The strongly robust simplicial complex of monomial curves

THE STRONGLY ROBUST SIMPLICIAL COMPLEX OF MONOMIAL CURVES

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ABSTRACT. To every simple toric ideal I_T one can associate the strongly robust simplicial complex Δ_T, which determines the strongly robust property for all ideals that have I_T as their bouquet ideal. We show that for the simple toric ideals of monomial curves in $\mathbb{A}^s$, the strongly robust simplicial complex Δ_T is either $\{\emptyset\}$ or contains exactly one 0-dimensional face. In the case of monomial curves in $\mathbb{A}^3$, the strongly robust simplicial complex Δ_T contains one 0-dimensional face if and only if the toric ideal I_T is a complete intersection ideal with exactly two Betti degrees. Finally, we provide a construction to produce infinitely many strongly robust ideals with bouquet ideal the ideal of a monomial curve and show that they are all produced this way.

1. INTRODUCTION

Let $A \in \mathbb{Z}^{m\times n}$ be an integer matrix such that $\mathrm{Ker}_{\mathbb{Z}}(A) \cap \mathbb{N}^n = \{\mathbf{0}\}$. The toric ideal of A is the ideal $I_A \subset K[x_1, \ldots, x_n]$ generated by the binomials $x^{\mathbf{u}^+} - x^{\mathbf{u}^-}$ where K is a field, $\mathbf{u} \in \mathrm{Ker}_{\mathbf{Z}}(A)$ and $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ is the unique expression of $\mathbf{u}$ as a difference of two non-negative vectors with disjoint support, see [21, Chapter 4]. A toric ideal is called robust if it is minimally generated by its Universal Gröbner basis, where the Universal Gröbner basis is the union of all reduced Gröbner bases, see [4]. A *strongly robust* toric ideal is a toric ideal I_A for which the Graver basis $\mathrm{Gr}(I_A)$ is a minimal system of generators, see [23]. The condition $\mathrm{Ker}_{\mathbb{Z}}(A) \cap \mathbb{N}^n = \{\mathbf{0}\}$ implies that any minimal binomial generating set is contained in the Graver basis, see [9, Theorem 2.3]. For strongly robust ideals then the Graver basis is the unique minimal system of generators and, thus, any reduced Gröbner basis as well as the Universal Gröbner basis are identical with the Graver basis, since all of them contain a minimal system of generators and they are subsets of the Graver basis (see [21, Chapter 4]). We conclude that for a strongly robust toric ideal I_A the following sets are identical: the set of indispensable elements, any minimal system of binomial generators, any reduced Gröbner basis, the Universal Gröbner basis and the Graver basis. Therefore strongly robust toric ideals are robust. The classical example of strongly robust ideals are the Lawrence ideals, see [21, Chapter 7]. There are several articles in the literature studying robust related properties of ideals; see [2, 3, 10, 11, 22] for robust

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ideals, [4, 5] for robust toric ideals, [12, 24] for generalized robust toric ideals and [14, 17, 19, 20, 21, 23] for strongly robust toric ideals.

To characterize combinatorially the strongly robust property of toric ideals which have in common the same bouquet ideal I_T, in [17], we defined a simplicial complex, the strongly robust simplicial complex Δ_T, the faces of which determine the strongly robust property. In particular, let I_A be a toric ideal with bouquet ideal I_T, the ideal I_A is strongly robust if and only if the set ω of indices i, such that the i-th bouquet of I_A is non-mixed, is a face of Δ_T, see [17, Theorem 3.6]. Thus, understanding the strongly robust property of toric ideals I_A is equivalent to understanding the strongly robust simplicial complex Δ_T for simple toric ideals I_T. Simple toric ideals are ideals for which every bouquet is a singleton. Bouquet ideals are always simple. A method for the computation of the strongly robust simplicial complex Δ_T for a particular simple toric ideal I_T was given in [17, Theorem 3.7], however an interesting problem is to understand the strongly robust simplicial complex Δ_T for classes of simple toric ideals. In this direction, we determine the strongly robust simplicial complex Δ_T for the simple toric ideals of monomial curves, which are toric ideals defined by $1 \times s$-matrices, where $s \geq 3$. For $s = 2$, the toric ideal I_T is principal and thus it is never simple but always strongly robust. For $s \geq 3$, the ideal of a monomial curve is never strongly robust, see Remark 4.6. Toric ideals with bouquet ideal the toric ideal of a monomial curve include toric ideals of varieties with any dimension as well as varieties with any codimension greater than one. For example, a toric ideal with bouquet ideal the toric ideal I_T of a monomial curve defined by the matrix $T = (24, 40, 41, 60, 80)$ is the ideal I_A of a toric variety of dimension 7 and codimension 4 in Example 6.4, where we explain why such an ideal I_A is strongly robust. In Section 6, we provide infinitely many examples of strongly robust toric ideals with bouquet ideal the toric ideal of a monomial curve and we prove that all such toric ideals are provided by this method.

In [23], Sullivant asked the question: does every strongly robust toric ideal I_A of codimension r have at least r mixed bouquets? Since bouquets preserve the codimension and s is the number of bouquets of I_A, Sullivant's question is equivalent to a question about the dimension of the strongly robust simplicial complex of its bouquet ideal I_T: is it true that simple toric ideals I_T of codimension r in the polynomial ring of s variables have $\dim \Delta_T < s - r$? In Section 3, we give an affirmative answer to the question of Sullivant for the simple toric ideals of monomial curves.

The structure of the paper is the following. In Section 2, we present the notation, give definitions and previous results that will be required throughout the paper. Then, in Section 3, we firstly proceed by showing that the dimension of the strongly robust simplicial complex for a monomial curve T is $\dim \Delta_T \leq 0$. Note that for monomial curves the codimension is $s - 1$ thus $\dim \Delta_T \leq 0 < 1 = s - (s - 1)$, which agrees with Sullivant's conjecture. Section 4 contains results that describe the circuits of $\Lambda(T)_i$ which lead to Theorem 4.4 that links the properties of complete intersection and strongly robustness for monomial curves in $\mathbb{A}^s$. Using this we give a full description of the strongly robust simplicial complex in the case of monomial curves in $\mathbb{A}^3$ in Theorem 4.7 and show that for monomial curves in $\mathbb{A}^s$ the strongly robust simplicial complex is either $\{\emptyset\}$ or contains exactly one 0-dimensional face in Theorem 4.5. In Section 5, we extend the notions of a primitive element as well as the Graver basis and give a necessary and sufficient condition in terms of primitive elements (or the Graver basis) on whether one specific element is the unique 0-dimensional face of Δ_T. Finally, in Section 6, we use generalized Lawrence matrices to describe completely all matrices A for which the toric ideal I_A is strongly robust and which have bouquet ideal the toric ideal of a monomial curve.

2. Preliminaries

Let $A = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ be an integer matrix in $\mathbb{Z}^{m\times n}$, with column vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ and such that $\mathrm{Ker}_{\mathbb{Z}}(A) \cap \mathbb{N}^n = \{\mathbf{0}\}$. We say that $\mathbf{u} = \mathbf{v} +_c \mathbf{w}$ is a conformal decomposition of the vector $\mathbf{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)$ if $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\mathbf{u}^+ = \mathbf{v}^+ + \mathbf{w}^+, \mathbf{u}^- = \mathbf{v}^- + \mathbf{w}^-$, where $\mathbf{v}, \mathbf{w} \in \mathrm{Ker}_{\mathbb{Z}}(A)$. The conformal decomposition is called proper if both $\mathbf{v}$ and $\mathbf{w}$ are not zero. For the conformality, in terms of signs, the corresponding notation is the following: $+ = \oplus +_c \oplus$, $- = \ominus +_c \ominus$, $0 = 0 +_c 0$. where the symbol $\ominus$ means that the corresponding integer is nonpositive and the symbol $\oplus$ nonnegative. By $\mathrm{Gr}(A)$ we denote the set of elements in $\mathrm{Ker}_{\mathbb{Z}}(A)$ that do not have a proper conformal decomposition. A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_A$ is called *primitive* if $\mathbf{u} \in \mathrm{Gr}(A)$. The set of the primitive binomials is finite and it is called the *Graver basis* of I_A and is denoted by $\mathrm{Gr}(I_A)$, [21, Chapter 4].

We recall from [16, Definition 3.9] that for vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathrm{Ker}_{\mathbb{Z}}(A)$ such that $\mathbf{u} = \mathbf{v} + \mathbf{w}$, the sum is said to be a *semiconformal decomposition* of $\mathbf{u}$, written $\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w}$, if $v_i > 0$ implies that $w_i \geq 0$, and $w_i < 0$ implies that $v_i \leq 0$, for all $1 \leq i \leq n$. The decomposition is called *proper* if both $\mathbf{v}, \mathbf{w}$ are nonzero. The set of indispensable elements $S(A)$ of A consists of all nonzero vectors in $\mathrm{Ker}_{\mathbb{Z}}(A)$ with no proper semiconformal decomposition. For the semiconformality, in terms of signs, the corresponding notation is the following: $+ = * +_{sc} \oplus$, $- = \ominus +_{sc} *$, $0 = \ominus +_{sc} \oplus$, where the symbol $*$ means that it can take any value.

A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_A$ is called *indispensable* binomial if it belongs to the intersection of all minimal systems of binomial generators of I_A, up to identification of opposite binomials. The set of indispensable binomials is $S(I_A) = \{\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} | \mathbf{u} \in S(A)\}$ by [16, Lemma 3.10] and [8, Proposition 1.1].

Circuits are irreducible binomials of a toric ideal I_A with minimal support. In vector notation, a vector $\mathbf{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)$ is called a circuit of the matrix A if $\mathrm{supp}(\mathbf{u})$ is minimal and the components of $\mathbf{u}$ are relatively prime.

To the matrix $A = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ we associate its Gale transform, which is the $n \times (n - r)$ matrix whose columns span the lattice $\mathrm{Ker}_{\mathbb{Z}}(A)$, where r is the rank of A. We will denote the set of ordered row vectors of the Gale transform by $\{G(\mathbf{a}_1), \dots, G(\mathbf{a}_n)\}$. The vector $\mathbf{a}_i$ is called *free* if its Gale transform $G(\mathbf{a}_i)$ is equal to the zero vector, which means that i is not contained in the support of any element in $\mathrm{Ker}_{\mathbb{Z}}(A)$. The *bouquet graph* G_A of I_A is the graph on the set of vertices $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$, whose edge set E_A consists of those $\{\mathbf{a}_i, \mathbf{a}_j\}$ for which $G(\mathbf{a}_i)$ is a rational multiple of $G(\mathbf{a}_j)$ and vice-versa. The connected components of the graph G_A are called *bouquets*.

It follows from the definition that the free vectors of A form one bouquet, which we call the *free bouquet* of G_A. The non-free bouquets are of two types: *mixed* and *non-mixed*. A non-free bouquet is mixed if contains an edge $\{\mathbf{a}_i, \mathbf{a}_j\}$ such that $G(\mathbf{a}_i) = \lambda G(\mathbf{a}_j)$ for some $\lambda < 0$, and is non-mixed if it is either an isolated vertex or for all of its edges $\{\mathbf{a}_i, \mathbf{a}_j\}$ we have $G(\mathbf{a}_i) = \lambda G(\mathbf{a}_j)$ with $\lambda > 0$, see [19, Lemma 1.2].

Let $B_1, B_2, \ldots, B_s$ be the bouquets of I_A. We reorder the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ to $\mathbf{a}_{11}, \mathbf{a}_{12}, \ldots, \mathbf{a}_{1k_1},\ \mathbf{a}_{21}, \mathbf{a}_{22}, \ldots, \mathbf{a}_{2k_2},\ \ldots, \mathbf{a}_{s1}, \mathbf{a}_{s2}, \ldots, \mathbf{a}_{sk_s}$ in such a way that the first k_1 vectors belong to the bouquet B_1, the next k_2 to B_2 and so on up to the last k_s that belong to the bouquet B_s. Note that $k_1 + k_2 + \cdots + k_s = n$. For each bouquet B_i we define two vectors $\mathbf{c}_{B_i}$ and $\mathbf{a}_{B_i}$. If the bouquet B_i is free then we set $\mathbf{c}_{B_i} \in \mathbb{Z}^n$ to be any nonzero vector such that $\text{supp}(\mathbf{c}_{B_i}) = \{i1, \ldots, ik_i\}$ and with the property that the first nonzero coordinate, c_{i1}, is positive. For a non-free bouquet B_i of A, consider the Gale transforms of the elements in B_i. All the Gale transforms are nonzero, since the bouquet is non-free, and pairwise linearly dependent, since they belong to the same bouquet. Therefore, there exists a nonzero coordinate l in all of them. Let $g_l = \gcd(G(\mathbf{a}_{i1})_l, G(\mathbf{a}_{i2})_l, \ldots, G(\mathbf{a}_{ik_i})_l)$, where $(\mathbf{w})_l$ is the l-th component of a vector $\mathbf{w}$. Then $\mathbf{c}_{B_i}$ is the vector in $\mathbb{Z}^n$ whose qj-th component is 0 if $q \neq i$, and $c_{ij} = \varepsilon_{i1}(G(\mathbf{a}_{ij})_l)/g_l$, where ε_{i1} represents the sign of the integer $G(\mathbf{a}_{ij})_l$. Note that c_{i1} is always positive. Then the vector $\mathbf{a}_{B_i}$ (see [19, Definition 1.7]), is defined as $\mathbf{a}_{B_i} = \sum_{j=1}^{n} (c_{B_i})_j \mathbf{a}_j \in \mathbb{Z}^m$.

If B_i is a non-free bouquet of A, then B_i is a mixed bouquet if and only if the vector $\mathbf{c}_{B_i}$ has a negative and a positive coordinate, and B_i is non-mixed if and only if the vector $\mathbf{c}_{B_i}$ has all nonzero coordinates positive, see [19, Lemma 1.6]. The toric ideal I_{A_B} associated to the matrix A_B, whose columns are the vectors $\mathbf{a}_{B_i}$, $1 \leq i \leq s$, is called the bouquet ideal of I_A.

Let $\mathbf{u} = (u_1, u_2, \ldots, u_s) \in \text{Ker}_{\mathbb{Z}}(A_B)$ then the linear map

$$D(\mathbf{u}) = (c_{11}u_1, c_{12}u_1, \ldots, c_{1k_1}u_1, c_{21}u_2, \ldots, c_{2k_2}u_2, \ldots, c_{s1}u_s, \ldots, c_{sk_s}u_s),$$

where all $c_{j1}, 1 \leq j \leq s$, are positive, is an isomorphism from $\text{Ker}_{\mathbb{Z}}(A_B)$ to $\text{Ker}_{\mathbb{Z}}(A)$, see [19, Theorem 1.9].

The cardinality of the sets of different toric bases depends only on the signatures of the bouquets and the bouquet ideal, see [17, Theorem 2.3, Theorem 2.5] and [19, Theorem 1.11].

Note that the bouquet ideal is simple: a toric ideal is called *simple* if every bouquet is a singleton, in other words if $I_T \subset K[x_1, \ldots, x_s]$ and has s bouquets. The bouquet ideal of a simple toric ideal I_A is I_A itself. Let I_A be the ideal of a monomial curve, where $A = (n_1, n_2, \ldots, n_s)$ is an $1 \times s$ matrix. In this case for any $i, j \in [s]$ there exists one circuit with support only $\{i, j\}$. Then for $s \geq 3$ and any two n_k, n_l there exists a circuit that is zero on the k^{th} component and nonzero on the l^{th} component and vice versa. Therefore the Gale transforms $G(n_k), G(n_l)$ are not the one multiple of the other. Thus all toric ideals of monomial curves are simple if $s \geq 3$.

Definition 2.1. Let $I_T \subset K[x_1, \ldots, x_s]$ be a simple toric ideal and $\omega \subset \{1, \ldots, s\}$. A toric ideal I_A is called T_ω-robust ideal if and only if

- the bouquet ideal of I_A is I_T and
- $\omega = \{i \in [s] | B_i$ is non-mixed$\}$.

We denote by

$$S_\omega(T) = \{\mathbf{u} \in \mathrm{Gr}(T) | \ D(\mathbf{u}) \in S(A)\}$$

and call $S_\omega(T)$ the T_ω-indispensable set, where I_A is an T_ω-robust toric ideal and $S(A)$ is the set of indispensable elements of A.

The second part of the definition is correctly defined, since in [17] we showed that the set of elements $\mathbf{u}$ which belong to $\mathrm{Gr}(T)$, such that $D(\mathbf{u})$ is indispensable in a T_ω-robust toric ideal I_A, does not depend on the I_A chosen, but only on T and ω.

In [17] we introduced a simplicial complex, which determines the strongly robust property for toric ideals.

Definition 2.2. The set $\Delta_T = \{\omega \subseteq [s] \mid S_\omega(T) = \mathrm{Gr}(T)\}$ is called the *strongly robust complex* of T.

According to [17, Corollary 3.5, Theorem 3.6], the set Δ_T is a simplicial complex, which determines the strongly robust property for toric ideals.

Theorem 2.3. [17, Theorem 3.6] *Let I_A be a T_ω-robust toric ideal. The toric ideal I_A is strongly robust if and only if ω is a face of the strongly robust complex Δ_T.*

The following theorem provides a way to compute the strongly robust complex of a simple toric ideal I_T. By $\Lambda(T)$ we denote the second Lawrence lifting of T, which is the $(m+s) \times 2s$ matrix $\begin{pmatrix} T & 0 \\ I_s & I_s \end{pmatrix}$. By $\Lambda(T)_\omega$ we denote the matrix taken from $\Lambda(T)$ by removing the $(m+i)$-th row and the $(s+i)$-th column for each $i \in \omega$.

Example 2.4. For $T = (n_1, n_2, n_3, n_4)$ and $\omega = \{3\}$, the $\Lambda(T)_\omega$ matrix is

$$\begin{pmatrix} n_1 & n_2 & n_3 & n_4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 2.5. [17, Theorem 3.7] *The set ω is a face of the strongly robust complex Δ_T if and only if $I_{\Lambda(T)_\omega}$ is strongly robust.*

3. On the dimension of the strongly robust complex

Suppose that $T = (n_1, n_2, n_3)$ with the property that $\gcd(n_1, n_2, n_3) = 1$. For an element $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{N}^3$, we define the T-degree of the monomial $x^{\mathbf{u}}$ to be $\deg_T(x^{\mathbf{u}}) := u_1 n_1 + u_2 n_2 + u_3 n_3$. A vector b is called a *Betti T-degree* if I_T has a minimal generating set of binomials containing an element of T-degree b. Betti T-degrees do not depend on the minimal set of binomial generators, [6, 21].

We know from J. Herzog [15], that if I_T is not complete intersection then the ideal I_T is minimally generated by three binomials, while a complete intersection

I_T is minimally generated by two binomials. Let c_i be the smallest multiple of n_i that belongs to the semigroup generated by n_j, n_k, where $\{i, j, k\} = \{1, 2, 3\}$. Then there exist non-negative integers c_{ij}, c_{ik} (not necessarily unique) such that $c_i n_i = c_{ij} n_j + c_{ik} n_k$. We have the following cases.

B3 If all six $c_{ij} \neq 0$, then I_T is a non complete intersection ideal, minimally generated by the three elements $x_1^{c_1} - x_2^{c_{12}} x_3^{c_{13}}$, $x_2^{c_2} - x_1^{c_{21}} x_3^{c_{23}}$, $x_3^{c_3} - x_1^{c_{31}} x_2^{c_{32}}$. All minimal generators have full support, therefore the ideal is generic and thus all elements are indispensable, see [18, Lemma 3.3, Remark 4.4]. Being indispensable binomials implies that they have different T-degrees, see [7]. Thus in the non complete intersection case we have 3 Betti T-degrees, $c_1 n_1 \neq c_2 n_2 \neq c_3 n_3$.

B2-B1 If at least one $c_{ij} = 0$, then I_T is a complete intersection ideal and is generated by $x_j^{c_j} - x_k^{c_k}$ and $x_i^{c_i} - x_j^{c_{ij}} x_k^{c_{ik}}$. There are two cases:

- B2 the two binomials have different T-degrees, i.e. $c_j n_j = c_k n_k \neq c_i n_i$. In this case we have two Betti T-degrees.
- B1 the two binomials have the same T-degree, i.e. $c_1 n_1 = c_2 n_2 = c_3 n_3$. In this case we have one Betti T-degree, see [13]. It is easy to see that in the binomial $x_i^{c_i} - x_j^{c_{ij}} x_k^{c_{ik}}$ the monomial $x_j^{c_{ij}} x_k^{c_{ik}}$ can only be $x_j^{c_j}$ or $x_k^{c_k}$, otherwise one can find smaller multiples of n_j or n_k than c_j or c_k that belong to the semigroup generated by n_i, n_k or n_i, n_j, contradicting the choice of c_j or c_k. Thus both binomials are circuits. None of the circuits is indispensable since any two of the three binomials $x_1^{c_1} - x_2^{c_2}$, $x_1^{c_1} - x_3^{c_3}$ and $x_2^{c_2} - x_3^{c_3}$ generate the ideal I_T.

In both cases B1, B2, the circuit $x_j^{c_j} - x_k^{c_k}$ is the same as $x_j^{n_k^{\#}} - x_k^{n_j^{\#}}$, where $n_k^{\#}, n_j^{\#}$ are just the integers n_k, n_j divided by $g_{jk} = g.c.d(n_k, n_j)$. Thus $c_j = n_k^{\#}$ and $c_k = n_j^{\#}$.

Definition 3.1. Let $T = (n_1, n_2, n_3)$, we say that I_T is a *complete intersection on* n_i if $c_j n_j = c_k n_k \neq c_i n_i$. We say that I_T is a *complete intersection on all* if $c_1 n_1 = c_2 n_2 = c_3 n_3$.

Remark that the condition $c_j n_j = c_k n_k \neq c_i n_i$ implies that I_T can be complete intersection on n_i for at most one n_i.

Example 3.2. Let $T_1 = (7, 15, 20)$, then $c_1 = 5, c_2 = 4, c_3 = 3$ and the Betti T_1-degrees are $5 \cdot 7 \neq 4 \cdot 15 = 3 \cdot 20$. Therefore I_{T_1} is complete intersection on $n_1 = 7$. Let $T_2 = (5, 6, 15)$, then $c_1 = 3, c_2 = 5, c_3 = 1$ and the Betti T_2-degrees are $3 \cdot 5 = 1 \cdot 15 \neq 5 \cdot 6$. Therefore I_{T_2} is complete intersection on $n_2 = 6$. If $T_3 = (6, 8, 11)$, then $c_1 = 4, c_2 = 3, c_3 = 2$, the Betti T_3-degrees are $4 \cdot 6 = 3 \cdot 8 \neq 2 \cdot 11$, and therefore I_{T_3} is complete intersection on $n_3 = 11$. Let $T_4 = (6, 10, 15)$, then $c_1 = 5, c_2 = 3, c_3 = 2$ and there is only one Betti T_4-degree $5 \cdot 6 = 3 \cdot 10 = 2 \cdot 15$. Therefore I_{T_4} is complete intersection on all. Finally, for $T_5 = (3, 5, 7)$ we have $c_1 = 4, c_2 = 2, c_3 = 2$ and there are three Betti T_5-degrees: $4 \cdot 3 \neq 2 \cdot 5 \neq 2 \cdot 7$. Therefore I_{T_5} is not a complete intersection.

Proposition 3.3. *Let $T = (n_1, n_2, n_3)$. In the toric ideal I_T at least two of the three circuits are not indispensable.*

Proof. In the case that I_T is not complete intersection the toric ideal is generated by three binomials of full support [15, Section 3]. Then the toric ideal is generic and by [18, Lemma 3.3, Remark 4.4] all three generators are indispensable. Since all generators have full support none of them is a circuit. Therefore none of the three circuits of the toric ideal I_T is indispensable. In the complete intersection case, the ideal I_T has two minimal generators, one of which is always a circuit [15, Proposition 3.5 and Theorem 3.8]. If exactly one minimal generator was a circuit, then the other two circuits would not be indispensable. If both of the minimal generators were circuits then without loss of generality we can assume they would be of the form $x_i^a - x_j^b$, $x_i^c - x_k^d$. Namely, two of the monomials will be powers of the same variable. We can distinguish two cases.

- Firstly, the two exponents a, c could be different, so assume that $a < c$. In this case, according to [7, Theorem 3.4], the binomial $x_i^c - x_k^d$ would not be indispensable, as c would not be a minimal binomial T-degree. Therefore, in this case there exists at most one indispensable circuit, and since we have three circuits in total at least two would not be indispensable.
- In the second case, the two exponents a, c are equal. Then both generators $x_i^a - x_j^b$ and $x_i^a - x_k^d$ would be of the same Betti T-degree. The ideal is generated by any two of the following three circuits $x_i^a - x_j^b$, $x_i^a - x_k^d$, $x_j^d - x_k^b$, since $x_j^d - x_k^b = (x_i^a - x_j^b) - (x_i^a - x_k^d)$. Therefore, there is no indispensable binomial and in particular none of the circuits is indispensable. □

Theorem 3.4. *Let* $T = (n_1, n_2, \ldots, n_s)$ *then* $\dim(\Delta_T) \leq 0$.

Proof. Suppose on the contrary that $\dim(\Delta_T) > 0$. Then there exist i, j such that the edge $\{i, j\}$ belongs to the strongly robust complex Δ_T. Therefore, the toric ideal $I_{\Lambda(T)_{\{i,j\}}}$ is strongly robust by Theorem 2.5. Consider the ideal $I_{(n_i,n_j,n_k)}$ for any $k \neq i, j$. Let $\mathbf{c}$ be a non indispensable circuit of the toric ideal of $I_{(n_i,n_j,n_k)}$ different from $(n_j^{'}, -n_i^{'}, 0)$, where $n_i^{'}, n_j^{'}$ are the n_i, n_j divided by their greatest common divisor. Then without loss of generality $\mathbf{c}$ will be in the form $\mathbf{c} = (n_k^*, 0, -n_i^*)$, where n_i^*, n_k^* are the n_i, n_k divided by their greatest common divisor. We know that there always exists such a circuit $\mathbf{c}$, since by the Proposition 3.3 we have that at least two of the three circuits of $I_{(n_i,n_j,n_k)}$ are not indispensable. As the circuit $\mathbf{c}$ is not indispensable, it has a proper semiconformal decomposition into two vectors with the following pattern of signs $(n_k^*, 0, -n_i^*) = (*, -, \ominus) +_{sc} (\oplus, +, *)$. The first $*$ is a positive number and the second $*$ is a negative number, since $\mathrm{Ker}_{\mathbb{Z}}(n_i, n_j, n_k) \cap \mathbb{N}^3 = \{\mathbf{0}\}$. Namely, $(n_k^*, 0, -n_i^*) = (a, -b, -c) +_{sc} (d, b, -e)$, where $a, b, c, d, e \in \mathbb{N}$ and $abe \neq 0$, so this is a proper semiconformal decomposition. We will show below that this lifts into a proper semiconformal decomposition in $I_{\Lambda(T)_{\{i,j\}}}$. Indeed, in the toric ideal $I_{\Lambda(T)_{\{i,j\}}}$ we have

$$(0, \ldots, n_k^*, \ldots, 0, \ldots, -n_i^*, \ldots, n_i^*, \ldots, 0) =$$

$$(0, \ldots, a, \ldots, -b, \ldots, -c, \ldots, c, \ldots, 0) +_{sc} (0, \ldots, d, \ldots, b, \ldots, -e, \ldots, e, \ldots, 0),$$

where the only nonzero components are in the i^{th},k^{th} and $(s + k - 2)^{th}$ positions in the first vector and in the i^{th}, j^{th}, k^{th} and $(s+k-2)^{th}$ positions in the last two. This decomposition is proper semiconformal, since $(n_k^*, 0, -n_i^*) = (a, -b, -c) +_{sc} (d, b, -e)$ is proper. We observe that the element $(0, \ldots, n_k^*, \ldots, 0, \ldots, -n_i^*, \ldots, n_i^*, \ldots, 0) = D((0, \ldots, n_k^*, \ldots, 0, \ldots, -n_i^*, \ldots, 0))$ is a circuit of $I_{\Lambda(T)_{\{i,j\}}}$, since D maps circuits to circuits [19, Theorem 1.11], and is not an indispensable element in $I_{\Lambda(T)_{\{i,j\}}}$, since it admits a proper semiconformal decomposition. However, we know that circuits are always contained in the Graver basis [21, Proposition 4.11], so this means that the Graver basis $\mathrm{Gr}\left(\Lambda(T)_{\{i,j\}}\right)$ is not equal to the set of indispensable elements $S\left(\Lambda(T)_{\{i,j\}}\right)$. Therefore, the toric ideal $I_{\Lambda(T)_{\{i,j\}}}$ is not strongly robust, a contradiction. We conclude that $\dim(\Delta_T) \leq 0$. □

In [23, Corollary 1.3], Sullivant proved that strongly robust codimension 2 toric ideals have at least 2 mixed bouquets. For the strongly robust complex, this result means that $\dim(\Delta_T) < s-2$. In the same paper, Sullivant poses a stronger question which can be translated to the following question: for every simple codimension r toric ideal I_T, is it true that $\dim(\Delta_T) < s - r$? Theorem 3.4 proves that the answer to this question is affirmative for the simple toric ideals of monomial curves. Note that toric ideals of monomial curves have codimension $r = s - 1$.

4. Complete Intersection and strongly robustness

Proposition 4.1. *Let $T = (n_1, n_2, \ldots, n_s)$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathrm{Ker}_{\mathbb{Z}}(T)$. If $D(\mathbf{u}) = D(\mathbf{v}) +_{sc} D(\mathbf{w})$ in $\mathrm{Ker}_{\mathbb{Z}}\left(\Lambda(T)_{\{i\}}\right)$, then $[\mathbf{u}]^i = [\mathbf{v}]^i +_c [\mathbf{w}]^i$, where $[\mathbf{u}]^i$ is the vector obtained from $\mathbf{u}$ by deleting its i^{th} component.*

Proof. Let $j \in \{1, \ldots, s\}$ be such that $j \neq i$. Then, for the vector $D(\mathbf{u})$ in the kernel $\mathrm{Ker}_{\mathbb{Z}}\left(\Lambda(T)_{\{i\}}\right)$, one of the components is equal to u_j and another is $-u_j$. Similarly, the corresponding two components of $D(\mathbf{v}), D(\mathbf{w}) \in \mathrm{Ker}_{\mathbb{Z}}\left(\Lambda(T)_{\{i\}}\right)$ are $v_j, -v_j$ and $w_j, -w_j$ respectively. The semiconformal decomposition $D(\mathbf{u}) = D(\mathbf{v}) +_{sc} D(\mathbf{w})$, implies that on those components we have

$$(u_j) = (v_j) +_{sc} (w_j), \tag{1}$$

$$(-u_j) = (-v_j) +_{sc} (-w_j). \tag{2}$$

If $u_j \geq 0$, then $w_j \geq 0$ by (1), while $-v_j \leq 0$ by (2). Therefore, both v_j, w_j are non-negative and so the sum $(u_j) = (v_j) +_c (w_j)$ is conformal. If on the other hand $u_j \leq 0$, then $v_j \leq 0$ by (1) and $-w_j \geq 0$ by (2). Therefore, both v_j, w_j are non-positive and the sum $(u_j) = (v_j) +_c (w_j)$ is again conformal. □

Lemma 4.2. *Any circuit of $\Lambda(T)_{\{i\}}$ that has i in its support is indispensable.*

Proof. Due to the one-to-one correspondence between circuits of $\Lambda(T)_{\{i\}}$ and circuits of T, a circuit of $\Lambda(T)_{\{i\}}$ can be written as $D(\mathbf{u})$, where $\mathbf{u}$ is a circuit of T, see [19, Theorem 1.11]. Therefore, $D(\mathbf{u})$ has the form $D(0, \ldots, 0, n_i^{\#}, 0, \ldots, 0, -n_j^{\#}, 0, \ldots, 0)$, where $n_i^{\#}, n_j^{\#}$ are the n_i, n_j divided by their greatest common divisor, $n_i^{\#}$ is the j^{th} component of $\mathbf{u}$ and $-n_j^{\#}$ is the i^{th} component of $\mathbf{u}$. Let $D(\mathbf{u}) = D(\mathbf{v}) +_{sc} D(\mathbf{w})$ be a semi-conformal decomposition of $D(\mathbf{u})$. Then, by Proposition 4.1, we have that $[\mathbf{u}]^i = [\mathbf{v}]^i +_c [\mathbf{w}]^i$. As the only conformal decomposition of 0 is $0 + 0$, the only possible non zero component of each of the three vectors $[\mathbf{u}]^i, [\mathbf{v}]^i, [\mathbf{w}]^i$ is the j^{th} and we have that $n_i^{\#} = v_j +_c w_j$. Thus, both v_j, w_j are non negative and at least one is positive. Without loss of generality we can assume that $v_j > 0$, so $\mathbf{v}$ is not zero and $[\mathbf{v}]^i$ has only one element v_j in its support. This means that $\mathbf{v}$ has minimal support $\{i, j\}$ and so it is a multiple of the circuit $\mathbf{u}$. Therefore, $\mathbf{v} = l\mathbf{u}$ and $l \geq 1$, since v_j is positive. Thus, $n_i^{\#} = v_j + w_j \geq v_j = ln_i^{\#}$, therefore $l = 1$, $\mathbf{v} = \mathbf{u}$ and for $\mathbf{w}$ the only option is that $\mathbf{w} = \mathbf{0}$, thus $D(\mathbf{w}) = \mathbf{0}$, leading to a non-proper decomposition. This proves that $D(\mathbf{u})$ is indispensable. □

Lemma 4.3. *Let $T = (n_1, n_2, \ldots, n_s)$ and $\mathbf{u}$ be the circuit with support on $j, k \in [s]$. Then the circuit $D(\mathbf{u})$ is indispensable in $\Lambda(T)_{\{i\}}$ if and only if $I_{(n_i,n_j,n_k)}$ is a complete intersection on n_i.*

Proof. The circuit with support on j, k is $\mathbf{u} = (0, \ldots, 0, n_k^{\#}, 0, \ldots, 0, -n_j^{\#}, 0, \ldots, 0)$, where the two nonzero elements $n_k^{\#}, n_j^{\#}$ are in the j^{th} and k^{th} position respectively and $n_k^{\#}, n_j^{\#}$ are the n_k, n_j divided by their greatest common divisor. To prove one implication, let $I_{(n_i,n_j,n_k)}$ be a complete intersection on n_i. Then $n_k^{\#} = c_j$ and $n_j^{\#} = c_k$ and thus g.c.d$(c_j, c_k) = 1$. Suppose that the circuit $D(\mathbf{u})$ is not indispensable in $\Lambda(T)_{\{i\}}$ and let $D(\mathbf{u}) = D(\mathbf{v}) +_{sc} D(\mathbf{w})$ be a proper semiconformal decomposition of $D(\mathbf{u})$. Then, by Proposition 4.1, we have that $[\mathbf{u}]^i = [\mathbf{v}]^i +_c [\mathbf{w}]^i$. Also, coordinate wise $c_j = v_j +_c w_j = a + b$ and $c_k = v_k +_c w_k = -c - d$, where $a, b, c, d \in \mathbb{N}$. Moreover, from the semiconformal decomposition of $D(\mathbf{u})$, we have that $0 = v_i +_{sc} w_i$, so $v_i = -e, w_i = e$, where $e \in \mathbb{N}$. The rest of the components of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are zero, since the only conformal decomposition of 0 is $0 + 0$, by Proposition 4.1. Then, since $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathrm{Ker}_{\mathbb{Z}}(T)$ and

$$\begin{aligned}
\mathbf{v} &= (0, \ldots, 0,\ a, 0, \ldots, -e, \ldots, 0, -c, 0, \ldots, 0), \\
\mathbf{w} &= (0, \ldots, 0,\ b, 0, \ldots,\ \ e, \ldots, 0, -d, 0, \ldots, 0),
\end{aligned}$$

we have that $an_j - en_i - cn_k = 0$ and $bn_j + en_i - dn_k = 0$, where the possible nonzero components of $\mathbf{v}, \mathbf{w}$ are in the j^{th}, i^{th} and k^{th} positions. This implies that $an_j = cn_k + en_i$ and we distinguish two cases: $a = 0$ and $a > 0$. If $a = 0$, then $an_j = 0 = cn_k + en_i$ implies that $c = 0$ and $e = 0$, which means that $\mathbf{v} = \mathbf{0}$ and $D(\mathbf{v}) = \mathbf{0}$, a contradiction. In the case that $a > 0$, then an_j belongs to the semigroup $< n_k, n_i >$. However, c_jn_j is the smallest multiple of n_j that belongs to the semigroup $< n_k, n_i >$. Thus, $a \geq c_j = a + b$, which implies that $a = c_j$ and $b = 0$. We similarly argue for $dn_k = bn_j + en_i$. If $d = 0$, then $dn_k = 0 = bn_j + en_i$ implies $b = 0$ and $e = 0$. This means that $\mathbf{w} = \mathbf{0}$ and $D(\mathbf{w}) = \mathbf{0}$, a contradiction. In the case that $d > 0$, then dn_k belongs to the semigroup $< n_i, n_j >$. However, c_kn_k is the smallest multiple of n_k that belongs to the semigroup $< n_i, n_j >$. Thus, $d \geq c_k = c + d$, which implies that $d = c_k$ and $c = 0$.

In conclusion, from $an_j = cn_k + en_i$ and $dn_k = bn_j + en_i$, we have that $b = c = 0$, $a = c_j$, $d = c_k$ and thus $c_jn_j = en_i$ and $c_kn_k = en_i$. The equation $c_jn_j = en_i$ implies that g.c.d$(c_j, e) = 1$, since otherwise a smaller multiple of n_j would belong to the semigroup $< n_k, n_i >$. Then e divides n_j and thus n_i is a multiple of c_j. Similarly from $c_kn_k = en_i$ we have g.c.d$(c_k, e) = 1$. Then e divides n_k and thus n_i

is a multiple of c_k. Since g.c.d$(c_j, c_k) = 1$ we have that $n_i = \lambda c_j c_k$. Then $c_j n_j = e n_i$ implies $n_j = \lambda e c_k$ and $c_k n_k = e n_i$ implies $n_k = \lambda e c_j$.

Summarizing we have $n_i = \lambda c_j c_k$, $n_j = \lambda e c_k$, $n_k = \lambda e c_j$, g.c.d$(c_j, e) = 1$, g.c.d$(c_k, e) = 1$ and g.c.d$(c_j, c_k) = 1$. Recall that $c_i n_i$ is the smallest multiple of n_i that belongs to the semigroup generated by n_j, n_k. Then $c_i n_i = c_{ij} n_j + c_{ik} n_k$ implies that $c_i \lambda c_j c_k = c_{ij} \lambda e c_k + c_{ik} \lambda e c_j$. We conclude that e divides c_i, since g.c.d$(c_j, e) = 1$, g.c.d$(c_k, e) = 1$. Therefore $e \leq c_i$, but from the defining property of c_i and the fact that $e n_i = c_j n_j$ we have $e \geq c_i$. Thus $c_i = e$ and $c_i n_i = c_j n_j = c_k n_k$. This means that $I_{(n_i, n_j, n_k)}$ is complete intersection on all, a contradiction. Thus, $D(\mathbf{u})$ is indispensable in $\Lambda(T)_{\{i\}}$.

To prove the other implication, suppose now that $I_{(n_i, n_j, n_k)}$ is not a complete intersection on n_i. It follows then that either is a complete intersection on all or is not complete intersection at all. If it is complete intersection on all then $c_j n_j = c_i n_i = c_k n_k$. Then, we have the proper semi-conformal decomposition $\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w}$, where

$$\begin{aligned} \mathbf{u} &= (0, \ldots, \ c_j \ , \ldots, 0, \ 0 \ , 0, \ldots, 0, -c_k, 0, \ldots, 0), \\ \mathbf{v} &= (0, \ldots, \ c_j \ , \ldots, 0, -c_i, 0, \ldots, 0, \ \ 0 \ , \ 0, \ldots, 0), \\ \mathbf{w} &= (0, \ldots, \ 0 \ , \ldots, 0, \ c_i \ , 0, \ldots, 0, -c_k, 0, \ldots, 0) \ , \end{aligned}$$

where the components that can be nonzero in at least one vector are in the j^{th}, i^{th} and k^{th}-positions. This implies that $D(\mathbf{u}) = D(\mathbf{v}) +_{sc} D(\mathbf{w})$, since on all components except the one in the i^{th} position the sum is conformal. Thus $D(\mathbf{u})$ is not indispensable in $\Lambda(T)_{\{i\}}$, a contradiction.

In the case that $I_{(n_j, n_i, n_k)}$ is not complete intersection, then the circuit $(c_j, 0, -c_k)$ is not indispensable in $I_{(n_j, n_i, n_k)}$, therefore it has a proper semiconformal decomposition. Note that in terms of signs $(c_j, 0, -c_k) = (*, \ominus, \ominus) +_{sc} (\oplus, \oplus, *) = \mathbf{v} +_{sc} \mathbf{w}$. The first $*$ is $+$ and the second $*$ is $-$, since $\mathrm{Ker}_{\mathbb{Z}}(n_j, n_i, n_k) \cap \mathbb{N}^3 = \{\mathbf{0}\}$. Thus, the sum is conformal in the first and the last component. Then

$$D((0, \ldots, c_j, \ldots, 0, 0, \ldots, -c_k, 0, \ldots, 0)) =$$

$$D((0, \ldots, v_j, \ldots, v_i, 0, \ldots, v_k, 0, \ldots, 0)) +_{sc} D((0, \ldots, w_j, \ldots, w_i, 0, \ldots, w_k, 0, \ldots, 0)).$$

Thus, $D(\mathbf{u})$ is not indispensable in $\Lambda(T)_{\{i\}}$, a contradiction, and therefore the other implication is also proved. □

The following theorem shows that if $\{i\}$ is a face of the strongly robust complex Δ_T, then n_i has a very special property. As we will see in Theorem 4.5, if n_i satisfies this property, then n_i is unique.

Theorem 4.4. *Let $T = (n_1, n_2, \ldots, n_s)$. If the strongly robust complex Δ_T contains $\{i\}$ as a face then for every $j, k \in [s]$ with $j, k \neq i$, $I_{(n_i, n_j, n_k)}$ is a complete intersection on n_i.*

Proof. Suppose that $\{i\}$ is a face of Δ_T, then $\Lambda(T)_{\{i\}}$ is strongly robust. Then all circuits are indispensable, since for strongly robust toric ideals the set of indispensable elements is equal to the Graver basis and the latter contains all circuits. Therefore, by Lemma 4.3, $I_{(n_i, n_j, n_k)}$ is a complete intersection on n_i. □

Theorem 4.5. *Let $T = (n_1, n_2, \ldots, n_s)$. The strongly robust complex Δ_T is either $\{\emptyset\}$ or $\{\emptyset, \{i\}\}$ for exactly one $i \in [s]$.*

Proof. By Theorem 3.4 $\dim(\Delta_T) \leq 0$, thus $\Delta_T = \{\emptyset\}$ or $\Delta_T = \{\emptyset\} \cup \{\{i\}|i \in \Sigma \subset [s]\}$. We claim that Σ contains at most one element. Suppose not, and let $i, j \in \Sigma$, $i \neq j$. Take any $k \in [s]$ different from i, j and apply two times Theorem 4.4. We have that $I_{(n_i,n_j,n_k)}$ is a complete intersection on n_i and n_j, and thus a contradiction, see remark after Definition 3.1. Thus the set Σ can have at most one element. □

Remark 4.6. It follows from Theorem 4.5 that toric ideals of monomial curves in A^s for $s \geq 3$ are never strongly robust, since in this case I_T is a simple toric ideal, therefore all of its bouquets are not mixed. Thus, I_T is a $T_{[s]}$-robust ideal and $[s] \notin \Delta_T$. The fact that the toric ideal of a monomial curve is not robust for $s \geq 3$, thus also not strongly robust, was also noticed in [12, Corollary 4.17].

Although the converse of Theorem 4.4 is not true in general, it is true for 1×3 matrices, as the following Theorem shows.

Theorem 4.7. *Let $T = (n_1, n_2, n_3)$. We have the following three cases*

- *if I_T is not complete intersection, then Δ_T is the empty complex;*
- *if I_T is complete intersection on n_i, for an $i \in [3]$, then $\Delta_T = \{\emptyset, \{i\}\}$;*
- *if I_T is complete intersection on all, then Δ_T is the empty complex.*

Proof. By Proposition 4.5, we have that the strongly robust complex Δ_T is either $\{\emptyset\}$ or $\{\emptyset, \{i\}\}$ for one $i \in [3]$. By Theorem 4.4, if $\{i\} \in \Delta_T$, then $I_{(n_i,n_j,n_k)}$ is a complete intersection on n_i. Therefore it remains to show that if I_T is complete intersection on n_i, for an $i \in [3]$, then $\Delta_T = \{\emptyset, \{i\}\}$, or equivalently that $\Lambda(T)_{\{i\}}$ is strongly robust. Without loss of generality we may suppose that $i = 1$. By Lemma 4.2 the two circuits with 1 in their support are indispensable, while by Lemma 4.3 the remaining circuit with support on $\{2, 3\}$ is indispensable in $\Lambda(T)_{\{1\}}$, as I_T is a complete intersection on n_1. Thus it remains to prove that elements $\mathbf{u}$ in the $\mathrm{Gr}(T)$ with full support are indispensable in $\Lambda(T)_{\{1\}}$. Taking $\mathbf{u}$ or $-\mathbf{u}$, we can suppose that the first component of $\mathbf{u}$ is positive.

There are three cases then for $\mathbf{u}$: (1) $\mathbf{u} = (a, -b, -c)$, (2) $\mathbf{u} = (a, b, -c)$, and (3) $\mathbf{u} = (a, -b, c)$, where $a, b, c \in \mathbb{N}$.

(1) $\mathbf{u} = (a, -b, -c)$. Let $D(\mathbf{u}) = D(\mathbf{v}) +_{sc} D(\mathbf{w})$ be a semiconformal decomposition of $D(\mathbf{u})$ in $\Lambda(T)_{\{1\}}$. Then, from Proposition 4.1, we have $[\mathbf{u}]^1 = [\mathbf{v}]^1 +_c [\mathbf{w}]^1$, but $[\mathbf{u}]^1 = (-b, -c)$ and the sum being conformal implies that all components of $[\mathbf{v}]^1$, $[\mathbf{w}]^1$ are non positive. Since $\mathbf{v}, \mathbf{w} \in \mathrm{Ker}_{\mathbb{Z}}(T)$, this means that their first component is non negative. However, then the sum $D(\mathbf{u}) = D(\mathbf{v}) +_c D(\mathbf{w})$ is conformal, and $D(\mathbf{u}) \in \mathrm{Gr}(\Lambda(T)_{\{1\}})$ since $\mathbf{u} \in \mathrm{Gr}(T)$, [19]. As elements of the Graver basis as characterised as those with no proper conformal decomposition, one of the $D(\mathbf{v}), D(\mathbf{w})$ is zero and thus $D(\mathbf{u})$ is indispensable.

(2) $\mathbf{u} = (a, b, -c)$. Let $D(\mathbf{u}) = D(\mathbf{v}) +_{sc} D(\mathbf{w})$ be a semiconformal decomposition of $D(\mathbf{u})$ in $\Lambda(T)_{\{1\}}$. Then, from Proposition 4.1, we have $[\mathbf{u}]^1 = [\mathbf{v}]^1 +_c [\mathbf{w}]^1$. However, $[\mathbf{u}]^1 = (b, -c)$ and the sum being conformal implies that the second component of each of the $\mathbf{v}, \mathbf{w}$ is non negative, while the third component is non positive. Taking into account that the sum $D(\mathbf{u}) = D(\mathbf{v}) +_{sc} D(\mathbf{w})$ is semiconformal, we get

that the sign patent of $\mathbf{v}, \mathbf{w}$ is $\mathbf{v} = (*, \oplus, \ominus)$ and $\mathbf{w} = (\oplus, \oplus, \ominus)$. If $* = \oplus$, then $\mathbf{u}$ will have a conformal decomposition and since $\mathbf{u} \in \mathrm{Gr}(T)$, one of the $\mathbf{v}, \mathbf{w}$ is zero. Which implies that one of the $D(\mathbf{v}), D(\mathbf{w})$ is zero and thus $D(\mathbf{u})$ is indispensable.

If $* = \ominus$ then $\mathbf{u} = (a, b, -c) = \mathbf{v} + \mathbf{w} = (-a_1, b_1, -c_1) + (a_2, b_2, -c_2)$, where $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{N}$. Then, $-a_1 n_1 + b_1 n_2 - c_1 n_3 = 0$ and $a_2 n_1 + b_2 n_2 - c_2 n_3 = 0$, since $\mathbf{v}, \mathbf{w} \in \mathrm{Ker}_{\mathbb{Z}}(T)$. In the first equation, $b_1 = 0$ implies that $a_1 = 0 = c_1$ thus $\mathbf{v} = \mathbf{0}$ and the proof is complete, as $D(\mathbf{u})$ is indispensable. Otherwise, $b_1 n_2$ belongs to the semigroup generated by n_1, n_3 and (n_1, n_2, n_3) is a complete intersection on n_1 implies $b = b_1 + b_2 \geq b_1 \geq n_3^{\#}$. Similarly from the second equation we get that either $\mathbf{w} = \mathbf{0}$ or $c = c_1 + c_2 \geq c_2 \geq n_2^{\#}$. But then the proper sum $(a, b, -c) = (a, b - n_3^{\#}, -c + n_2^{\#}) + (0, n_3^{\#}, -n_2^{\#})$ is conformal, which is a contradiction since $\mathbf{u} \in \mathrm{Gr}(T)$.

(3) $\mathbf{u} = (a, -b, c)$. For the third case we argue in a similar manner as in the second case.

Thus in all cases $D(\mathbf{u})$ is indispensable in $\Lambda(T)_{\{1\}}$ and thus $\Lambda(T)_{\{1\}}$ is strongly robust. □

A different proof of Theorem 4.7 can be given using Lemmata 4.2, 4.3 and [17, Corollary 4.6], since for $T = (n_1, n_2, n_3)$ the ideal $I_{\Lambda(T)_{\{1\}}}$ has codimension 2. We have preferred the above proof, as it follows the style of the remaining proofs in this article.

5. Primitive elements and strongly robustness

In this section, we generalise the notion of a primitive element and that of a Graver basis and use it to give a necessary and sufficient criterion for a vertex $\{i\}$ to be a face of the strongly robust simplicial complex.

Definition 5.1. *An element $\mathbf{u} \in S \subset \mathbb{Z}^n$ is called primitive in S if there is no $\mathbf{v} \in S$, $\mathbf{v} \neq \mathbf{u}$ such that $\mathbf{v}^+ \leq \mathbf{u}^+$ and $\mathbf{v}^- \leq \mathbf{u}^-$. The set of primitive elements in S is denoted by* $\mathrm{Graver}(S)$.

Primitive elements in $S = \mathrm{Ker}_{\mathbb{Z}}(A)$ constitute the Graver basis of A.

Definition 5.2. *Let $T = (n_1, n_2, \dots, n_s)$, we define*

$$\mathrm{Gr}(T)^i = \{[\mathbf{u}]^i | \mathbf{u} \in \mathrm{Gr}(T)\} \subset \mathbb{Z}^{s-1}.$$

Proposition 5.3. *Let $\mathbf{u} \in \mathrm{Gr}(T)$. Then $D(\mathbf{u})$ is indispensable in $\Lambda(T)_{\{i\}}$ if and only if $[\mathbf{u}]^i$ is primitive in $\mathrm{Gr}(T)^i$.*

Proof. Suppose $[\mathbf{u}]^i$ is not primitive in $\mathrm{Gr}(T)^i$. Then there exists an element $\mathbf{v} \in \mathrm{Gr}(T)$ such that $[\mathbf{v}^+]^i \leq [\mathbf{u}^+]^i$ and $[\mathbf{v}^-]^i \leq [\mathbf{u}^-]^i$. Set $\mathbf{w} = \mathbf{u} - \mathbf{v}$. Then it follows that $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $[\mathbf{u}]^i = [\mathbf{v}]^i +_c [\mathbf{w}]^i$. Looking at the i-components of $\mathbf{v}, \mathbf{w}$ and putting in front the negative one we get that one of the sums $D(\mathbf{u}) = D(\mathbf{v}) + D(\mathbf{w})$ or $D(\mathbf{u}) = D(\mathbf{w}) + D(\mathbf{v})$ is semiconformal. Thus $D(\mathbf{u})$ is not indispensable in $\Lambda(T)_{\{i\}}$.

Suppose that $D(\mathbf{u})$ is not indispensable in $\Lambda(T)_{\{i\}}$. Then there is a proper semi-conformal decomposition $D(\mathbf{u}) = D(\mathbf{v}) +_{sc} D(\mathbf{w})$, note that in any decomposition like that the sum is conformal in every component except possible at the i-th component, see Proposition 4.1. But $\mathbf{u} \in \mathrm{Gr}(T)$ therefore $D(\mathbf{u}) \in \mathrm{Gr}(\Lambda(T)_{\{i\}})$, thus it does not have a proper conformal decomposition. We conclude that the i-th component of $\mathbf{v}$ is negative and the i-th component of $\mathbf{w}$ is positive. From all semiconformal decompositions $D(\mathbf{u}) = D(\mathbf{v}) +_{sc} D(\mathbf{w})$ choose one with w_i smallest. Then, for these choices of $\mathbf{v}, \mathbf{w}$, we claim that $\mathbf{w} \in \mathrm{Gr}(T)$. Suppose not, then there is a proper conformal decomposition $\mathbf{w} = \mathbf{w}' +_c \mathbf{w}'' = \mathbf{w}'' +_c \mathbf{w}'$. Choose $\mathbf{w}''$ to be one with $w_i > w''_i$. But then it is easy to see that $D(\mathbf{u}) = D(\mathbf{v} + \mathbf{w}') +_{sc} D(\mathbf{w}'')$, which is a contradiction, since $w_i > w''_i$. Thus $\mathbf{w} \in \mathrm{Gr}(T)$. Therefore $[\mathbf{w}]^i$ is in $\mathrm{Gr}(T)^i$ and $[\mathbf{w}^+]^i \leq [\mathbf{u}^+]^i$ and $[\mathbf{w}^-]^i \leq [\mathbf{u}^-]^i$. Thus $[\mathbf{u}]^i$ is not primitive in $\mathrm{Gr}(T)^i$. □

Theorem 5.4. *Let $T = (n_1, n_2, \dots, n_s)$. The strongly robust complex Δ_T contains $\{i\}$ as a face if and only if for every element $\mathbf{u} \in \mathrm{Gr}(T)$ the $[\mathbf{u}]^i$ is primitive in $\mathrm{Gr}(T)^i$ (or equivalently if and only if $\mathrm{Graver}(\mathrm{Gr}(T)^i) = \mathrm{Gr}(T)^i$).*

Proof. Suppose that $\{i\}$ is a face of Δ_T then $\Lambda(T)_{\{i\}}$ is strongly robust. Therefore $D(\mathbf{u})$ is indispensable in $\Lambda(T)_{\{i\}}$ for every element $\mathbf{u} \in \mathrm{Gr}(T)$. Then by Proposition 5.3 for every element $\mathbf{u} \in \mathrm{Gr}(T)$ the $[\mathbf{u}]^i$ is primitive in $\mathrm{Gr}(T)^i$.

Suppose now that for every element $\mathbf{u} \in \mathrm{Gr}(T)$ the $[\mathbf{u}]^i$ is primitive in $\mathrm{Gr}(T)^i$. Then Proposition 5.3 implies that $D(\mathbf{u})$ is indispensable in $\Lambda(T)_{\{i\}}$ for every $\mathbf{u} \in \mathrm{Gr}(T)$. But [19, Theorem 1.11] says that all elements in the Graver basis of $\Lambda(T)_{\{i\}}$ are in the form $D(\mathbf{u})$ with $\mathbf{u} \in \mathrm{Gr}(T)$. Thus $\Lambda(T)_{\{i\}}$ is strongly robust and so $\{i\}$ is a face of Δ_T. □

6. Generalized Lawrence matrices

In [19, Section 2], Petrović et al generalized the notion of a Lawrence matrix, see [21, Chapter 7], by introducing generalized Lawrence matrices. For every matrix A there exists a generalized Lawrence matrix with the same kernel up to permutation of columns, see [19, Corollary 2.3]. Let $(c_1, \dots, c_m) \in \mathbb{Z}^m$ be any vector having full support and with the greatest common divisor of all its components equal to 1. Then there exist integers $\lambda_1, \dots, \lambda_m$ such that $1 = \lambda_1 c_1 + \cdots + \lambda_m c_m$. For any choice of $\lambda_1, \dots, \lambda_m$ and any integer n we define the matrix $A(n, (c_1, \dots, c_m)) = (\lambda_1 n, \dots, \lambda_m n) \in \mathbb{Z}^{1 \times m}$ and the matrix

$$C(c_1, \dots, c_m) = \begin{pmatrix} -c_2 & c_1 & & & \\ -c_3 & & c_1 & & \\ & & & \ddots & \\ -c_m & & & & c_1 \end{pmatrix} \in \mathbb{Z}^{(m-1) \times m}.$$

Theorem 6.1. *Let $T = (n_1, n_2, \dots, n_s) \in \mathbb{Z}^{1 \times s}$. Let $\mathbf{c}_1, \dots, \mathbf{c}_s$ be any set of vectors having full support and each with the greatest common divisor of all its components equal to 1, with $\mathbf{c}_j \in \mathbb{Z}^{m_j}$ for some $m_j \geq 1$. In the case that $\Delta_T = \{\emptyset\}$ each $\mathbf{c}_j = (c_{j1}, \dots, c_{jm_j}) \in \mathbb{Z}^{m_j}$ has the first component positive and at least one component negative, while in the case that $\Delta_T = \{\emptyset, \{i\}\}$ then the same is true for all $\mathbf{c}_j$ with $j \neq i$. Define $p = 1 + \sum_{i=1}^{s}(m_i - 1)$ and $q = \sum_{i=1}^{s} m_i$. Then the toric ideal I_A is* *strongly robust, where*

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_s \end{pmatrix} \in \mathbb{Z}^{p\times q},$$

$A_j = A(n_j, (c_{j1}, \ldots, c_{jm_j}))$ *and* $C_j = C(c_{j1}, \ldots, c_{jm_j})$ *for all* $j = 1, \ldots, s$.

Proof. The matrix A is a generalized Lawrence matrix, see [19, Theorem 2.1] with bouquet ideal the toric ideal of the monomial curve I_T, with all bouquets mixed except possibly the B_i, in the case that $\Delta_T = \{\emptyset, \{i\}\}$. Therefore the toric ideal is T_ω-robust, where ω is either the empty set or $\{i\}$. In both cases ω is a face of the strongly robust simplicial complex Δ_T. Thus the toric ideal is strongly robust, see Theorem 2.3. □

Example 6.2. Theorem 6.1 provides a way to produce examples of strongly robust ideals with bouquet ideal the ideal of any monomial curve. Take for example the monomial curve with defining matrix $T = (4, 5, 6)$. The toric ideal I_T is a complete intersection on 5 thus according to Theorem 4.7 the strongly robust complex Δ_T is equal to $\{\emptyset, \{2\}\}$. Choose three integer vectors, one for each bouquet, of any dimension with full support and the greatest common divisor of all its components equal to 1 such that they have a positive first component and at least one component negative except possibly for the second vector which may have all components positive. For example choose $\mathbf{c}_1 = (2, -1, -2023)$, $\mathbf{c}_2 = (10, 2024, 7, 4)$ and $\mathbf{c}_3 = (5, 3, -2029)$. For each vector $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{Z}^m$ choose integers $\lambda_1, \ldots, \lambda_m$ such that $1 = \lambda_1 c_1 + \cdots + \lambda_m c_m$. For example $1 = 0\cdot 2 + (-1)\cdot(-1) + 0\cdot(-2023)$, $1 = (-1)\cdot 10 + 0\cdot 2024 + 1\cdot 7 + 1\cdot 4$ and $1 = 2\cdot 5 + (-3)\cdot 3 + 0\cdot(-2029)$. Then Theorem 6.1 says that the toric ideal I_A is strongly robust, where

$$A = \begin{pmatrix} 0 & -4 & 0 & -5 & 0 & 5 & 5 & 12 & -18 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2023 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2024 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2029 & 0 & 5 \end{pmatrix}.$$

According to the following theorem, *all* strongly robust ideals with bouquet ideal the ideal of a monomial curve are produced like in the above example.

Theorem 6.3. *Let* I_A *be any toric ideal which is strongly robust and such that its bouquet ideal is the ideal of a monomial curve. Then there exists a generalized Lawrence matrix* A' *such that* $I_A = I_{A'}$*, up to permutation of column indices, where*

$$A' = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_s \end{pmatrix} \in \mathbb{Z}^{p\times q},$$

with matrices $A_j = A(n_j, (c_{j1}, \dots, c_{jm_j}))$ *and* $C_j = C(c_{j1}, \dots, c_{jm_j})$ *for some* $T = (n_1, n_2, \dots, n_s) \in \mathbb{Z}^{1\times s}$*, and* $\mathbf{c}_j \in \mathbb{Z}^{m_j}$ *for some* $m_j \geq 1$*, for all* $j = 1, \dots, s$*, are integer vectors having full support and each one with the greatest common divisor of all its components equal to* 1*. In the case that* $\Delta_T = \{\emptyset\}$ *each* $\mathbf{c}_j = (c_{j1}, \dots, c_{jm_j}) \in \mathbb{Z}^{m_j}$ *has the first component positive and at least one component negative. In the case that* $\Delta_T = \{\emptyset, \{i\}\}$ *each* $\mathbf{c}_j$ *with* $j \neq i$ *has the first component positive and at least one component negative, while* $\mathbf{c}_i$ *may have all components positive.*

Proof. By hypothesis, the ideal I_A is strongly robust with its bouquet ideal I_T, where $T = (n_1, n_2, \dots, n_s) \in \mathbb{Z}^{1\times s}$. By Theorem 4.5 the strongly robust complex Δ_T is either $\{\emptyset\}$ or $\{\emptyset, \{i\}\}$ for one $i \in [s]$. Then the ideal I_A is either T or $T_{\{i\}}$-robust, therefore all bouquets of I_A are mixed with the possible exception of the i-th bouquet.

For any integer matrix A, there exists a generalized Lawrence matrix A' such that $I_A = I_{A'}$, up to permutation of column indices, by [19, Corollary 2.3]. Since the bouquet ideal is the ideal I_T and all bouquets of I_A are mixed with the possible exception of the i-th bouquet, the vectors $\mathbf{c}_j$ have a positive first component and at least one component negative, with the possible exception of $\mathbf{c}_i$ which may have all components positive. □

Example 6.4. Consider the matrix $A = (\mathbf{a}_1, \dots, \mathbf{a}_{11}) \in \mathbb{Z}^{8\times 11}$, given by

$$A = \begin{pmatrix} 36 & 60 & 4 & 40 & 64 & 39 & 1 & 72 & 84 & 12 & 4 \\ 12 & 20 & 4 & 8 & 24 & -2 & 1 & 12 & 4 & 0 & 4 \\ 36 & 80 & 4 & 48 & 88 & 39 & 1 & 84 & 84 & 12 & 4 \\ 60 & 100 & 12 & 16 & 112 & 33 & 4 & 120 & 104 & 36 & 24 \\ 24 & 40 & 8 & 24 & 48 & -4 & 2 & 36 & 8 & 0 & 8 \\ 12 & 20 & 4 & -12 & 24 & -2 & 1 & 24 & 8 & 12 & 8 \\ 12 & 20 & 4 & -12 & 24 & -2 & 1 & 24 & 12 & 12 & 12 \\ 24 & 40 & 0 & 4 & 40 & 39 & 1 & 60 & 84 & 24 & 4 \end{pmatrix}.$$

Using `4ti2` [1] we can see that the toric ideal I_A is strongly robust and a Gale transform of the matrix A is

$$\begin{pmatrix} 5 & 40 & 779 & -13642 \\ -18 & -162 & -3198 & 56004 \\ -15 & -120 & -2337 & 40926 \\ 0 & 6 & 123 & -2154 \\ 15 & 135 & 2665 & -46670 \\ 0 & 0 & 4 & -72 \\ 0 & 0 & 8 & -144 \\ 0 & -4 & -82 & 1436 \\ 0 & 0 & 0 & 1 \\ 0 & 14 & 287 & -5026 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The toric ideal I_A has five bouquets $B_1 = \{a_1, a_3\}$, $B_2 = \{a_2, a_5\}$, $B_3 = \{a_6, a_7\}$, $B_4 = \{a_4, a_8, a_{10}\}$, $B_5 = \{a_9, a_{11}\}$ all of them being mixed except the third one, which is non-mixed. The corresponding $\mathbf{c}_B$ vectors are: $\mathbf{c}_{B_1} = (1, 0, -3, 0, 0, 0, 0, 0, 0, 0, 0)$

$$\mathbf{c}_{B_2} = (0, 6, 0, 0, -5, 0, 0, 0, 0, 0, 0), \quad \mathbf{c}_{B_3} = (0, 0, 0, 0, 0, 1, 2, 0, 0, 0, 0),$$

$$\mathbf{c}_{B_4} = (0,0,0,3,0,0,0,-2,0,7,0), \quad \mathbf{c}_{B_5} = (0,0,0,0,0,0,0,0,1,0,-1).$$

The bouquet ideal is the toric ideal of the matrix $A_B = (\mathbf{a}_{B_1}, \mathbf{a}_{B_2}, \mathbf{a}_{B_3}, \mathbf{a}_{B_4}, \mathbf{a}_{B_5})$, that is

$$A_B = \begin{pmatrix} 24 & 40 & 41 & 60 & 80 \\ 0 & 0 & 0 & 0 & 0 \\ 24 & 40 & 41 & 60 & 80 \\ 24 & 40 & 41 & 60 & 80 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 24 & 40 & 41 & 60 & 80 \end{pmatrix}.$$

The bouquet ideal I_{A_B} is exactly the same as the toric ideal of the monomial curve for $T = (24, 40, 41, 60, 80)$ for which we know that the strongly robust complex is $\Delta_T = \{\emptyset, \{3\}\}$. Note that only the third bouquet is not mixed and thus I_A is a $T_{\{3\}}$-robust ideal which explains why it is strongly robust.

Then, the generalized Lawrence matrix with the same kernel, after permutation of column indices, is

$$A' = \begin{pmatrix} 24 & 0 & 40 & 40 & 41 & 0 & 60 & 60 & 0 & 80 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Note that the permutation of column indices to bring the vectors of the same bouquet together gives the following isomorphism of the two kernels $\phi : \mathrm{Ker}_{\mathbb{Z}}(A) \mapsto \mathrm{Ker}_{\mathbb{Z}}(A')$,

$$\phi(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}) = (u_1, u_3, u_2, u_5, u_6, u_7, u_4, u_8, u_{10}, u_9, u_{11}).$$

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References

[1] 4ti2 team, 4ti2 - a software package for algebraic, geometric and combinatorial problems on linear spaces, available at www.4ti2.de (2007).

[2] D. Bernstein, A. Zelevinsky, Combinatorics of maximal minors, J. Algebraic Combin. 2(2), 111–121 (1993).

[3] A. Boocher, Free resolutions and sparse determinantal ideals, Math. Res. Letters 19(4) 805–821 (2012).

[4] A. Boocher, E. Robeva, Robust toric ideals, J. Symbolic Comput., 68(1), 254–264 (2015).

[5] A. Boocher, B. C. Brown, T. Duff, L. Lyman, T. Murayama, A. Nesky and K. Schaefer, Robust graph ideals, Ann. Comb. 19(4), 641–660 (2015).

[6] A. Campillo and P. Pison, L'idéal d'un semi-group de type fini, Comptes Rendues Acad. Sci. Paris, Série I, 316(1993) 1303-1306.

[7] H. Charalambous, A. Katsabekis, A. Thoma, Minimal systems of binomial generators and the indispensable complex of a toric ideal, Proc. Amer. Math. Soc. 135, 3443–3451 (2007).

[8] H. Charalambous, A. Thoma, M. Vladoiu, Markov complexity of monomial curves, J. Algebra 417, 391–411 (2014).
[9] H. Charalambous, A. Thoma, M. Vladoiu, Markov bases and generalized Lawrence liftings, Ann. Comb. 19(4), 661–669 (2015).
[10] A. Conca, E. De Negri, E. Gorla, Universal Gröbner bases for maximal minors, International Mathematics Research Notices 11, 3245–3262 (2015).
[11] A. Conca, E. De Negri, E. Gorla, Cartwright–Sturmfels ideals associated to graphs and linear spaces. J. Comb. Algebra 2 (3), 231–257 (2018).
[12] I. Garcia-Marco, Ch. Tatakis, On robustness and related properties on Toric ideals, J Algebr Comb 57, 21–52 (2023).
[13] P. A. García Sánchez, I. Ojeda, J. C. Rosales, Affine semigroups having a unique Betti element, J. Algebra Appl. 12 (3) 1250177,(2013).
[14] E. Gross, S. Petrović, Combinatorial degree bound for toric ideals of hypergraphs, Internat. J. Algebra Comput. 23 (6) 1503–1520 (2013).
[15] J. Herzog, Generators and relations of abelian semigroups and semigroup rings, Manuscripta Math. 3, 175–193 (1970).
[16] S. Hoşten, S. Sullivant, A finiteness theorem for Markov bases of hierarchical models, J. Combin. Theory Ser. A 114, 311–321 (2007).
[17] D. Kosta, A. Thoma, M. Vladoiu, On the strongly robust property of toric ideals, J. Algebra 616, 1–25 (2023).
[18] I. Peeva, B. Sturmfels, Generic lattice ideals, J. Amer. Math. Soc. 11(2), 363–373 (1998).
[19] S. Petrović, A. Thoma, M. Vladoiu, Bouquet algebra of toric ideals, J. Algebra 512, 493–525 (2018).
[20] S. Petrović, A. Thoma, M. Vladoiu, Hypergraph encodings of arbitrary toric ideals, J. Combin. Theory Ser. A 166, 11–41 (2019).
[21] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, University Lecture Series, No 8, MAS, R.I. (1995).
[22] B. Sturmfels, A. Zelevinsky, Maximal minors and their leading terms, Adv. Math. 98(1), 65–112 (1993).
[23] S. Sullivant, Strongly Robust Toric Ideals in Codimension 2, J. Algebr. Stat. 10, 128–136 (2019).
[24] Ch. Tatakis, Generalized robust toric ideals, J. Pure Appl. Algebra 220, 263–277 (2016).

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