcd}
\mathbb{F} & 0 \arrow[l] & 0 \arrow[l] \\
\mathbb{F} \arrow[u, "\mathsf{id}"] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & 0 \arrow[u] &
\end{tikzcd}
$$

Note that the supports of the above thin indecomposable representations are intervals (as posets). Therefore, $PH_1((\mathsf{Star}_\mathsf{k})^2_{-,-};\mathbb{F})$ is interval decomposable. $\square$

Theorem 4.13. *$PH_0((\hat{\mathsf{Star}}_\mathsf{k})^2_{-,-};\mathbb{F}) \cong M_1 \oplus M_2 \oplus (\bigoplus_{i=2}^{k-1} E_i) \oplus (\bigoplus_{i=2}^{k-1} \hat{E}_i) \oplus (\bigoplus_{j=2}^{k^2-k} F_j)$ where*

$$
M_1 = \begin{tikzcd}
\mathbb{F} & \mathbb{F}^2 \arrow[l, "{\begin{bmatrix}1 & 1\end{bmatrix}}"'] & \mathbb{F} \arrow[l, "{\begin{bmatrix}1\\0\end{bmatrix}}"'] \\
\mathbb{F} \arrow[u, "\mathsf{id}"] & \mathbb{F}^2 \arrow[l, "{\begin{bmatrix}1 & 1\end{bmatrix}}"'] \arrow[u, "\mathsf{id}"] & \mathbb{F} \arrow[l, "{\begin{bmatrix}1\\0\end{bmatrix}}"] \arrow[u, "\mathsf{id}"'] \\
 & \mathbb{F} \arrow[u, "{\begin{bmatrix}0\\1\end{bmatrix}}"] &
\end{tikzcd}
$$

$$
M_2 = \begin{tikzcd}
0 & \mathbb{F} \arrow[l] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u, "\mathsf{id}"] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u, "\mathsf{id}"'] \\
 & 0 \arrow[u] &
\end{tikzcd}
$$

and for $i = 2, \ldots, k-1$,

$$E_i = \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] \\ & 0 \arrow[u] & \end{tikzcd}$$

$$\hat{E}_i = \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] \\ & 0 \arrow[u] & \end{tikzcd}$$

and for $j = 2, \ldots, k^2-k$,

$$F_j = \begin{tikzcd} 0 & \mathbb{F} \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u, "\mathsf{id}"] & 0 \arrow[l] \arrow[u] \\ & \mathbb{F} \arrow[u, "\mathsf{id}"] & \end{tikzcd}$$

Proof. Equation 4.57 provides us with the behavior of each arrow with given basis elements. Our goal is to find a new basis of $PH_0((\mathsf{Star_k})^2_{r,L_{e_1}};\mathbb{F})$ for all $(r, L_{e_1}) \in \mathbb{R}\times\mathbb{R}$ such that each morphism in the diagram maps every basis to another basis or zero, depending on the geometry of $(\hat{\mathsf{Star_k}})^2_{r,L_{e_1}}$. We choose the basis for each vector space provided in the diagram below:

$$
\begin{tikzcd}
<f> & \left\langle e_1, \hat{e}_1 - e_1, f_1, f_2 - f_1, \dots, f_{k^2-k} - f_1 \right\rangle \arrow[l, "\alpha"'] & < e_1, \hat{e}_1 - e_1 > \arrow[l, hook'] \\
<f> \arrow[u] & \left\langle e_1, e_2 - e_1, \dots, e_{k-1} - e_1, \hat{e}_1 - e_1, \hat{e}_2 - \hat{e}_1, \dots, \hat{e}_{k-1} - \hat{e}_1, f_1, f_2 - f_1, \dots, f_{k^2-k} - f_1 \right\rangle \arrow[l, "\gamma"'] \arrow[u, "\beta"] & \left\langle e_1, e_2 - e_1, \dots, e_{k-1} - e_1, \hat{e}_1 - e_1, \hat{e}_2 - \hat{e}_1, \dots, \hat{e}_{k-1} - \hat{e}_1 \right\rangle \arrow[l, hook'] \arrow[u, "\epsilon"'] \\
& < f_1, f_2 - f_1, \dots, f_{k^2-k} - f_1 > \arrow[u, hook] &
\end{tikzcd}
$$

Let

$$
M_1 := \begin{tikzcd}
< f_1 > & < e_1, f_1 > \arrow[l, "\alpha"'] & < e_1 > \arrow[l, hook'] \\
< f_1 > \arrow[u] & < e_1, f_1 > \arrow[l, "\gamma"'] \arrow[u, "\beta"] & < e_1 > \arrow[l, hook'] \arrow[u, "\epsilon"'] \\
& < f_1 > \arrow[u, hook] &
\end{tikzcd}
\cong
\begin{tikzcd}
\mathbb{F} & \mathbb{F}^2 \arrow[l, "{\begin{bmatrix} 1 & 1 \end{bmatrix}}"'] & \mathbb{F} \arrow[l, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"'] \\
\mathbb{F} \arrow[u, "\mathsf{id}"] & \mathbb{F}^2 \arrow[l, "{\begin{bmatrix} 1 & 1 \end{bmatrix}}"'] \arrow[u, "\mathsf{id}"] & \mathbb{F} \arrow[l, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"] \arrow[u, "\mathsf{id}"'] \\
& \mathbb{F} \arrow[u, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}"] &
\end{tikzcd}
\tag{4.59}
$$

$$
M_2 := \begin{tikzcd}
0 & < \hat{e}_1 - e_1 > \arrow[l, "\alpha"'] & < \hat{e}_1 - e_1 > \arrow[l, hook'] \\
0 \arrow[u] & < \hat{e}_1 - e_1 > \arrow[l, "\gamma"'] \arrow[u, "\beta"] & < \hat{e}_1 - e_1 > \arrow[l, hook'] \arrow[u, "\epsilon"'] \\
& 0 \arrow[u, hook] &
\end{tikzcd}
\cong
\begin{tikzcd}
0 & \mathbb{F} \arrow[l] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u, "\mathsf{id}"] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u, "\mathsf{id}"'] \\
& 0 \arrow[u] &
\end{tikzcd}
\tag{4.60}
$$

For $i = 2, \dots, k-1$

$$E_i := \begin{tikzcd} 0 & 0 \arrow[l, "\alpha"'] & 0 \arrow[l, hook'] \\ 0 \arrow[u] & < e_i - e_1 > \arrow[l, "\gamma"'] \arrow[u, "\beta"] & < e_i - e_1 > \arrow[l, hook'] \arrow[u, "\epsilon"'] \\ & 0 \arrow[u, hook] & \end{tikzcd} \cong \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] \\ & 0 \arrow[u] & \end{tikzcd} \tag{4.61}$$

$$\hat{E}_i := \begin{tikzcd} 0 & 0 \arrow[l, "\alpha"'] & 0 \arrow[l, hook'] \\ 0 \arrow[u] & < \hat{e}_i - \hat{e}_1 > \arrow[l, "\gamma"'] \arrow[u, "\beta"] & < \hat{e}_i - \hat{e}_1 > \arrow[l, hook'] \arrow[u, "\epsilon"'] \\ & 0 \arrow[u, hook] & \end{tikzcd} \cong \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] \\ & 0 \arrow[u] & \end{tikzcd} \tag{4.62}$$

For $j = 2, \dots, k^2 - k$

$$F_j := \begin{tikzcd} 0 & < f_j - f_1 > \arrow[l, "\alpha"'] & 0 \arrow[l, hook'] \\ 0 \arrow[u] & < f_j - f_1 > \arrow[l, "\gamma"'] \arrow[u, "\beta"] & 0 \arrow[l, hook'] \arrow[u, "\epsilon"'] \\ & < f_j - f_1 > \arrow[u, hook] & \end{tikzcd} \cong \begin{tikzcd} 0 & \mathbb{F} \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u, "\mathsf{id}"] & 0 \arrow[l] \arrow[u] \\ & \mathbb{F} \arrow[u, "\mathsf{id}"] & \end{tikzcd} \tag{4.63}$$

Note that M_2, E_i, $\hat{E}_i$, and F_j (where $i = 2, \dots, k-1$ and $j = 2, \dots, k^2 - k$) are interval modules, hence by Lemma 3.1, they are indecomposable. Now we are going to show that M_1 is indecomposable. By the Fitting's lemma (Theorem 2.36), it suffices to show that $\mathsf{End}(M_1)$ does not contain any idempotents except 0 and id.

Note that any linear transformation $\mathbb{F} \to \mathbb{F}$ is a scalar multiplication. We use $x : \mathbb{F} \to \mathbb{F}$ to denote the linear transformation that sends f to xf for all $f \in \mathbb{F}$. Let $\phi \in \mathsf{End}(M_1)$. It consists of the following data:

- $x : \mathbb{F} \to \mathbb{F}$;
- $y : \mathbb{F} \to \mathbb{F}$;
- $z : \mathbb{F} \to \mathbb{F}$;
- $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the morphism $\mathbb{F}^2 \to \mathbb{F}^2$ under the given basis,

such that

$$x \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.64}$$

$$y \circ [1 \quad 1] = A \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.65}$$

$$z \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.66}$$

Equation 4.64 implies

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \tag{4.67}$$

Hence $a = x$ and $c = 0$. Update $A = \begin{bmatrix} x & b \\ 0 & d \end{bmatrix}$.

Equation 4.65 implies

$$[x \quad b+d] = [y \quad y] \tag{4.68}$$

Hence $x = y$ and $b + d = y$. Update $A = \begin{bmatrix} x & b \\ 0 & x - b \end{bmatrix}$.

Equation 4.66 implies

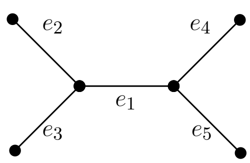
$$\begin{bmatrix} b \\ x - b \end{bmatrix} = \begin{bmatrix} 0 \\ z \end{bmatrix} \tag{4.69}$$

Hence $b = 0$ and $x = z$. Update $A = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$. Therefore, $\mathsf{End}(M_1) \cong \mathbb{F}$. Since $\mathbb{F}$ is a field, it does not contain any idempotents except 0 and id. Thus M_1 is indecomposable.

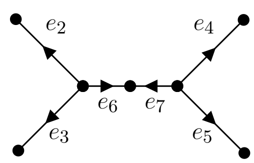
$\square$

4.5 Configuration Spaces of the H Graph

An H graph is a tree shown in Figure 4.6a. We want to assign each edge an orientation that matches the orientation we assigned for the Y graph. Hence we subdivide the bridge of H graph by introducing an artificial vertex, as shown in Figure 4.6b. We denote the geometric realization of the resulting graph by $\mathcal{H}$ and use $\hat{\mathcal{H}}$ to denote the metric graph $\mathcal{H}$ where the length of the bridge is L_{e_1} and the length of any other edge is 1. In this section, we describe the second configuration spaces of $\hat{\mathcal{H}}$ with restraint parameters r and L_{e_1}.



(a)



(b)

Figure 4.6

Let L_{e_i} denote the length of the e_i. The parametric polytope of $\mathcal{H}^2_{r,\vec{L}}$ is given by the following inequalities, where $\vec{L} = (L_{e_2}, \ldots, L_{e_7})$ and $L_{e_6} + L_{e_7} = L_{e_1}$:

For $i, j \in \{2,3\}$**,** $i \neq j$**:**

$$\begin{aligned} 0 &\leq x \leq L_{e_i} \\ 0 &\leq y \leq L_{e_j} \\ x + y &\geq r \end{aligned} \tag{4.70}$$

For $i, j \in \{4,5\}$**,** $i \neq j$**:**

$$\begin{aligned} 0 &\leq x \leq L_{e_i} \\ 0 &\leq y \leq L_{e_j} \\ x + y &\geq r \end{aligned} \tag{4.71}$$

For $i \in \{2,3\}$**,** $j = 6$**:**

$$\begin{aligned} 0 &\leq x \leq L_{e_i} \\ 0 &\leq y \leq L_{e_6} \\ x + y &\geq r \end{aligned} \tag{4.72}$$

For $j \in \{2,3\}$**,** $i = 6$**:**

$$\begin{aligned} 0 &\leq x \leq L_{e_i} \\ 0 &\leq y \leq L_{e_j} \\ x + y &\geq r \end{aligned} \tag{4.73}$$

For $i \in \{2,3\}$**,** $j = 7$**:**

$$\begin{aligned} 0 &\leq x \leq L_{e_i} \\ 0 &\leq y \leq L_{e_7} \\ x + L_{e_1} - y &\geq r \end{aligned} \tag{4.74}$$

For $j \in \{2,3\}$**,** $i = 7$**:**

$$\begin{aligned} 0 \leq x &\leq L_{e_7} \\ 0 \leq y &\leq L_{e_j} \\ -x + L_{e_1} + y &\geq r \end{aligned} \tag{4.75}$$

For $i, j \in \{6,7\}$**,** $i \neq j$**:**

$$\begin{aligned} 0 \leq x &\leq L_{e_i} \\ 0 \leq y &\leq L_{e_j} \\ -x + L_{e_1} - y &\geq r \end{aligned} \tag{4.76}$$

For $i \in \{2,3\}$**,** $j \in \{4,5\}$**:**

$$\begin{aligned} 0 \leq x &\leq L_{e_i} \\ 0 \leq y &\leq L_{e_j} \\ x + L_{e_1} + y &\geq r \end{aligned} \tag{4.77}$$

For $i \in \{4,5\}$**,** $j \in \{2,3\}$**:**

$$\begin{aligned} 0 \leq x &\leq L_{e_i} \\ 0 \leq y &\leq L_{e_j} \\ x + L_{e_1} + y &\geq r \end{aligned} \tag{4.78}$$

For $i \in \{4,5\}$**,** $j = 7$**:**

$$\begin{aligned} 0 \leq x &\leq L_{e_i} \\ 0 \leq y &\leq L_{e_j} \\ x + y &\geq r \end{aligned} \tag{4.79}$$

For $j \in \{4,5\}$**,** $i = 7$**:**

$$\begin{aligned} 0 \leq x \leq L_{e_i} \\ 0 \leq y \leq L_{e_j} \\ x + y \geq r \end{aligned} \tag{4.80}$$

For $i \in \{4,5\}$**,** $j = 6$**:**

$$\begin{aligned} 0 \leq x \leq L_{e_i} \\ 0 \leq y \leq L_{e_j} \\ x + L_{e_1} - y \geq r \end{aligned} \tag{4.81}$$

For $j \in \{4,5\}$**,** $i = 6$**:**

$$\begin{aligned} 0 \leq x \leq L_{e_i} \\ 0 \leq y \leq L_{e_j} \\ -x + L_{e_1} + y \geq r \end{aligned} \tag{4.82}$$

For $i, j \in \{2,3,4,5,6,7\}$**,** $i = j$**,** $x < y$**:**

$$\begin{aligned} 0 \leq x \leq L_{e_i} \\ 0 \leq y \leq L_{e_i} \\ -x + y \geq r \end{aligned} \tag{4.83}$$

For $i, j \in \{2,3,4,5,6,7\}$**,** $i = j$**,** $x > y$**:**

$$\begin{aligned} 0 \leq x \leq L_{e_i} \\ 0 \leq y \leq L_{e_i} \\ x - y \geq r \end{aligned} \tag{4.84}$$

When $\vec{L} = (1, \ldots, 1)$ and $L_{e_6} = L_{e_7} = \frac{1}{2} L_{e_1}$,

$$\hat{\mathcal{H}} = \hat{\mathsf{Star}}_3 \vee \hat{\mathsf{Star}}_3$$

and the critical hyperplanes in the parameter space are:

$$\begin{aligned} r &= 1 \\ r &= 2 \\ r &= \frac{1}{2} L_{e_1} \\ r &= 1 + \frac{1}{2} L_{e_1} \\ r &= 1 + L_{e_1} \\ r &= L_{e_1} \\ r &= 2 + L_{e_1} \end{aligned} \tag{4.85}$$

Consider the following cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$:

$$\begin{aligned} U_{11} &= (\hat{\mathsf{Star}}_3)^2_r \\ U_{12} &= \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 - \{(x,y) \in \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 \mid \delta(x,y) < r\} \\ U_{21} &= \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 - \{(y,x) \in \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 \mid \delta(x,y) < r\} \\ U_{22} &= (\hat{\mathsf{Star}}_3)^2_r \end{aligned} \tag{4.86}$$

Note that there are natural inclusions

$$\begin{aligned}
U_{11} \cap U_{12} &\hookrightarrow U_{11} \\
U_{11} \cap U_{12} &\hookrightarrow U_{12} \\
U_{11} \cap U_{21} &\hookrightarrow U_{11} \\
U_{11} \cap U_{21} &\hookrightarrow U_{21} \\
U_{22} \cap U_{12} &\hookrightarrow U_{22} \\
U_{22} \cap U_{12} &\hookrightarrow U_{12} \\
U_{22} \cap U_{21} &\hookrightarrow U_{22} \\
U_{22} \cap U_{21} &\hookrightarrow U_{21}
\end{aligned} \tag{4.87}$$

Lemma 4.1. $\vec{L} = (1, \dots, 1)$ and $L_{e_6} = L_{e_7} = \frac{1}{2} L_{e_1}$. Then

$$H_0(\hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 - \{(x,y) \in \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 \mid \delta(x,y) < r\}) \cong \begin{cases} \mathbb{Z}, & \text{if } 0 < r \leq L_{e_1} + 1 \\ \mathbb{Z}^4, & \text{if } L_{e_1} + 1 < r \leq L_{e_1} + 2 \\ 0, & \text{else} \end{cases}$$

and

$$H_1(\hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 - \{(x,y) \in \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 \mid \delta(x,y) < r\}) \cong \begin{cases} \mathbb{Z}, & \text{if } L_{e_1} < r \leq L_{e_1} + 1 \\ 0, & \text{else} \end{cases}$$

Proof. Consider the following cover of $\hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 - \{(x,y) \in \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 \mid \delta(x,y) < r\}$:

$$\begin{aligned}
V_1 &= \hat{\mathsf{Star}}_3 \times e_7 - \{(x,y) \in \hat{\mathsf{Star}}_3 \times e_7 \mid \delta(x,y) < r\} \\
V_2 &= \hat{\mathsf{Star}}_3 \times (e_4 \vee e_5) - \{(x,y) \in \hat{\mathsf{Star}}_3 \times (e_4 \vee e_5) \mid \delta(x,y) < r\}
\end{aligned} \tag{4.88}$$

Note that

$$V_1 \simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq L_{e_1} \\ \{*_1, *_2\}, & \text{if} L_{e_1} < r \leq L_{e_1} + 1 \\ \emptyset & \text{else} \end{cases}$$

$$V_2 \simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq L_{e_1} \\ S^1, & \text{if} L_{e_1} < r \leq L_{e_1} + 1 \\ \{*_1, *_2, *_3, *_4\}, & \text{if} L_{e_1} + 1 < r \leq L_{e_1} + 2 \\ \emptyset & \text{else} \end{cases} \tag{4.89}$$

and

$$V_1 \cap V_2 \simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq L_{e_1} \\ \{*_1, *_2\}, & \text{if} L_{e_1} < r \leq L_{e_1} + 1 \\ \emptyset & \text{else} \end{cases} \tag{4.90}$$

When $0 < r \leq L_{e_1}$**:** The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|cccc} q & & & & \\ 2 & \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & 0 & \\ 0 & \mathbb{Z}^2 & \xleftarrow{d^1} \mathbb{Z} & 0 & \\ \hline & 0 & 1 & 2 & p \end{array}$$

We want to understand the behavior of d^1, where d^1 is induced from the following maps:

$$V_1 \cap V_2 \hookrightarrow V_1$$

$$V_1 \cap V_2 \hookrightarrow V_2$$

Let a be the generator of $H_0(V_1 \cap V_2)$. Choose a generator b of $H_0(V_1)$ and a generator c of $H_0(V_2)$ such that $d^1(a) = b + c$. Thus $\dim \mathsf{Im}\, d^1 = 1$ and $\dim \mathsf{ker} d^1 = 0$, and we obtain the E^2 page of the spectral sequence:

$$
\begin{array}{c|ccc}
q & & & \\
2 & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0(\hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 - \{(x,y) \in \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 \mid \delta(x,y) < r\}) \cong \mathbb{Z}$$

and

$$H_1(\hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 - \{(x,y) \in \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 \mid \delta(x,y) < r\}) = 0$$

When $L_{e_1} < r \leq L_{e_1} + 1$: The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z} & 0 & 0 \\
0 & \mathbb{Z}^3 & \xleftarrow{d^1} \mathbb{Z}^2 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

We want to understand the behavior of d^1, where d^1 is induced from the following maps:

$$V_1 \cap V_2 \hookrightarrow V_1$$

$$V_1 \cap V_2 \hookrightarrow V_2$$

Let a_1, a_2 be the generators of $H_0(V_1 \cap V_2)$. Choose generators b_1 and b_2 for $H_0(V_1)$ and a generator c for $H_0(V_2)$ such that

- b_1 and a_1 lie in the same path component;
- b_2 and a_2 lie in the same path component.

Hence

$$d^1(a_1) = b_1 + c$$

$$d^1(a_2) = b_2 + c$$

Consider the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The rank of the matrix is 2, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Thus $\dim \mathsf{Im}\, d^1 = 2$ and $\dim \mathsf{ker} d^1 = 1$, and we obtain the E^2 page of the spectral sequence:

$$
\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & \\
1 & \mathbb{Z} & 0 & 0 & \\
0 & \mathbb{Z} & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0(\hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 - \{(x,y) \in \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 \mid \delta(x,y) < r\}) \cong \mathbb{Z}$$

and

$$H_1(\hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 - \{(x,y) \in \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 \mid \delta(x,y) < r\}) \cong \mathbb{Z}$$

When $L_{e_1} + 1 < r \leq L_{e_1} + 2$: The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|cccc}
q & & & & \\
2 & \vdots & \vdots & \vdots & \\
1 & 0 & 0 & 0 & \\
0 & \mathbb{Z}^4 & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_0(\hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 - \{(x,y) \in \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 \mid \delta(x,y) < r\}) \cong \mathbb{Z}^4$$

and

$$H_1(\hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 - \{(x,y) \in \hat{\mathsf{Star}}_3 \times \hat{\mathsf{Star}}_3 \mid \delta(x,y) < r\}) = 0$$

$\square$

Proposition 4.14. Let L_{e_1} be a positive number. If $r \leq \frac{1}{2}L_{e_1}$ and $r \leq 1$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^3$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & \\
1 & \mathbb{Z}^2 & 0 & 0 & \\
0 & \mathbb{Z}^4 & \xleftarrow{d^1} \mathbb{Z}^4 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.87). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let b be the generator of $H_0(U_{22} \cap U_{12})$ and $\hat{b}$ be the generator of $H_0(U_{22} \cap U_{21})$. Choose a generator (denoted by c) for $H_0(U_{11})$, (denoted by $\hat{c}$) for $H_0(U_{22})$, and let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$\begin{aligned}
d^1(a) &= c + e \\
d^1(\hat{a}) &= \hat{c} + \hat{e} \\
d^1(b) &= c + e \\
d^1(\hat{b}) &= \hat{c} + \hat{e}
\end{aligned} \tag{4.91}$$

Consider the following matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The rank of this matrix is 3, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^3$ and $\mathsf{ker} d^1 \cong \mathbb{Z}$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z}^2 & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^3$$

□

Proposition 4.15. Let L_{e_1} be a positive number. If $\frac{1}{2}L_{e_1} < r \leq L_{e_1}$ and $r \leq 1$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^3$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z}^6 & \xleftarrow{d^1} \mathbb{Z}^8 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.87). Let a_1, a_2 be the generators of $H_0(U_{11} \cap U_{12})$ such that a_1 and a_2 lie in different path components of $U_{11} \cap U_{12}$. Similarly, let $\hat{a}_1, \hat{a}_2$ be the generators of $H_0(U_{11} \cap U_{21})$, let b_1, b_2 be the generators of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_1, \hat{b}_2$ be the generators of $H_0(U_{22} \cap U_{21})$ such that the generators of each abelian group lie in different path components. Choose generators (denoted by c_3 and c_6) for $H_0(U_{11})$, (denoted by $\hat{c}_3$ and $\hat{c}_6$) for $H_0(U_{22})$, and let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$
\begin{aligned}
d^1(a_1) &= c_3 + e \\
d^1(a_2) &= c_6 + e \\
d^1(\hat{a}_1) &= c_3 + \hat{e} \\
d^1(\hat{a}_2) &= c_6 + \hat{e} \\
d^1(b_1) &= \hat{c}_3 + e \\
d^1(b_2) &= \hat{c}_6 + e \\
d^1(\hat{b}_1) &= \hat{c}_3 + \hat{e} \\
d^1(\hat{b}_2) &= \hat{c}_6 + \hat{e}
\end{aligned}
\tag{4.92}
$$

Consider the following matrix

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 5, and the diagonal elements of the Smith Normal form of the matrix above are all equal to ± 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^5$ and $\mathsf{ker} d^1 \cong \mathbb{Z}^3$. Hence the E^2 page of the Mayer-Vietoris spectral sequence is:

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}^3 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^3$$

$\square$

Proposition 4.16. Let L_{e_1} be a positive number. If $L_{e_1} < r \leq 1$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^5$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & \\
1 & \mathbb{Z}^2 & 0 & 0 & \\
0 & \mathbb{Z}^6 & \xleftarrow{d^1} \mathbb{Z}^8 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.87). Let a_1, a_2 be the generators of $H_0(U_{11} \cap U_{12})$ such that a_1 and a_2 lie in different path components of $U_{11} \cap U_{12}$. Similarly, let $\hat{a}_1, \hat{a}_2$ be the generators of $H_0(U_{11} \cap U_{21})$ such that $\hat{a}_1$ and $\hat{a}_2$ lie in different path components of $U_{11} \cap U_{21}$. Let b_1, b_2 be the generators of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_1, \hat{b}_2$ be the generators of $H_0(U_{22} \cap U_{21})$ such that the generators of each abelian group lie in different path components. Choose generators for $H_0(U_{11})$ (denoted by c_3 and c_6) and $H_0(U_{22})$ (denoted by $\hat{c}_3$ and $\hat{c}_6$). Let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$
\begin{aligned}
d^1(a_1) &= c_3 + e \\
d^1(a_2) &= c_6 + e \\
d^1(\hat{a}_1) &= c_3 + \hat{e} \\
d^1(\hat{a}_2) &= c_6 + \hat{e} \\
d^1(b_1) &= \hat{c}_3 + e \\
d^1(b_2) &= \hat{c}_6 + e \\
d^1(\hat{b}_1) &= \hat{c}_3 + \hat{e} \\
d^1(\hat{b}_2) &= \hat{c}_6 + \hat{e}
\end{aligned}
\tag{4.93}
$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 5, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^5$ and $\mathsf{ker} d^1 \cong \mathbb{Z}^3$. Hence the E^2 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z}^2 & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}^3 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^5$$

$\square$

Proposition 4.17. Let L_{e_1} be a positive number. If $1 < r \leq 2$ and $r \leq \frac{1}{2}L_{e_1}$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^6 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z}^{10} & \xleftarrow{d^1} \mathbb{Z}^4 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.87). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let b be the generator of $H_0(U_{22} \cap U_{12})$ and $\hat{b}$ be the generator of $H_0(U_{22} \cap U_{21})$. Choose generators (denoted by c_1, c_3, c_4 and c_6) for $H_0(U_{11})$ such that no two generators lie in the same path component of U_{11}. Similarly, choose generators (denoted by $\hat{c}_1$, $\hat{c}_3$, $\hat{c}_4$ and $\hat{c}_6$) for $H_0(U_{22})$ such that no two generators lie in the same path component of U_{22}. Moreover, let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$\begin{aligned} d^1(a) &= c_1 + e \\ d^1(\hat{a}) &= c_4 + \hat{e} \\ d^1(b) &= \hat{c}_1 + e \\ d^1(\hat{b}) &= \hat{c}_4 + \hat{e} \end{aligned} \tag{4.94}$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 4, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^4$ and $\mathsf{ker} d^1 = 0$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & \\
1 & 0 & 0 & 0 & \\
0 & \mathbb{Z}^6 & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^6 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

$\square$

Proposition 4.18. Let L_{e_1} be a positive number. If $1 < r \leq 2$ and $\frac{1}{2}L_{e_1} < r \leq L_{e_1}$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^6 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & \\
1 & 0 & 0 & 0 & \\
0 & \mathbb{Z}^{14} \xleftarrow{d^1} & \mathbb{Z}^{8} & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.87). Let a_1, a_2 be the generators of $H_0(U_{11} \cap U_{12})$ and $\hat{a}_1, \hat{a}_2$ be the generators of $H_0(U_{11} \cap U_{21})$. Let b_1, b_2 be the generators of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_1, \hat{b}_2$ be the generators of $H_0(U_{22} \cap U_{21})$. Choose generators (denoted by $c_1, \ldots, c_6$) for $H_0(U_{11})$ such that no two generators lie in the same path component of U_{11}. Similarly, choose generators (denoted by $\hat{c}_1, \ldots, \hat{c}_6$) for $H_0(U_{22})$ such that no two generators lie in the same path component of U_{22}. Moreover, let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$
\begin{aligned}
d^1(a_1) &= c_1 + e \\
d^1(a_2) &= c_2 + e \\
d^1(\hat{a}_1) &= c_4 + \hat{e} \\
d^1(\hat{a}_2) &= c_5 + \hat{e} \\
d^1(b_1) &= \hat{c}_1 + e \\
d^1(b_2) &= \hat{c}_2 + e \\
d^1(\hat{b}_1) &= \hat{c}_4 + \hat{e} \\
d^1(\hat{b}_2) &= \hat{c}_5 + \hat{e}
\end{aligned}
\tag{4.95}
$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 8, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^8$ and $\mathsf{ker} d^1 = 0$. Hence we obtain the

E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & \\
1 & 0 & 0 & 0 & \\
0 & \mathbb{Z}^6 & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^6 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

$\square$

Proposition 4.19. Let L_{e_1} be a positive number. If $1 < r \leq 2$ and $L_{e_1} < r \leq \frac{1}{2}L_{e_1} + 1$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^6 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & \\
1 & \mathbb{Z}^2 & 0 & 0 & \\
0 & \mathbb{Z}^{14} \xleftarrow{d^1} & \mathbb{Z}^8 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.87). Let a_1, a_2 be the generators of $H_0(U_{11} \cap U_{12})$ and $\hat{a}_1, \hat{a}_2$ be the generators of $H_0(U_{11} \cap U_{21})$. Let b_1, b_2 be the generators of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_1, \hat{b}_2$ be the generators of $H_0(U_{22} \cap U_{21})$. Choose generators (denoted by $c_1, \ldots, c_6$) for $H_0(U_{11})$ such that no two generators lie in the same path component of U_{11}. Similarly, choose generators (denoted by $\hat{c}_1, \ldots, \hat{c}_6$) for $H_0(U_{22})$ such that no two generators lie in the same path component of U_{22}. Moreover, let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$\begin{aligned}
d^1(a_1) &= c_1 + e \\
d^1(a_2) &= c_2 + e \\
d^1(\hat{a}_1) &= c_4 + \hat{e} \\
d^1(\hat{a}_2) &= c_5 + \hat{e} \\
d^1(b_1) &= \hat{c}_1 + e \\
d^1(b_2) &= \hat{c}_2 + e \\
d^1(\hat{b}_1) &= \hat{c}_4 + \hat{e} \\
d^1(\hat{b}_2) &= \hat{c}_5 + \hat{e}
\end{aligned} \tag{4.96}$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 8, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^8$ and $\mathsf{ker} d^1 = 0$. Hence we obtain the

E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & \\
1 & \mathbb{Z}^2 & 0 & 0 & \\
0 & \mathbb{Z}^6 & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^6 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

□

Proposition 4.20. Let L_{e_1} be a positive number. If $1 < r \leq 2$ and $\frac{1}{2}L_{e_1} + 1 < r \leq L_{e_1} + 1$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^6 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z}^2 & 0 & 0 \\
0 & \mathbb{Z}^6 & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^6 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}$$

$\square$

Proposition 4.21. Let L_{e_1} be a positive number. If $1 < r \leq 2$ and $L_{e_1} + 1 < r$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{12} \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z}^{12} & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{12} \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

□

Proposition 4.22. Let L_{e_1} be a positive number. If $2 < r$ and $r \leq \frac{1}{2}L_{e_1}$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2} \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z}^6 & \xleftarrow{d^1} \mathbb{Z}^4 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.87). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let b be the generator of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_1$ be the generator of $H_0(U_{22} \cap U_{21})$. Choose generators (denoted by c_1, c_4) for $H_0(U_{11})$ such that no two generators lie in the same path component of U_{11}. Similarly, choose generators (denoted by $\hat{c}_1, \hat{c}_4$) for $H_0(U_{22})$ such that no two generators lie in the same path component of U_{22}. Moreover, let e be the generator of $H_0(U_{12})$ and $\hat{e}$

be the generator of $H_0(U_{21})$ such that

$$\begin{aligned} d^1(a) &= c_1 + e \\ d^1(\hat{a}) &= c_4 + \hat{e} \\ d^1(b) &= \hat{c}_1 + e \\ d^1(\hat{b}) &= \hat{c}_4 + \hat{e} \end{aligned} \tag{4.97}$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 4, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^4$ and $\mathsf{ker} d^1 = 0$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z}^2 & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

□

Proposition 4.23. Let L_{e_1} be a positive number. If $2 < r$ and $\frac{1}{2}L_{e_1} < r \leq \frac{1}{2}L_{e_1} + 1$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z}^{10} \xleftarrow{d^1} & \mathbb{Z}^8 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

We want to understand the behavior of d^1, where d^1 is induced by inclusions (4.87). Let a_1, a_2 be the generators of $H_0(U_{11} \cap U_{12})$ and $\hat{a}_1, \hat{a}_2$ be the generators of $H_0(U_{11} \cap U_{21})$. Let b_1, b_2 be the generators of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_1, \hat{b}_2$ be the generators of $H_0(U_{22} \cap U_{21})$. Choose generators (denoted by c_1, c_2, c_4, c_5) for $H_0(U_{11})$ such that no two generators lie in the same path component of U_{11}. Similarly, choose generators (denoted by $\hat{c}_1, \hat{c}_2, \hat{c}_4, \hat{c}_5$) for $H_0(U_{22})$ such that no two generators lie in the same path component of U_{22}. Moreover, let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$\begin{aligned}
d^1(a_1) &= c_1 + e \\
d^1(a_2) &= c_2 + e \\
d^1(\hat{a}_1) &= c_4 + \hat{e} \\
d^1(\hat{a}_2) &= c_5 + \hat{e} \\
d^1(b_1) &= \hat{c}_1 + e \\
d^1(b_2) &= \hat{c}_2 + e \\
d^1(\hat{b}_1) &= \hat{c}_4 + \hat{e} \\
d^1(\hat{b}_2) &= \hat{c}_5 + \hat{e}
\end{aligned} \tag{4.98}$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Run the row reduction, we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 8, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^8$ and $\mathsf{ker} d^1 = 0$. Hence we obtain the

E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & \\
1 & 0 & 0 & 0 & \\
0 & \mathbb{Z}^2 & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

and

$$H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

$\square$

Proposition 4.24. Let L_{e_1} be a positive number. If $2 < r$ and $\frac{1}{2}L_{e_1} + 1 < r \leq L_{e_1}$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). Note that when $2 < r$ and $\frac{1}{2}L_{e_1} + 1 < r \leq L_{e_1}$,

$$U_{11} = \emptyset = U_{22}$$

The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z}^2 & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

$\square$

Proposition 4.25. Let L_{e_1} be a positive number. If $2 < r$ and $L_{e_1} < r \leq L_{e_1} + 1$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). Note that when $2 < r$ and $L_{e_1} < r \leq L_{e_1} + 1$, $U_{11} = \emptyset = U_{22}$. The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z}^2 & 0 & 0 \\
0 & \mathbb{Z}^2 & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

□

Proposition 4.26. Let L_{e_1} be a positive number. If $2 < r$ and $L_{e_1} + 1 < r \leq L_{e_1} + 2$, then

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^8 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}})^2_{r,L_{e_1}}$ given in Equation (4.86). Note that when $2 < r$ and $L_{e_1} + 1 < r \leq L_{e_1} + 2$, $U_{11} = \emptyset = U_{22}$. The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z}^8 & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}}) \cong \mathbb{Z}^8 \quad \text{and} \quad H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}}) = 0$$

□

In summary, the rank of $H_0((\hat{\mathcal{H}})^2_{r,L_{e_1}})$ for all $r > 0$ and $L_{e_1} > 0$ is shown in Figure 4.7, and the rank of $H_1((\hat{\mathcal{H}})^2_{r,L_{e_1}})$ for all $r > 0$ and $L_{e_1} > 0$ is shown in Figure 4.8.

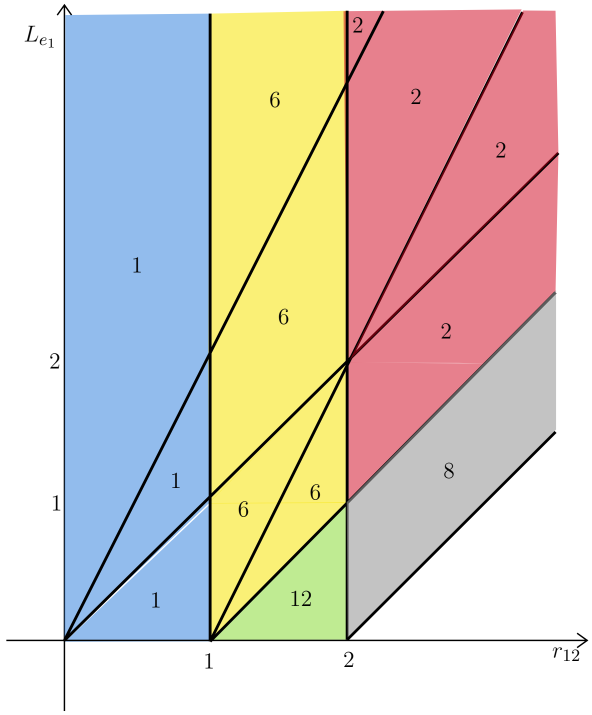


Figure 4.7

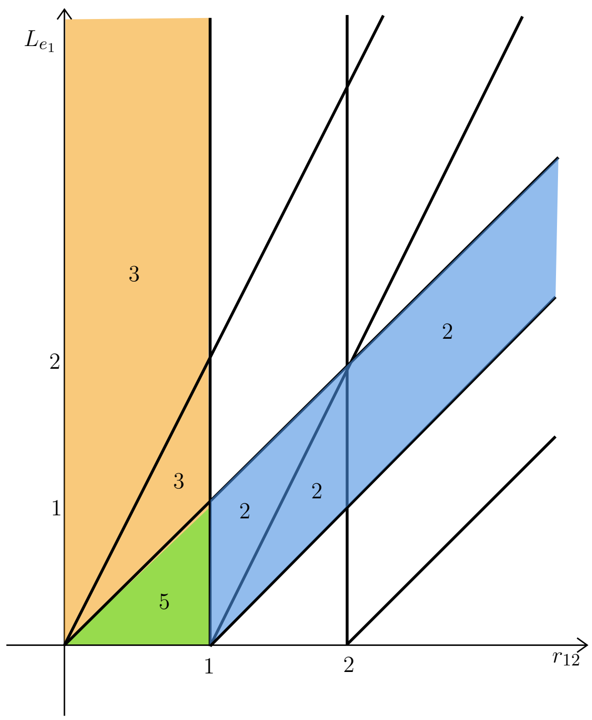


Figure 4.8

4.6 Decomposition of $PH_i(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$

In the previous section, we computed $H_i(\hat{\mathcal{H}}^2_{r,L_{e_1}})$ for all $r > 0$ and $L_{e_1} > 0$. Note that the hyperplane arrangement of $\hat{\mathcal{H}}^2_{r,L_{e_1}}$ can be interpreted as a functor

$$\hat{\mathcal{H}}^2_{-,-} : (\mathbb{R}_{>0}, \leq)^{\mathsf{op}} \times (\mathbb{R}_{>0}, \leq) \to \mathbf{Top}$$

where $\hat{\mathcal{H}}^2_{-,-}$ sends $(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ to $\hat{\mathcal{H}}^2_{a,b}$ and sends the unique arrow $(a,b) \to (a',b')$ to the inclusion map $\iota : \hat{\mathcal{H}}^2_{a,b} \to \hat{\mathcal{H}}^2_{a',b'}$, for all $a' \leq a$ and $b \leq b'$. Post-composing the i-th homology functor $H_i(-)$ with $\hat{\mathcal{H}}^2_{-,-}$, we obtain

$$PH_i(\hat{\mathcal{H}}^2_{-,-}) : (\mathbb{R}_{>0}, \leq)^{\mathsf{op}} \times (\mathbb{R}_{>0}, \leq) \to \mathbf{Ab}$$

At the object level, for each $(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$,

$$PH_i(\hat{\mathcal{H}}^2_{a,b}) = H_i(\hat{\mathcal{H}}^2_{a,b})$$

At the morphism level, $PH_i(\hat{\mathcal{H}}^2_{-,-})$ sends each morphism $(a,b) \to (a',b')$ to a group homomorphism

$$\iota_* : H_i(\hat{\mathcal{H}}^2_{a,b}) \to H_i(\hat{\mathcal{H}}^2_{a',b'})$$

where ι_* is induced by the inclusion map $\iota : \hat{\mathcal{H}}^2_{a,b} \to \hat{\mathcal{H}}^2_{a',b'}$ in $\mathbf{Top}$.

One natural question is whether or not it can be written as a direct sum of polycodes. In this section, we give the decompositions of $PH_0(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$ and $PH_1(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$.

Because there are finitely many chambers in the hyperplane arrangement of $\hat{\mathcal{H}}^2_{r,L_{e_1}}$, we may associate the hyperplane arrangement with the Hasse diagram of a poset (denoted by

$(P,\leq))$ as follows:

- each chamber of the hyperplane arrangement is an element of $(P,\leq)$;
- each arrow corresponds to a wall between two chambers, and the orientation of the arrow is given by the filtration of the spaces $\hat{\mathcal{H}}^2_{r,L_{e_1}}$.

We associate $PH_0(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$ with a representation over $(P,\leq)$:

$$
\begin{tikzcd}
\mathbb{F} & \mathbb{F}^6 \arrow[l] & \mathbb{F}^2 \arrow[l] \\
\mathbb{F} \arrow[u] & \mathbb{F}^6 \arrow[l] \arrow[u] & \mathbb{F}^2 \arrow[l] \arrow[u] \\
\mathbb{F} \arrow[u] & \mathbb{F}^6 \arrow[l] \arrow[u] & \mathbb{F}^2 \arrow[u] \\
 & \mathbb{F}^6 \arrow[u] & \mathbb{F}^2 \arrow[l] \arrow[u] \\
 & \mathbb{F}^{12} \arrow[u] & \mathbb{F}^8 \arrow[l] \arrow[u]
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
< e > & < c_3, c_6, \hat{c}_3, \hat{c}_6, e, \hat{e} > \arrow[l, "\alpha"'] & < e, \hat{e} > \arrow[l, "\iota"'] \\
< e > \arrow[u] & < c_3, c_6, \hat{c}_3, \hat{c}_6, e, \hat{e} > \arrow[l, "\alpha"'] \arrow[u] & < e, \hat{e} > \arrow[l, "\iota"'] \arrow[u] \\
< e > \arrow[u] & < c_3, c_6, \hat{c}_3, \hat{c}_6, e, \hat{e} > \arrow[l, "\alpha"'] \arrow[u] & < e, \hat{e} > \arrow[u] \\
 & < c_3, c_6, \hat{c}_3, \hat{c}_6, e, \hat{e} > \arrow[u] & < e, \hat{e} > \arrow[l, "\iota"'] \arrow[u] \\
 & \left\langle \begin{matrix} c_3, c_6, \hat{c}_3, \hat{c}_6, e_1, \dots, e_4, \\ \hat{e}_1, \dots, \hat{e}_4 \end{matrix} \right\rangle \arrow[u, "\beta"] & < e_1, \dots, e_4, \hat{e}_1, \dots, \hat{e}_4 > \arrow[l, "\varsigma"'] \arrow[u, "\gamma"']
\end{tikzcd}
\tag{4.99}
$$

where

- α maps every basis element of $< c_3, c_6, \hat{c}_3, \hat{c}_6, e, \hat{e} >$ to e;
- β maps $e_1, \ldots, e_4$ to e, maps $\hat{e}_1, \ldots, \hat{e}_4$ to $\hat{e}$, maps c_i to c_i, and maps $\hat{c}_i$ to $\hat{c}_i$ for $i = 3, 6$;
- γ maps $e_1, \ldots, e_4$ to e and maps $\hat{e}_1, \ldots, \hat{e}_4$ to $\hat{e}$;
- ι is an inclusion map;
- ς is an inclusion map;
- unlabeled vertical maps are the identity maps.

By an abuse of notation, we use $PH_0(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$ to denote the P-indexed persistence module given by (4.99).

We associate $PH_1(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$ with a representation over $(P, \leq)$:

$$\begin{tikzcd}
\mathbb{F}^3 & 0 \arrow[l] & 0 \arrow[l] \\
\mathbb{F}^3 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
\mathbb{F}^5 \arrow[u, "\eta"] & \mathbb{F}^2 \arrow[l, "\zeta"'] \arrow[u] & 0 \arrow[u] \\
 & \mathbb{F}^2 \arrow[u] & \mathbb{F}^2 \arrow[l] \arrow[u] \\
 & 0 \arrow[u] & 0 \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.100}$$

where

- ζ is the inclusion map;

- $\eta \circ \zeta = 0$;
- unlabeled vertical maps are the identity maps.

By an abuse of notation, we use $PH_1(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$ to denote the P-indexed persistence module given by (4.100).

Theorem 4.27. *$PH_1(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$ is interval decomposable.*

Proof. We denote the support of $PH_1(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$ by M. (See (4.101).)

$$\begin{tikzcd}
\mathbb{F}^3 & & \\
\mathbb{F}^3 \arrow[u] & & \\
\mathbb{F}^5 \arrow[u, "\eta"] & \mathbb{F}^2 \arrow[l, "\zeta"'] & \\
& \mathbb{F}^2 \arrow[u] & \mathbb{F}^2 \arrow[l]
\end{tikzcd} \tag{4.101}$$

Note that $\eta \circ \zeta = 0$, and all unlabeled morphisms are identity maps, see (4.100). Moreover, ζ is the inclusion map, hence M can be decompose into two subrepresentations M_1 and M_2:

$$M_1 = \begin{tikzcd}
0 & & \\
0 \arrow[u] & & \\
\mathbb{F}^2 \arrow[u] & \mathbb{F}^2 \arrow[l, "\mathsf{id}"'] & \\
& \mathbb{F}^2 \arrow[u] & \mathbb{F}^2 \arrow[l]
\end{tikzcd} \tag{4.102}$$

$$M_2 = \begin{tikzcd} \mathbb{F}^3 & & \\ \mathbb{F}^3 \arrow[u] & & \\ \mathbb{F}^3 \arrow[u] & 0 \arrow[l] & \\ & 0 \arrow[u] & 0 \arrow[l] \end{tikzcd} \tag{4.103}$$

Since M_1 is an A_4-quiver and M_2 is an A_3-quiver, they are interval decomposable. Let P_1 denote the underlying poset of the support of M_1, and P_2 denote the underlying poset of the support of M_2. Note that the intervals in P_i (where $i = 1, 2$) are also intervals in P. Therefore, we can extend each indecomposable representation of M_i (where $i = 1, 2$) to a subrepresentation of $PH_1(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$ by putting 0 to every vertex which is not in the support of the subrepresentation and trivial morphism between two vertices where at least one vertex is not in the support of the subrepresentation.

□

Theorem 4.28. *$PH_0(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$ is interval decomposable.*

Proof. Equation (4.99) provides us with the behavior of each arrow with given basis elements. Our goal is to find a new basis of each vector space $PH_0(\hat{\mathcal{H}}^2_{r,L_{e_1}};\mathbb{F})$ such that each morphism in the diagram maps every basis to another basis or zero, depending on the geometry of $\hat{\mathcal{H}}^2_{r,L_{e_1}}$.

We choose a basis for each vector space $PH_0(\hat{\mathcal{H}}^2_{r,L_{e_1}};\mathbb{F})$:

$$
\begin{tikzcd}
< e > & \left\langle \begin{matrix} c_3-e, c_6-e, \hat{c}_3- \\ \hat{e}, \hat{c}_6-\hat{e}, e, \hat{e}-e \end{matrix} \right\rangle \arrow[l, "\alpha"'] & < e, \hat{e}-e > \arrow[l, "\iota"'] \\
< e > \arrow[u] & \left\langle \begin{matrix} c_3-e, c_6-e, \hat{c}_3- \\ \hat{e}, \hat{c}_6-\hat{e}, e, \hat{e}-e \end{matrix} \right\rangle \arrow[l, "\alpha"'] \arrow[u] & < e, \hat{e}-e > \arrow[l, "\iota"'] \arrow[u] \\
< e > \arrow[u] & \left\langle \begin{matrix} c_3-e, c_6-e, \hat{c}_3- \\ \hat{e}, \hat{c}_6-\hat{e}, e, \hat{e}-e \end{matrix} \right\rangle \arrow[l, "\alpha"'] \arrow[u] & < e, \hat{e}-e > \arrow[u] \\
 & \left\langle \begin{matrix} c_3-e, c_6-e, \hat{c}_3- \\ \hat{e}, \hat{c}_6-\hat{e}, e, \hat{e}-e \end{matrix} \right\rangle \arrow[u] & < e, \hat{e}-e > \arrow[l, "\iota"'] \arrow[u] \\
 & \left\langle \begin{matrix} c_3-e_1, c_6-e_1, \hat{c}_3- \\ \hat{e}_1, \hat{c}_6-\hat{e}_1, e_1, e_2- \\ e_1, \dots, e_4-e_1, \hat{e}_1- \\ e_1, \hat{e}_2-\hat{e}_1, \dots, \hat{e}_4-\hat{e}_1 \end{matrix} \right\rangle \arrow[u, "\beta"] & \left\langle \begin{matrix} e_1, e_2-e_1, \dots, e_4- \\ e_1, \hat{e}_1-e_1, \hat{e}_2- \\ \hat{e}_1, \dots, \hat{e}_4-\hat{e}_1 \end{matrix} \right\rangle \arrow[l, "\varsigma"'] \arrow[u, "\gamma"']
\end{tikzcd}
\tag{4.104}
$$

$$
N_1 = \begin{tikzcd}
< e > & < e > \arrow[l, "\mathsf{id}"'] & < e > \arrow[l, "\mathsf{id}"'] \\
< e > \arrow[u] & < e > \arrow[l, "\mathsf{id}"'] \arrow[u] & < e > \arrow[l, "\mathsf{id}"'] \arrow[u] \\
< e > \arrow[u] & < e > \arrow[l, "\mathsf{id}"'] \arrow[u] & < e > \arrow[u] \\
 & < e > \arrow[u] & < e > \arrow[l, "\mathsf{id}"'] \arrow[u] \\
 & < e_1 > \arrow[u, "\beta"] & < e_1 > \arrow[l, "\mathsf{id}"'] \arrow[u, "\gamma"']
\end{tikzcd}
\cong
\begin{tikzcd}
\mathbb{F} & \mathbb{F} \arrow[l, "\mathsf{id}"'] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \\
\mathbb{F} \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] \\
\mathbb{F} \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] & \mathbb{F} \arrow[u, "\mathsf{id}"'] \\
 & \mathbb{F} \arrow[u, "\mathsf{id}"] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] \\
 & \mathbb{F} \arrow[u, "\beta"] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u, "\gamma"']
\end{tikzcd}
\tag{4.105}
$$

$$N_2 = \begin{tikzcd}
0 & < \hat{e} - e > \arrow[l] & < \hat{e} - e > \arrow[l, "\mathsf{id}"'] \\
0 \arrow[u] & < \hat{e} - e > \arrow[l] \arrow[u] & < \hat{e} - e > \arrow[l, "\mathsf{id}"'] \arrow[u] \\
0 \arrow[u] & < \hat{e} - e > \arrow[l] \arrow[u] & < \hat{e} - e > \arrow[u] \\
 & < \hat{e} - e > \arrow[u] & < \hat{e} - e > \arrow[l, "\mathsf{id}"'] \arrow[u] \\
 & < \hat{e}_1 - e_1 > \arrow[u, "\beta"] & < \hat{e}_1 - e_1 > \arrow[l, "\mathsf{id}"'] \arrow[u, "\gamma"']
\end{tikzcd}
\cong
\begin{tikzcd}
0 & \mathbb{F} \arrow[l] & \mathbb{F} \arrow[l] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & \mathbb{F} \arrow[u] \\
 & \mathbb{F} \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] \\
 & \mathbb{F} \arrow[u, "\beta"] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u, "\gamma"']
\end{tikzcd} \tag{4.106}$$

For $i = 3, 6$

$$E_i = \begin{tikzcd}
0 & < c_i - e > \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & < c_i - e > \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
0 \arrow[u] & < c_i - e > \arrow[l] \arrow[u] & 0 \arrow[u] \\
 & < c_i - e > \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & < c_i - e_1 > \arrow[u, "\beta"] & 0 \arrow[l] \arrow[u]
\end{tikzcd}
\cong
\begin{tikzcd}
0 & \mathbb{F} \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[u] \\
 & \mathbb{F} \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & \mathbb{F} \arrow[u, "\beta"] & 0 \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.107}$$

and

$$\hat{E}_i = \begin{tikzcd} 0 & < \hat{c}_i - \hat{e} > \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & < \hat{c}_i - \hat{e} > \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & < \hat{c}_i - \hat{e} > \arrow[l] \arrow[u] & 0 \arrow[u] \\ & < \hat{c}_i - \hat{e} > \arrow[u] & 0 \arrow[l] \arrow[u] \\ & < \hat{c}_i - \hat{e}_1 > \arrow[u, "\beta"] & 0 \arrow[l] \arrow[u] \end{tikzcd} \cong \begin{tikzcd} 0 & \mathbb{F} \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[u] \\ & \mathbb{F} \arrow[u] & 0 \arrow[l] \arrow[u] \\ & \mathbb{F} \arrow[u, "\beta"] & 0 \arrow[l] \arrow[u] \end{tikzcd} \tag{4.108}$$

For $j = 2, 3, 4$

$$F_j = \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\ & < e_i - e_1 > \arrow[u] & < e_i - e_1 > \arrow[l, "\mathsf{id}"'] \arrow[u] \end{tikzcd} \cong \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\ & \mathbb{F} \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] \end{tikzcd} \tag{4.109}$$

and

$$\hat{F}_j = \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\ & <\hat{e}_i - \hat{e}_1> \arrow[u] & <\hat{e}_i - \hat{e}_1> \arrow[l, "\mathsf{id}"'] \arrow[u] \end{tikzcd} \cong \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\ & \mathbb{F} \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] \end{tikzcd} \tag{4.110}$$

It is clear that $PH_0(\hat{\mathcal{H}}^2_{-,-};\mathbb{F}) \cong N_1 \oplus N_2 \oplus E_3 \oplus E_6 \oplus \hat{E}_3 \oplus \hat{E}_6 \oplus (\bigoplus_{j=2}^{4} F_j) \oplus (\bigoplus_{j=2}^{4} \hat{F}_j)$. Note that N_1, N_2, E_i, $\hat{E}_i$, F_j, and $\hat{F}_j$ (where $i = 3, 6$ and $j = 2, 3, 4$) are interval modules, hence by Lemma 3.1, they are indecomposable. Therefore, $PH_0(\hat{\mathcal{H}}^2_{-,-};\mathbb{F})$ is interval decomposable. $\square$

4.7 Configuration Spaces of the Generalized H Graph

Let $m, n \geq 3$. A generalized H graph, denoted by $\mathcal{H}_{m,n}$, is a tree shown in Figure 4.9a. One can obtain $\mathcal{H}_{m,n}$ by concatenating Star_m and Star_n at a degree 1 vertex x_0. We want to assign each edge an orientation where the orientation is matched with the orientation that we assigned for the star graph. Hence we subdivide the bridge of $\mathcal{H}_{m,n}$ by introducing a new vertex (and replacing the bridge with two new edges), as shown in Figure 4.9b. We use f to denote the new edge incident to the center of Star_m and use f' to denote the new edge incident to the center of Star_n. By an abuse of notation, we denote the resulting graph by $\mathcal{H}_{m,n}$, and we use $\hat{\mathcal{H}}_{m,n}$ to denote the special $\mathcal{H}$ where the length of the bridge (before the subdivision) is L_{e_1} and the length of any other edge is 1. In this section, we describe the second configuration spaces of $\hat{\mathcal{H}}_{m,n}$ with restraint parameters r and L_{e_1}.

(a) (b)

Figure 4.9

Let L_{e_i} denote the length of the e_i for $i = 2, \ldots, m$ and $L_{e'_i}$ denote the length of the e'_i for $i = 2, \ldots, n$. In addition, we use L_f denote the length of the f and use $L_{f'}$ denote the length of the f'. The parametric polytope of $(\mathcal{H}_{m,n})^2_{r,\vec{L}}$ is given by the following inequalities, where $\vec{L} = (L_{e_2}, \ldots, L_{e_m}, L_{e'_2}, \ldots, L_{e'_n}, L_f, L_{f'})$ and $L_f + L_{f'} = L_{e_1}$:

For $i, j \in \{2, \ldots, m\}$**,** $i \neq j$**:**

$$\begin{aligned} &0 \leq x \leq L_{e_i} \\ &0 \leq y \leq L_{e_j} \\ &x + y \geq r \end{aligned} \tag{4.111}$$

For $i, j \in \{2, \ldots, n\}$**,** $i \neq j$**:**

$$\begin{aligned} &0 \leq x \leq L_{e'_i} \\ &0 \leq y \leq L_{e'_j} \\ &x + y \geq r \end{aligned} \tag{4.112}$$

For $i \in \{2, \ldots, m\}$**,** $j = f$**:**

$$\begin{aligned} &0 \leq x \leq L_{e_i} \\ &0 \leq y \leq L_f \\ &x + y \geq r \end{aligned} \tag{4.113}$$

For $j \in \{2, \ldots, m\}$**,** $i = f$**:**

$$\begin{aligned} 0 \le x &\le L_f \\ 0 \le y &\le L_{e_j} \\ x + y &\ge r \end{aligned} \tag{4.114}$$

For $i \in \{2, \ldots, m\}$**,** $j = f'$**:**

$$\begin{aligned} 0 \le x &\le L_{e_i} \\ 0 \le y &\le L_{f'} \\ x + L_{e_1} - y &\ge r \end{aligned} \tag{4.115}$$

For $j \in \{2, \ldots, m\}$**,** $i = f'$**:**

$$\begin{aligned} 0 \le x &\le L_{f'} \\ 0 \le y &\le L_{e_j} \\ -x + L_{e_1} + y &\ge r \end{aligned} \tag{4.116}$$

For $i, j \in \{2, \ldots, n\}$**,** $i \neq j$**:**

$$\begin{aligned} 0 \le x &\le L_{e'_i} \\ 0 \le y &\le L_{e'_j} \\ -x + L_{e_1} - y &\ge r \end{aligned} \tag{4.117}$$

For $i \in \{2, \ldots, m\}$**,** $j \in \{2, \ldots, n\}$**:**

$$\begin{aligned} 0 \le x &\le L_{e_i} \\ 0 \le y &\le L_{e'_j} \\ x + L_{e_1} + y &\ge r \end{aligned} \tag{4.118}$$

For $i \in \{2, \ldots, n\}$**,** $j \in \{2, \ldots, m\}$**:**

$$\begin{aligned} 0 \leq x \leq L_{e'_i} \\ 0 \leq y \leq L_{e_j} \\ x + L_{e_1} + y \geq r \end{aligned} \tag{4.119}$$

For $i \in \{2, \ldots, n\}$**,** $j = f'$**:**

$$\begin{aligned} 0 \leq x \leq L_{e'_i} \\ 0 \leq y \leq L_{f'} \\ x + y \geq r \end{aligned} \tag{4.120}$$

For $j \in \{2, \ldots, n\}$**,** $i = f'$**:**

$$\begin{aligned} 0 \leq x \leq L_{f'} \\ 0 \leq y \leq L_{e'_j} \\ x + y \geq r \end{aligned} \tag{4.121}$$

For $i \in \{2, \ldots, n\}$**,** $j = f$**:**

$$\begin{aligned} 0 \leq x \leq L_{e'_i} \\ 0 \leq y \leq L_f \\ x + L_{e_1} - y \geq r \end{aligned} \tag{4.122}$$

For $j \in \{2, \ldots, n\}$**,** $i = f$**:**

$$\begin{aligned} 0 \leq x \leq L_f \\ 0 \leq y \leq L_{e'_j} \\ -x + L_{e_1} + y \geq r \end{aligned} \tag{4.123}$$

For $i, j \in \{2, \ldots, m\}$**,** $i = j$**,** $x < y$**:**

$$\begin{aligned} 0 \leq x \leq L_{e_i} \\ 0 \leq y \leq L_{e_i} \\ -x + y \geq r \end{aligned} \tag{4.124}$$

For $i, j \in \{2, \ldots, m\}$**,** $i = j$**,** $x > y$**:**

$$\begin{aligned} 0 \leq x \leq L_{e_i} \\ 0 \leq y \leq L_{e_i} \\ x - y \geq r \end{aligned} \tag{4.125}$$

For $i, j \in \{2, \ldots, n\}$**,** $i = j$**,** $x < y$**:**

$$\begin{aligned} 0 \leq x \leq L_{e'_i} \\ 0 \leq y \leq L_{e'_i} \\ -x + y \geq r \end{aligned} \tag{4.126}$$

For $i, j \in \{2, \ldots, n\}$**,** $i = j$**,** $x > y$**:**

$$\begin{aligned} 0 \leq x \leq L_{e'_i} \\ 0 \leq y \leq L_{e'_i} \\ x - y \geq r \end{aligned} \tag{4.127}$$

For $i = j = f$**,** $x > y$**:**

$$\begin{aligned} 0 \leq x \leq L_f \\ 0 \leq y \leq L_f \\ x - y \geq r \end{aligned} \tag{4.128}$$

For $i = j = f$, $x < y$:

$$\begin{aligned} &0 \le x \le L_f \\ &0 \le y \le L_f \\ &y - x \ge r \end{aligned} \tag{4.129}$$

For $i = j = f'$, $x > y$:

$$\begin{aligned} &0 \le x \le L_{f'} \\ &0 \le y \le L_{f'} \\ &x - y \ge r \end{aligned} \tag{4.130}$$

For $i = j = f'$, $x < y$:

$$\begin{aligned} &0 \le x \le L_{f'} \\ &0 \le y \le L_{f'} \\ &y - x \ge r \end{aligned} \tag{4.131}$$

Compare the inequality systems (4.70)-(4.84) with inequality systems (4.111)-(4.131) associated to $(\mathcal{H}_{m,n})^2_{r,\vec{L}}$ when $\vec{L} = (1,\dots,1,1,\dots,1,L_f,L_{f'})$ and $L_f = L_{f'} = \frac{1}{2}L_{e_1}$, we conclude that $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ and $\hat{\mathcal{H}}^2_{r,L_{e_1}}$ have the same type of parametric polytopes. Therefore, $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ and $\hat{\mathcal{H}}^2_{r,L_{e_1}}$ have the same critical hyperplanes.

Consider the following cover of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$:

$$\begin{aligned} U_{11} &= (e_m)^2_r \\ U_{12} &= e_m \times \hat{\mathcal{H}}_{m-1,n} - \{(x,y) \in e_m \times \hat{\mathcal{H}}_{m-1,n} \mid \delta(x,y) < r\} \\ U_{21} &= \hat{\mathcal{H}}_{m-1,n} \times e_m - \{(y,x) \in \hat{\mathcal{H}}_{m-1,n} \times e_m \mid \delta(x,y) < r\} \\ U_{22} &= (\hat{\mathcal{H}}_{m-1,n})^2_r \end{aligned} \tag{4.132}$$

Note that there are natural inclusion maps

$$\begin{aligned}
U_{11} \cap U_{12} &\hookrightarrow U_{11} \\
U_{11} \cap U_{12} &\hookrightarrow U_{12} \\
U_{11} \cap U_{21} &\hookrightarrow U_{11} \\
U_{11} \cap U_{21} &\hookrightarrow U_{21} \\
U_{22} \cap U_{12} &\hookrightarrow U_{22} \\
U_{22} \cap U_{12} &\hookrightarrow U_{12} \\
U_{22} \cap U_{21} &\hookrightarrow U_{22} \\
U_{22} \cap U_{21} &\hookrightarrow U_{21}
\end{aligned} \tag{4.133}$$

Now we are going to calculate $H_i(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ in 2 steps. First we calculate $H_i(\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}$ for all $m \geq 3$, then we calculate $H_i(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ for all $n \geq 3$.

Step 1: Calculating $H_i(\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}$.

Lemma 4.2. $\vec{L} = (1, \dots, 1, L_f, L_{f'}) \in (\mathbb{R}_{>0})^{m+3}$ where $L_f = L_{f'} = \frac{1}{2}L_{e_1}$. Then

$$H_0(U_{12}) \cong \begin{cases} \mathbb{Z}, & \text{if } 0 < r \leq 1 \\ \mathbb{Z}^{m-1}, & \text{if } 1 < r \leq 2 \text{ and } r \leq 1 + L_{e_1} \\ \mathbb{Z}^{m}, & \text{if } 1 < r \leq 2 \text{ and } r > 1 + L_{e_1} \\ \mathbb{Z}, & \text{if } 2 < r \text{ and } r \leq 1 + L_{e_1} \\ \mathbb{Z}^{2}, & \text{if } 2 < r \text{ and } 1 + L_{e_1} < r \leq 2 + L_{e_1} \\ 0, & \text{else} \end{cases}$$

and

$$H_1(U_{12}) \cong 0$$

Proposition 4.29. Let L_{e_1} be a positive number. If $r \leq L_{e_1}$ and $r \leq 1$, then

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m^2-3m+3}$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^5 \xleftarrow{\quad d^1 \quad} & \mathbb{Z}^{2m} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let $b_2, \ldots, b_{m-1}$ and b' be the generators of $H_0(U_{22} \cap U_{12})$. Let $\hat{b}_2, \ldots, \hat{b}_{m-1}$ and $\hat{b}'$ be the generators of $H_0(U_{22} \cap U_{21})$. Choose generators (denoted by c_1 and c_2) for $H_0(U_{11})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}$ and we choose a generator (denoted by f) for $H_0(U_{22})$.

Let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$\begin{aligned} d^1(a) &= c_1 + e \\ d^1(\hat{a}) &= c_2 + \hat{e} \\ d^1(b_i) &= e + f, \forall i = 2, \dots, m-1 \\ d^1(b') &= e + f \\ d^1(\hat{b}_i) &= \hat{e} + f, \forall i = 2, \dots, m-1 \\ d^1(\hat{b}') &= \hat{e} + f \end{aligned} \tag{4.134}$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix}$$

The rank of this matrix is 4, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^4$ and $\mathsf{ker} d^1 \cong \mathbb{Z}^{2m-4}$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|ccc} q & & & \\ 3 & \vdots & \vdots & \vdots \\ 2 & 0 & 0 & 0 \\ 1 & \mathbb{Z}^{m^2-5m+7} & 0 & 0 \\ 0 & \mathbb{Z} & \mathbb{Z}^{2m-4} & 0 \\ \hline & 0 & 1 & 2 \quad p \end{array}$$

Here we applied the induction hypothesis $H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m^2-5m+7}$. Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}$$

and

$$H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m^2-5m+7} \oplus \mathbb{Z}^{2m-4} \cong \mathbb{Z}^{m^2-3m+3}$$

$\square$

Proposition 4.30. Let L_{e_1} be a positive number. If $L_{e_1} < r \leq 1$, then

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m^2-m-1}$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^5 & \xleftarrow{d^1} \mathbb{Z}^{2m+2} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let $b_2, \ldots, b_{m-1}, b_1'$ and b_2' be the generators of $H_0(U_{22} \cap U_{12})$. Let $\hat{b}_2, \ldots, \hat{b}_{m-1}, \hat{b}_1'$ and $\hat{b}_2'$ be the generators of $H_0(U_{22} \cap U_{21})$. Choose generators (denoted by c_1 and c_2) for $H_0(U_{11})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}$ and we choose a generator (denoted by f) for $H_0(U_{22})$. Let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$\begin{aligned}
d^1(a) &= c_1 + e \\
d^1(\hat{a}) &= c_2 + \hat{e} \\
d^1(b_i) &= e + f, \forall i = 2, \ldots, m-1 \\
d^1(b'_j) &= e + f, \forall j = 1, 2 \\
d^1(\hat{b}_i) &= \hat{e} + f, \forall i = 2, \ldots, m-1 \\
d^1(\hat{b}'_j) &= \hat{e} + f, \forall j = 1, 2
\end{aligned} \tag{4.135}$$

Consider the following matrix

$$\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1
\end{bmatrix}$$

The rank of this matrix is 4, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^4$ and $\mathsf{ker} d^1 \cong \mathbb{Z}^{2m-2}$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z}^{(m-1)^2-(m-1)-1} & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}^{2m-2} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Here we applied the induction hypothesis $H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{(m-1)^2-(m-1)-1}$. Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}$$

and

$$H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{(m-1)^2-(m-1)-2} \oplus \mathbb{Z}^{2m-2} \cong \mathbb{Z}^{m^2-m-1}$$

□

Proposition 4.31. Let L_{e_1} be a positive number. If $1 < r \leq 2$ and $r \leq L_{e_1}$, then

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m^2-3m+6} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & & \vdots & \vdots \\
2 & 0 & & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) & & 0 & 0 \\
0 & \mathbb{Z}^{2m-2} \oplus H_0((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) & \xleftarrow{d^1} & \mathbb{Z}^2 & 0 \\
\hline
& 0 & & 1 & 2 \quad p
\end{array}
$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Since $H_0(U_{11} \cap U_{12}) = H_0(U_{11} \cap U_{21}) = \emptyset$, we do not need to choose generators for their 0-th homology groups. Let b be the generator of $H_0(U_{22} \cap U_{12})$ and $\hat{b}$ be the generator of $H_0(U_{22} \cap U_{21})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}^{(m-1)(m-4)+6}$ and we choose a collection of generators (denoted by f_{ij} and $\hat{f}_{ij}$, where $2 \leq i < j \leq m-1$ or $m+1 \leq i < j \leq m+2$, f', and $\hat{f}'$) for $H_0(U_{22})$. Let $e_2, \ldots, e_{m-1}, e'$ be the generators of $H_0(U_{12})$ and $\hat{e}_2, \ldots, \hat{e}_{m-1}, \hat{e}'$ be the generators of $H_0(U_{21})$ such that

$$
\begin{aligned}
d^1(b) &= e' + f' \\
d^1(\hat{b}) &= \hat{e}' + \hat{f}'
\end{aligned}
\tag{4.136}
$$

Consider the following matrix

$$
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix}
$$

The rank of this matrix is 2, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^2$ and $\mathsf{coker}\, d^1 \cong \mathbb{Z}^{m^2-3m+6}$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z}^{m^2-3m+6} & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Here we applied the induction hypothesis

$$H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) = 0$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m^2-3m+6}$$

and

$$H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) = 0$$

$\square$

Proposition 4.32. Let L_{e_1} be a positive number. If $1 < r \leq 2$ and $L_{e_1} < r \leq L_{e_1} + 1$, then

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m^2-3m+6} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) = \mathbb{Z}^{2m-4}$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^{2m-2} \oplus H_0((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) \xleftarrow{d^1} & \mathbb{Z}^4 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Since $H_0(U_{11} \cap U_{12}) = H_0(U_{11} \cap U_{21}) = \emptyset$, we do not need to choose generators for their 0-th homology groups. Let b_1 and b_2 be the generators of $H_0(U_{22} \cap U_{12})$. Let $\hat{b}_1$ and $\hat{b}_2$ be the generators of $H_0(U_{22} \cap U_{21})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}^{(m-1)(m-4)+6}$ and we choose a collection of generators (denoted by f_{ij} and $\hat{f}_{ij}$, where $2 \leq i < j \leq m-1$ or $m+1 \leq i < j \leq m+2$, f', and $\hat{f}'$) for $H_0(U_{22})$. Let $e_2, \ldots, e_{m-1}, e'$ be the generators of $H_0(U_{12})$ and $\hat{e}_2, \ldots, \hat{e}_{m-1}, \hat{e}'$ be the generators of $H_0(U_{21})$ such that for $i = 1, 2$,

$$
\begin{aligned}
d^1(b_i) &= e' + f' \\
d^1(\hat{b}_i) &= \hat{e}' + \hat{f}'
\end{aligned}
\tag{4.137}
$$

Consider the following matrix

$$
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
$$

The rank of this matrix is 2, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^2$ and $\mathsf{coker}\, d^1 \cong \mathbb{Z}^{2m-2+(m-1)(m-4)+6-2} \cong \mathbb{Z}^{m^2-3m+6}$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z}^{2(m-1)-4} & 0 & 0 \\
0 & \mathbb{Z}^{m^2-3m+6} & \mathbb{Z}^2 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Here we applied the induction hypothesis

$$H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) = \mathbb{Z}^{2(m-1)-4}$$

and

$$H_0((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) = \mathbb{Z}^{(m-1)(m-4)+6}$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m^2-3m+6}$$

and

$$H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) = \mathbb{Z}^{2m-4}$$

□

Proposition 4.33. Let L_{e_1} be a positive number. If $1 < r \leq 2$ and $L_{e_1} + 1 < r \leq L_{e_1} + 2$, then

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m^2+m} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the

Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^{2m} \oplus H_0((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

By induction hypothesis,

$$H_0(U_{22}) \cong \mathbb{Z}^{(m-1)^2+(m-1)}$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m^2+m}$$

and

$$H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) = 0$$

$\square$

Proposition 4.34. Let L_{e_1} be a positive number. If $2 < r$ and $r \leq L_{e_1}$, then

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2 \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^2 \oplus H_0((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) \xleftarrow{d^1} & \mathbb{Z}^2 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Since $H_0(U_{11} \cap U_{12}) = H_0(U_{11} \cap U_{21}) = \emptyset$, we do not need to choose generators for their 0-th homology groups. Let b be the generator of $H_0(U_{22} \cap U_{12})$ and $\hat{b}$ be the generator of $H_0(U_{22} \cap U_{21})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}^{(m-1)(m-4)+6}$ and we choose a set of generators (denoted by f' and $\hat{f}'$) for $H_0(U_{22})$. Let e be the generator of $H_0(U_{12})$ and $\hat{e}'$ be the generator of $H_0(U_{21})$ such that for $i = 1, 2$,

$$
\begin{aligned}
d^1(b_i) &= e' + f' \\
d^1(\hat{b}_i) &= \hat{e}' + \hat{f}'
\end{aligned}
\tag{4.138}
$$

Consider the following matrix

$$
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix}
$$

The rank of this matrix is 2, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^2$ and $\mathsf{coker}\, d^1 \cong \mathbb{Z}^2$.

Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & \\
1 & 0 & 0 & 0 & \\
0 & \mathbb{Z}^2 & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

Here we applied the induction hypothesis

$$H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) = 0$$

and

$$H_0((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) = \mathbb{Z}^2$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

and

$$H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) = 0$$

□

Proposition 4.35. Let L_{e_1} be a positive number. If $2 < r$ and $L_{e_1} < r \leq L_{e_1} + 1$, then

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2 \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) = \mathbb{Z}^{2m-4}$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the

Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^2 \oplus H_0((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) \xleftarrow{d^1} & \mathbb{Z}^4 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Since $H_0(U_{11} \cap U_{12}) = H_0(U_{11} \cap U_{21}) = \emptyset$, we do not need to choose generators for their 0-th homology groups. Let b_1 and b_2 be the generators of $H_0(U_{22} \cap U_{12})$. Let $\hat{b}_1$ and $\hat{b}_2$ be the generators of $H_0(U_{22} \cap U_{21})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}^2$ and we choose a set of generators (denoted by f' and $\hat{f}'$) for $H_0(U_{22})$. Let e be the generator of $H_0(U_{12})$ and $\hat{e}'$ be the generator of $H_0(U_{21})$ such that for $i = 1, 2$,

$$
\begin{aligned}
d^1(b_i) &= e' + f' \\
d^1(\hat{b}_i) &= \hat{e}' + \hat{f}'
\end{aligned}
\tag{4.139}
$$

Consider the following matrix

$$
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
$$

The rank of this matrix is 2, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^2$ and $\mathsf{coker}\, d^1 \cong \mathbb{Z}^2$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z}^{2(m-1)-4} & 0 & 0 \\
0 & \mathbb{Z}^2 & \mathbb{Z}^2 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Here we applied the induction hypothesis

$$H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2(m-1)-4}$$

and

$$H_0((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) = \mathbb{Z}^2$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

and

$$H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2m-4}$$

$\square$

Proposition 4.36. Let L_{e_1} be a positive number. If $2 < r$ and $L_{e_1} + 1 < r \leq L_{e_1} + 2$, then

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{4(m-3)+8} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the

Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^4 \oplus H_0((\hat{\mathcal{H}}_{m-1,3})^2_{r,L_{e_1}}) & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}^{4(m-4)+8}$ and $H_1(U_{22}) = 0$. Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{4(m-3)+8}$$

and

$$H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) = 0$$

$\square$

Step 2: Calculating $H_i(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ **where** $n \geq 4$

Lemma 4.3. $\vec{L} = (1, \ldots, 1, L_f, L_{f'}) \in (\mathbb{R}_{>0})^{m+3}$ where $L_f = L_{f'} = \frac{1}{2}L_{e_1}$. Then

$$H_0(U_{12}) \cong \begin{cases} \mathbb{Z}, & \text{if } 0 < r \leq 1 \\ \mathbb{Z}^{n-1}, & \text{if } 1 < r \leq 2 \text{ and } r \leq 1 + L_{e_1} \\ \mathbb{Z}^{m+n-3}, & \text{if } 1 < r \leq 2 \text{ and } r > 1 + L_{e_1} \\ \mathbb{Z}, & \text{if } 2 < r \text{ and } r \leq 1 + L_{e_1} \\ \mathbb{Z}^{m-1}, & \text{if } 2 < r \text{ and } 1 + L_{e_1} < r \leq 2 + L_{e_1} \\ 0, & \text{else} \end{cases}$$

and

$$H_1(U_{12}) \cong 0$$

Proposition 4.37. Let L_{e_1} be a positive number. If $r \leq L_{e_1}$ and $r \leq 1$, then

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,3})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m(m-3)+n(n-3)+3}$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & & \vdots & \vdots \\
2 & 0 & & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) & & 0 & 0 \\
0 & \mathbb{Z}^4 \oplus H_0((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) & \xleftarrow{d^1} & \mathbb{Z}^{2n} & 0 \\
\hline
 & 0 & & 1 & 2 \quad p
\end{array}$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let $b'_2, \ldots, b'_{n-1}$ and b be the generators of $H_0(U_{22} \cap U_{12})$. Let $\hat{b}'_2, \ldots, \hat{b}'_{n-1}$ and $\hat{b}$ be the generators of $H_0(U_{22} \cap U_{21})$. Choose generators (denoted by c_1 and c_2) for $H_0(U_{11})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}$ and we choose a generator (denoted by f) for $H_0(U_{22})$. Let e' be the generator of $H_0(U_{12})$ and $\hat{e}'$ be the generator of $H_0(U_{21})$ such that

$$
\begin{aligned}
&d^1(a) = c_1 + e' \\
&d^1(\hat{a}) = c_2 + \hat{e}' \\
&d^1(b_i) = e' + f, \forall i = 2, \ldots, n-1 \\
&d^1(b') = e' + f \\
&d^1(\hat{b}_i) = \hat{e}' + f, \forall i = 2, \ldots, n-1 \\
&d^1(\hat{b}') = \hat{e}' + f
\end{aligned}
\tag{4.140}
$$

Consider the following matrix

$$
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1
\end{bmatrix}
$$

The rank of this matrix is 4, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^4$ and $\mathsf{ker} d^1 \cong \mathbb{Z}^{2n-4}$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z}^{(n-1)(n-4)+m(m-3)+3} & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}^{2n-4} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Here we applied the induction hypothesis

$$H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{(n-1)(n-4)+m(m-3)+3}$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}$$

and

$$H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m^2-5m+7} \oplus \mathbb{Z}^{2m-4} \cong \mathbb{Z}^{m(m-3)+n(n-3)+3}$$

$\square$

Proposition 4.38. Let L_{e_1} be a positive number. If $L_{e_1} < r \leq 1$, then

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{(m+n)(m+n-7)+11}$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ given in Equation (4.132).

The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^4 \oplus H_0((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \xleftarrow{d^1} & \mathbb{Z}^{2m+2n-4} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Let a be the generator of $H_0(U_{11} \cap U_{12})$ and $\hat{a}$ be the generator of $H_0(U_{11} \cap U_{21})$. Let $b'_2, \ldots, b'_{n-1}$, $b_2, \ldots, b_{m-1}$ and b_m be the generators of $H_0(U_{22} \cap U_{12})$. Let $\hat{b}'_2, \ldots, \hat{b}'_{n-1}$, $\hat{b}_2, \ldots, \hat{b}_{m-1}$ and $\hat{b}_m$ be the generators of $H_0(U_{22} \cap U_{21})$. Choose generators (denoted by c_1 and c_2) for $H_0(U_{11})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}$ and we choose a generator (denoted by f) for $H_0(U_{22})$. Let e' be the generator of $H_0(U_{12})$ and $\hat{e}'$ be the generator of $H_0(U_{21})$ such that

$$
\begin{aligned}
d^1(a) &= c_1 + e' \\
d^1(\hat{a}) &= c_2 + \hat{e}' \\
d^1(b_i) &= e' + f, \forall i = 2, \ldots, n-1 \\
d^1(b'_j) &= e' + f, \forall j = 2, \ldots, m \\
d^1(\hat{b}_i) &= \hat{e}' + f, \forall i = 2, \ldots, n-1 \\
d^1(\hat{b}'_j) &= \hat{e}' + f, \forall j = 2, \ldots, m
\end{aligned}
\tag{4.141}
$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix}$$

The rank of this matrix is 4, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^4$ and $\mathsf{ker} d^1 \cong \mathbb{Z}^{2m+2n-8}$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z}^{(m+n-1)(m+n-8)+11} & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}^{2m+2n-8} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Here we applied the induction hypothesis $H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{(m+n-1)(m+n-8)+11}$. Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}$$

and

$$H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{(m+n-1)(m+n-8)+10} \oplus \mathbb{Z}^{2m+2n-8} \cong \mathbb{Z}^{(m+n)(m+n-7)+11}$$

$\square$

Proposition 4.39. Let L_{e_1} be a positive number. If $1 < r \leq 2$ and $r \leq L_{e_1}$, then

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m(m-3)+n(n-3)+6} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^{2n-2} \oplus H_0((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \xleftarrow{d^1} & \mathbb{Z}^2 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Let b be the generator of $H_0(U_{22} \cap U_{12})$ and $\hat{b}$ be the generator of $H_0(U_{22} \cap U_{21})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}^{m(m-3)+(n-1)(n-4)+6}$ and we choose generators (denoted by f_{ij}, $\hat{f}_{ij}$ for $2 \leq i < j \leq m$ and $m+2 \leq i < j \leq m+n-1$, $f_{2,m+2}$ and $\hat{f}_{2,m+2}$) for $H_0(U_{22})$. Let $e'_2, \ldots, e'_{n-1}, e$ be the generators of $H_0(U_{12})$ and $\hat{e}'_1, \ldots, \hat{e}'_{n-1}, \hat{e}$ be the generator of $H_0(U_{21})$ such that

$$\begin{aligned} d^1(b) &= e' + f_{2,m+2} \\ d^1(\hat{b}) &= \hat{e}' + \hat{f}_{2,m+2} \end{aligned} \tag{4.142}$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The rank of this matrix is 2, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^2$ and $\mathsf{ker} d^1 = 0$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z}^{m(m-3)+n(n-3)+6} & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Here we applied the induction hypothesis $H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) = 0$. Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m(m-3)+n(n-3)+6}$$

and

$$H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) = 0$$

□

Proposition 4.40. Let L_{e_1} be a positive number. If $1 < r \leq 2$ and $L_{e_1} < r \leq L_{e_1} + 1$, then

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m(m-3)+n(n-3)+6} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2(m-2)(n-2)}$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^{2n-2} \oplus H_0((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \xleftarrow{d^1} & \mathbb{Z}^{2m-2} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Let $b_2, \ldots, b_m$ be the generators of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_2, \ldots, \hat{b}_m$ be the generators of $H_0(U_{22} \cap U_{21})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}^{m(m-3)+(n-1)(n-4)+6}$ and we choose generators (denoted by f_{ij}, $\hat{f}_{ij}$ for $2 \leq i < j \leq m$ and $m+2 \leq i < j \leq m+n-1$, $f_{2,m+2}$ and $\hat{f}_{2,m+2}$) for $H_0(U_{22})$. Let $e'_2, \ldots, e'_{n-1}, e$ be the generators of $H_0(U_{12})$ and $\hat{e}'_1, \ldots, \hat{e}'_{n-1}, \hat{e}$ be the generator of $H_0(U_{21})$ such that

$$\begin{aligned} d^1(b) &= e' + f_{2,m+2} \\ d^1(\hat{b}) &= \hat{e}' + \hat{f}_{2,m+2} \end{aligned} \tag{4.143}$$

Consider the following matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The rank of this matrix is 2, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^2$ and $\mathsf{ker} d^1 = 2m-4$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z}^{2(m-2)(n-3)} & 0 & 0 \\
0 & \mathbb{Z}^{m(m-3)+n(n-3)+6} & \mathbb{Z}^{2m-4} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Here we applied the induction hypothesis $H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2(m-2)(n-3)}$. Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{m(m-3)+n(n-3)+6}$$

and

$$H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2(m-2)(n-2)}$$

□

Proposition 4.41. Let L_{e_1} be a positive number. If $1 < r \leq 2$ and $L_{e_1}+1 < r \leq L_{e_1}+2$, then

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{(m+n)(m+n-5)+6} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^{2m+2n-6} \oplus H_0((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{(m+n)(m+n-5)+6}$$

and

$$H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) = 0$$

□

Proposition 4.42. Let L_{e_1} be a positive number. If $2 < r$ and $r \leq L_{e_1}$, then

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2 \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^2 \oplus H_0((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \xleftarrow{d^1} & \mathbb{Z}^2 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Let b be the generator of $H_0(U_{22} \cap U_{12})$ and $\hat{b}$ be the generator of $H_0(U_{22} \cap U_{21})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}^2$ and we choose generators (denoted by $f_{2,m+2}$ and $\hat{f}_{2,m+2}$) for $H_0(U_{22})$. Let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$
\begin{aligned}
d^1(b) &= e + f_{2,m+2} \\
d^1(\hat{b}) &= \hat{e} + \hat{f}_{2,m+2}
\end{aligned}
\tag{4.144}
$$

Consider the following matrix

$$
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix}
$$

The rank of this matrix is 2, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore,

$$\mathsf{Im}\, d^1 \cong \mathbb{Z}^2$$

and

$$\mathsf{ker} d^1 = 0$$

Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & \vdots & \vdots & \\
2 & 0 & 0 & 0 & \\
1 & 0 & 0 & 0 & \\
0 & \mathbb{Z}^2 & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

Here we applied the induction hypothesis

$$H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) = 0$$

and

$$H_0((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

and

$$H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) = 0$$

$\square$

Proposition 4.43. Let L_{e_1} be a positive number. If $2 < r$ and $L_{e_1} < r \leq L_{e_1} + 1$, then

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2 \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2(m-2)(n-2)}$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) & 0 & 0 \\
0 & \mathbb{Z}^2 \oplus H_0((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \xleftarrow{d^1} & \mathbb{Z}^{2m-2} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

We need to understand the behavior of d^1, where d^1 is induced by the inclusion maps (4.133). Let $b_2, \ldots, b_m$ be the generators of $H_0(U_{22} \cap U_{12})$ and $\hat{b}_2, \ldots, \hat{b}_m$ be the generators of $H_0(U_{22} \cap U_{21})$. By induction hypothesis, $H_0(U_{22}) \cong \mathbb{Z}^2$ and we choose generators (denoted by $f_{2,m+2}$ and $\hat{f}_{2,m+2}$) for $H_0(U_{22})$. Let e be the generator of $H_0(U_{12})$ and $\hat{e}$ be the generator of $H_0(U_{21})$ such that

$$
\begin{aligned}
d^1(b_i) &= e + f_{2,m+2}, \forall i = 2, \ldots, m \\
d^1(\hat{b}_i) &= \hat{e} + \hat{f}_{2,m+2}, \forall i = 2, \ldots, m
\end{aligned}
\tag{4.145}
$$

Consider the following matrix

$$
\begin{bmatrix}
1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 1 \\
1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 1
\end{bmatrix}
$$

The rank of this matrix is 2, and the diagonal elements of the Smith Normal form of the matrix above are all equal to 1. Therefore, $\mathsf{Im}\, d^1 \cong \mathbb{Z}^2$ and $\mathsf{ker} d^1 = 2m - 4$. Hence we obtain the E^2 page of the Mayer-Vietoris spectral sequence:

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & \mathbb{Z}^{2(m-2)(n-3)} & 0 & 0 \\
0 & \mathbb{Z}^2 & \mathbb{Z}^{2m-4} & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Here we applied the induction hypothesis

$$H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2(m-2)(n-3)}$$

and

$$H_0((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^2$$

and

$$H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2(m-2)(n-2)}$$

□

Proposition 4.44. Let L_{e_1} be a positive number. If $2 < r$ and $L_{e_1} + 1 < r \leq L_{e_1} + 2$, then

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2(m-1)(n-1)} \quad \text{and} \quad H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) = 0$$

Proof. Consider the cover of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ given in Equation (4.132). The E^1 page of the

Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \vdots & \vdots & \vdots \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z}^{2m-2} \oplus \mathbb{Z}^{2(m-1)(n-2)} & 0 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Here we applied the induction hypothesis

$$H_1((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) = 0$$

and

$$H_0((\hat{\mathcal{H}}_{m,n-1})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2(m-1)(n-2)}$$

Since there is no non-trivial arrow on the E^1 page, $E^1 = E^\infty$. Therefore,

$$H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) \cong \mathbb{Z}^{2(m-1)(n-1)}$$

and

$$H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}) = 0$$

$\square$

In summary, when $m, n \geq 3$, the rank of $H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}})$ for all $r > 0$ and $L_{e_1} > 0$ is shown in Figure 4.10. When $m, n \geq 3$, the rank of $H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}})$ for all $r > 0$ and $L_{e_1} > 0$ is shown in Figure 4.11.

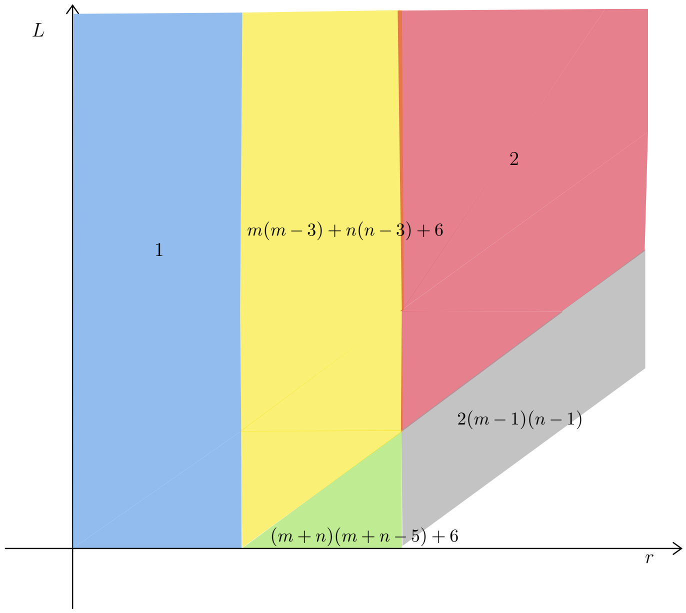


Figure 4.10: Rank of $H_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}})$

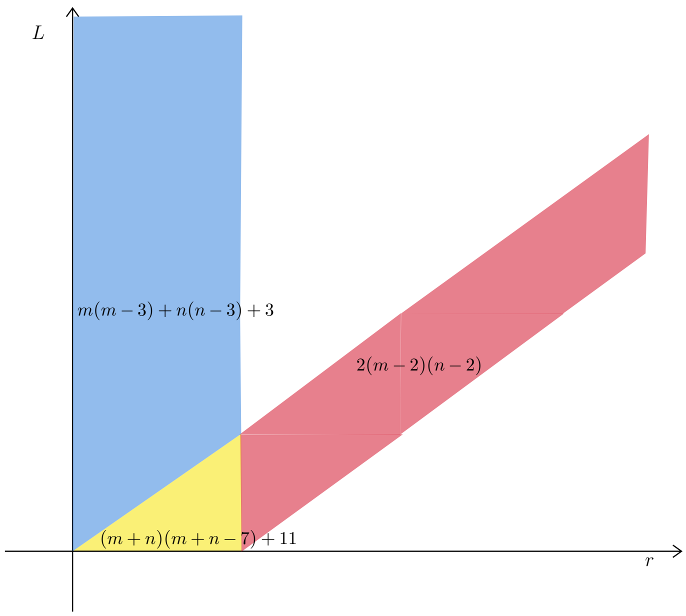


Figure 4.11: Rank of $H_1((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}})$

4.8 Decomposition of $PH_i((\hat{\mathcal{H}}_{m,n})^2_{-,-};\mathbb{F})$

In the previous section, we computed the persistence module $H_i((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}})$ for all $r, L_{e_1} \in \mathbb{R}_{>0}$. Note that the hyperplane arrangement of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$ can be interpreted as a functor

$$(\hat{\mathcal{H}}_{m,n})^2_{-,-} : (\mathbb{R}_{>0}, \leq)^{\mathsf{op}} \times (\mathbb{R}_{>0}, \leq) \to \textbf{Top}$$

where $(\hat{\mathcal{H}}_{m,n})^2_{-,-}$ sends $(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ to $(\hat{\mathcal{H}}_{m,n})^2_{a,b}$ and sends the unique arrow $(a,b) \to (a',b')$ to the inclusion map $\iota : (\hat{\mathcal{H}}_{m,n})^2_{a,b} \to (\hat{\mathcal{H}}_{m,n})^2_{a',b'}$, for all $a' \leq a$ and $b \leq b'$. Post-composing the i-th homology functor $H_i(-)$ with $(\hat{\mathcal{H}}_{m,n})^2_{-,-}$, we obtain

$$PH_i((\hat{\mathcal{H}}_{m,n})^2_{-,-}) : (\mathbb{R}_{>0}, \leq)^{\mathsf{op}} \times (\mathbb{R}_{>0}, \leq) \to \textbf{Ab}$$

In other words, at the object level, for each $(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$,

$$PH_i((\hat{\mathcal{H}}_{m,n})^2_{a,b}) = H_i((\hat{\mathcal{H}}_{m,n})^2_{a,b})$$

At the morphism level, $PH_i((\hat{\mathcal{H}}_{m,n})^2_{-,-})$ sends each morphism $(a,b) \to (a',b')$ to a group homomorphism

$$\iota_* : H_i((\hat{\mathcal{H}}_{m,n})^2_{a,b}) \to H_i((\hat{\mathcal{H}}_{m,n})^2_{a',b'})$$

where ι_* is induced by the inclusion map $\iota : (\hat{\mathcal{H}}_{m,n})^2_{a,b} \to (\hat{\mathcal{H}}_{m,n})^2_{a',b'}$ in **Top**.

One natural question is whether or not it can be written as a direct sum of polycodes. In this section, we give the decompositions of $PH_i((\hat{\mathcal{H}}_{m,n})^2_{-,-};\mathbb{F})$ where $i = 0, 1$.

Since there are finitely many chambers in the hyperplane arrangement of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$, we may associate the hyperplane arrangement with the Hasse diagram of a poset (denoted by

$(P,\leq))$ as follows:

- each chamber of the hyperplane arrangement is an element of $(P,\leq)$;
- each arrow corresponds to a wall between two chambers, and the orientation of the arrow is given by the filtration of the spaces $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$.

A poset representation over $(P,\leq)$ related to $PH_0((\hat{\mathcal{H}}_{m,n})^2_{-,-};\mathbb{F})$ is given in (4.146):

$$\begin{tikzcd}
\mathbb{F} & \mathbb{F}^{m^2+n^2-3m-3n+6} \arrow[l] & \mathbb{F}^2 \arrow[l] \\
\mathbb{F} \arrow[u] & \mathbb{F}^{m^2+n^2-3m-3n+6} \arrow[l] \arrow[u] & \mathbb{F}^2 \arrow[l] \arrow[u] \\
\mathbb{F} \arrow[u] & \mathbb{F}^{m^2+n^2-3m-3n+6} \arrow[l] \arrow[u] & \mathbb{F}^2 \arrow[u] \\
 & \mathbb{F}^{m^2+n^2-3m-3n+6} \arrow[u] & \mathbb{F}^2 \arrow[l] \arrow[u] \\
 & \mathbb{F}^{m^2+n^2+2mn-5m-5n+6} \arrow[u] & \mathbb{F}^{2(m-1)(n-1)} \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.146}$$

By an abuse of notation, we use $PH_0((\hat{\mathcal{H}}_{-,-})^2_{r,L_{e_1}};\mathbb{F})$ to denote the 2-parameter persistence module given in (4.146). We want to understand the behavior of each arrow in the diagram.

$$
\begin{tikzcd}
<f> & \left\langle \begin{array}{l} e'_k, \hat{e}'_k, f_{2,m+2}, \hat{f}_{2,m+2}, f_{ij}, \hat{f}_{ij} \mid 2 \leq k \leq \\ n-1, 2 \leq i < j \leq m, m+2 \leq i < j \leq \\ m+n-1 \end{array} \right\rangle \arrow[l, "\alpha"'] & < f_{2,m+2}, \hat{f}_{2,m+2} > \arrow[l, "\iota"'] \\
<f> \arrow[u] & \left\langle \begin{array}{l} e'_k, \hat{e}'_k, f_{2,m+2}, \hat{f}_{2,m+2}, f_{ij}, \hat{f}_{ij} \mid 2 \leq k \leq \\ n-1, 2 \leq i < j \leq m, m+2 \leq i < j \leq \\ m+n-1 \end{array} \right\rangle \arrow[l, "\alpha"'] \arrow[u] & < f_{2,m+2}, \hat{f}_{2,m+2} > \arrow[l, "\iota"'] \arrow[u] \\
<f> \arrow[u] & \left\langle \begin{array}{l} e'_k, \hat{e}'_k, f_{2,m+2}, \hat{f}_{2,m+2}, f_{ij}, \hat{f}_{ij} \mid 2 \leq k \leq \\ n-1, 2 \leq i < j \leq m, m+2 \leq i < j \leq \\ m+n-1 \end{array} \right\rangle \arrow[l, "\alpha"'] \arrow[u] & < f_{2,m+2}, \hat{f}_{2,m+2} > \arrow[u] \\
 & \left\langle \begin{array}{l} e'_k, \hat{e}'_k, f_{2,m+2}, \hat{f}_{2,m+2}, f_{ij}, \hat{f}_{ij} \mid 2 \leq k \leq \\ n-1, 2 \leq i < j \leq m, m+2 \leq i < j \leq \\ m+n-1 \end{array} \right\rangle \arrow[u] & < f_{2,m+2}, \hat{f}_{2,m+2} > \arrow[l, "\iota"'] \arrow[u] \\
 & \left\langle \begin{array}{l} e'_k, \hat{e}'_k, f_{l,m+2}, \ldots, f_{l,m+n}, \\ \hat{f}_{l,m+2}, \ldots, \hat{f}_{l,m+n}, f_{ij}, \hat{f}_{ij} \mid 2 \leq k \leq \\ n-1, 2 \leq i < j \leq m, m+2 \leq i < \\ j \leq m+n-1, 2 \leq l \leq m \end{array} \right\rangle \arrow[u, "\beta"] & \left\langle \begin{array}{l} f_{l,m+2}, \ldots, \\ f_{l,m+n}, \hat{f}_{l,m+2}, \\ \ldots, \hat{f}_{l,m+n} \mid 2 \leq \\ l \leq m \end{array} \right\rangle \arrow[l, "\varsigma"'] \arrow[u, "\gamma"']
\end{tikzcd}
\tag{4.147}
$$

where

- α maps every basis element of $< e'_k, \hat{e}'_k, f_{2,m+2}, \hat{f}_{2,m+2}, f_{ij}, \hat{f}_{ij} \mid 2 \leq k \leq n-1, 2 \leq i < j \leq m, m+2 \leq i < j \leq m+n-1 >$ to f;

- β maps e_k to e_k and maps $\hat{e}_k$ to $\hat{e}_k$ for all $2 \leq k \leq n-1$. In addition, β maps f_{ij} to f_{ij} and $\hat{f}_{ij}$ to $\hat{f}_{ij}$ for all $2 \leq i < j \leq m$ and $m+2 \leq i < j \leq m+n-1$. Moreover, β maps $f_{l,m+k}$ to $f_{2,m+k}$ and $\hat{f}_{l,m+k}$ to $\hat{f}_{2,m+k}$ for all $2 \leq k \leq n$ and $2 \leq l \leq m$.
- γ maps $f_{l,m+k}$ to $f_{2,m+2}$ and maps $\hat{f}_{l,m+k}$ to $\hat{f}_{2,m+2}$ for all $2 \leq k \leq n$ and $2 \leq l \leq m$;
- ι is an inclusion map;
- ς is an inclusion map;
- unlabeled vertical maps are the identity maps.

A poset representation over $(P, \leq)$ related to $PH_1((\hat{\mathcal{H}}_{-,-})^2_{r,L_{e_1}};\mathbb{F})$ is given by (4.148):

$$\begin{tikzcd}
\mathbb{F}^{m^2-3m+n^2-3n+3} & 0 \arrow[l] & 0 \arrow[l] \\
\mathbb{F}^{m^2-3m+n^2-3n+3} \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
\mathbb{F}^{m^2+n^2+2mn-7m-7n+11} \arrow[u, "\eta"] & \mathbb{F}^{2(m-2)(n-2)} \arrow[l, "\zeta"'] \arrow[u] & 0 \arrow[u] \\
& \mathbb{F}^{2(m-2)(n-2)} \arrow[u] & \mathbb{F}^{2(m-2)(n-2)} \arrow[l] \arrow[u] \\
& 0 \arrow[u] & 0 \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.148}$$

where

- ζ is the inclusion map;
- $\eta \circ \zeta = 0$;
- unlabeled non-trivial vertical maps are the identity maps.

By an abuse of notation, we use $PH_1((\hat{\mathcal{H}}_{-,-})^2_{r,L_{e_1}};\mathbb{F})$ to denote the P-indexed persistence module given by (4.148).

Theorem 4.45. $PH_1((\hat{\mathcal{H}}_{-,-})^2_{a,b};\mathbb{F})$ *is interval decomposable.*

Proof. We denote the support of $PH_1((\hat{\mathcal{H}}_{-,-})^2_{r,L_{e_1}};\mathbb{F})$ by M. (See (4.149).)

$$\begin{tikzcd}
\mathbb{F}^{m^2-3m+n^2-3n+3} & & \\
\mathbb{F}^{m^2-3m+n^2-3n+3} \arrow[u] & & \\
\mathbb{F}^{m^2+n^2+2mn-7m-7n+11} \arrow[u, "\eta"] & \mathbb{F}^{2(m-2)(n-2)} \arrow[l, "\zeta"'] & \\
& \mathbb{F}^{2(m-2)(n-2)} \arrow[u] & \mathbb{F}^{2(m-2)(n-2)} \arrow[l]
\end{tikzcd} \tag{4.149}$$

Note that $\eta \circ \zeta = 0$ and the endomorphism of $\mathbb{F}^{2(m-2)(n-2)}$ is the identity map, see (4.148). Moreover, ζ is the inclusion map, hence M can be decompose into two subrepresentations M_1 and M_2:

$$M_1 = \begin{tikzcd}
0 & & \\
0 \arrow[u] & & \\
\mathbb{F}^{2(m-2)(n-2)} \arrow[u] & \mathbb{F}^{2(m-2)(n-2)} \arrow[l, "\mathrm{id}"'] & \\
& \mathbb{F}^{2(m-2)(n-2)} \arrow[u] & \mathbb{F}^{2(m-2)(n-2)} \arrow[l]
\end{tikzcd} \tag{4.150}$$

$$M_2 = \begin{tikzcd} \mathbb{F}^{m^2-3m+n^2-3n+3} & & \\ \mathbb{F}^{m^2-3m+n^2-3n+3} \arrow[u] & & \\ \mathbb{F}^{m^2-3m+n^2-3n+3} \arrow[u] & 0 \arrow[l] & \\ & 0 \arrow[u] & 0 \arrow[l] \end{tikzcd} \tag{4.151}$$

Since M_1 is an A_4-quiver and M_2 is an A_3-quiver, they are interval decomposable. Let P_1 denote the underlying poset of the support of M_1, and P_2 denote the underlying poset of the support of M_2. Note that the intervals in P_i (where $i = 1, 2$) are also intervals in P. Therefore, we can extend each indecomposable representation of M_i (where $i = 1, 2$) to a subrepresentation of $PH_1((\hat{\mathcal{H}}_{-,-})^2_{r,L_{e_1}};\mathbb{F})$ by putting 0 to every vertex which is not in the support of the subrepresentation and trivial morphism between two vertices where at least one vertex is not in the support of the subrepresentation. $\square$

Theorem 4.46. *$PH_0((\hat{\mathcal{H}}_{-,-})^2_{r,L_{e_1}};\mathbb{F})$ is interval decomposable.*

Proof. Equation (4.147) provides us with the behavior of each arrow with given basis elements. Our goal is to find a new basis of each vector space $PH_0((\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}};\mathbb{F})$ such that each morphism in (4.147) maps every basis to another basis or zero, depending on the geometry of $(\hat{\mathcal{H}}_{m,n})^2_{r,L_{e_1}}$. We choose the basis for each vector space provided in the diagram below:

$$
\begin{tikzcd}[column sep=small]
< f > & \left\langle \begin{array}{l} e'_k - f_{2,m+2}, \hat{e}'_k - \hat{f}_{2,m+2}, f_{2,m+2}, \\ \hat{f}_{2,m+2} - f_{2,m+2}, f_{ij} - f_{2,m+2}, \hat{f}_{ij} - \\ \hat{f}_{2,m+2} \mid 2 \leq k \leq n-1, 2 \leq i < j \leq \\ m, m+2 \leq i < j \leq m+n-1 \end{array} \right\rangle \arrow[l, "\alpha"'] & < f_{2,m+2}, \hat{f}_{2,m+2} - f_{2,m+2} > \arrow[l, "\iota"'] \\
< f > \arrow[u] & \left\langle \begin{array}{l} e'_k - f_{2,m+2}, \hat{e}'_k - \hat{f}_{2,m+2}, f_{2,m+2}, \\ \hat{f}_{2,m+2} - f_{2,m+2}, f_{ij} - f_{2,m+2}, \hat{f}_{ij} - \\ \hat{f}_{2,m+2} \mid 2 \leq k \leq n-1, 2 \leq i < j \leq \\ m, m+2 \leq i < j \leq m+n-1 \end{array} \right\rangle \arrow[l, "\alpha"'] \arrow[u] & < f_{2,m+2}, \hat{f}_{2,m+2} - f_{2,m+2} > \arrow[l, "\iota"'] \arrow[u] \\
< f > \arrow[u] & \left\langle \begin{array}{l} e'_k - f_{2,m+2}, \hat{e}'_k - \hat{f}_{2,m+2}, f_{2,m+2}, \\ \hat{f}_{2,m+2} - f_{2,m+2}, f_{ij} - f_{2,m+2}, \hat{f}_{ij} - \\ \hat{f}_{2,m+2} \mid 2 \leq k \leq n-1, 2 \leq i < j \leq \\ m, m+2 \leq i < j \leq m+n-1 \end{array} \right\rangle \arrow[l, "\alpha"'] \arrow[u] & < f_{2,m+2}, \hat{f}_{2,m+2} - f_{2,m+2} > \arrow[u] \\
 & \left\langle \begin{array}{l} e'_k - f_{2,m+2}, \hat{e}'_k - \hat{f}_{2,m+2}, f_{2,m+2}, \\ \hat{f}_{2,m+2} - f_{2,m+2}, f_{ij} - f_{2,m+2}, \hat{f}_{ij} - \\ \hat{f}_{2,m+2} \mid 2 \leq k \leq n-1, 2 \leq i < j \leq \\ m, m+2 \leq i < j \leq m+n-1 \end{array} \right\rangle \arrow[u] & < f_{2,m+2}, \hat{f}_{2,m+2} - f_{2,m+2} > \arrow[l, "\iota"'] \arrow[u] \\
 & < A > \arrow[u, "\beta"] & < B > \arrow[l, "\varsigma"'] \arrow[u, "\gamma"']
\end{tikzcd}
\tag{4.152}
$$

where A consists of the following elements:

1. $e'_k - f_{2,m+2}$ and $\hat{e}'_k - \hat{f}_{2,m+2}$ for all $2 \leq k \leq n-1$;

2. $f_{ij} - f_{2,m+2}, \hat{f}_{ij} - \hat{f}_{2,m+2}$ for all $2 \leq i < j \leq m$ and $m+2 \leq i < j \leq m+n-1$;

3. $f_{2,m+2}, f_{2,m+3} - f_{2,m+2}, \ldots, f_{2,m+n} - f_{2,m+2}$;

4. $f_{s,m+t} - f_{2,m+t}$ for all $3 \leq s \leq m$ and $2 \leq t \leq n$;

5. $\hat{f}_{l,m+t} - f_{l,m+t}$ for all $2 \leq l \leq m$ and $2 \leq t \leq n$

and B consists of the following elements:

1. $f_{2,m+2}, f_{2,m+3} - f_{2,m+2}, \ldots, f_{2,m+n-1} - f_{2,m+2}$;

2. $f_{s,m+t} - f_{2,m+t}$ for all $3 \leq s \leq m$ and $2 \leq t \leq n$;

3. $\hat{f}_{l,m+t} - f_{l,m+t}$ for all $2 \leq l \leq m$ and $2 \leq t \leq n$

Define

$$N_1 = \begin{tikzcd}
< f > & < f_{2,m+2} > \arrow[l, "\mathsf{id}"'] & < f_{2,m+2} > \arrow[l, "\mathsf{id}"'] \\
< f > \arrow[u] & < f_{2,m+2} > \arrow[l, "\mathsf{id}"'] \arrow[u] & < f_{2,m+2} > \arrow[l, "\mathsf{id}"'] \arrow[u] \\
< f > \arrow[u] & < f_{2,m+2} > \arrow[l, "\mathsf{id}"'] \arrow[u] & < f_{2,m+2} > \arrow[u] \\
& < f_{2,m+2} > \arrow[u] & < f_{2,m+2} > \arrow[l, "\mathsf{id}"'] \arrow[u] \\
& < f_{2,m+2} > \arrow[u, "\beta"] & < f_{2,m+2} > \arrow[l, "\mathsf{id}"'] \arrow[u, "\gamma"']
\end{tikzcd}
\cong
\begin{tikzcd}
\mathbb{F} & \mathbb{F} \arrow[l, "\mathsf{id}"'] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \\
\mathbb{F} \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] \\
\mathbb{F} \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] & \mathbb{F} \arrow[u, "\mathsf{id}"'] \\
& \mathbb{F} \arrow[u, "\mathsf{id}"] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] \\
& \mathbb{F} \arrow[u, "\beta"] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u, "\gamma"']
\end{tikzcd}
\tag{4.153}$$

$$N_2 = \begin{tikzcd}
0 & \arrow[l] <\hat{f}_{2,m+2} - f_{2,m+2}> & \arrow[l, "\mathsf{id}"'] <\hat{f}_{2,m+2} - f_{2,m+2}> \\
0 \arrow[u] & \arrow[l] <\hat{f}_{2,m+2} - f_{2,m+2}> \arrow[u] & \arrow[l, "\mathsf{id}"'] <\hat{f}_{2,m+2} - f_{2,m+2}> \arrow[u] \\
0 \arrow[u] & \arrow[l] <\hat{f}_{2,m+2} - f_{2,m+2}> \arrow[u] & <\hat{f}_{2,m+2} - f_{2,m+2}> \arrow[u] \\
 & <\hat{f}_{2,m+2} - f_{2,m+2}> \arrow[u] & \arrow[l, "\mathsf{id}"'] <\hat{f}_{2,m+2} - f_{2,m+2}> \arrow[u] \\
 & <\hat{f}_{2,m+2} - f_{2,m+2}> \arrow[u, "\beta"] & \arrow[l, "\mathsf{id}"'] <\hat{f}_{2,m+2} - f_{2,m+2}> \arrow[u, "\gamma"']
\end{tikzcd}
\cong
\begin{tikzcd}
0 & \arrow[l] \mathbb{F} & \arrow[l] \mathbb{F} \\
0 \arrow[u] & \arrow[l] \mathbb{F} \arrow[u] & \arrow[l] \mathbb{F} \arrow[u] \\
0 \arrow[u] & \arrow[l] \mathbb{F} \arrow[u] & \mathbb{F} \arrow[u] \\
 & \mathbb{F} \arrow[u] & \arrow[l] \mathbb{F} \arrow[u] \\
 & \mathbb{F} \arrow[u, "\beta"] & \arrow[l, "\mathsf{id}"'] \mathbb{F} \arrow[u, "\gamma"']
\end{tikzcd} \tag{4.154}$$

For $2 \leq k \leq n-1$

$$E_k = \begin{tikzcd}
0 & \arrow[l] <e'_k - f_{2,m+2}> & \arrow[l] 0 \\
0 \arrow[u] & \arrow[l] <e'_k - f_{2,m+2}> \arrow[u] & \arrow[l] 0 \arrow[u] \\
0 \arrow[u] & \arrow[l] <e'_k - f_{2,m+2}> \arrow[u] & 0 \arrow[u] \\
 & <e'_k - f_{2,m+2}> \arrow[u] & \arrow[l] 0 \arrow[u] \\
 & <e'_k - f_{2,m+2}> \arrow[u, "\beta"] & \arrow[l] 0 \arrow[u]
\end{tikzcd}
\cong
\begin{tikzcd}
0 & \arrow[l] \mathbb{F} & \arrow[l] 0 \\
0 \arrow[u] & \arrow[l] \mathbb{F} \arrow[u] & \arrow[l] 0 \arrow[u] \\
0 \arrow[u] & \arrow[l] \mathbb{F} \arrow[u] & 0 \arrow[u] \\
 & \mathbb{F} \arrow[u] & \arrow[l] 0 \arrow[u] \\
 & \mathbb{F} \arrow[u, "\beta"] & \arrow[l] 0 \arrow[u]
\end{tikzcd} \tag{4.155}$$

and

$$\hat{E}_k = \begin{tikzcd}
0 & < \hat{e}'_k - \hat{f}_{2,m+2} > \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & < \hat{e}'_k - \hat{f}_{2,m+2} > \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
0 \arrow[u] & < \hat{e}'_k - \hat{f}_{2,m+2} > \arrow[l] \arrow[u] & 0 \arrow[u] \\
 & < \hat{e}'_k - \hat{f}_{2,m+2} > \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & < \hat{e}'_k - \hat{f}_{2,m+2} > \arrow[u, "\beta"] & 0 \arrow[l] \arrow[u]
\end{tikzcd} \cong \begin{tikzcd}
0 & \mathbb{F} \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[u] \\
 & \mathbb{F} \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & \mathbb{F} \arrow[u, "\beta"] & 0 \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.156}$$

For $2 \leq i < j \leq m, m+2 \leq i < j \leq m+n-1$

$$F_{ij} = \begin{tikzcd}
0 & < f_{ij} - f_{2,m+2} > \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & < f_{ij} - f_{2,m+2} > \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
0 \arrow[u] & < f_{ij} - f_{2,m+2} > \arrow[l] \arrow[u] & 0 \arrow[u] \\
 & < f_{ij} - f_{2,m+2} > \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & < f_{ij} - f_{2,m+2} > \arrow[u, "\beta"] & 0 \arrow[l] \arrow[u]
\end{tikzcd} \cong \begin{tikzcd}
0 & \mathbb{F} \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[u] \\
 & \mathbb{F} \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & \mathbb{F} \arrow[u, "\beta"] & 0 \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.157}$$

and

$$\hat{F}_{ij} = \begin{tikzcd} 0 & < \hat{f}_{ij} - \hat{f}_{2,m+2} > \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & < \hat{f}_{ij} - \hat{f}_{2,m+2} > \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & < \hat{f}_{ij} - \hat{f}_{2,m+2} > \arrow[l] \arrow[u] & 0 \arrow[u] \\ & < \hat{f}_{ij} - \hat{f}_{2,m+2} > \arrow[u] & 0 \arrow[l] \arrow[u] \\ & < \hat{f}_{ij} - \hat{f}_{2,m+2} > \arrow[u, "\beta"] & 0 \arrow[l] \arrow[u] \end{tikzcd} \cong \begin{tikzcd} 0 & \mathbb{F} \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & \mathbb{F} \arrow[l] \arrow[u] & 0 \arrow[u] \\ & \mathbb{F} \arrow[u] & 0 \arrow[l] \arrow[u] \\ & \mathbb{F} \arrow[u, "\beta"] & 0 \arrow[l] \arrow[u] \end{tikzcd} \tag{4.158}$$

For all $3 \leq t \leq n$

$$G_{2,t} = \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & 0 \arrow[l] & 0 \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\ & < f_{2,m+t} - f_{2,m+2} > \arrow[u] & < f_{2,m+t} - f_{2,m+2} > \arrow[l, "\mathsf{id}"'] \arrow[u] \end{tikzcd} \cong \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\ & \mathbb{F} \arrow[u] & \mathbb{F} \arrow[l, "\mathsf{id}"'] \arrow[u] \end{tikzcd} \tag{4.159}$$

and

$$\hat{G}_{2,t} = \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\ & < \hat{f}_{2,m+t} - f_{2,m+t} > \arrow[u] & < \hat{f}_{2,m+t} - f_{2,m+t} > \arrow[l, "\mathrm{id}"'] \arrow[u] \end{tikzcd} \cong \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\ & \mathbb{F} \arrow[u] & \mathbb{F} \arrow[l, "\mathrm{id}"'] \arrow[u] \end{tikzcd} \tag{4.160}$$

For all $2 \leq t \leq n$ and $3 \leq s \leq m$,

$$G_{s,t} = \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\ & < f_{s,m+t} - f_{2,m+t} > \arrow[u] & < f_{s,m+t} - f_{2,m+t} > \arrow[l, "\mathrm{id}"'] \arrow[u] \end{tikzcd} \cong \begin{tikzcd} 0 & 0 \arrow[l] & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\ & \mathbb{F} \arrow[u] & \mathbb{F} \arrow[l, "\mathrm{id}"'] \arrow[u] \end{tikzcd} \tag{4.161}$$

and

$$\hat{G}_{s,t} = \begin{tikzcd}
0 & 0 \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[u] \\
 & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & < \hat{f}_{s,m+t} - \hat{f}_{2,m+t} > \arrow[u] & < \hat{f}_{s,m+t} - \hat{f}_{2,m+t} > \arrow[l, "\mathrm{id}"'] \arrow[u]
\end{tikzcd} \cong \begin{tikzcd}
0 & 0 \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[u] \\
 & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \\
 & \mathbb{F} \arrow[u] & \mathbb{F} \arrow[l, "\mathrm{id}"'] \arrow[u]
\end{tikzcd} \tag{4.162}$$

It is clear that

$$\begin{aligned} PH_0((\hat{\mathcal{H}}_{m,n})^2_{-,-};\mathbb{F}) \cong N_1 \oplus N_2 \oplus (\bigoplus_{k=2}^{n-1} E_k) \oplus (\bigoplus_{k=2}^{n-1} \hat{E}_k) \oplus (\bigoplus_{2\leq i<j\leq m} F_{ij}) \oplus (\bigoplus_{2\leq i<j\leq m} \hat{F}_{ij}) \\ \oplus (\bigoplus_{m+2\leq i<j\leq m+n-1} F_{ij}) \oplus (\bigoplus_{m+2\leq i<j\leq m+n-1} \hat{F}_{ij}) \\ \oplus (\bigoplus_{3\leq t\leq n-1} G_{2,t}) \oplus (\bigoplus_{3\leq t\leq n-1} \hat{G}_{2,t}) \\ \oplus (\bigoplus_{3\leq s\leq m} \bigoplus_{2\leq t\leq n-1} G_{s,t}) \oplus (\bigoplus_{3\leq s\leq m} \bigoplus_{2\leq t\leq n-1} \hat{G}_{s,t}) \end{aligned} \tag{4.163}$$

Note that N_1, N_2, E_k, $\hat{E}_k$, F_{ij}, $\hat{F}_{ij}$, $G_{2,t}$, $\hat{G}_{2,t}$, $G_{s,t}$, and $\hat{G}_{s,t}$ are interval modules, hence by Lemma 3.1, they are indecomposable. Therefore, $PH_0((\hat{\mathcal{H}}_{-,-})^2_{r,L_{e_1}};\mathbb{F})$ is interval decomposable. □

4.9 $PH_i(\mathbb{T}^2_{r,L_{e_1}};\mathbb{F})$

Let $\mathbb{T} = (V, E)$ be a tree, and by abuse of notation, we use $\mathbb{T}$ to denote a finite tree whose underlying graph is $\mathbb{T}$ and the length of the edge e_1 has length L_{e_1} while other edges have length 1. Note that T can be written as a union of the star graphs and generalized H graphs, say $\mathbb{T} = \mathbb{T}_1 \cup \cdots \cup \mathbb{T}_n$, then we can calculate $H_i(\mathbb{T}^2_{r,L_{e_1}})$: first calculate $H_i((\mathbb{T}_1 \cup \mathbb{T}_2)^2_{r,l_{e_1}})$, using the cover

$$
\begin{aligned}
W_{11} &= (\mathbb{T}_1)^2_{r,L_{e_1}} \\
W_{12} &= \mathbb{T}_1 \times \mathbb{T}_2 - \{(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2 \mid \delta(x, y) < r\} \\
W_{21} &= \mathbb{T}_2 \times \mathbb{T}_1 - \{(x, y) \in \mathbb{T}_2 \times \mathbb{T}_1 \mid \delta(x, y) < r\} \\
W_{22} &= (\mathbb{T}_2)^2_{r,L_{e_1}}
\end{aligned}
\tag{4.164}
$$

Applying the Mayer-Vietoris spectral sequence, we can calculate $H_i((\mathbb{T}_1 \cup \mathbb{T}_2)^2_{r,L_{e_1}})$. We will obtain $H_i(\mathbb{T}^2_{r,L_{e_1}})$ after finitely many steps, adding a star graph or a generalized H graph at each step. Since $PH_i(\mathbb{T}^2_{r,L_{e_1}}) = H_i(\mathbb{T}^2_{r,L_{e_1}})$ for all $r, L_{e_1} \in \mathbb{R}_{>0}$, we can compute the functor $PH_i(\mathbb{T}^2_{-,-})$ at the object level.

4.10 $H_i(\mathsf{Tree}^2_{r,\mathbf{L}})$ of Trees with Arbitrary Edge Lengths

In this section, we use $\hat{\mathsf{Star}}_k$ to denote the metric star graph where the length of an edge e_i is L_{e_i} (assume $L_{e_i} > 0$) for all $i = 1, \ldots, k$ and use $\hat{\mathbb{T}}$ to denote a finite tree $\mathbb{T} = (V, E)$ where the length of an edge e_i is L_{e_i} (assume $L_{e_i} > 0$) for all $i = 1, \ldots, |E|$

Theorem 4.47. *Let $k \geq 1$. Then $H_2((\hat{\mathsf{Star}}_k)^2_r) = 0$ and $H_1((\hat{\mathsf{Star}}_k)^2_r)$ is torsion-free.*

Proof. Induction on k.

When $k = 1$. Note that

$$(\hat{\mathsf{Star}}_1)^2_r \simeq \begin{cases} \{*_1, *_2\}, & \text{if } 0 < r \leq L_{e_1} \\ \emptyset, & \text{if } \ L_{e_1} < r \end{cases} \tag{4.165}$$

Since the spaces that are homotopy equivalent to $(\hat{\mathsf{Star}}_1)^2_r$ do not contain 1-cells and 2-cells, $H_2((\hat{\mathsf{Star}}_1)^2_r) = H_1((\hat{\mathsf{Star}}_1)^2_r) = 0$.

When $k = 2$, Without loss of generality we assume $L_{e_1} \leq L_{e_2}$. Note that

$$(\hat{\mathsf{Star}}_2)^2_r \simeq \begin{cases} \{*_1, *_2\}, & \text{if } 0 < r \leq L_{e_1} \\ \{*_1, *_2\}, & \text{if } L_{e_1} < r \leq L_{e_2} \\ \{*_1, *_2\}, & \text{if } L_{e_2} < r \leq L_{e_1} + L_{e_2} \\ \emptyset, & \text{if } \ L_{e_1} + L_{e_2} < r \end{cases} \tag{4.166}$$

Since the spaces that are homotopy equivalent to $(\hat{\mathsf{Star}}_2)^2_r$ do not contain 1-cells and 2-cells, $H_2((\hat{\mathsf{Star}}_2)^2_r) = H_1((\hat{\mathsf{Star}}_2)^2_r) = 0$.

When $k = N + 1$, we consider the following cover of $(\hat{\mathsf{Star}}_{N+1})^2_r$:

$$\begin{aligned} U_{11} &= (e_1)^2_r \\ U_{12} &= e_1 \times \hat{\mathsf{Star}}_N - \{(x, y) \in e_1 \times \hat{\mathsf{Star}}_N \mid \delta(x, y) < r\} \\ U_{21} &= \hat{\mathsf{Star}}_N \times e_1 - \{(x, y) \in \hat{\mathsf{Star}}_N \times e_1 \mid \delta(x, y) < r\} \\ U_{22} &= (\hat{\mathsf{Star}}_N)^2_r \end{aligned} \tag{4.167}$$

Let $S := \{L_{e_i} \mid i = 2, \ldots, N + 1\}$ with cardinality $m := |S|$. Note that S is a subset of $\mathbb{R}$ hence S is a totally-ordered set. Without loss of generality, we assume $s_1 < s_2 < \cdots < s_m$

where $s_i \in S$ for $i = 1, \ldots, m$. Moreover, for each $i = 1, \ldots, m$, we use c_i to denote cardinality of the set $\{j = 2, \ldots, N+1 \mid L_{e_j} = s_i\}$.

Note that $(e_1)^2_r \simeq \begin{cases} \{*_1, *_2\}, & \text{if } 0 < r \leq L_{e_1} \\ \emptyset, & \text{if } L_{e_1} < r \end{cases}$, where $\{*_1, *_2\}$ is a subspace of $(e_1)^2_r$ consists of two points.

On the other hand,

$$U_{12} \simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq L_{e_1} \\ \{*_2, *_3, \ldots, *_{N+1}\}, & \text{if } L_{e_1} < r \leq L_{e_1} + s_1 \\ \{*_2, *_3, \ldots, *_{N+1-c_1}\}, & \text{if } L_{e_1} + s_1 < r \leq L_{e_1} + s_2 \\ \{*_2, *_3, \ldots, *_{N+1-c_1-c_2}\}, & \text{if } L_{e_1} + s_2 < r \leq L_{e_1} + s_3 \\ \vdots & \vdots \\ \emptyset, & \text{if } L_{e_1} + s_m < r \end{cases}$$

Similarly,

$$U_{21} \simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq L_{e_1} \\ \{*_2, *_3, \ldots, *_{N+1}\}, & \text{if } L_{e_1} < r \leq L_{e_1} + s_1 \\ \{*_2, *_3, \ldots, *_{N+1-c_1}\}, & \text{if } L_{e_1} + s_1 < r \leq L_{e_1} + s_2 \\ \{*_2, *_3, \ldots, *_{N+1-c_1-c_2}\}, & \text{if } L_{e_1} + s_2 < r \leq L_{e_1} + s_3 \\ \vdots & \vdots \\ \emptyset, & \text{if } L_{e_1} + s_m < r \end{cases}$$

Therefore, for all $i \geq 1$,

$$H_i(U_{11}) = H_i(U_{12}) = H_i(U_{21}) = 0 \tag{4.168}$$

Now let's consider the intersections of U_{ij}. Note that

$$\begin{aligned}
U_{11} \cap U_{12} &\simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq L_{e_1} \\ \emptyset, & \text{if } L_{e_1} < r \end{cases} \\
U_{11} \cap U_{21} &\simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq L_{e_1} \\ \emptyset, & \text{if } L_{e_1} < r \end{cases} \\
U_{22} \cap U_{12} &= (\hat{\mathsf{Star}}_N)^2_r \cap (e_1 \times \hat{\mathsf{Star}}_N - \{(x,y) \in e_1 \times \hat{\mathsf{Star}}_N \mid \delta(x,y) < r\}) \\
&\simeq \begin{cases} \{*_2, *_3, \dots, *_{N+1}\}, & \text{if } 0 < r \leq s_1 \\ \{*_2, *_3, \dots, *_{N+1-c_1}\}, & \text{if } s_1 < r \leq s_2 \\ \{*_2, *_3, \dots, *_{N+1-c_1-c_2}\}, & \text{if } s_2 < r \leq s_3 \\ \vdots & \vdots \\ \emptyset, & \text{if } s_m < r \end{cases} \\
U_{22} \cap U_{21} &= (\hat{\mathsf{Star}}_N)^2_r \cap \hat{\mathsf{Star}}_N \times e_1 - \{(x,y) \in \hat{\mathsf{Star}}_N \times e_1 \mid \delta(x,y) < r\} \\
&\simeq \begin{cases} \{*_2, *_3, \dots, *_{N+1}\}, & \text{if } 0 < r \leq s_1 \\ \{*_2, *_3, \dots, *_{N+1-c_1}\}, & \text{if } s_1 < r \leq s_2 \\ \{*_2, *_3, \dots, *_{N+1-c_1-c_2}\}, & \text{if } s_2 < r \leq s_3 \\ \vdots & \vdots \\ \emptyset, & \text{if } s_m < r \end{cases}
\end{aligned} \tag{4.169}$$

Therefore, for all $i \geq 1$,

$$H_i(U_{11} \cap U_{12}) = H_i(U_{11} \cap U_{21}) = H_i(U_{22} \cap U_{12}) = H_i(U_{22} \cap U_{21}) = 0 \tag{4.170}$$

The E^1 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & 0 & 0 \\
2 & H_2((\hat{\mathsf{Star}}_N)^2_r) & 0 & 0 \\
1 & H_1((\hat{\mathsf{Star}}_N)^2_r) & 0 & 0 \\
0 & * & \xleftarrow{d^1} \; * & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Hence the E^2 page of the Mayer-Vietoris spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
3 & \vdots & 0 & 0 \\
2 & H_2((\hat{\mathsf{Star}}_N)^2_r) & 0 & 0 \\
1 & H_1((\hat{\mathsf{Star}}_N)^2_r) & 0 & 0 \\
0 & * & * & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Note that $E^2_{10} = \mathsf{ker} d^1$ is a subgroup of a free abelian group, hence E^2_{10} is free.

Since there is no non-trivial differential on the E^2 page, $E^2 = E^\infty$. Hence

$$H_2((\hat{\mathsf{Star}}_{N+1})^2_r) \cong H_2((\hat{\mathsf{Star}}_N)^2_r)$$

By the induction hypothesis, $H_2((\hat{\mathsf{Star}}_N)^2_r) = 0$, hence

$$H_2((\hat{\mathsf{Star}}_{N+1})^2_r) = 0$$

By the induction hypothesis, $H_1((\hat{\mathsf{Star}}_N)^2_r)$ is torsion-free, hence

$$H_1((\hat{\mathsf{Star}}_{N+1})^2_r) \cong H_1((\hat{\mathsf{Star}}_N)^2_r) \oplus E^2_{10}$$

is torsion-free. □

Lemma 4.4. Let $\mathbb{T}$ be a metric tree with at least one essential vertices such that all its edges have length 1. Let $\{v\}$ be a leaf of $\mathbb{T}$. Then for all $r > 0$,

$$H_1(\mathbb{T} \times \{v\} - \{(x, y) \in \mathbb{T} \times \{v\} \mid \delta(x, y) < r\}) = 0$$

Proof. Without loss of generality, we assume $\mathbb{T}$ does not have vertices with degree 2. Induction on the number of essential vertices of $\mathbb{T}$.

When $n = 1$, then $\mathbb{T} = \mathsf{Star}_k$ for some $k \in \mathbb{N}$. Hence

$$\mathbb{T} \times \{v\} - \{(x, y) \in \mathbb{T} \times \{v\} \mid \delta(x, y) < r\} \simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq 1 \\ \{*_1, *_2, \dots, *_{k-1}\}, & \text{if } 1 < r \leq 2 \\ \emptyset, & \text{if } 2 < r \end{cases} \tag{4.171}$$

Since the spaces that are homotopy equivalent to $\mathbb{T} \times \{v\} - \{(x, y) \in \mathbb{T} \times \{v\} \mid \delta(x, y) < r\}$ do not have 2-cells, $H_1(\mathbb{T} \times \{v\} - \{(x, y) \in \mathbb{T} \times \{v\} \mid \delta(x, y) < r\}) = 0$ for all $r > 0$.

Assume the statement is true when $n = N$. When $n = N + 1$, let u be an essential vertex of $\mathbb{T}$ such that it is a leaf of the graph $\mathbb{T}'$ obtaining by deleting all the leaves of $\mathbb{T}$ and u is not adjacent to v.

Now we construct $\mathbb{T}$ as a wedge sum of two trees at vertex u: a tree (denoted by $\tilde{\mathbb{T}}$) obtained by deleting all leaves that incident to u and a tree (denoted by Star_{k-1}, where $k = \deg u$ in $\mathbb{T}$) with 1 essential vertex u which is not adjacent to v. Let

$$V_1 = \tilde{\mathbb{T}} \times \{v\} - \{(x, y) \in \tilde{\mathbb{T}} \times \{v\} \mid \delta(x, y) < r\}$$

and

$$V_2 = \mathsf{Star}_{k-1} \times \{v\} - \{(x, y) \in \mathsf{Star}_{k-1} \times \{v\} \mid \delta(x, y) < r\}$$

Let $a = \delta(u, v)$. Note that

$$V_1 \cap V_2 = \begin{cases} \{u\} \times \{v\}, & \text{if } 0 < r \leq a \\ \emptyset, & \text{if } a < r \end{cases} \tag{4.172}$$

Hence the reduced Mayer-Vietoris sequence implies

$$H_1(\mathbb{T} \times \{v\} - \{(x, y) \in \mathbb{T} \times \{v\} \mid \delta(x, y) < r\}) \cong H_1(V_1) \oplus H_1(V_2) \tag{4.173}$$

Since $\tilde{\mathbb{T}}$ has N essential vertices, by the induction hypothesis,

$$H_1(V_1) = H_1(\tilde{\mathbb{T}} \times \{v\} - \{(x, y) \in \tilde{\mathbb{T}} \times \{v\} \mid \delta(x, y) < r\}) = 0$$

Recall $a = \delta(u,v)$. Note that

$$V_2 \simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq a \\ \{*_1, *_2, \dots, *_{k-1}\}, & \text{if } a < r \leq a+1 \\ \emptyset, & \text{if } a+1 < r \end{cases} \tag{4.174}$$

Since the spaces that are homotopy equivalent to $V_2 = \mathsf{Star}_{k-1} \times \{v\} - \{(x,y) \in \mathsf{Star}_{k-1} \times \{v\} \mid \delta(x,y) < r\}$ do not have 1-cells,

$$H_1(\mathsf{Star}_{k-1} \times \{v\} - \{(x,y) \in \mathsf{Star}_{k-1} \times \{v\} \mid \delta(x,y) < r\}) = 0$$

Thus $H_1(V_2) = H_1(\mathbb{T} \times \{v\} - \{(x,y) \in \mathbb{T} \times \{v\} \mid \delta(x,y) < r\}) = 0$ for all $r > 0$ when $\mathbb{T}$ has $N+1$ essential vertices.

In conclusion, $H_1(\mathbb{T} \times \{v\} - \{(x,y) \in \mathbb{T} \times \{v\} \mid \delta(x,y) < r\}) = 0$ for all $r > 0$. □

Lemma 4.5. Let $\hat{\mathbb{T}} = (V,E)$ be a metric tree with at least one essential vertex, where the length of each edge e_i is L_{e_i} for all $i = 1, \dots, |E|$. Assume there exists $v_0 \in V$ and a subtree $\tilde{\mathbb{T}}$ of $\hat{\mathbb{T}}$ such that $\mathbb{T} = \tilde{\mathbb{T}} \vee_{v_0} \mathsf{Star}_{k-1}$ for some $k \geq 2$, where v_0 is the center of Star_{k-1} and a leaf of $\tilde{\mathbb{T}}$, then for all $r > 0$,

$$H_2(\tilde{\mathbb{T}} \times \mathsf{Star}_{k-1} - \{(x,y) \in \tilde{\mathbb{T}} \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\}) = 0$$

and $H_1(\tilde{\mathbb{T}} \times \mathsf{Star}_{k-1} - \{(x,y) \in \tilde{\mathbb{T}} \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\})$ is torsion-free.

Proof. Let $U_{12} := \tilde{\mathbb{T}} \times \mathsf{Star}_{k-1} - \{(x,y) \in \tilde{\mathbb{T}} \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\}$. Induction on the number of vertices of $\tilde{\mathbb{T}}$. When $n = 2$, then $\hat{\mathbb{T}} = \hat{\mathsf{Star}}_k$ and $\tilde{\mathbb{T}}$ is an edge of $\hat{\mathbb{T}}$, denoted by e_1. We denote the edges of Star_{k-1} by $e_2, \dots, e_k$. Let $S := \{L_{e_i} \mid i = 2, \dots, k\}$ with cardinality

$m := |S|$. Note that S is a subset of $\mathbb{R}$ hence S is a totally-ordered set. Without loss of generality, we assume $s_1 < s_2 < \cdots < s_m$ where $s_i \in S$ for $i = 1, \ldots, m$. Moreover, for each $i = 1, \ldots, m$, we use c_i to denote cardinality of the set $\{j = 2, \ldots, k \mid L_{e_j} = s_i\}$.

$$U_{12} \simeq \begin{cases} \{*\}, & \text{if } 0 < r \leq L_{e_1} \\ \{*_2, *_3, \ldots, *_k\}, & \text{if } L_{e_1} < r \leq L_{e_1} + s_1 \\ \{*_2, *_3, \ldots, *_{k-c_1}\}, & \text{if } L_{e_1} + s_1 < r \leq L_{e_1} + s_2 \\ \{*_2, *_3, \ldots, *_{k-c_1-c_2}\}, & \text{if } L_{e_1} + s_2 < r \leq L_{e_1} + s_3 \\ \vdots & \vdots \\ \emptyset, & \text{if } L_{e_1} + s_m < r \end{cases}$$

Therefore, $H_2(U_{12}) = H_1(U_{12}) = 0$.

Now we assume $\tilde{\mathbb{T}}$ has $N+1$ vertices. Since $\tilde{\mathbb{T}}$ is a tree, it has at least 2 leaves. Hence there exists a leaf u of $\tilde{\mathbb{T}}$ other than v_0 such that $\mathsf{diam}(\tilde{\mathbb{T}}) = \delta(u, v_0)$. Let e_1 denote the unique edge that incident to u and let w be the unique vertex adjacent to u. Consider the following cover of U_{12}:

$$\begin{aligned} V_1 &= (\tilde{\mathbb{T}} - \{u\}) \times \mathsf{Star}_{k-1} - \{(x,y) \in (\tilde{\mathbb{T}} - \{u\}) \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \\ V_2 &= e_1 \times \mathsf{Star}_{k-1} - \{(x,y) \in e_1 \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \end{aligned} \tag{4.175}$$

The Mayer-Vietoris long exact sequence of this cover is

$$\begin{aligned} \cdots \to H_2(V_1 \cap V_2) \to H_2(V_1) \oplus H_2(V_2) \to H_2(U_{12}) \\ \to H_1(V_1 \cap V_2) \to H_1(V_1) \oplus H_1(V_2) \to H_1(U_{12}) \\ \to H_0(V_1 \cap V_2) \to H_0(V_1) \oplus H_0(V_2) \to H_0(U_{12}) \to 0 \end{aligned} \tag{4.176}$$

Note that

$$
\begin{aligned}
V_1 \cap V_2 &= \{w\} \times \mathsf{Star}_{k-1} - \{(x,y) \in \{w\} \times \mathsf{Star}_{k-1} \mid \delta(x,y) < r\} \\
&\simeq \begin{cases}
\{*_2, *_3, \ldots, *_k\}, & \text{if } 0 < r \leq s_1 \\
\{*_2, *_3, \ldots, *_{k-c_1}\}, & \text{if } s_1 < r \leq s_2 \\
\{*_2, *_3, \ldots, *_{k-c_1-c_2}\}, & \text{if } s_2 < r \leq s_3 \\
\vdots & \vdots \\
\emptyset, & \text{if } s_m < r
\end{cases}
\end{aligned} \tag{4.177}
$$

Hence $H_2(V_1 \cap V_2) = H_1(V_1 \cap V_2) = 0$ and $H_0(V_1 \cap V_2) = 0$ is free. Note that, by the induction hypothesis, $H_2(V_1) = 0$ and $H_i(V_1)$ is torsion-free $(i = 0, 1)$ since $\tilde{\mathbb{T}} - \{u\}$ has N vertices. On the other hand, $H_2(V_2) = 0$ and $H_i(V_2)$ is torsion-free $(i = 0, 1)$ since V_2 is the base case of the induction. Hence the Mayer-Vietoris long exact sequence becomes

$$
\begin{aligned}
\cdots \to 0 &\to 0 \oplus 0 \to H_2(U_{12}) \\
0 &\to H_1(V_1) \oplus H_1(V_2) \xrightarrow{j_*} H_1(U_{12}) \\
&\xrightarrow{\partial_1} H_0(V_1 \cap V_2) \xrightarrow{i_*} H_0(V_1) \oplus H_0(V_2) \xrightarrow{j_*} H_0(U_{12}) \to 0
\end{aligned} \tag{4.178}
$$

Hence

$$H_2(U_{12}) = 0 \tag{4.179}$$

and

$$0 \to H_1(V_1) \oplus H_1(V_2) \xrightarrow{j_*} H_1(U_{12}) \to \operatorname{\mathsf{Im}} \partial_1 \to 0 \tag{4.180}$$

Since $H_0(V_1) \oplus H_0(V_2)$ is free and $\operatorname{\mathsf{Im}} \partial_1$ is a subgroup of $H_0(V_1) \oplus H_0(V_2)$, $\operatorname{\mathsf{Im}} \partial_1$ is free. Hence (4.180) splits and

$$H_1(U_{12}) \cong H_1(V_1) \oplus H_1(V_2) \oplus \operatorname{\mathsf{Im}} \partial_1 \tag{4.181}$$

Since $H_1(V_1)$, $H_1(V_2)$ and $\operatorname{Im}\partial_1$ are torsion-free, so is $H_1(U_{12})$.

In conclusion, for all $r > 0$, $H_2(U_{12}) = 0$ and $H_1(U_{12})$ is torsion-free.

□

Theorem 4.48. *Let $\hat{\mathbb{T}}$ be a metric tree (with at least one essential vertex) where the length of an edge e_i is L_{e_i} for all $i = 1, \ldots, |E|$. Then*

$$H_i(\hat{\mathbb{T}}^2_r) = 0$$

for all $i \geq 2$ and $H_1(\hat{\mathbb{T}}^2_r)$ is torsion-free all possible values of r.

Proof. Without loss of generality, we assume $\hat{\mathbb{T}}$ does not have vertices with degree 2. Induction on the number of essential vertices of $\hat{\mathbb{T}}$.

When $n = 1$, then $\hat{\mathbb{T}} = \hat{\mathsf{Star}}_k$ for some $k \in \mathbb{N}$. Note that for all possible r and L_{e_1}, $H_i((\hat{\mathsf{Star}}_k)^2_r; \mathbb{F}) = 0$ for all $i \geq 2$, and $H_1((\hat{\mathsf{Star}}_k)^2_r; \mathbb{F})$ is torsion-free. (See Theorem 4.47.)

Assume the statement is true when $n = N$. When $n = N + 1$, let v be an essential vertex of $\hat{\mathbb{T}}$ such that it is a leaf of the graph $\hat{\mathbb{T}}'$ which is obtained by deleting all the leaves of $\hat{\mathbb{T}}$ and let u be the essential vertex of $\hat{\mathbb{T}}$ such that u is adjacent to v. Now we can obtain $\hat{\mathbb{T}}$ as a wedge sum of two trees at vertex v: a tree (denoted by $\tilde{\mathbb{T}}$) obtained by deleting all leaves that incident to the vertex v where $k = \deg v$ and a tree (denoted by $\hat{\mathsf{Star}}_{k-1}$) with 1 essential vertex. Note that $\tilde{\mathbb{T}}$ has N essential vertices.

Consider the following cover of $\hat{\mathbb{T}}^2_r$:

$$\begin{aligned}
U_{11} &= (\tilde{\mathbb{T}})^2_r \\
U_{12} &= \tilde{\mathbb{T}} \times \hat{\mathsf{Star}}_{k-1} - \{(x,y) \in \tilde{\mathbb{T}} \times \hat{\mathsf{Star}}_{k-1} \mid \delta(x,y) < r\} \\
U_{21} &= \hat{\mathsf{Star}}_{k-1} \times \tilde{\mathbb{T}} - \{(x,y) \in \hat{\mathsf{Star}}_{k-1} \times \tilde{\mathbb{T}} \mid \delta(x,y) < r\} \\
U_{22} &= (\hat{\mathsf{Star}}_{k-1})^2_r
\end{aligned} \tag{4.182}$$

By Theorem 4.47, we know $H_2(U_{22}) = 0$. By Lemma 4.5 and its dual, we know $H_2(U_{12}) = 0$ and $H_2(U_{21}) = 0$. On the other hand, by the induction hypothesis, $H_i(\tilde{\mathbb{T}}^2_r; \mathbb{F}) = 0$ for all $i \geq 2$ and all possible values of r and L_{e_1}. Now let's consider the intersections of U_{ij}.

$$\begin{aligned}
U_{11} \cap U_{12} &= \tilde{\mathbb{T}} \times \{v\} - \{(x,y) \in \tilde{\mathbb{T}} \times \{v\} \mid \delta(x,y) < r\} \\
U_{11} \cap U_{21} &= \{v\} \times \tilde{\mathbb{T}} - \{(x,y) \in \{v\} \times \tilde{\mathbb{T}} \mid \delta(x,y) < r\} \\
U_{22} \cap U_{12} &\simeq \begin{cases} \{*_1, *_2, \ldots, *_{k-1}\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases} \\
U_{22} \cap U_{21} &\simeq \begin{cases} \{*_1, *_2, \ldots, *_{k-1}\}, & \text{if } 0 < r \leq 1 \\ \emptyset, & \text{if } 1 < r \end{cases} \\
U_{11} \cap U_{22} &= \emptyset \\
U_{12} \cap U_{21} &= \emptyset
\end{aligned} \tag{4.183}$$

By Lemma 4.4, $H_1(U_{11} \cap U_{12}) = 0 = H_1(U_{11} \cap U_{21})$. In addition, all the triple intersections of distinct U_{ij} are empty.

The E^1 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|cccc}
q & & & & \\
3 & \vdots & & 0 & 0 \\
2 & H_2((\tilde{\mathbb{T}})^2_r) \oplus H_2((\hat{\mathsf{Star}}_{k-1})^2_r) & & 0 & 0 \\
1 & H_1((\tilde{\mathbb{T}})^2_r) \oplus H_1((\hat{\mathsf{Star}}_{k-1})^2_r) & & 0 & 0 \\
0 & * & \xleftarrow{\quad d^1 \quad} & * & 0 \\
\hline
 & 0 & & 1 & 2 \quad p
\end{array}
$$

The E^2 page of the Mayer-Vietoris spectral sequence is

$$
\begin{array}{c|cccc}
q & & & & \\
 & \vdots & \vdots & \vdots & \vdots \\
2 & H_2((\tilde{\mathbb{T}})^2_r) \oplus H_2((\hat{\mathsf{Star}}_{k-1})^2_r) & 0 & 0 & 0 \\
1 & H_1((\tilde{\mathbb{T}})^2_r) \oplus H_2((\hat{\mathsf{Star}}_{k-1})^2_r) & 0 & 0 & 0 \\
0 & * & \mathsf{ker} d^1 & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 \quad p
\end{array}
$$

Since there is no non-trivial arrow on the E^2 page, $E^2 = E^\infty$. Therefore, for all possible values of r,

1. $H_i(\hat{\mathbb{T}}^2_r; \mathbb{F}) = 0$ for all $i \geq 3$ and all possible values of r;

2. $H_2(\hat{\mathbb{T}}^2_r; \mathbb{F}) \cong H_2((\tilde{\mathbb{T}})^2_r) \oplus H_2((\hat{\mathsf{Star}}_{k-1})^2_r)$. Applying the induction hypothesis and Theorem 4.47, we conclude that $H_2(\hat{\mathbb{T}}^2_r; \mathbb{F}) = 0$;

3. $H_1(\hat{\mathbb{T}}^2_r; \mathbb{F}) \cong H_1((\tilde{\mathbb{T}})^2_r) \oplus H_1((\hat{\mathsf{Star}}_{k-1})^2_r) \oplus \mathsf{ker} d^1$. Note that $\mathsf{ker} d^1$ is a subgroup of a

free abelian group. Hence $\mathsf{ker} d^1$ is free. By induction hypothesis and Theorem 4.47, we then conclude that $H_1(\hat{\mathbb{T}}^2_r;\mathbb{F})$ is torsion-free.

$\square$

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