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QUASI-INVARIANT GAUSSIAN MEASURES FOR THE CUBIC FOURTH ORDER NONLINEAR SCHRÖDINGER EQUATION IN NEGATIVE SOBOLEV SPACES

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Abstract. We continue the study on the transport properties of the Gaussian measures on Sobolev spaces under the dynamics of the cubic fourth order nonlinear Schrödinger equation. By considering the renormalized equation, we extend the quasi-invariance results in [30, 27] to Sobolev spaces of negative regularity. Our proof combines the approach introduced by Planchon, Tzvetkov, and Visciglia [35] with the normal form approach in [30, 27].

Contents

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1. Introduction

1.1. **Main result.** In this paper, we study the statistical properties of solutions to the cubic fourth order nonlinear Schrödinger equation (4NLS) on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:[1]

$$i\partial_t u = \partial_x^4 u + |u|^2 u, \qquad (x,t) \in \mathbb{T}\times\mathbb{R}. \tag{1.1}$$

Let us first introduce some notations. Given $s \in \mathbb{R}$, we consider the Gaussian measures μ_s, formally written as

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2}\, du = \prod_{n\in\mathbb{Z}} Z_{s,n}^{-1} e^{-\frac{1}{2}\langle n\rangle^{2s}|\widehat{u}_n|^2}\, d\widehat{u}_n. \tag{1.2}$$

Namely, μ_s is the induced probability measure under the random Fourier series:[2]

$$\omega \in \Omega \longmapsto u^\omega(x) = u(x;\omega) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^s} e^{inx}, \tag{1.3}$$

where $\langle\,\cdot\,\rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables[3] on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is easy to see that the random distribution (1.3) belongs almost surely to $H^\sigma(\mathbb{T})$ if and only if

$$\sigma < s - \frac{1}{2}. \tag{1.4}$$

In [30, 27], with Tzvetkov and Sosoe, the first author studied the transport properties of Gaussian measures μ_s in (1.2) under the 4NLS dynamics and proved quasi-invariance[4] of μ_s, $s > \frac{1}{2}$. Our main goal in this paper is to extend the quasi-invariance results in [30, 27] to Gaussian measures on periodic distributions of negative regularity.

It is known [30] that the cubic 4NLS (1.1) is globally well-posed in $L^2(\mathbb{T})$. Moreover, this well-posedness result is sharp in the sense that (1.1) is known to be ill-posed in negative Sobolev spaces [19, 33]. Thus, in view of (1.4), the quasi-invariance result for $s > \frac{1}{2}$ is optimal since for $s \le \frac{1}{2}$, the cubic 4NLS (1.1) is almost surely ill-posed with respect to the initial data given by the random Fourier series (1.3). In order to study the dynamical problem in negative Sobolev spaces, we consider the following renormalized 4NLS:

$$i\partial_t u = \partial_x^4 u + \big(|u|^2 - 2 \textstyle\fint_{\mathbb{T}} |u|^2\, dx\big)u, \tag{1.5}$$

where $\fint f(x)dx = \frac{1}{2\pi}\int f(x)dx$. For smooth functions, the equation (1.5) is equivalent to (1.1) via the following invertible gauge transform:

$$\mathcal{G}(u)(t) := e^{2it\fint |u(t)|^2 dx} u(t).$$

Namely, $u \in C(\mathbb{R}; L^2(\mathbb{T}))$ satisfies (1.1) if and only if $\mathcal{G}(u)$ satisfies (1.5). On the other hand, the gauge transform $\mathcal{G}$ does not make sense outside $L^2(\mathbb{T})$ and thus these equations describe genuinely different dynamics, if any, outside $L^2(\mathbb{T})$. As mentioned above, the original equation (1.1) is ill-posed in negative Sobolev spaces. As for the renormalized cubic 4NLS (1.5), the first author and Y. Wang [33] proved its global well-posedness in $H^s(\mathbb{T})$ for $s > -\frac{1}{3}$. See also [21] for local

[1] The defocusing / focusing nature of the equation does not play any role and thus we only consider the defocusing case. The main result also applies to the focusing case.

[2] In the following, we often drop the harmless factor of 2π.

[3] By convention, we set $\mathrm{Var}(g_n) = 1$, $n \in \mathbb{Z}$.

[4] Given a measure space (X, μ), we say that μ is quasi-invariant under a measurable transformation $T : X \to X$ if the transported measure $T_*\mu = \mu \circ T^{-1}$ and μ are equivalent, i.e. mutually absolutely continuous with respect to each other.

well-posedness of (1.5) for $s = -\frac{1}{3}$. See [4, 6, 16, 28, 34] for an analogous renormalization in the context of the usual nonlinear Schrödinger equation (NLS) with the second order dispersion. Before proceeding further, we point out that the solution map to (1.5), constructed in [33, 21], is not locally uniformly continuous in negative Sobolev spaces [8, 30]. Namely, we can not construct solutions by a contraction argument. This point will be important in our study; see Proposition 3.2 below.

We now state our main result.

Theorem 1.1. *Let $s > \frac{3}{10}$. Then, the Gaussian measure μ_s in* (1.2) *is quasi-invariant under the dynamics of the renormalized cubic 4NLS* (1.5).

The transport properties of Gaussian measures have been studied extensively in probability theory; see, for example, [5, 36, 9, 10]. In [39], Tzvetkov initiated the study of transport properties of Gaussian measures on functions / distributions under nonlinear Hamiltonian PDEs and there has been a significant progress in this direction [39, 30, 31, 27, 29, 35, 17, 13, 37, 11]. In particular, Theorem 1.1 extends the quasi-invariance results in [30, 27][5] to negative Sobolev spaces $H^\sigma(\mathbb{T})$, $\sigma > -\frac{1}{5}$.

The general strategy, as introduced in [39], is to study quasi-invariance of the Gaussian measures μ_s indirectly by studying weighted Gaussian measures ρ_s, where the weight corresponds to correction terms that arise due to the presence of the nonlinearity. The two key steps in this strategy are (i) the construction of the weighted Gaussian measure ρ_s and (ii) an energy estimate on the time derivative of the modified energy (that is, the energy of the Gaussian measure plus the correction terms). It is crucial to choose good correction terms in order to establish an effective energy estimate. In the context of 4NLS (1.1), this general strategy was applied in [30, 27]. In [30], the correction term was obtained by applying a normal form reduction (i.e. integration by parts in time) in the spirit of [38, 26, 1, 25]. In the second work [27], Sosoe, Tzvetkov, and the first author employed an infinite iteration of normal form reductions, introduced in [18], to compute an infinite series of correction terms to the H^s-energy functional. Such an infinite iteration of normal form reductions has turned out to be a useful tool in constructing solutions to PDEs and establishing energy estimates; see [18, 33, 22, 34, 20, 12].

In order to prove Theorem 1.1 for the renormalized 4NLS (1.5) in negative Sobolev spaces, we also apply an infinite iteration of normal form reductions to the H^s-energy functional and introduce infinitely many correction terms. In [27], the multilinear forms appearing in normal form reductions were shown to be bounded in $L^2(\mathbb{T})$. The main task here is to extend the boundedness of these multilinear forms to negative Sobolev spaces $H^\sigma(\mathbb{T})$, $-\frac{1}{5} < \sigma < 0$. See also Remark 5.2. This gives rise to the modified energies $\mathcal{E}_N(u)$ in (3.4) whose time derivatives are uniformly controlled on bounded sets in the support of the Gaussian measure μ_s (see Proposition 3.4), provided that $s > \frac{3}{10}$. We point out that, as in the previous works [30, 27], the regularity restriction in Theorem 1.1 comes from the energy estimate.

The next step is to construct weighted Gaussian measures. In [30, 27], the weighted Gaussian measures were normalized to be probability measures thanks to the (conserved) L^2-cutoff. For our current problem in negative Sobolev spaces, however, an L^2-cutoff is not available and thus the weighted Gaussian measures associated with the modified energies $\mathcal{E}_N(v)$ are not probability measures. An important observation is that our proof of quasi-invariance is entirely

[5]The quasi-invariance results in [30, 27] were proved for (1.1) but they equally apply to the renormalized 4NLS (1.5).

local in $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ (see Subsection 3.5). This allows us to work with the weighted Gaussian measures restricted to compact sets in $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$, for which we prove strong convergence (see Proposition 3.6). We then follow the approach introduced by Planchon, Tzvetkov, and Visciglia [35], where they established local-in-time (and also local in the phase space) quasi-invariance properties, and close the argument.

Since our argument is based on the study of frequency-truncated dynamics (see (3.1)), an approximation property of the truncated dynamics (Proposition 3.2) also plays a key role. In $L^2(\mathbb{T})$, a standard contraction argument yields local well-posedness of (1.5). By a slight variation of this contraction argument, one can easily prove the desired approximation properties of the truncated dynamics in $L^2(\mathbb{T})$ (see [30, Appendix B]). In negative Sobolev spaces, however, we can not use a contraction argument to establish local well-posedness of (1.5) due to the failure of local uniform continuity of the solution map [8, 30]. Hence, a more careful argument is required in studying approximation properties of the truncated dynamics. In fact, in negative Sobolev spaces, we only prove a weaker approximation property of the truncated dynamics. See Remark 3.3. In Section 4, we discuss in detail the approximation property of the truncated dynamics in negative Sobolev spaces.

Remark 1.2. In [35], the authors compared their approach and the normal form approach in [30, 27] and stated "It would be interesting to find situations where the approaches of [[30, 27]] and the one used in [[35]] can collaborate." Our proof of Theorem 1.1 provides the first such example, combining the methods from [35] and [30, 27].

Remark 1.3. In [32], Tzvetkov, Y. Wang, and the first author constructed global-in-time dynamics for (1.5) almost surely with respect to the white noise, i.e. the Gaussian measure μ_s with $s = 0$. They also proved invariance of the white noise μ_0 under (1.5), which in particular implies its quasi-invariance. Thus, it is an interesting question to fill in the gap $0 < s \le \frac{3}{10}$ between Theorem 1.1 and the result in [32].

Remark 1.4. In [23], the second author with G. Li and Zine recently proved global well-posedness of the following renormalized fractional NLS on $\mathbb{T}$ (for $\alpha > 2$):

$$i\partial_t u = (-\partial_x^2)^{\frac{\alpha}{2}} u + \big(|u|^2 - 2 ⨍_{\mathbb{T}} |u|^2\, dx\big)u \tag{1.6}$$

in $H^\sigma(\mathbb{T})$ for $\sigma > \frac{2-\alpha}{6}$. While we only consider the renormalized 4NLS (1.5) in this paper for simplicity of presentation, our argument can be easily adapted to study the quasi-invariance property of μ_s under the dynamics of (1.6) for some range of $s \le \frac{1}{2}$.

Remark 1.5. At each step of normal form reductions, we introduce a correction term. This is precisely how correction terms are introduced in the I-method [7]. In order to prove the energy estimate (Proposition 3.4), we implement an infinite iteration of normal form reductions and thus introduce an infinite series of correction terms. In other words, the modified energies $\mathcal{E}_N(v)$ defined in (3.4) can be viewed as modified energies of an infinite order in the I-method terminology. Finally, we remark that this infinite iteration of normal form reductions allows us to encode multilinear dispersion in the structure of the modified energy and thus to exchange analytical difficulty with algebraic / combinatorial difficulty.

1.2. **Organization.** In Section 2, we introduce some notations. In Section 3, by assuming the approximation property of the truncated dynamics (Proposition 3.2) and the energy estimate (Proposition 3.4) with the related normal form reductions, we prove Theorem 1.1. In Section 4, we discuss the approximation property of the truncated dynamics. In Section 5, we then establish the energy estimate (Proposition 3.4) by implementing an infinite iteration of normal form reductions.

2. Notations

In the following, we fix small $\varepsilon > 0$ and set

$$\sigma = s - \frac{1}{2} - \varepsilon \tag{2.1}$$

such that (1.4) is satisfied. Given $R > 0$, we use B_R to denote the ball of radius R in $H^\sigma(\mathbb{T})$ centered at the origin.

Given $N \in \mathbb{N} \cup \{\infty\}$, we use $\mathbf{P}_{\leq N}$ to denote the Dirichlet projection onto the frequencies $\{|n| \leq N\}$ and set $\mathbf{P}_{>N} := \mathrm{Id} - \mathbf{P}_{\leq N}$. When $N = \infty$, it is understood that $\mathbf{P}_{\leq N} = \mathrm{Id}$. Define E_N by

$$E_N = \mathbf{P}_{\leq N} H^\sigma(\mathbb{T}) = \operatorname{span}\{e^{inx} : |n| \leq N\}$$

and let $E_N^\perp$ be the orthogonal complement of E_N in $H^\sigma(\mathbb{T})$.

Given $s \in \mathbb{R}$, let μ_s be the Gaussian measure on $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ defined in (1.2). Then, we can write μ_s as

$$\mu_s = \mu_{s,N} \otimes \mu_{s,N}^\perp, \tag{2.2}$$

where $\mu_{s,N}$ and $\mu_{s,N}^\perp$ are the marginal distributions of μ_s restricted onto E_N and $E_N^\perp$, respectively. In other words, $\mu_{s,N}$ and $\mu_{s,N}^\perp$ are induced probability measures under the following random Fourier series:

$$\mathbf{P}_{\leq N} u : \omega \in \Omega \longmapsto \mathbf{P}_{\leq N} u(x;\omega) = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^s} e^{inx},$$

$$\mathbf{P}_{>N} u : \omega \in \Omega \longmapsto \mathbf{P}_{>N} u(x;\omega) = \sum_{|n| > N} \frac{g_n(\omega)}{\langle n \rangle^s} e^{inx},$$

respectively. Formally, we can write $\mu_{s,N}$ and $\mu_{s,N}^\perp$ as

$$d\mu_{s,N} = Z_{s,N}^{-1} e^{-\frac{1}{2}\|\mathbf{P}_{\leq N} u\|_{H^s}^2} du_N \qquad \text{and} \qquad d\mu_{s,N}^\perp = \widehat{Z}_{s,N}^{-1} e^{-\frac{1}{2}\|\mathbf{P}_{>N} u\|_{H^s}^2} du_N^\perp, \tag{2.3}$$

where du_N and $du_N^\perp$ are (formally) the products of the Lebesgue measures on the Fourier coefficients:

$$du_N = \prod_{|n| \leq N} d\widehat{u}(n) \qquad \text{and} \qquad du_N^\perp = \prod_{|n| > N} d\widehat{u}(n). \tag{2.4}$$

Given a function $u \in H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$, we may use u_n to denote the Fourier coefficient $\widehat{u}(n)$ of u, when there is no confusion. This shorthand notation is useful in Section 5.

We use $S(t)$ to denote the linear propagator for the fourth order Schrödinger equation:

$$S(t) = e^{-it\partial_x^4}.$$

We denote by $\mathcal{N}(u)$ the renormalized nonlinearity in (1.5):

$$\mathcal{N}(u) = \big(|u|^2 - 2 \textstyle\fint_{\mathbb{T}} |u|^2 \, dx\big) u. \tag{2.5}$$

We also define the phase function $\phi(\bar{n})$ by

$$\phi(\bar{n}) = \phi(n_1, n_2, n_3, n) = n_1^4 - n_2^4 + n_3^4 - n^4. \tag{2.6}$$

Then, recall from [30] that

$$\phi(\bar{n}) = (n_1 - n_2)(n_1 - n)\Big(n_1^2 + n_2^2 + n_3^2 + n^2 + 2(n_1 + n_3)^2\Big) \tag{2.7}$$

under $n = n_1 - n_2 + n_3$. Lastly, given $n \in \mathbb{Z}$ and $N \in \mathbb{N}$, we define the index sets $\Gamma(n)$ and $\Gamma_N(n)$ by

$$\Gamma(n) = \big\{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3 \text{ and } n_1, n_3 \neq n\big\} \tag{2.8}$$

and

$$\Gamma_N(n) = \big\{(n_1, n_2, n_3) \in \mathbb{Z}^3 : |n_j| \leq N,\ n = n_1 - n_2 + n_3 \text{ and } n_1, n_3 \neq n\big\}. \tag{2.9}$$

Note that $\phi(\bar{n}) \neq 0$ on $\Gamma(n)$ and $\Gamma_N(n)$.

Given $T > 0$, we use the following shorthand notation: $C_T H_x^\sigma = C([0, T]; H^\sigma(\mathbb{T}))$, etc.

In view of the time reversibility of the equation (1.5), we only consider positive times in the following.

3. Proof of the main result

In this section, we go over the proof of Theorem 1.1 by assuming (i) the approximation property of the truncated dynamics (Proposition 3.2) and (ii) the energy estimate (Proposition 3.4) and the analysis on the correction terms (Lemma 3.5). We present the proofs of Propositions 3.2 and 3.4 in Sections 4 and 5, respectively. While we follow closely the structure of Section 3 in [27], we avoid using the interaction representation $v(t) = S(-t)u(t)$ in this section so that the modified energies and the associated weighted Gaussian measures are not time-dependent. Compare this with [30, 27], where the modified energies and the associated weighted Gaussian measures were time-dependent.

In the following, we fix $\frac{3}{10} < s \leq \frac{1}{2}$ and set $\sigma = s - \frac{1}{2} - \varepsilon$ for some small $\varepsilon > 0$, unless otherwise stated.

3.1. **Truncated dynamics.**

Given $N \in \mathbb{N}$, we consider the following truncated version of the renormalized 4NLS:

$$i\partial_t u = \partial_x^4 u + \mathbf{P}_{\leq N}\mathcal{N}(\mathbf{P}_{\leq N} u), \tag{3.1}$$

where $\mathcal{N}(u)$ is as in (2.5). Note that (3.1) is not a finite-dimensional system of ODEs, when written on the Fourier side. The higher frequency part $\mathbf{P}_{>N} u$ is propagated by the linear flow.

Given initial data $u_0 \in H^\sigma(\mathbb{T})$, we can write $u_0 = \mathbf{P}_{\leq N} u_0 + \mathbf{P}_{>N} u_0$. Then, the L^2-global well-posedness of the (renormalized) 4NLS [30] yields a global-in-time solution u_N to the low frequency dynamics:

$$\begin{cases} i\partial_t u_N = \partial_x^4 u_N + \mathbf{P}_{\leq N}\mathcal{N}(u_N) \\ u_N|_{t=0} = \mathbf{P}_{\leq N} u_0, \end{cases} \tag{3.2}$$

while the high frequency dynamics with initial data $\mathbf{P}_{>N} u_0$ evolves linearly and hence is globally well-posed. We denote by $\Phi_N(t)$ the flow map of the truncated dynamics (3.1) at time t: $u(0) \in H^\sigma(\mathbb{T}) \to u(t) \in H^\sigma(\mathbb{T})$. We also denote by $\Phi(t)$ the flow map to the renormalized 4NLS (1.5), constructed in [33].

We first record the following uniform (in N) growth bound. This estimate essentially follows from the growth bound on solutions to the renormalized cubic 4NLS (1.5) in negative Sobolev spaces [33].

Lemma 3.1. *Let $\sigma > -\frac{1}{3}$. Given any $R > 0$ and $T > 0$, there exists $C(R,T) > 0$ such that*

$$\Phi_N(t)(B_R) \subset B_{C(R,T)}$$

for any $t \in [0,T]$ and $N \in \mathbb{N} \cup \{\infty\}$, with the understanding that $\Phi_\infty = \Phi$. Here, B_R denotes the ball of radius R in $H^\sigma(\mathbb{T})$ centered at the origin.

Next, we state the approximation property of the truncated dynamics (3.1).

Proposition 3.2. *Let $\sigma > -\frac{1}{3}$. Given $R > 0$, let $A \subset B_R$ be a compact set in $H^\sigma(\mathbb{T})$. Given $t \in \mathbb{R}$, $u_0 \in A$, and small $\delta > 0$, there exists $N_0 = N_0(t, R, u_0, \delta) \in \mathbb{N}$ such that*

$$\Phi(t)(u_0) \in \Phi_N(t)(A + B_\delta)$$

for any $N \geq N_0$.

We present the proof of Proposition 3.2 in Section 4.

Remark 3.3. (i) It is possible to state Proposition 3.2 without referring to a compact set A. In fact, there exists $N_0 = N_0(t, u_0, \delta) \in \mathbb{N}$ such that $\Phi(t)(u_0) \in \Phi_N(t)(u_0 + B_\delta)$ for any $N \geq N_0$. We, however, stated Proposition 3.2 as above so that the statement can be easily compared with the corresponding statement in the L^2-setting; see [30, Proposition B.3/6.21].

(ii) We point out that Proposition 3.2 is weaker than the approximation property of the truncated dynamics in $L^2(\mathbb{T})$, which played a key role in the previous works [30, 27]. Due to the lack of local uniform continuity of the solution map in negative Sobolev spaces, the rate of approximation N_0 depends on the initial data u_0 in Proposition 3.2, while, in $L^2(\mathbb{T})$, N_0 does not depend on $u_0 \in A$; see [30, Proposition B.3/6.21]. In particular, we do not know if we have $\Phi(t)(A) \subset \Phi_N(t)(A + B_\delta)$ for any sufficiently large $N \gg 1$. This is different from the situation considered in [35], thus requiring a careful implementation of the argument. See Subsection 3.5.

We also point out that, in [30], the continuity of the solution map from $L^2(\mathbb{T})$ to the (local-in-time) $X^{0,b}$-space was implicitly used to control the high frequency part $\mathbf{P}_{>N}\Phi(t)(u_0)$ of the solution, uniformly in u_0 belonging to a compact set $A \subset L^2(\mathbb{T})$; see [30, Lemma B.1/6.19]. In negative Sobolev spaces, however, we do not know[6] how to obtain such a uniform control on the high frequency part $\mathbf{P}_{>N}\Phi(t)(u_0)$ for u_0 belonging to a compact set $A \subset H^\sigma(\mathbb{T})$.

3.2. **Energy estimate.** In this subsection, we introduce a modified H^s-energy functional and state the crucial energy estimate in negative Sobolev spaces (Proposition 3.4) whose proof is presented in Section 5.

Let $N \in \mathbb{N} \cup \{\infty\}$. We say that u is a solution to (3.2) if u is a solution to (3.2) when $N \in \mathbb{N}$ and to (1.5) when $N = \infty$. Then, by iteratively applying normal form reductions as in [27], we

[6]Recall that the solutions constructed in [33] belong to the short-time $X^{\sigma,b}$-space, while those constructed in [21] belong to the modified $X^{\sigma,b}$-space which depends on initial data; see (4.10) below. In particular, we do not know if the solution map is continuous from $H^\sigma(\mathbb{T})$ into the standard $X^{\sigma,b}$-space if $\sigma < 0$.

formally[7] obtain the following identity:[8]

$$\frac{d}{dt}\bigg(\frac{1}{2}\|u(t)\|_{H^s}^2\bigg) = \frac{d}{dt}\bigg(\sum_{j=2}^{\infty}\mathcal{N}_{0,N}^{(j)}(u)(t)\bigg) + \sum_{j=2}^{\infty}\mathcal{N}_{1,N}^{(j)}(u)(t) + \sum_{j=2}^{\infty}\mathcal{R}_{N}^{(j)}(u)(t) \tag{3.3}$$

for any (smooth) solution u to the finite-dimensional truncated dynamics (3.2) (i.e. the low frequency part of (3.1)). Here, $\mathcal{N}_{0,N}^{(j)}$ is a $2j$-linear form and $\mathcal{N}_{1,N}^{(j)}$ and $\mathcal{R}_{N}^{(j)}$ are $(2j+2)$-linear forms. This motivates us to define the following modified energy:

$$\mathcal{E}_N(u) := \frac{1}{2}\|u\|_{H^s}^2 - \sum_{j=2}^{\infty}\mathcal{N}_{0,N}^{(j)}(u)(t). \tag{3.4}$$

When $N=\infty$, we simply denote $\mathcal{E}_\infty(u)$ by $\mathcal{E}(u)$ and also drop the subscript $N=\infty$ from the multilinear forms; for example, we write $\mathcal{N}_{0}^{(j)}$ for $\mathcal{N}_{0,\infty}^{(j)}$.

We now state the energy estimate.

Proposition 3.4 (energy estimate)**.** *Let $\frac{3}{10} < s \le \frac{1}{2}$ and $\sigma = s - \frac{1}{2} - \varepsilon$ for some small $\varepsilon > 0$. Then, given any $R > 0$ and $T > 0$, the following energy estimate holds uniformly in $N \in \mathbb{N}\cup\{\infty\}$:*

$$\sup_{t\in[0,T]}\bigg|\frac{d}{dt}\mathcal{E}_N(u)(t)\bigg| \le C_s(R)$$

for any solution $u \in C(\mathbb{R}; H^\sigma(\mathbb{T}))$ to (3.2), satisfying the growth bound:

$$\sup_{t\in[0,T]}\|u(t)\|_{H^\sigma} \le R. \tag{3.5}$$

We also record the following bound on the correction terms. Set

$$\mathfrak{S}_N(u) := \sum_{j=2}^{\infty}\mathcal{N}_{0,N}^{(j)}(\mathbf{P}_{\le N}u) \tag{3.6}$$

for $N \in \mathbb{N}\cup\{\infty\}$.

Lemma 3.5. *Let $\frac{1}{6} < s \le \frac{1}{2}$. Then, given any $R > 0$, there exists $C_s = C_s(R) > 0$ such that*

$$|\mathfrak{S}_N(u)| \le C_s(R) \tag{3.7}$$

for any $u \in B_R \subset H^\sigma(\mathbb{T})$ and $N \in \mathbb{N}\cup\{\infty\}$. Furthermore, $\mathfrak{S}_N(u)$ converges to $\mathfrak{S}_\infty(u)$ as $N\to\infty$ for each $u \in B_R$.

In Section 5, we present the proofs of Proposition 3.4 and Lemma 3.5. The main tool is an infinite iteration of normal form reductions from [27], where such an argument was implemented in $L^2(\mathbb{T})$. For our problem, however, we need to prove boundedness of each multilinear term by a product of the H^σ-norm of u with $\sigma = s - \frac{1}{2} - \varepsilon < 0$. For this purpose, we adapt the argument from [33], where an infinite iteration of normal form reductions was implemented in negative Sobolev spaces. Indeed, the only essential difference between our argument and that in [33] is the presence of the weight $\langle n\rangle^{2s}$, coming from the H^s-norm squared on the left-hand side of (3.3).

[7] For each finite $N\in\mathbb{N}$, any solution to (3.2) is smooth and thus the computation leading to (3.3) does not require any justification. See Section 5.

[8] Hereafter, we use the following shorthand notation for multilinear form: $\mathcal{N}_{0,N}^{(j)}(u) = \mathcal{N}_{0,N}^{(j)}(u,\dots,u)$, etc.

3.3. **Weighted Gaussian measures.** As in [30, 27], we prove quasi-invariance of the Gaussian measure μ_s indirectly by first establishing quasi-invariance of weighted Gaussian measures associated with the modified energies $\mathcal{E}(u)$ and $\mathcal{E}_N(u)$ in (3.4). In [30, 27], the weighted Gaussian measures were normalized to be probability measures thanks to the conserved L^2-cutoff. Due to the unavailability of a cutoff based on a conservation law in negative regularity, we do not normalize our weighted Gaussian measures (which is precisely the setting for the approach in [35]). See Subsection 3.5.

We define the following measures:

$$d\rho_s(u) = F_s(u)d\mu_s(u) \qquad \text{and} \qquad d\rho_{s,N}(u) = F_{s,N}(u)d\mu_s(u), \tag{3.8}$$

where $F_s(u)$ and $F_{s,N}(u)$ are given by

$$F_s(u) := \exp\bigg(-\mathcal{E}(u) + \frac{1}{2}\|u\|_{H^s}^2\bigg) = \exp\bigg(\sum_{j=2}^{\infty}\mathcal{N}_0^{(j)}(u)\bigg), \tag{3.9}$$

$$F_{s,N}(u) := \exp\bigg(-\mathcal{E}_N(\mathbf{P}_{\le N}u) + \frac{1}{2}\|\mathbf{P}_{\le N}u\|_{H^s}^2\bigg) = \exp\bigg(\sum_{j=2}^{\infty}\mathcal{N}_{0,N}^{(j)}(\mathbf{P}_{\le N}u)\bigg). \tag{3.10}$$

We also write $\rho_{s,\infty} = \rho_s$ and $F_{s,\infty}(u) = F_s(u)$.

Note that the quasi-invariance property is a local property in the sense we only need to work on compact sets in $H^\sigma(\mathbb{T})$. Thus, in proving quasi-invariance of ρ_s and $\rho_{s,N}$, we only require $F_{s,N} \in L^1_{\text{loc}}(\mu_s)$, uniformly in $N \in \mathbb{N}\cup\{\infty\}$, (i.e. $F_{s,N}$ is locally integrable with a uniform (in N) bound on each compact set) and $F_{s,N} \to F_s$ in $L^1_{\text{loc}}(\mu_s)$.

Proposition 3.6. *Let $\frac{1}{6} < s \le \frac{1}{2}$ and $\sigma = s - \frac{1}{2} - \varepsilon$ for some small $\varepsilon > 0$. Given any $R > 0$, there exists $C = C(s,R) > 0$ such that*

$$\rho_{s,N}(B_R) = \int_{B_R} F_{s,N}(u)d\mu_s(u) \le C(s,R) \tag{3.11}$$

for any $N \in \mathbb{N}\cup\{\infty\}$. Namely, $F_{s,N} \in L^1_{\text{loc}}(\mu_s)$, uniformly in $N \in \mathbb{N}\cup\{\infty\}$. Moreover, we have

$$\lim_{N\to\infty}\int_{B_R}|F_{s,N}(u) - F_s(u)|d\mu_s(u) = 0. \tag{3.12}$$

Proof. The bound (3.11) follows from (3.7) in Lemma 3.5. Furthermore, it follows from Lemma 3.5 that $\mathfrak{S}_N(u)$ converges to $\mathfrak{S}_\infty(u)$ as $N \to \infty$ for each $u \in B_R$. Then, from (3.6), (3.9), and (3.10), we see that $F_{s,N}$ converges to F_s almost surely with respect to μ_s. Together with the bound (3.7) in Lemma 3.5, the bounded convergence theorem yields (3.12). □

3.4. **A change-of-variable formula.** Next, we go over a global aspect of the proof of Theorem 1.1. From (2.2) and (2.3), we can write $\rho_{s,N}$ in (3.8) as

$$d\rho_{s,N} = Z_{s,N}^{-1}\exp\big(-\mathcal{E}_N(\mathbf{P}_{\le N}u)\big)\,du_N \otimes d\mu_{s,N}^{\perp}, \tag{3.13}$$

where du_N is as in (2.4) (= the Lebesgue measure on $E_N \cong \mathbb{C}^{2N+1}$) and $Z_{s,N}^{-1}$ is the normalizing constant for $\mu_{s,N}$. Proceeding as in [30] with (3.13), we have the following change-of-variable formula.

Lemma 3.7. *Let $\frac{1}{6} < s \le \frac{1}{2}$ and $\sigma = s - \frac{1}{2} - \varepsilon$ for some small $\varepsilon > 0$. Then, we have*

$$\begin{aligned}\rho_{s,N}(\Phi_N(t)(A)) &= \int_{\Phi_N(t)(A)} e^{\sum_{j=2}^\infty \mathcal{N}_{0,N}^{(j)}(\mathbf{P}_{\le N} u)} d\mu_s(u)\\ &= Z_{s,N}^{-1} \int_A e^{-\mathcal{E}_N(\mathbf{P}_{\le N}\Phi_N(t)(u))} du_N \otimes d\mu_{s,N}^\perp\end{aligned}$$

for any $t \in \mathbb{R}$, $N \in \mathbb{N}$, and any measurable set $A \subset H^\sigma(\mathbb{T})$.

3.5. **Proof of Theorem 1.1.** We are now ready to present the proof of Theorem 1.1. We follow the argument in [35] but due to the weaker approximation property of the truncated dynamics in negative Sobolev spaces, more care is needed to close the argument. Fix $\frac{3}{10} < s \le \frac{1}{2}$ and set $\sigma = s - \frac{1}{2} - \varepsilon$ for some small $\varepsilon > 0$.

In the following, we only consider the positive times. Fix $t > 0$. Then, by the inner regularity of the measure μ_s, it suffices to show that

$$A \subset H^\sigma(\mathbb{T}) \text{ compact and } \mu_s(A) = 0 \qquad \Longrightarrow \qquad \mu_s(\Phi(t)(A)) = 0.$$

Fix a compact set $A \subset H^\sigma(\mathbb{T})$ such that $\mu_s(A) = 0$. From Lemma 3.5 with Lemma 3.1, we have

$$0 < \exp\bigg(\sum_{j=2}^\infty \mathcal{N}_0^{(j)}(u)\bigg) < \infty \tag{3.14}$$

for all $u \in A \cup \Phi(t)(A)$. Then, it follows from (3.8) and (3.9) with $\mu_s(A) = 0$ that $\rho_s(A) = 0$. Our goal is to prove

$$\rho_s(\Phi(t)(A)) = 0. \tag{3.15}$$

Once we prove (3.15), we then conclude from (3.14) that $\mu_s(\Phi(t)(A)) = 0$.

Since A is compact, we have $A \subset B_R \subset H^\sigma(\mathbb{T})$ for some $R > 0$. By Lemma 3.1, there exists $C(R) > 0$ such that

$$\Phi_N(\tau)(B_{2R}) \subset B_{C(R)} \tag{3.16}$$

for any $\tau \in [0, t]$ and $N \in \mathbb{N} \cup \{\infty\}$.

Fix a measurable set $D \subset B_{2R}$. Then, from (3.13) and Lemma 3.7, we have

$$\begin{aligned}\bigg|\frac{d}{d\tau}\rho_{s,N}(\Phi_N(\tau)(D))\bigg| &= \bigg|\frac{d}{d\tau} Z_{s,N}^{-1}\int_{\Phi_N(\tau)(D)} \exp\big(-\mathcal{E}_N(\mathbf{P}_{\le N}u)\big)\, du_N \otimes d\mu_{s,N}^\perp\bigg|\\ &= \bigg|Z_{s,N}^{-1}\int_D \frac{d}{d\tau}\exp\big(-\mathcal{E}_N(\mathbf{P}_{\le N}\Phi_N(\tau)(u))\big)\, du_N \otimes d\mu_{s,N}^\perp\bigg|.\end{aligned} \tag{3.17}$$

From Proposition 3.4 with (3.16), we also have

$$\bigg|\frac{d}{d\tau}\exp\big(-\mathcal{E}_N(\mathbf{P}_{\le N}\Phi_N(\tau)(u))\big)\bigg| \le C'(R)\exp\big(-\mathcal{E}_N(\mathbf{P}_{\le N}\Phi_N(\tau)(u))\big) \tag{3.18}$$

for any $\tau \in [0,t]$ and $u \in D$. Hence, from (3.17), (3.18), and Lemma 3.7 with (3.8) and (3.13), we have

$$\begin{aligned}\left|\frac{d}{d\tau}\rho_{s,N}(\Phi_N(\tau)(D))\right| &\le Z_{s,N}^{-1}\, C'(R)\int_D \exp\big(-\mathcal{E}_N(\mathbf{P}_{\le N}\Phi_N(\tau)(u))\big)\, du_N \otimes d\mu_{s,N}^{\perp}\\ &= Z_{s,N}^{-1}\, C'(R)\int_{\Phi_N(\tau)(D)} \exp\big(-\mathcal{E}_N(\mathbf{P}_{\le N}u)\big)\, du_N \otimes d\mu_{s,N}^{\perp}\\ &= C'(R)\int_{\Phi_N(\tau)(D)} F_{s,N}(u)d\mu_s\\ &= C'(R)\,\rho_{s,N}(\Phi_N(\tau)(D))\end{aligned}$$

for any $\tau \in [0,t]$. Then, by Gronwall's inequality, we obtain

$$\rho_{s,N}(\Phi_N(\tau)(D)) = \int_{\Phi_N(\tau)(D)} F_{s,N}(u)d\mu_s \le \exp(C'(R)\tau)\rho_{s,N}(D) \tag{3.19}$$

for any $\tau \in [0,t]$ and $N \in \mathbb{N}$. Note that the estimate (3.19) allows us to conclude quasi-invariance of $\rho_{s,N}$ (and μ_s) under the truncated dynamics $\Phi_N(t)$.

Next, by a limiting argument, we prove quasi-invariance of ρ_s under $\Phi(t)$. From Proposition 3.6, we have

$$\lim_{N\to\infty}\int_{B_{C(R)}} |F_{s,N}(u) - F_s(u)|d\mu_s(u) = 0, \tag{3.20}$$

where $C(R)$ is as in (3.16). Thus, given small $\delta > 0$, we have

$$\begin{aligned}\rho_s(\Phi(t)(A)) &= \int_{\Phi(t)(A)} F_s(u)d\mu_s\\ &= \int\limits_{\Phi(t)(A)\cap\Phi_N(t)(A+B_\delta)} F_s(u)d\mu_s + \int\limits_{\Phi(t)(A)\setminus\Phi_N(t)(A+B_\delta)} F_s(u)d\mu_s\\ &\le \int\limits_{\Phi_N(t)(A+B_\delta)} F_s(u)d\mu_s + \int\limits_{\Phi(t)(A)\setminus\Phi_N(t)(A+B_\delta)} F_s(u)d\mu_s\\ &\le \int\limits_{\Phi_N(t)(A+B_\delta)} F_{s,N}(u)d\mu_s + \int\limits_{\Phi(t)(A)\setminus\Phi_N(t)(A+B_\delta)} F_s(u)d\mu_s + \delta\end{aligned} \tag{3.21}$$

for any sufficiently large $N \gg 1$. Then, by applying (3.19) (with $D = A + B_\delta$ for $\delta < R$) to (3.21) and then applying (3.20) again, we have

$$\begin{aligned}\rho_s(\Phi(t)(A)) &\le \exp(C'(R)t)\int\limits_{A+B_\delta} F_{s,N}(u)d\mu_s + \int\limits_{\Phi(t)(A)\setminus\Phi_N(t)(A+B_\delta)} F_s(u)d\mu_s + \delta\\ &\le \exp(C'(R)t)\int\limits_{A+B_\delta} F_s(u)d\mu_s + \int\limits_{\Phi(t)(A)\setminus\Phi_N(t)(A+B_\delta)} F_s(u)d\mu_s + 2\delta.\end{aligned} \tag{3.22}$$

By Proposition 3.6 and the Lebesgue dominated convergence theorem, we have

$$\lim_{\delta\to 0}\int_{A+B_\delta} F_s(u)d\mu_s = \int_A F_s(u)d\mu_s = \rho_s(A) = 0. \tag{3.23}$$

Next, we consider the second term on the right-hand side of (3.22). Let $A_N := \Phi(t)(A) \setminus \Phi_N(t)(A + B_\delta)$. Then, it follows from Proposition 3.2 that

$$\limsup A_N = \bigcap_{k=1}^{\infty} \bigcup_{N=k}^{\infty} A_N = \varnothing. \tag{3.24}$$

Indeed, if (3.24) did not hold, then there would be at least one element $u \in \limsup A_N$, namely, $u \in A_N$ for infinitely many N. This is clearly a contradiction to Proposition 3.2 since, given any such u (which in particular belongs to $\Phi(t)(A)$), we have

$$u \in \Phi_N(t)(A + B_\delta) \subset A_N^c$$

for all $N \geq N_0(t, R, u, \delta)$. This implies that $\lim_{N\to\infty} \mathbf{1}_{A_N}(u) = 0$ for any $u \in \Phi(t)(A)$ (and thus for any $u \in H^\sigma(\mathbb{T})$). Hence, by Lemma 3.5 and the Lebesgue dominated convergence theorem, we have

$$\lim_{N\to\infty} \int_{A_N} F_s(u) d\mu_s = 0. \tag{3.25}$$

Finally, putting (3.22), (3.23), and (3.25) together and taking $\delta \to 0$, we conclude (3.15). This completes the proof of Theorem 1.1.

4. On the approximation property of the truncated dynamics

In this section, we study the approximation property of the truncated dynamics (3.1) and present the proof of Proposition 3.2.

4.1. **Gauged 4NLS.** We first go over the basic reduction of the problem. Fix $\sigma > -\frac{1}{3}$. Let $u \in C(\mathbb{R}; H^\sigma(\mathbb{T}))$ be the global solution to the renormalized 4NLS (1.5) with $u|_{t=0} = u_0$. The main obstruction in carrying out analysis in negative Sobolev spaces is the resonant part of the nonlinearity. In order to weaken the resonant interaction, we introduce the following gauge transform $\mathcal{J} = \mathcal{J}_{u_0}$ as in [32, 23]:

$$\mathcal{J}(u)(x,t) = \mathcal{J}_{u_0}(u)(x,t) = \sum_{n\in\mathbb{Z}} e^{inx - it|\widehat{u}_0(n)|^2} \widehat{u}(n,t). \tag{4.1}$$

This gauge transform is clearly invertible and leaves the H^s-norm invariant. A direct computation shows that the gauged function $v = \mathcal{J}(u)$ satisfies the following gauged 4NLS:

$$\begin{cases} i\partial_t v = \partial_x^4 v + \mathcal{N}_1(v) + \mathcal{N}_2(v), \\ v|_{t=0} = u_0. \end{cases} \tag{4.2}$$

Here, the first nonlinearity $\mathcal{N}_1(v)$ is defined by

$$\mathcal{N}_1(v)(x,t) := \sum_{n\in\mathbb{Z}} e^{inx} \sum_{\Gamma(n)} e^{it\Theta(\bar{n})} \widehat{v}(n_1,t)\overline{\widehat{v}(n_2,t)}\widehat{v}(n_3,t), \tag{4.3}$$

where $\Gamma(n)$ is as in (2.8) and the phase function $\Theta(\bar{n}) = \Theta_{u_0}(\bar{n})$ is given by

$$\Theta(\bar{n}) := \Theta(n_1,n_2,n_3,n) = |\widehat{u}_0(n_1)|^2 - |\widehat{u}_0(n_2)|^2 + |\widehat{u}_0(n_3)|^2 - |\widehat{u}_0(n)|^2. \tag{4.4}$$

The second nonlinearity $\mathcal{N}_2(v)$ is defined by

$$\mathcal{N}_2(v)(x,t) := -\sum_{n\in\mathbb{Z}} e^{inx} \Big(|\widehat{v}(n,t)|^2 - |\widehat{u}_0(n)|^2 \Big) \widehat{v}(n,t). \tag{4.5}$$

In the following, we often view $\mathcal{N}_1$ as a trilinear operator and, with a slight abuse of notations, we write $\mathcal{N}_1(v_1, v_2, v_3)$ to denote the right-hand side of (4.3), where we replace the jth occurrence of

v by v_j, $j = 1, 2, 3$. Given a trilinear operator $\mathcal{M}(v_1, v_2, v_3)$, we write $\mathcal{M}(v)$ to mean $\mathcal{M}(v, v, v)$. We apply this convention in the following.

Next, we apply the gauge transform $\mathcal{J}$ in (4.1) to the truncated dynamics (3.1). Let $u_N \in C(\mathbb{R}; H^\sigma(\mathbb{T}))$ be the global solution to the truncated equation (3.1) with the same initial data $u_N|_{t=0} = u_0$. Then, the gauged function $v_N = \mathcal{J}(u_N)$ satisfies the following gauged truncated 4NLS:

$$\begin{cases} i\partial_t v_N = \partial_x^4 v_N + \mathcal{N}_1^N(v_N) + \mathcal{N}_2^N(v_N) \\ v_N|_{t=0} = u_0, \end{cases} \tag{4.6}$$

where $\mathcal{N}_1^N(v_N)$ and $\mathcal{N}_2^N(v_N)$ are given by

$$\begin{aligned} \mathcal{N}_1^N(v_N)(x,t) &:= \mathbf{P}_{\le N}\mathcal{N}_1(\mathbf{P}_{\le N} v)(x,t) \\ &= \sum_{|n|\le N} e^{inx} \sum_{\Gamma_N(n)} e^{it\Theta(\overline{n})} \widehat{v}_N(n_1,t)\overline{\widehat{v}_N(n_2,t)}\widehat{v}_N(n_3,t), \\ \mathcal{N}_2^N(v_N)(x,t) &:= -\sum_{|n|\le N} e^{inx}\Big(|\widehat{v}_N(n,t)|^2 - |\widehat{u}_0(n)|^2\Big)\widehat{v}_N(n,t) \\ &\quad\; + \sum_{|n|>N} e^{inx}|\widehat{u}_0(n)|^2\widehat{v}_N(n,t). \end{aligned} \tag{4.7}$$

Note that the high frequency part of the solution to the gauged truncated 4NLS (4.6) is given by

$$\mathbf{P}_{>N} v_N(t) = S_{u_0}(t)\mathbf{P}_{>N} u_0,$$

where $S_{u_0}(t)$ is the modified linear propagator defined by

$$S_{u_0}(t) f := \sum_{n\in\mathbb{Z}} e^{-it(n^4 + |\widehat{u}_0(n)|^2)} \widehat{f}(n) e^{inx}. \tag{4.8}$$

When $N = \infty$, the equation (4.6) formally reduces to (4.2) and thus we use the notations v_∞, $\mathcal{N}_1^\infty(v)$, and $\mathcal{N}_2^\infty(v)$ for v, $\mathcal{N}_1(v)$, and $\mathcal{N}_2(v)$ in the following.

4.2. **Function spaces and nonlinear estimates.** We recall the definition of the basic function spaces and the key estimates in proving local well-posedness of the renormalized 4NLS (1.5) in negative Sobolev spaces.

We first recall the Fourier restriction norm method introduced by Bourgain [3]. Given $s, b \in \mathbb{R}$, we define the $X^{s,b}$-space as the completion of $\mathcal{S}(\mathbb{T}\times\mathbb{R})$ under the following norm:

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau + n^4\rangle^b \widehat{u}(n,\tau)\|_{\ell^2_n L^2_\tau}.$$

Given a time interval $I \subset \mathbb{R}$, we define the local-in-time version $X^{s,b}(I)$ by setting

$$\|u\|_{X^{s,b}(I)} = \inf\big\{\|\widetilde{u}\|_{X^{s,b}} : \widetilde{u}|_I = u\big\}.$$

When $I = [0, T]$, we also set $X^{s,b}_T = X^{s,b}(I)$. We use the same notation for the time restriction of other function spaces. Recall that

$$\|u\|_{C_T H^s_x} \lesssim \|u\|_{X^{s,b}_T} \tag{4.9}$$

for $b > \frac{1}{2}$. Using the $X^{s,b}$-space, local well-posedness in $L^2(\mathbb{T})$ of 4NLS (1.1) (and the renormalized 4NLS (1.5)) follows from the L^4-Strichartz estimate and a contraction argument. See [30].

Due to the lack of local uniform continuity of the solution map, one can not use a contraction argument to prove local well-posedness of the renormalized 4NLS (1.5) in negative Sobolev spaces. In [33], the short-time Fourier restriction norm method and the normal form approach were used to overcome this issue. Following the previous works [38, 26, 24] on the modified KdV equation and the third order NLS, Kwak [21] used the modified $X^{s,b}$-space, defined by the norm:

$$\|u\|_{Y^{s,b}_{u_0}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau + n^4 - |\widehat{u}_0(n)|^2\rangle^b \widehat{u}(n,\tau)\|_{\ell^2_n L^2_\tau} \tag{4.10}$$

for $u|_{t=0} = u_0$ and proved local well-posedness of (1.5) by a compactness argument. In [23], Li, Zine, and the second author proved local well-posedness of the fractional NLS (1.6) (for $\alpha > 2$) below $L^2(\mathbb{T})$ by studying the gauged formulation (as in (4.2)). We point out that

$$\|\mathcal{J}_{u_0}(u)\|_{X^{s,b}} = \|u\|_{Y^{s,b}_{u_0}} \tag{4.11}$$

and thus studying the renormalized 4NLS (1.5) in the $Y^{s,b}_{u_0}$-spaces is equivalent to studying the gauged renormalized 4NLS (4.2) in the standard $X^{s,b}$-spaces.

Let $\Psi(t)$ and $\Psi_N(t)$ be the solution maps to (4.2) and (4.6), respectively, with the understanding that $\Psi_\infty(t) = \Psi(t)$. Then, as a consequence of the aforementioned well-posedness results, we have the following uniform growth bound for $\sigma > -\frac{1}{3}$; given any $R > 0$ and $T > 0$, there exists $C_0(T,R) > 0$ such that

$$\sup_{N\in\mathbb{N}\cup\{\infty\}} \sup_{u_0\in B_R} \|\Psi_N(t)(u_0)\|_{X_T^{\sigma,\frac{1}{2}+\varepsilon}} \le C_0(T,R) \tag{4.12}$$

for some small $\varepsilon > 0$, where $B_R \subset H^\sigma(\mathbb{T})$ denotes the ball of radius R centered at the origin. For small $T = T(R) > 0$, the bound (4.12) follows from the uniform (in N) local well-posedness result in [21, 23].[9] For large $T > 0$, the bound (4.12) follows from the same bound over short time intervals together with the global-in-time control and the strong uniqueness statement in [33] (which guarantees that the solutions constructed in [33, 21, 23] all agree) and the subadditivity of the local-in-time $X^{\sigma,\frac{1}{2}+\varepsilon}$-norms over disjoint time intervals as in Lemma A.4 in [2].

Next, we recall the linear estimates. See [3, 14].

Lemma 4.1. *Let $s \in \mathbb{R}$ and $0 < T \le 1$.*

(i) *For any $b \in \mathbb{R}$, we have*

$$\|S(t)u_0\|_{X_T^{s,b}} \le C_b \|u_0\|_{H^s}.$$

(ii) *Let $-\frac{1}{2} < b' \le 0 \le b \le b'+1$. Then, we have*

$$\bigg\| \int_0^t S(t-t')F(t')dt' \bigg\|_{X_T^{s,b}} \le C_{b,b'} T^{1-b+b'} \|F\|_{X_T^{s,b'}}.$$

We now state the nonlinear estimates, which essentially follow from [21, 23]. In the remaining part of this section, we fix small $\varepsilon > 0$.

Lemma 4.2. *Let $-\frac{1}{2} < \sigma < 0$ and $T > 0$. Then, we have*

$$\|\mathcal{N}_1^N(v_1,v_2,v_3)\|_{X_T^{\sigma,-\frac{1}{2}+2\varepsilon}} \lesssim \prod_{j=1}^3 \|v_j\|_{X_T^{\sigma,\frac{1}{2}+\varepsilon}}, \tag{4.13}$$

uniformly in $N \in \mathbb{N}\cup\{\infty\}$.

[9]In [21, 23], only the untruncated equation (1.5) was considered. In view of the uniform (in N) boundedness of $\mathbf{P}_{\le N}$ on the relevant function spaces, the local well-posedness argument in [21, 23] also applies to the truncated equation (3.1), uniformly in $N \in \mathbb{N}$.

Proof. This is a direct consequence of Proposition 3.1 in [21]. Indeed, in terms of our notations, Proposition 3.1 in [21] establishes the following trilinear bound:

$$\|\mathsf{N}_1(u_1,u_2,u_3)\|_{Y^{\sigma,-\frac{1}{2}+2\varepsilon}_{u_0,T}} \lesssim \prod_{j=1}^{3}\|u_j\|_{Y^{\sigma,\frac{1}{2}}_{u_0,T}} \tag{4.14}$$

for $-\frac{1}{2}<\sigma<0$ and $0<T\le 1$, where N_1 is defined by

$$\mathsf{N}_1(u_1,u_2,u_3)(x,t) := \sum_{n\in\mathbb{Z}} e^{inx}\sum_{\Gamma(n)}\widehat{u}_1(n_1,t)\overline{\widehat{u}_2(n_2,t)}\widehat{u}_3(n_3,t). \tag{4.15}$$

We first note that the restriction $T\le 1$ in (4.14) does not play any role in the proof presented in [21] and thus we can drop the restriction $T\le 1$. A similar comment applies to the lemmas below.

From (4.7) and (4.15) with (4.1) and (4.4), we have

$$\mathcal{N}_1^N(v_1,v_2,v_3) = \mathbf{P}_{\le N}\mathcal{J}\big(\mathsf{N}_1(\mathbf{P}_{\le N}u_1,\mathbf{P}_{\le N}u_2,\mathbf{P}_{\le N}u_3)\big), \tag{4.16}$$

where $u_j=\mathcal{J}^{-1}(v_j)$. Then, from (4.16), (4.11), and (4.14) together with the uniform (in N) boundedness of $\mathbf{P}_{\le N}$ on the $X^{s,b}_T$- and $Y^{s,b}_{u_0,T}$-spaces, we have

$$\begin{aligned}\|\mathcal{N}_1^N(v_1,v_2,v_3)\|_{X_T^{\sigma,-\frac{1}{2}+2\varepsilon}} &= \|\mathbf{P}_{\le N}\mathsf{N}_1(\mathbf{P}_{\le N}u_1,\mathbf{P}_{\le N}u_2,\mathbf{P}_{\le N}u_3)\|_{Y^{\sigma,-\frac{1}{2}+2\varepsilon}_{u_0,T}}\\ &\lesssim \prod_{j=1}^{3}\|\mathbf{P}_{\le N}u_j\|_{Y^{\sigma,\frac{1}{2}}_{u_0,T}} \le \prod_{j=1}^{3}\|v_j\|_{X_T^{\sigma,\frac{1}{2}+\varepsilon}},\end{aligned}$$

where, in the last step, we used the monotonicity of the $X^{s,b}$-norm in the parameter b. This yields (4.13). $\square$

Lemma 4.3. *Let $-\frac{1}{3}<\sigma<0$ and $T>0$. Given $N\in\mathbb{N}\cup\{\infty\}$, let v_N be the smooth solution to (4.6) with $v_N|_{t=0}=u_0\in C^\infty(\mathbb{T})$. Then, we have*

$$\begin{aligned}\sup_{|n|\le N}\bigg|\operatorname{Im}\bigg(\int_0^T\sum_{\Gamma_N(n)} e^{it\Theta(\bar n)}\widehat{v}_N(n_1,t)\overline{\widehat{v}_N(n_2,t)}\widehat{v}_N(n_3,t)\overline{\widehat{v}_N(n,t)}dt\bigg)\bigg|\\ \lesssim \|v_N\|^4_{X_T^{\sigma,\frac{1}{2}+\varepsilon}} + \|v_N\|^6_{X_T^{\sigma,\frac{1}{2}+\varepsilon}} + \|v_N\|^8_{X_T^{\sigma,\frac{1}{2}+\varepsilon}},\end{aligned} \tag{4.17}$$

where $\Gamma_N(n)$ is as in (2.9) and $\Gamma_N(n)=\Gamma(n)$ when $N=\infty$. Here, the implicit constant in (4.17) is independent of $N\in\mathbb{N}$.

Proof. • **Case 1:** $N=\infty$. We first recall Proposition 3.4 in [21]; given $-\frac{1}{3}\le\sigma<0$ and $0<T\le 1$, we have

$$\begin{aligned}\sup_{n\in\mathbb{Z}}\bigg|\operatorname{Im}\bigg(\int_0^T\sum_{\Gamma(n)}\widehat{u}(n_1,t)\overline{\widehat{u}(n_2,t)}\widehat{u}(n_3,t)\overline{\widehat{u}(n,t)}dt\bigg)\bigg|\\ \lesssim \|u_0\|^4_{H^\sigma} + \Big(\|u_0\|^2_{H^\sigma}+\|u\|^4_{Y^{\sigma,\frac{1}{2}}_{u_0,T}}\Big)^2 + \|u\|^4_{Y^{\sigma,\frac{1}{2}}_{u_0,T}} + \|u\|^6_{Y^{\sigma,\frac{1}{2}}_{u_0,T}}\end{aligned} \tag{4.18}$$

for any smooth solution u to (1.5) with $u|_{t=0}=u_0\in C^\infty(\mathbb{T})$. As mentioned in the proof of Lemma 4.2, we can drop the restriction $T\le 1$ and the estimate (4.18) indeed holds for any $T>0$ (at least for *smooth* solutions; see Remark 4.5 below).

Given $u_0 \in C^\infty(\mathbb{T})$, let v be the solution to (4.2) with $v|_{t=0} = u_0$. Then, $u = \mathcal{J}^{-1}(v)$ satisfies (1.5) with $u|_{t=0} = u_0$. Hence, from (4.18) with (4.1), (4.4), and (4.11), we obtain

$$\begin{aligned}
&\sup_{n\in\mathbb{Z}}\bigg|\operatorname{Im}\bigg(\int_0^T \sum_{\Gamma(n)} e^{it\Theta(\bar n)}\widehat{v}(n_1,t)\overline{\widehat{v}(n_2,t)}\widehat{v}(n_3,t)\overline{\widehat{v}(n,t)}dt\bigg)\bigg| \\
&\quad\lesssim \|u_0\|_{H^\sigma}^4 + \Big(\|u_0\|_{H^\sigma}^2 + \|v\|_{X_T^{\sigma,\frac12}}^4\Big)^2 + \|v\|_{X_T^{\sigma,\frac12}}^4 + \|v\|_{X_T^{\sigma,\frac12}}^6 \\
&\quad\lesssim \|v\|_{X_T^{\sigma,\frac12+\varepsilon}}^4 + \|v\|_{X_T^{\sigma,\frac12+\varepsilon}}^6 + \|v\|_{X_T^{\sigma,\frac12+\varepsilon}}^8,
\end{aligned}$$

where, by relaxing the temporal regularity from $b = \frac12$ to $b = \frac12 + \varepsilon$, we used (4.9) in the last step. This proves (4.17) for $N = \infty$.

• **Case 2:** $N < \infty$. As in the case $N = \infty$, we establish (4.17) for $N < \infty$ by reducing the estimate to an analogue of (4.18). For this purpose, we first recall the proof of Proposition 3.4 in [21] (namely, the estimate (4.18)). First, we divide the domain $\Gamma(n)$ into a good region $\Gamma^{\text{good}}(n)$ and a bad region $\Gamma^{\text{bad}}(n)$.[10] Then, the good part, i.e. the contribution to (4.18) from $\Gamma^{\text{good}}(n)$, is treated by establishing 4-linear estimates (Cases II and III in the proof of [21, Proposition 3.4]), yielding the third term on the right-hand side of (4.18). In handling the bad part, i.e. the contribution to (4.18) from $\Gamma^{\text{bad}}(n)$ (corresponding to Case I in the proof of [21, Proposition 3.4]), we first apply integration by parts in time (as in [38, 26, 24]) and write

$$\begin{aligned}
&\int_0^T \sum_{\Gamma^{\text{bad}}(n)} \widehat{u}(n_1,t)\overline{\widehat{u}(n_2,t)}\widehat{u}(n_3,t)\overline{\widehat{u}(n,t)}dt \\
&\quad= \int_0^T \sum_{\Gamma^{\text{bad}}(n)} e^{-i\phi(\bar n)t}\widehat{w}(n_1,t)\overline{\widehat{w}(n_2,t)}\widehat{w}(n_3,t)\overline{\widehat{w}(n,t)}dt \\
&\quad= \sum_{\Gamma^{\text{bad}}(n)} \frac{e^{-i\phi(\bar n)t}}{-i\phi(\bar n)}\widehat{w}(n_1,t)\overline{\widehat{w}(n_2,t)}\widehat{w}(n_3,t)\overline{\widehat{w}(n,t)}\bigg|_{t=0}^T \\
&\qquad+ \int_0^T \sum_{\Gamma^{\text{bad}}(n)} \frac{e^{-i\phi(\bar n)t}}{i\phi(\bar n)}\partial_t\Big(\widehat{w}(n_1,t)\overline{\widehat{w}(n_2,t)}\widehat{w}(n_3,t)\overline{\widehat{w}(n,t)}\Big)dt \\
&\quad=: \mathrm{I}_n + \mathrm{II}_n,
\end{aligned} \tag{4.19}$$

where $\phi(\bar n)$ is as in (2.6) and $w(t) = S(-t)u(t)$ denotes the interaction representation of u. As for I_n, a simple 4-linear estimate yields

$$\sup_{n\in\mathbb{Z}}|\mathrm{I}_n| \lesssim \|u_0\|_{H^\sigma}^4 + \|u(T)\|_{H^\sigma}^4.$$

Combining this with the following bound (see Corollary 3.3 in [21]):

$$\|u(T)\|_{H^\sigma}^2 \lesssim \|u_0\|_{H^\sigma}^2 + \|u\|_{Y_{u_0,T}^{\sigma,\frac12}}^4,$$

we obtain

$$\sup_{n\in\mathbb{Z}}|\mathrm{I}_n| \lesssim \|u_0\|_{H^\sigma}^4 + \Big(\|u_0\|_{H^\sigma}^2 + \|u\|_{Y_{u_0,T}^{\sigma,\frac12}}^4\Big)^2,$$

yielding the first two terms on the right-hand side of (4.18).

[10] The precise definitions of $\Gamma^{\text{good}}(n)$ and a bad region $\Gamma^{\text{bad}}(n)$ are not important for our purpose.

As for II_n, recalling that u satisfies (1.5), we see that $w(t) = S(-t)u(t)$ satisfies

$$i\partial_t w = S(-t)\mathcal{N}(S(t)u). \tag{4.20}$$

See also (5.1) below. By applying the product rule in taking a time derivative in (4.19) and substituting (4.20), we express II_n as a sum of 6-linear terms, each of which can be bounded by establishing 6-linear estimates. This yields the fourth term on the right-hand side of (4.18).

In establishing (4.17) for $N < \infty$, we repeat the argument in Case 1 and first reduce the proof of (4.17) to establishing the following analogue of (4.18):

$$\begin{aligned} &\sup_{|n|\le N}\bigg|\operatorname{Im}\bigg(\int_0^T\sum_{\Gamma_N(n)}\widehat{u}_N(n_1,t)\overline{\widehat{u}_N(n_2,t)}\widehat{u}_N(n_3,t)\overline{\widehat{u}_N(n,t)}dt\bigg)\bigg| \\ &\qquad \lesssim \|u_0\|_{H^\sigma}^4 + \Big(\|u_0\|_{H^\sigma}^2 + \|u_N\|^4_{Y^{\sigma,\frac12}_{u_0,T}}\Big)^2 + \|u_N\|^4_{Y^{\sigma,\frac12}_{u_0,T}} + \|u_N\|^6_{Y^{\sigma,\frac12}_{u_0,T}} \end{aligned} \tag{4.21}$$

for any smooth solution u_N to (3.1) with $u|_{t=0} = u_0 \in C^\infty(\mathbb{T})$. Once (4.21) is established, we can simply repeat the reduction in Case 1 (with $u_N = \mathcal{J}^{-1}(v_N)$) and obtain (4.17) for $N < \infty$.

Lastly, note that the only difference between the equations (3.1) and (1.5) is the presence of the frequency cutoff $\mathbf{P}_{\le N}$. Hence, in view of the uniform (in N) boundedness of $\mathbf{P}_{\le N}$ on the relevant spaces, we see that the proof of (4.18) described above (namely, the proof of [21, Proposition 3.4]) can be directly applied[11] to establish (4.21) for $N < \infty$. This concludes the proof of Lemma 4.3. □

Lemma 4.4. *Let $-\frac13 < \sigma < 0$ and $T > 0$. Given $N \in \mathbb{N}$, let v and v_N be the smooth solutions to* (4.2) *and* (4.6)*, respectively, with $v|_{t=0} = v_N|_{t=0} = u_0 \in C^\infty(\mathbb{T})$. Then, we have*

$$\begin{aligned} &\sup_{|n|\le N}\bigg|\operatorname{Im}\bigg(\int_0^T\sum_{\Gamma_N(n)}e^{it\Theta(\bar n)}\Big(\widehat{v}(n_1,t)\overline{\widehat{v}(n_2,t)}\widehat{v}(n_3,t)\overline{\widehat{v}(n,t)} \\ &\qquad\qquad\qquad - \widehat{v}_N(n_1,t)\overline{\widehat{v}_N(n_2,t)}\widehat{v}_N(n_3,t)\overline{\widehat{v}_N(n,t)}\Big)dt\bigg)\bigg| \\ &\quad \le C\Big(\|v\|_{X_T^{\sigma,\frac12+\varepsilon}}, \|v_N\|_{X_T^{\sigma,\frac12+\varepsilon}}\Big)\Big(\|v-v_N\|_{X_T^{\sigma,\frac12+\varepsilon}} + \|\mathbf{P}_{>\frac N3}v\|_{X_T^{\sigma,\frac12+\varepsilon}}\Big), \end{aligned} \tag{4.22}$$

where the implicit constant in (4.22) *is independent of $N \in \mathbb{N}$.*

Proof. This lemma follows from a slight modification of the proof of Proposition 3.8 in [21] which states the following difference estimate; given $-\frac13 \le \sigma < 0$ and $0 < T \le 1$, we have

$$\begin{aligned} &\sup_{n\in\mathbb{Z}}\bigg|\operatorname{Im}\bigg(\int_0^T\sum_{\Gamma(n)}\Big(\widehat{u}_1(n_1,t)\overline{\widehat{u}_1(n_2,t)}\widehat{u}_1(n_3,t)\overline{\widehat{u}_1(n,t)} \\ &\qquad\qquad\qquad - \widehat{u}_2(n_1,t)\overline{\widehat{u}_2(n_2,t)}\widehat{u}_2(n_3,t)\overline{\widehat{u}_2(n,t)}\Big)dt\bigg)\bigg| \\ &\quad \le C\Big(\|u_0\|_{H^\sigma}, \|u_1\|_{Y^{\sigma,\frac12}_{u_0,T}}, \|u_2\|_{Y^{\sigma,\frac12}_{u_0,T}}\Big)\|u_1-u_2\|_{Y^{\sigma,\frac12}_{u_0,T}} \end{aligned} \tag{4.23}$$

for any smooth solutions u_1, u_2 to (1.5) with $u_1|_{t=0} = u_2|_{t=0} = u_0 \in C^\infty(\mathbb{T})$. As before, we can drop the restriction $T \le 1$ and the estimate (4.23) holds for any $T > 0$ (at least for *smooth* solutions; see Remark 4.5 below). The proof of (4.23) is analogous to that of (4.18)

[11]including the integration-by-parts argument in (4.19). We just need to insert $\mathbf{P}_{\le N}$ in appropriate places.

(i.e. Proposition 3.4 in [21]). Namely, divide the domain $\Gamma(n)$ into a good region $\Gamma^{\text{good}}(n)$ and a bad region $\Gamma^{\text{bad}}(n)$. Then, the good part is estimated by the same 4-linear estimates as in the proof of (4.18), while, as for the bad part, we apply integration by parts at the level of the interaction representation (as in (4.19)) and rewrite the 4-linear terms into the 4-linear boundary terms and the 6-linear terms.

In order to prove (4.22), we aim to bound the following difference:

$$\begin{aligned} \sup_{|n|\le N}\bigg|\operatorname{Im}\bigg(\int_0^T\sum_{\Gamma_N(n)}\Big(&\widehat{u}(n_1,t)\overline{\widehat{u}(n_2,t)}\widehat{u}(n_3,t)\overline{\widehat{u}(n,t)}\\ &-\widehat{u}_N(n_1,t)\overline{\widehat{u}_N(n_2,t)}\widehat{u}_N(n_3,t)\overline{\widehat{u}_N(n,t)}\Big)dt\bigg)\bigg|, \end{aligned} \tag{4.24}$$

where u and u_N are solutions to (1.5) and (3.1), respectively, with $u|_{t=0}=u_N|_{t=0}=u_0\in C^\infty(\mathbb{T})$. We proceed as in the proof of (4.23) (= Proposition 3.8 in [21]) described above. In studying (4.23), a difference appears in the integration-by-parts step (in estimating the contribution from the bad region $\Gamma^{\text{bad}}(n)$). After applying integration by parts to the first summand in (4.24), the non-boundary looks like

$$\int_0^T\frac{e^{i\phi(\bar n)t}}{i\phi(\bar n)}\partial_t\Big(\widehat{w}(n_1,t)\overline{\widehat{w}(n_2,t)}\widehat{w}(n_3,t)\overline{\widehat{w}(n,t)}\Big)dt, \tag{4.25}$$

where $\phi(\bar n)$ is as in (2.6) and $w=S(-t)u(t)$ is the interaction representation of u. See (4.19). We then apply the product rule and use (4.20) to replace $\partial_t\widehat{w}$ by (the Fourier transform of) the cubic nonlinearity: $\mathcal{M}(w)(t):=S(-t)\mathcal{N}(S(t)w(t))$. Write

$$\mathcal{M}(w)=\mathbf{P}_{\le N}\mathcal{M}(\mathbf{P}_{\le N}w)+\mathbf{P}_{\le N}\big(\mathcal{M}(w)-\mathcal{M}(\mathbf{P}_{\le N}w)\big)+\mathbf{P}_{>N}\mathcal{M}(w). \tag{4.26}$$

The first term on the right-hand side of (4.26) can be put together with the analogous contribution for $w_N(t)=S(-t)u_N(t)$ coming from the second summand in (4.24), yielding

$$\begin{aligned} \mathbf{P}_{\le N}&\mathcal{M}(\mathbf{P}_{\le N}w)-\mathbf{P}_{\le N}\mathcal{M}(\mathbf{P}_{\le N}w_N)\\ &=\mathbf{P}_{\le N}\mathcal{M}(\mathbf{P}_{\le N}(w-w_N),\mathbf{P}_{\le N}w,\mathbf{P}_{\le N}w)\\ &\quad+\mathbf{P}_{\le N}\mathcal{M}(\mathbf{P}_{\le N}w_N,\mathbf{P}_{\le N}(w-w_N),\mathbf{P}_{\le N}w)\\ &\quad+\mathbf{P}_{\le N}\mathcal{M}(\mathbf{P}_{\le N}w_N,\mathbf{P}_{\le N}w_N,\mathbf{P}_{\le N}(w-w_N)) \end{aligned} \tag{4.27}$$

Then, by substituting (4.27) (for $\partial_t\widehat{w}$) in (4.25) and applying the 6-linear estimate from the proof of Proposition 3.4 in [21], we bound the contribution from this term to (4.24) by

$$C\Big(\|u_0\|_{H^\sigma},\|u\|_{Y^{\sigma,\frac12}_{u_0,T}},\|u_N\|_{Y^{\sigma,\frac12}_{u_0,T}}\Big)\|u-u_N\|_{Y^{\sigma,\frac12}_{u_0,T}}. \tag{4.28}$$

As for the second term on the right-hand side of (4.26), we first write

$$\begin{aligned} \mathbf{P}_{\le N}&\big(\mathcal{M}(w)-\mathcal{M}(\mathbf{P}_{\le N}w)\big)\\ &=\mathbf{P}_{\le N}\mathcal{M}(\mathbf{P}_{>N}w,w,w)+\mathbf{P}_{\le N}\mathcal{M}(\mathbf{P}_{\le N}w,\mathbf{P}_{>N}w,w)\\ &\quad+\mathbf{P}_{\le N}\mathcal{M}(\mathbf{P}_{\le N}w,\mathbf{P}_{\le N}w,\mathbf{P}_{>N}w). \end{aligned} \tag{4.29}$$

Namely, one of the factors is given by $\mathbf{P}_{>N}w$. Then, by substituting (4.29) (for $\partial_t\widehat{w}$) in (4.25) and applying the 6-linear estimate from the proof of Proposition 3.4 in [21] as before, we bound the contribution from this term to (4.24) by

$$C\Big(\|u_0\|_{H^\sigma},\|u\|_{Y^{\sigma,\frac12}_{u_0,T}}\Big)\|\mathbf{P}_{>N}u\|_{Y^{\sigma,\frac12}_{u_0,T}}. \tag{4.30}$$

As for the the third term on the right-hand side of (4.26):

$$\mathbf{P}_{>N}\mathcal{M}(w) = \mathbf{P}_{>N}\mathcal{M}(w,w,w),$$

we first note that this term vanishes unless one of the factors has frequencies greater than $\frac{N}{3}$. Then, proceeding as above, we bound the contribution from this term to (4.24) by

$$C\Big(\|u_0\|_{H^\sigma}, \|u\|_{Y^{\sigma,\frac12}_{u_0,T}}\Big)\|\mathbf{P}_{>\frac{N}{3}}u\|_{Y^{\sigma,\frac12}_{u_0,T}}. \tag{4.31}$$

Then, putting (4.28), (4.30), and (4.31) together, we obtain

$$\begin{aligned}(4.24) &\le C\Big(\|u_0\|_{H^\sigma}, \|u\|_{Y^{\sigma,\frac12}_{u_0,T}}, \|u_N\|_{Y^{\sigma,\frac12}_{u_0,T}}\Big)\Big(\|u-u_N\|_{Y^{\sigma,\frac12}_{u_0,T}} + \|\mathbf{P}_{>\frac{N}{3}}u\|_{Y^{\sigma,\frac12}_{u_0,T}}\Big)\\ &\le C'\Big(\|u\|_{Y^{\sigma,\frac12+\varepsilon}_{u_0,T}}, \|u_N\|_{Y^{\sigma,\frac12+\varepsilon}_{u_0,T}}\Big)\Big(\|u-u_N\|_{Y^{\sigma,\frac12+\varepsilon}_{u_0,T}} + \|\mathbf{P}_{>\frac{N}{3}}u\|_{Y^{\sigma,\frac12+\varepsilon}_{u_0,T}}\Big)\end{aligned} \tag{4.32}$$

for any $\varepsilon > 0$. Here, in the second inequality, we used the embedding (4.9) (for the $Y^{\sigma,\frac12+\varepsilon}_{u_0,T}$-space).

Finally, given the smooth solutions v and v_N to (4.2) and (4.6), respectively, with $v|_{t=0} = v_N|_{t=0} = u_0 \in C^\infty(\mathbb{T})$, let $u = \mathcal{J}^{-1}(v)$ and $u_N = \mathcal{J}^{-1}(v_N)$. Then, the desired bound (4.22) follows from (4.32) with (4.1), (4.4), and (4.11). This concludes the proof of Lemma 4.4. □

Remark 4.5. As pointed out in [24], the smoothness assumption in Lemmas 4.3 and 4.4 is not necessary. In view of Lemma 4.2, it suffices to assume that $v, v_N \in X^{\sigma,\frac12+\varepsilon}_T$ for $\sigma > -\frac12$. See [23] for details. We also point out that, in Lemmas 4.3 and 4.4, the endpoint $\sigma = -\frac13$ is excluded so that the estimates in these lemmas hold for *rough* solutions in $C([0,T];H^\sigma(\mathbb{T}))$, $-\frac13 < \sigma < 0$, for *any* $T > 0$, using the global-in-time control (4.12), which is valid only for $\sigma > -\frac13$.

4.3. **Proof of Proposition 3.2.** We now establish the approximation property of the truncated dynamics (3.1) (Proposition 3.2). In view of the approximation result in $L^2(\mathbb{T})$ (see [30]), we restrict our attention to the range $-\frac13 < \sigma < 0$. We first establish the following preliminary lemma.

Lemma 4.6. *Let $-\frac13 < \sigma < 0$ and $u_0 \in H^\sigma(\mathbb{T})$. Then, for any $T > 0$ and $\delta > 0$, there exists $N_0 = N_0(T, u_0, \delta) \in \mathbb{N}$ such that*

$$\|\Psi(t)(u_0) - \Psi_N(t)(u_0)\|_{H^\sigma} < \delta$$

for any $t \in [0,T]$ and $N \ge N_0$.

Proof. We first consider the high frequency part of the dynamics. Recalling that $\mathbf{P}_{>N}\Psi_N(t)(u_0) = S_{u_0}(t)\mathbf{P}_{>N}u_0$, where $S_{u_0}(t)$ is as in (4.8). Hence, there exists $N_1 = N_1(u_0,\delta) \in \mathbb{N}$ such that

$$\|\mathbf{P}_{>N}\Psi_N(t)(u_0)\|_{L^\infty_T H^\sigma_x} = \|S_{u_0}(t)\mathbf{P}_{>N}u_0\|_{L^\infty_T H^\sigma_x} = \|\mathbf{P}_{>N}u_0\|_{H^\sigma} < \frac{\delta}{4}$$

for any $N \ge N_1$. From (4.12) with $N = \infty$ and the Lebesgue dominated convergence theorem, we have

$$\|\mathbf{P}_{>N}\Psi(t)(u_0)\|_{L^\infty_T H^\sigma_x} \lesssim \|\mathbf{P}_{>N}\Psi(t)(u_0)\|_{X^{\sigma,\frac12+\varepsilon}_T} < \frac{\delta}{4}$$

for any $N \ge N_2 = N_2(T,u_0,\delta) \in \mathbb{N}$.

Hence, it suffices to show that there exists $N_3 = N_3(T,u_0,\delta) \in \mathbb{N}$ such that

$$\|\mathbf{P}_{\le N}\Psi(t)(u_0) - \mathbf{P}_{\le N}\Psi_N(t)(u_0)\|_{L^\infty_T H^\sigma_x} < \frac{\delta}{2} \tag{4.33}$$

for any $N \geq N_3$. By writing (4.2) and (4.6) in the Duhamel formulations with $v(t) = \Psi(t)(u_0)$ and $v_N(t) = \Psi_N(t)(u_0)$, we have

$$\begin{aligned}\mathbf{P}_{\leq N} v(t) - \mathbf{P}_{\leq N} v_N(t) &= -i \sum_{j=1}^{2} \int_0^t S(t-t') \big(\mathbf{P}_{\leq N}\mathcal{N}_j(v) - \mathbf{P}_{\leq N}\mathcal{N}_j^N(v_N)\big)(t')dt' \\ &=: \mathrm{I} + \mathrm{II}.\end{aligned} \tag{4.34}$$

We set $w_N = \mathbf{P}_{\leq N} v - \mathbf{P}_{\leq N} v_N$. We first estimate I. From (4.3) and (4.7), we have

$$\begin{aligned}&\mathbf{P}_{\leq N}\mathcal{N}_1(v) - \mathbf{P}_{\leq N}\mathcal{N}_1^N(v_N) \\ &\quad = \sum_{|n|\leq N} e^{inx} \sum_{\Gamma_N(n)} e^{it\Theta(\bar{n})} \Big(\widehat{w}_N(n_1,t)\overline{\widehat{v}(n_2,t)}\widehat{v}(n_3,t) \\ &\qquad + \widehat{v}_N(n_1,t)\overline{\widehat{w}_N(n_2,t)}\widehat{v}(n_3,t) + \widehat{v}_N(n_1,t)\overline{\widehat{v}_N(n_2,t)}\widehat{w}_N(n_3,t)\Big) \\ &\qquad + \sum_{|n|\leq N} e^{inx} \sum_{\substack{\Gamma(n) \\ \max_{j=1,2,3}|n_j|>N}} e^{it\Theta(\bar{n})} \widehat{v}(n_1,t)\overline{\widehat{v}(n_2,t)}\widehat{v}(n_3,t).\end{aligned}$$

Hence, from Lemmas 4.1 and 4.2 with (4.12), we have

$$\begin{aligned}\|\mathrm{I}\|_{X_\tau^{\sigma,\frac12+\varepsilon}} &\lesssim \tau^\varepsilon \|\mathbf{P}_{\leq N}\mathcal{N}_1(v) - \mathbf{P}_{\leq N}\mathcal{N}_1^N(v_N)\|_{X_\tau^{\sigma,-\frac12+2\varepsilon}} \\ &\leq \tau^\varepsilon C(T,R)\Big(\|w_N\|_{X_\tau^{\sigma,\frac12+\varepsilon}} + \|\mathbf{P}_{>N} v\|_{X_\tau^{\sigma,\frac12+\varepsilon}}\Big)\end{aligned} \tag{4.35}$$

for any $\tau \in [0,T]$, where $R = \|u_0\|_{H^\sigma}$.

Next, we consider II in (4.34). From (4.5) and (4.7), we have

$$\begin{aligned}\mathrm{II} &= i\int_0^t S(t-t') \sum_{|n|\leq N} e^{inx}\Big(|\widehat{v}(n,t')|^2 - |\widehat{u}_0(n)|^2\Big)\widehat{w}_N(n,t')\,dt' \\ &\quad + i\int_0^t S(t-t') \sum_{|n|\leq N} e^{inx}\Big(|\widehat{v}(n,t')|^2 - |\widehat{v}_N(n,t')|^2\Big)\widehat{v}_N(n,t')\,dt' \\ &=: \mathrm{II}_1 + \mathrm{II}_2.\end{aligned} \tag{4.36}$$

By Lemma 4.1, the fundamental theorem of calculus, (4.2), Lemma 4.3, and (4.12), we have

$$\begin{aligned}\|\mathrm{II}_1\|_{X_\tau^{\sigma,\frac12+\varepsilon}} &\lesssim \tau^\varepsilon \|(i\partial_t - \partial_x^4)\mathrm{II}_1\|_{X_\tau^{\sigma,-\frac12+2\varepsilon}} \leq \tau^\varepsilon \|(i\partial_t - \partial_x^4)\mathrm{II}_1\|_{L^2_\tau H^\sigma_x} \\ &\lesssim \tau^\varepsilon \sup_{\substack{t\in[0,\tau] \\ |n|\leq N}} \bigg| \mathrm{Re}\int_0^t \partial_t \widehat{v}(n,t')\overline{\widehat{v}(n,t')}dt'\bigg| \cdot \|w_N\|_{X_\tau^{\sigma,\frac12+\varepsilon}} \\ &= \tau^\varepsilon \sup_{\substack{t\in[0,\tau] \\ |n|\leq N}} \bigg| \mathrm{Im}\bigg(\int_0^t \sum_{\Gamma(n)} e^{it\Theta(\bar{n})}\widehat{v}(n_1,t')\overline{\widehat{v}(n_2,t')}\widehat{v}(n_3,t')\overline{\widehat{v}(n,t')}dt'\bigg)\bigg| \cdot \|w_N\|_{X_\tau^{\sigma,\frac12+\varepsilon}} \\ &\leq \tau^\varepsilon C(T,R)\|w_N\|_{X_\tau^{\sigma,\frac12+\varepsilon}}\end{aligned} \tag{4.37}$$

for any $\tau \in [0,T]$.

Recalling that $v|_{t=0} = v_N|_{t=0} = u_0$, it follows from the fundamental theorem of calculus, (4.2), (4.6), Lemmas 4.4 and 4.3, and (4.12) that

$$\begin{aligned}
\Big||\widehat{v}(n,t)|^2 - |\widehat{v}_N(n,t)|^2\Big| &\le \Big||\widehat{v}(n,t)|^2 - |\widehat{u}_0(n)|^2\Big| + \Big||\widehat{v}_N(n,t)|^2 - |\widehat{u}_0(n)|^2\Big| \\
&\le 2\bigg|\operatorname{Im}\bigg(\int_0^t \sum_{\Gamma_N(n)} e^{it\Theta(\bar{n})}\Big(\widehat{v}(n_1,t')\overline{\widehat{v}(n_2,t')}\widehat{v}(n_3,t')\overline{\widehat{v}(n,t')} \\
&\qquad\qquad - \widehat{v}_N(n_1,t')\overline{\widehat{v}_N(n_2,t')}\widehat{v}_N(n_3,t')\overline{\widehat{v}_N(n,t')}\Big)dt'\bigg)\bigg| \\
&\quad + 2\bigg|\operatorname{Im}\bigg(\int_0^t \sum_{\substack{\Gamma(n)\\ \max_{j=1,2,3}|n_j|>N}} e^{it\Theta(\bar{n})}\widehat{v}(n_1,t')\overline{\widehat{v}(n_2,t')}\widehat{v}(n_3,t')\overline{\widehat{v}(n,t')}dt'\bigg)\bigg| \\
&\le C(T,R)\Big(\|w_N\|_{X_\tau^{\sigma,\frac{1}{2}+\varepsilon}} + \|\mathbf{P}_{>\frac{N}{3}}v\|_{X_\tau^{\sigma,\frac{1}{2}+\varepsilon}}\Big),
\end{aligned} \tag{4.38}$$

uniformly in $|n| \le N$ and $0 \le t \le \tau \le T$. Then, from (4.36), Lemma 4.1, (4.38), and (4.12), we obtain

$$\begin{aligned}
\|\mathrm{II}_2\|_{X_\tau^{\sigma,\frac{1}{2}+\varepsilon}} &\lesssim \tau^\varepsilon \|(i\partial_t - \partial_x^4)\mathrm{II}_2\|_{X_\tau^{\sigma,-\frac{1}{2}+2\varepsilon}} \le \tau^\varepsilon \|(i\partial_t - \partial_x^4)\mathrm{II}_2\|_{L^2_\tau H^\sigma_x} \\
&\lesssim \tau^\varepsilon \sup_{\substack{t\in[0,\tau]\\ |n|\le N}} \Big||\widehat{v}(n,t)|^2 - |\widehat{v}_N(n,t)|^2\Big| \cdot \|v_N\|_{X_\tau^{\sigma,\frac{1}{2}+\varepsilon}} \\
&\le \tau^\varepsilon C(T,R)\Big(\|w_N\|_{X_\tau^{\sigma,\frac{1}{2}+\varepsilon}} + \|\mathbf{P}_{>\frac{N}{3}}v\|_{X_\tau^{\sigma,\frac{1}{2}+\varepsilon}}\Big).
\end{aligned} \tag{4.39}$$

Therefore, from (4.34), (4.35), (4.36), (4.37), and (4.39), we have

$$\|w_N\|_{X_\tau^{\sigma,\frac{1}{2}+\varepsilon}} \lesssim \tau^\varepsilon C_*(T,R)\Big(\|w_N\|_{X_\tau^{\sigma,\frac{1}{2}+\varepsilon}} + \|\mathbf{P}_{>\frac{N}{3}}v\|_{X_\tau^{\sigma,\frac{1}{2}+\varepsilon}}\Big) \tag{4.40}$$

for any $\tau \in [0,T]$. By choosing $\tau = \tau(T,R) > 0$ sufficiently small such that

$$\tau^\varepsilon C_*(T,R) \le \frac{1}{2}, \tag{4.41}$$

we obtain, from (4.40) with (4.9),

$$\|w_N\|_{L^\infty_\tau H^\sigma_x} \lesssim \|w_N\|_{X_\tau^{\sigma,\frac{1}{2}+\varepsilon}} \le C_1(T,R)\|\mathbf{P}_{>\frac{N}{3}}v\|_{X_T^{\sigma,\frac{1}{2}+\varepsilon}}. \tag{4.42}$$

We now consider the second time interval $I_2 = [\tau, 2\tau]$. The estimates (4.35) on I and (4.37) on II_1 also hold on $[\tau, 2\tau]$. As for the analysis on II_2, we need to make the following modification in (4.38). By writing

$$\begin{aligned}
|\widehat{v}(n,t)|^2 - |\widehat{v}_N(n,t)|^2 &= \Big(|\widehat{v}(n,t)|^2 - |\widehat{v}(n,\tau)|^2\Big) - \Big(|\widehat{v}_N(n,t)|^2 - |\widehat{v}_N(n,\tau)|^2\Big) \\
&\quad + \Big(|\widehat{v}(n,\tau)|^2 - |\widehat{v}_N(n,\tau)|^2\Big),
\end{aligned} \tag{4.43}$$

we estimate the first two terms on the right-hand side of (4.43) by using the fundamental theorem of calculus as in (4.38), while the last term on the right-hand side of (4.43) is already controlled by (4.38) with $t = \tau$. Together with (4.42), this gives

$$\begin{aligned}
|\widehat{v}(n,t)|^2 - |\widehat{v}_N(n,t)|^2 &\le C(T,R)\Big(\|w_N\|_{X^{\sigma,\frac{1}{2}+\varepsilon}([\tau,2\tau])} + \|\mathbf{P}_{>\frac{N}{3}}v\|_{X^{\sigma,\frac{1}{2}+\varepsilon}([\tau,2\tau])}\Big) \\
&\quad + C_1'(T,R)\|\mathbf{P}_{>\frac{N}{3}}v\|_{X_\tau^{\sigma,\frac{1}{2}+\varepsilon}}
\end{aligned} \tag{4.44}$$

uniformly in $|n| \le N$ and $0 \le \tau \le t \le 2\tau \le T$. Therefore, proceeding as before with (4.44), we have

$$\|w_N\|_{X^{\sigma,\frac{1}{2}+\varepsilon}([\tau,2\tau])} \lesssim \tau^\varepsilon C_*(T,R)\|w_N\|_{X^{\sigma,\frac{1}{2}+\varepsilon}([\tau,2\tau])} + \tau^\varepsilon C_1''(T,R)\|\mathbf{P}_{>\frac{N}{3}}v\|_{X_T^{\sigma,\frac{1}{2}+\varepsilon}}. \tag{4.45}$$

Hence, from (4.45) and (4.41), we obtain

$$\|w_N\|_{L^\infty([\tau,2\tau];H^\sigma)} \lesssim \|w_N\|_{X^{\sigma,\frac{1}{2}+\varepsilon}([\tau,2\tau])} \le C_2(T,R)\|\mathbf{P}_{>\frac{N}{3}}v\|_{X_T^{\sigma,\frac{1}{2}+\varepsilon}}.$$

By repeating this argument, we have

$$\|w_N\|_{L^\infty(I_j;H^\sigma)} \lesssim \|w_N\|_{X^{\sigma,\frac{1}{2}+\varepsilon}(I_j)} \le C_j(T,R)\|\mathbf{P}_{>\frac{N}{3}}v\|_{X_T^{\sigma,\frac{1}{2}+\varepsilon}}.$$

on the jth time interval $I_j = [(j-1)\tau, j\tau] \cap [0,T]$. Note that while $C_j(T,R)$ is increasing in j, it follows from our choice of τ in (4.41) that $\max_{j=1,\dots,[\frac{T}{\tau}]+1} C_j(T,R) \le C^*(T,R)$ for some $C^*(T,R) > 0$. Therefore, we conclude that

$$\|w_N\|_{L_T^\infty H_x^\sigma} \le C^*(T,R)\|\mathbf{P}_{>\frac{N}{3}}v\|_{X_T^{\sigma,\frac{1}{2}+\varepsilon}}. \tag{4.46}$$

Then, the desired bound (4.33) follows from (4.46) and the Lebesgue dominated convergence theorem with (4.12). This completes the proof of Lemma 4.6. □

Remark 4.7. Due to the lack of local uniform continuity of the solution map in negative Sobolev spaces, it is crucial that $\Psi(t)(u_0)$ and $\Psi_N(t)(u_0)$ have the same initial condition u_0 in the proof of Lemma 4.6; see (4.38).

We conclude this section by presenting the proof of Proposition 3.2. We follow [39, Proposition 2.10] and [30, Proposition B.3/6.21].

Proof of Proposition 3.2. Let $u_0 \in A$, $t \in \mathbb{R}$, and small $\delta > 0$. Write

$$\Phi(t)(u_0) = \Phi_N(t)(\Phi_N(-t)\Phi(t)(u_0)).$$

By setting $w_N = \Phi_N(-t)\Phi(t)(u_0)$, it suffices to show that there exists $N_0(t,R,u_0,\delta) \in \mathbb{N}$ such that

$$w_N \in A + B_\delta$$

for every $N \ge N_0$. Define z_N by

$$z_N = \Phi_N(-t)\Phi(t)(u_0) - u_0$$

such that $w_N = u_0 + z_N$. Since $u_0 \in A$, we only need to check that $z_N \in B_\delta$ for all $N \gg 1$. By writing

$$z_N = \Phi_N(-t)\big(\Phi(t)(u_0) - \Phi_N(t)(u_0)\big),$$

it follows from the uniform (in N) growth bound on the H^σ-norm of solutions to (3.1) (see [33, Proposition 6.6] for the case $N = \infty$) that

$$\begin{aligned} \|z_N\|_{H^\sigma} &= \big\|\Phi_N(-t)\big(\Phi(t)(u_0) - \Phi_N(t)(u_0)\big)\big\|_{H^\sigma} \\ &\le C(t)\|\Phi(t)(u_0) - \Phi_N(t)(u_0)\|_{H^\sigma}^{c(\sigma)} \end{aligned}$$

for some $c(\sigma) > 0$. By the unitarity of the gauge transform $\mathcal{J}$ in (4.1) (for fixed $t \in \mathbb{R}$) and Lemma 4.6, we have

$$\|\Phi(t)(u_0) - \Phi_N(t)(u_0)\|_{H^\sigma} \longrightarrow 0$$

as $N \to \infty$. This implies that $z_N \in B_\delta$ for $N \geq N_0(t, R, u_0, \delta) \in \mathbb{N}$. This proves Proposition 3.2. □

5. Normal form reductions

In this section, we present the proof of Proposition 3.4 and Lemma 3.5 by implementing an infinite iteration of normal form reductions as in [27, 33]. This procedure allows us to construct an infinite sequences of correction terms and thus build the desired modified energies $\mathcal{E}_N(u)(t)$ and $\mathcal{E}(u)(t)$ in (3.4).

5.1. **Main proposition.** In this subsection, by expressing the multilinear terms in the series expansion (3.3) in terms of the interaction representation, we state the bounds on these multilinear terms (Proposition 5.1). By assuming these bounds, we then present the proofs of Proposition 3.4 and Lemma 3.5.

In order to encode multilinear dispersion in an effective manner, it is convenient to work with the following interaction representation of u defined by

$$v(t) := S(-t)u(t).$$

On the Fourier side, we have

$$v_n(t) = e^{itn^4} u_n(t),$$

where, for simplicity of notation, we set $v_n(t) = \widehat{v}(n, t)$, etc. We use this short-hand notation in the remaining part of this section. If u is a solution to (1.5), then $\{v_n\}_{n\in\mathbb{Z}}$ satisfies the following equation:

$$\begin{aligned} \partial_t v_n &= -i \sum_{\Gamma(n)} e^{-i\phi(\bar{n})t} v_{n_1} \overline{v_{n_2}} v_{n_3} + i|v_n|^2 v_n \\ &=: \mathcal{N}(v)_n + \mathcal{R}(v)_n, \end{aligned} \tag{5.1}$$

where $\phi(\bar{n})$ and $\Gamma(n)$ are as in (2.6) and (2.8). By writing (3.2) in terms of the interaction representation, we have the following finite dimensional system of ODEs:

$$\partial_t v_n = -i \sum_{\Gamma_N(n)} e^{-i\phi(\bar{n})t} v_{n_1} \overline{v_{n_2}} v_{n_3} + i|v_n|^2 v_n, \qquad |n| \leq N \tag{5.2}$$

with $v|_{t=0} = \mathbf{P}_{\leq N} v|_{t=0}$, namely, $v_n|_{t=0} = 0$ for $|n| > N$.

In the following, we simply say that v is a solution to (5.2) if v is a solution to (5.2) when $N \in \mathbb{N}$ and to (5.1) when $N = \infty$. We state out main result in this section.

Proposition 5.1. *Let $\frac{3}{10} < s \leq \frac{1}{2}$ and $\sigma = s - \frac{1}{2} - \varepsilon$ for some small $\varepsilon > 0$. Then, given $N \in \mathbb{N} \cup \{\infty\}$, there exist multilinear forms $\{\mathsf{N}^{(j)}_{0,N}(t)\}_{j=2}^\infty$, $\{\mathsf{N}^{(j)}_{1,N}(t)\}_{j=2}^\infty$, and $\{\mathsf{R}^{(j)}_{N}(t)\}_{j=2}^\infty$, depending on $t \in \mathbb{R}$, such that*

$$\frac{d}{dt}\bigg(\frac{1}{2}\|v(t)\|_{H^s}^2\bigg) = \frac{d}{dt}\bigg(\sum_{j=2}^\infty \mathsf{N}^{(j)}_{0,N}(t)(v(t))\bigg) + \sum_{j=2}^\infty \mathsf{N}^{(j)}_{1,N}(t)(v(t)) + \sum_{j=2}^\infty \mathsf{R}^{(j)}_{N}(t)(v(t)) \tag{5.3}$$

for any solution $v \in C(\mathbb{R}; H^\sigma(\mathbb{T}))$ to (5.2).[12] *Here, $\mathsf{N}^{(j)}_{0,N}(t)$ are $2j$-linear forms, while $\mathsf{N}^{(j)}_{1,N}$ and $\mathsf{R}^{(j)}_{N}$ are $(2j+2)$-linear forms, satisfying the following bounds in $H^\sigma(\mathbb{T})$; there exist positive*

[12]Note that the left-hand side of (5.3) does not a priori make sense for $v \in C(\mathbb{R}; H^\sigma(\mathbb{T}))$. The identity (5.3) is to be understood in the limiting sense for rough solutions.

constants $C_0(j)$, $C_1(j)$, and $C_2(j)$, decaying faster than any exponential rate[13] *as $j \to \infty$ such that*

$$\sup_{t\in\mathbb{R}} \big|\mathsf{N}^{(j)}_{0,N}(t)(f_1,\dots,f_{2j})\big| \le C_0(j)\prod_{k=1}^{2j}\|f_k\|_{H^\sigma}, \tag{5.4}$$

$$\sup_{t\in\mathbb{R}} \big|\mathsf{N}^{(j)}_{1,N}(t)(f_1,\dots,f_{2j+2})\big| \le C_1(j)\prod_{k=1}^{2j+2}\|f_k\|_{H^\sigma}, \tag{5.5}$$

$$\sup_{t\in\mathbb{R}} \big|\mathsf{R}^{(j)}_{N}(t)(f_1,\dots,f_{2j+2})\big| \le C_2(j)\prod_{k=1}^{2j+2}\|f_k\|_{H^\sigma} \tag{5.6}$$

for $j = 2, 3, \dots$. Note that these constants $C_0(j)$, $C_1(j)$, and $C_2(j)$ are independent of the cutoff size $N \in \mathbb{N}\cup\{\infty\}$.

We now present the proofs of Proposition 3.4 and Lemma 3.5 by assuming Proposition 5.1. First, we prove Proposition 3.4. Given $N \in \mathbb{N}\cup\{\infty\}$, let $u \in C(\mathbb{R}; H^\sigma(\mathbb{T}))$ be a solution to (3.2), satisfying the growth bound (3.5). Then, we define the multilinear form $\mathcal{N}^{(j)}_{0,N}$, $\mathcal{N}^{(j)}_{1,N}$, and $\mathcal{R}^{(j)}_{N}$ by setting

$$\begin{aligned} \mathcal{N}^{(j)}_{0,N}(u(t)) &:= \mathsf{N}^{(j)}_{0,N}(t)(S(-t)u(t)),\\ \mathcal{N}^{(j)}_{1,N}(u(t)) &:= \mathsf{N}^{(j)}_{1,N}(t)(S(-t)u(t)),\\ \mathcal{R}^{(j)}_{N}(u(t)) &:= \mathsf{R}^{(j)}_{N}(t)(S(-t)u(t)). \end{aligned} \tag{5.7}$$

While the multilinear forms $\mathsf{N}^{(j)}_{0,N}$, $\mathsf{N}^{(j)}_{1,N}$, and $\mathsf{R}^{(j)}_{N}$ appearing in Proposition 5.1 are non-autonomous (i.e. they depend on $t \in \mathbb{R}$), it is easy to see from the construction of these multilinear forms carried out in the remaining part of this section that the multilinear forms $\mathcal{N}^{(j)}_{0,N}$, $\mathcal{N}^{(j)}_{1,N}$, and $\mathcal{R}^{(j)}_{N}$ defined in (5.7) are indeed autonomous.

From (5.3) and (5.7) with the unitarity of $S(t)$, we obtain (3.3). By defining the modified energy $\mathcal{E}_N(u)$ as in (3.4), it follows from (3.3) and (5.7)

$$\frac{d}{dt}\mathcal{E}_N(u)(t) = \sum_{j=2}^{\infty}\mathsf{N}^{(j)}_{1,N}(t)(S(-t)u(t)) + \sum_{j=2}^{\infty}\mathsf{R}^{(j)}_{N}(t)(S(-t)u(t)). \tag{5.8}$$

Then, from (5.8) and Proposition 5.1 together with the growth bound (3.5) and the fast decay (in j) of the constants $C_0(j)$, $C_1(j)$, and $C_2(j)$, we obtain

$$\begin{aligned} \sup_{t\in[0,T]}\bigg|\frac{d}{dt}\mathcal{E}_N(u)(t)\bigg| &\le \sum_{j=2}^{\infty}\big(C_1(j)+C_2(j)\big)R^{2j+2}\\ &\le C_s(R). \end{aligned}$$

This proves Proposition 3.4.

[13]In fact, by slightly modifying the proof, we can make $C_0(j)$, $C_1(j)$, and $C_2(j)$ decay as fast as we want as $j \to \infty$.

We now turn to the proof of Lemma 3.5. Let $u \in B_R \subset H^\sigma(\mathbb{R})$. Then, from (3.6), (5.7), and (5.4) in Proposition 5.1, we have

$$\begin{aligned}|\mathfrak{S}_N(u)| &= \bigg|\sum_{j=2}^\infty \mathcal{N}_{0,N}^{(j)}(\mathbf{P}_{\le N}u)\bigg| = \bigg|\sum_{j=2}^\infty \mathsf{N}_{0,N}^{(j)}(t)(\mathbf{P}_{\le N}S(-t)u)\bigg| \\ &\le \sum_{j=2}^\infty C_0(j)R^{2j} \le C_s(R)\end{aligned}$$

for any $N \in \mathbb{N} \cup \{\infty\}$ (and any $t \in \mathbb{R}$). As for the convergence part, we refer the readers to Subsection 4.7 in [27] for details. This completes the proofs of Proposition 3.4 and Lemma 3.5.

Remark 5.2. In [27], Proposition 5.1 was shown for $\sigma = 0$ (and $\frac{1}{2} < s < 1$), where the divisor counting argument played an important role. In the current setting with $\sigma < 0$, we need to make use of the fourth order dispersion to gain derivatives and, for this purpose, we follow the argument in [33]. In particular, we do not rely on the divisor counting argument. The essential difference between our argument and that in [33] is the presence of the weight $\langle n\rangle^{2s}$, coming from the H^s-norm squared on the left-hand side of (5.3). Namely, for our problem, we need to exhibit a stronger smoothing property than that in [33], resulting in a worse regularity restriction $\sigma > -\frac{1}{5}$ in Proposition 5.1.

5.2. **Notations: index by ordered bi-trees.** In this subsection, we go over notations from [18, 27, 33] for implementing an infinite iteration of normal form reductions. Our main goal is to apply normal form reductions to the H^s-energy functional[14] and thus we need tree-like structures that grow in two directions. For our analysis, ordered bi-trees in Definition 5.4 play an essential role.

Definition 5.3. (i) Given a partially ordered set $\mathcal{T}$ with partial order $\le$, we say that $b \in \mathcal{T}$ with $b \le a$ and $b \ne a$ is a child of $a \in \mathcal{T}$, if $b \le c \le a$ implies either $c = a$ or $c = b$. If the latter condition holds, we also say that a is the parent of b.

(ii) A tree $\mathcal{T}$ is a finite partially ordered set satisfying the following properties:

- (a) Let $a_1, a_2, a_3, a_4 \in \mathcal{T}$. If $a_4 \le a_2 \le a_1$ and $a_4 \le a_3 \le a_1$, then we have $a_2 \le a_3$ or $a_3 \le a_2$,
- (b) A node $a \in \mathcal{T}$ is called terminal, if it has no child. A non-terminal node $a \in \mathcal{T}$ is a node with exactly three ordered[15] children denoted by a_1, a_2, and a_3,
- (c) There exists a maximal element $r \in \mathcal{T}$ (called the root node) such that $a \le r$ for all $a \in \mathcal{T}$,
- (d) $\mathcal{T}$ consists of the disjoint union of $\mathcal{T}^0$ and $\mathcal{T}^\infty$, where $\mathcal{T}^0$ and $\mathcal{T}^\infty$ denote the collections of non-terminal nodes and terminal nodes, respectively.

(iii) A *bi-tree* $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ is a disjoint union of two trees $\mathcal{T}_1$ and $\mathcal{T}_2$, where the root nodes r_j of $\mathcal{T}_j$, $j = 1, 2$, are joined by an edge. A bi-tree $\mathcal{T}$ consists of the disjoint union of $\mathcal{T}^0$ and $\mathcal{T}^\infty$, where $\mathcal{T}^0$ and $\mathcal{T}^\infty$ denote the collections of non-terminal nodes and terminal nodes, respectively.

[14] More precisely, to the evolution equation satisfied by the H^s-energy functional.

[15] For example, we simply label the three children as a_1, a_2, and a_3 by moving from left to right in the planar graphical representation of the tree $\mathcal{T}$. As we see below, we assign the Fourier coefficients of the interaction representation v at a_1 and a_3, while we assign the complex conjugate of the Fourier coefficients of v at the second child a_2.

By convention, we assume that the root node r_1 of the tree $\mathcal{T}_1$ is non-terminal, while the root node r_2 of the tree $\mathcal{T}_2$ may be terminal.

(iv) Given a bi-tree $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, we define a projection Π_j, $j = 1, 2$, onto a tree by setting

$$\Pi_j(\mathcal{T}) = \mathcal{T}_j.$$

Note that the number $|\mathcal{T}|$ of nodes in a bi-tree $\mathcal{T}$ is $3j + 2$ for some $j \in \mathbb{N}$, where $|\mathcal{T}^0| = j$ and $|\mathcal{T}^\infty| = 2j + 2$. Let us denote the collection of trees in the jth generation (namely, with j parental nodes) by $BT(j)$, i.e.

$$BT(j) := \{\mathcal{T} : \mathcal{T} \text{ is a bi-tree with } |\mathcal{T}| = 3j + 2\}.$$

Next, we recall the notion of ordered bi-trees, for which we keep track of how a bi-tree "grew" into a given shape.

Definition 5.4. (i) We say that a sequence $\{\mathcal{T}_j\}_{j=1}^J$ is a chronicle of J generations, if

- (a) $\mathcal{T}_j \in BT(j)$ for each $j = 1, \dots, J$,
- (b) $\mathcal{T}_{j+1}$ is obtained by changing one of the terminal nodes in $\mathcal{T}_j$ into a non-terminal node (with three children), $j = 1, \dots, J - 1$.

Given a chronicle $\{\mathcal{T}_j\}_{j=1}^J$ of J generations, we refer to $\mathcal{T}_J$ as an *ordered bi-tree* of the Jth generation. We denote the collection of the ordered trees of the Jth generation by $\mathfrak{BT}(J)$. Note that the cardinality of $\mathfrak{BT}(J)$ is given by $|\mathfrak{BT}(1)| = 1$ and

$$|\mathfrak{BT}(J)| = 4 \cdot 6 \cdot 8 \cdot \dots \cdot 2J = 2^{J-1} \cdot J! =: c_J, \quad J \geq 2. \tag{5.9}$$

(ii) Given an ordered bi-tree $\mathcal{T}_J \in \mathfrak{BT}(J)$ as above, we define projections π_j, $j = 1, \dots, J - 1$, onto the previous generations by setting

$$\pi_j(\mathcal{T}_J) = \mathcal{T}_j \in \mathfrak{BT}(j).$$

We stress that the notion of ordered bi-trees comes with associated chronicles. For example, given two ordered bi-trees $\mathcal{T}_J$ and $\widetilde{\mathcal{T}}_J$ of the Jth generation, it may happen that $\mathcal{T}_J = \widetilde{\mathcal{T}}_J$ as bi-trees (namely as planar graphs) according to Definition 5.3, while $\mathcal{T}_J \neq \widetilde{\mathcal{T}}_J$ as ordered bi-trees according to Definition 5.4. In the following, when we refer to an ordered bi-tree $\mathcal{T}_J$ of the Jth generation, it is understood that there is an underlying chronicle $\{\mathcal{T}_j\}_{j=1}^J$.

Given a bi-tree $\mathcal{T}$, we associate each terminal node $a \in \mathcal{T}^\infty$ with the Fourier coefficient (or its complex conjugate) of the interaction representation v and sum over all possible frequency assignments. For this purpose, we recall the notion of index functions, assigning integers to *all* the nodes in $\mathcal{T}$ in a consistent manner.

Definition 5.5. (i) Given a bi-tree $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, we define an index function $\mathbf{n} : \mathcal{T} \to \mathbb{Z}$ such that

- (a) $n_{r_1} = n_{r_2}$, where r_j is the root node of the tree $\mathcal{T}_j$, $j = 1, 2$,
- (b) $n_a = n_{a_1} - n_{a_2} + n_{a_3}$ for $a \in \mathcal{T}^0$, where a_1, a_2, and a_3 denote the children of a,
- (c) $\{n_a, n_{a_2}\} \cap \{n_{a_1}, n_{a_3}\} = \emptyset$ for $a \in \mathcal{T}^0$,

where we identified $\mathbf{n} : \mathcal{T} \to \mathbb{Z}$ with $\{n_a\}_{a \in \mathcal{T}} \in \mathbb{Z}^{\mathcal{T}}$. We use $\mathfrak{N}(\mathcal{T}) \subset \mathbb{Z}^{\mathcal{T}}$ to denote the collection of such index functions $\mathbf{n}$ on $\mathcal{T}$.

(ii) Given a tree $\mathcal{T}$, we also define an index function $\mathbf{n} : \mathcal{T} \to \mathbb{Z}$ by omitting the condition (a) and denote by $\mathfrak{N}(\mathcal{T}) \subset \mathbb{Z}^{\mathcal{T}}$ the collection of index functions $\mathbf{n}$ on $\mathcal{T}$.

Remark 5.6. (i) In view of the consistency condition (a), we can refer to $n_{r_1} = n_{r_2}$ as the frequency at the root node without ambiguity. We shall simply denote it by n_r in the following.
(ii) Given a bi-tree $\mathcal{T} \in BT(J)$ and $n \in \mathbb{Z}$, consider the summation over all possible frequency assignments $\{\mathbf{n} \in \mathfrak{N}(\mathcal{T}) : n_r = n\}$. While $|\mathcal{T}^\infty| = 2J + 2$, there are $2J$ free variables in this summation. Namely, the condition $n_r = n$ reduces two summation variables. It is easy to see this by separately considering the cases $\Pi_2(\mathcal{T}) = \{r_2\}$ and $\Pi_2(\mathcal{T}) \neq \{r_2\}$.

Given an ordered bi-tree $\mathcal{T}_J$ of the Jth generation with a chronicle $\{\mathcal{T}_j\}_{j=1}^J$ and associated index functions $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J)$, we use superscripts to denote such generations of frequencies.

Fix $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J)$. Consider $\mathcal{T}_1 = \pi_1(\mathcal{T}_J)$ of the first generation. Its nodes consist of the two root nodes r_1, r_2, and the children r_{11}, r_{12}, and r_{13} of the first root node r_1. We define the first generation of frequencies by

$$\big(n^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}\big) := (n_{r_1}, n_{r_{11}}, n_{r_{12}}, n_{r_{13}}).$$

The ordered bi-tree $\mathcal{T}_2 = \pi_2(\mathcal{T}_J)$ of the second generation is constructed from $\mathcal{T}_1$ by changing one of its terminal nodes $a \in \mathcal{T}_1^\infty = \{r_2, r_{11}, r_{12}, r_{13}\}$ into a non-terminal node. Then, we define the second generation of frequencies by setting

$$\big(n^{(2)}, n_1^{(2)}, n_2^{(2)}, n_3^{(2)}\big) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}).$$

As we see below, this corresponds to introducing a new set of frequencies after the first differentiation by parts.

In general, we construct an ordered bi-tree $\mathcal{T}_j = \pi_j(\mathcal{T}_J)$ of the jth generation from $\mathcal{T}_{j-1}$ by changing one of its terminal nodes $a \in \mathcal{T}_{j-1}^\infty$ into a non-terminal node. Then, we define the jth generation of frequencies by

$$\big(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}\big) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}).$$

We denote by ϕ_j the phase function for the frequencies introduced at the jth generation:

$$\phi_j = \phi_j\big(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}\big) := \big(n_1^{(j)}\big)^4 - \big(n_2^{(j)}\big)^4 + \big(n_3^{(j)}\big)^4 - \big(n^{(j)}\big)^4.$$

Note that we have $|\phi_j| \geq 1$ in view of Definition 5.5 and (2.7). We also denote by μ_j the phase function corresponding to the usual cubic NLS (at the jth generation):

$$\begin{aligned}\mu_j = \mu_j\big(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}\big) &:= \big(n_1^{(j)}\big)^2 - \big(n_2^{(j)}\big)^2 + \big(n_3^{(j)}\big)^2 - \big(n^{(j)}\big)^2 \\ &= -2\big(n^{(j)} - n_1^{(j)}\big)\big(n^{(j)} - n_3^{(j)}\big).\end{aligned}$$

Then, from (2.7), we have

$$|\phi_j| \sim (n_{\max}^{(j)})^2 \cdot \big|\big(n^{(j)} - n_1^{(j)}\big)\big(n^{(j)} - n_3^{(j)}\big)\big| \sim (n_{\max}^{(j)})^2 \cdot |\mu_j|, \tag{5.10}$$

where $n_{\max}^{(j)}$ is defined by

$$n_{\max}^{(j)} := \max\big(|n^{(j)}|, |n_1^{(j)}|, |n_2^{(j)}|, |n_3^{(j)}|\big).$$

Lastly, given an ordered bi-tree $\mathcal{T} \in \mathfrak{BT}(J)$ for some $J \in \mathbb{N}$, define $A_j \subset \mathfrak{N}(\mathcal{T})$ by

$$A_j = \big\{|\widetilde{\phi}_{j+1}| \lesssim (2J+4)^3 |\widetilde{\phi}_j|\big\} \cup \big\{|\widetilde{\phi}_{j+1}| \lesssim (2J+4)^3 |\phi_1|\big\}, \tag{5.11}$$

where $\widetilde{\phi}_j$ is defined by

$$\widetilde{\phi}_j = \sum_{k=1}^{j} \phi_k. \tag{5.12}$$

In Subsections 5.3 and 5.4, we perform normal form reductions in an iterative manner. At each step, we divide multilinear forms into nearly resonant part (corresponding to the frequencies belonging to A_j) and highly non-resonant part (corresponding to the frequencies belonging to A_j^c) and apply a normal form reduction only to the highly non-resonant part. Then, we prove the multilinear estimates (5.4), (5.5), and (5.6) for a solution v to (5.2), uniformly in $N \in \mathbb{N} \cup \{\infty\}$. For simplicity of presentation, however, we only consider the $N = \infty$ case and work on the equation (5.1) without the frequency cutoff $\mathbf{1}_{|n| \le N}$ in the following. We point out that the same normal form reductions and estimates hold for the truncated equation (5.2), uniformly in $N \in \mathbb{N}$, with straightforward modifications: (i) set $\widehat{v}_n = 0$ for all $|n| > N$ and (ii) the multilinear forms for (5.2) are obtained by inserting the frequency cutoff $\mathbf{1}_{|n| \le N}$ in appropriate places.[16] In the following, we introduce multilinear forms such as $\mathsf{N}_0^{(j)}$, $\mathsf{N}_1^{(j)}$, $\mathsf{N}_2^{(j)}$, and $\mathsf{R}^{(j)}$ for the untruncated equation (5.1). With a small modification, these multilinear forms give rise to $\mathsf{N}_{0,N}^{(j)}$, $\mathsf{N}_{1,N}^{(j)}$, $\mathsf{N}_{2,N}^{(j)}$, and $\mathsf{R}_N^{(j)}$, $N \in \mathbb{N}$, for the truncated equation (5.2), appearing in Proposition 5.1.

We point out that given finite $N \in \mathbb{N}$, a solution to the truncated equation (5.2) is smooth and therefore the formal computations presented in Subsections 5.3 and 5.4 can be easily justified for solutions to (5.2). When $N = \infty$, we need to impose the regularity condition $v \in C(\mathbb{R}; H^\sigma(\mathbb{T}))$, $\sigma \ge \frac{1}{6}$, to justify the normal form procedure. See [18, 33] for details. Hence, given a solution $v \in C(\mathbb{R}; H^\sigma(\mathbb{T}))$ to (5.1) with $-\frac{1}{5} < \sigma \le 0$ as in Proposition 5.1, we need to go through a limiting argument to obtain the identity (5.3). This argument, however, is standard and thus we omit details.

5.3. **First few steps of normal form reductions.** In this section and the next section, we go over normal form reductions. The formal computation at each step and the resulting multilinear forms are essentially the same as those appearing in [27] (modulo the slightly different frequency sets A_j defined in (5.11)). In terms of the actual estimates on the multilinear forms, however, we closely follow the argument in [33]. For readers' convenience, we present essentially the full details.

In this section, we go over the first few steps. Let v be a smooth global solution to (5.1). With $\phi(\bar{n})$ and $\Gamma(n)$ as in (2.6) and (2.8), we have

$$\begin{aligned} \frac{d}{dt}\bigg(\frac{1}{2}\|v(t)\|_{H^s}^2\bigg) &= -\operatorname{Re} i \sum_{n\in\mathbb{Z}} \sum_{\Gamma(n)} \langle n\rangle^{2s} e^{-i\phi(\bar{n})t} v_{n_1}(t)\overline{v_{n_2}}(t) v_{n_3}(t)\overline{v_n}(t) \\ &=: \mathsf{N}^{(1)}(t)(v(t)). \end{aligned} \tag{5.13}$$

Remark 5.7. (i) Due to the presence of the phase factors in their definitions, the multilinear forms such as $\mathsf{N}^{(1)}(t)(v(t))$ are non-autonomous in t. In the following, however, we establish nonlinear estimates on these multilinear forms, uniformly in $t \in \mathbb{R}$, by simply using $|e^{-i\phi(\bar{n})t}| = 1$. Hence, we suppress such t-dependence when there is no confusion.

(ii) The complex conjugate signs on v_{n_j} do not play any significant role. Hereafter, we drop the complex conjugate sign.

[16] Using the bi-tree notation, it follows from (5.2) that we simply need to insert the frequency cutoff $\mathbf{1}_{|n^{(j)}| \le N}$ on the parental frequency $n^{(j)}$ assigned to each non-terminal node $a \in \mathcal{T}^0$.

In view of (2.7) and (2.8), we have $|\phi(\bar{n})| \geq 1$ in (5.13). Then, by performing a normal form reduction, namely, differentiating by parts, and substituting the equation (5.1), we obtain

$$\begin{aligned}
\mathsf{N}^{(1)}(v)(t) &= \operatorname{Re} \partial_t \bigg[\sum_{\mathcal{T}_1 \in \mathfrak{BT}(1)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1)} \langle n_r \rangle^{2s} \frac{e^{-i\phi_1 t}}{\phi_1} \prod_{a \in \mathcal{T}_1^\infty} v_{n_a} \bigg] \\
&\quad - \operatorname{Re} \sum_{\mathcal{T}_1 \in \mathfrak{BT}(1)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1)} \langle n_r \rangle^{2s} \frac{e^{-i\phi_1 t}}{\phi_1} \partial_t \bigg(\prod_{a \in \mathcal{T}_1^\infty} v_{n_a} \bigg) \\
&= \operatorname{Re} \partial_t \bigg[\sum_{\mathcal{T}_1 \in \mathfrak{BT}(1)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1)} \langle n_r \rangle^{2s} \frac{e^{-i\phi_1 t}}{\phi_1} \prod_{a \in \mathcal{T}_1^\infty} v_{n_a} \bigg] \\
&\quad - \operatorname{Re} \sum_{\mathcal{T}_1 \in \mathfrak{BT}(1)} \sum_{b \in \mathcal{T}_1^\infty} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1)} \langle n_r \rangle^{2s} \frac{e^{-i\phi_1 t}}{\phi_1} \mathcal{R}(v)_{n_b} \prod_{a \in \mathcal{T}_1^\infty \setminus \{b\}} v_{n_a} \\
&\quad - \operatorname{Re} \sum_{\mathcal{T}_2 \in \mathfrak{BT}(2)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_2)} \langle n_r \rangle^{2s} \frac{e^{-i(\phi_1+\phi_2)t}}{\phi_1} \prod_{a \in \mathcal{T}_2^\infty} v_{n_a} \\
&=: \partial_t \mathsf{N}_0^{(2)}(v)(t) + \mathsf{R}^{(2)}(v)(t) + \mathsf{N}^{(2)}(v)(t).
\end{aligned} \tag{5.14}$$

In the second equality, we used the equation (5.1) to replace $\partial_t v_{n_b}$ by the resonant part $\mathcal{R}(v)_{n_b}$ and the non-resonant part $\mathcal{N}(v)_{n_b}$. Note that the substitution of $\mathcal{N}(v)_{n_b}$ amounts to extending the tree $\mathcal{T}_1 \in \mathfrak{BT}(1)$ (and $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1)$) to $\mathcal{T}_2 \in \mathfrak{BT}(2)$ (and to $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_2)$, respectively) by replacing the terminal node $b \in \mathcal{T}_1^\infty$ into a non-terminal node with three children b_1, b_2, and b_3.

Remark 5.8. Strictly speaking, the phase factor appearing in $\mathsf{N}^{(2)}(v)$ may be $\phi_1 - \phi_2$ when the time derivative falls on the terms with the complex conjugate. In the following, however, we simply write it as $\phi_1 + \phi_2$ since it does not make any difference in our analysis. Also, we often replace ± 1 and $\pm i$ by 1 for simplicity when they do not play an important role. Lastly, for notational simplicity, we drop the real part symbol on multilinear forms with the understanding that all the multilinear forms appear with the real part symbol.

We first estimate the boundary term $\mathsf{N}_0^{(2)}$. In the remaining part of this section, we set $\sigma = \sigma(s) = s - \frac{1}{2} - \varepsilon$ for some small $\varepsilon > 0$ as in (2.1).

Lemma 5.9. *Let $\mathsf{N}_0^{(2)}$ be as in (5.14). Then, for $s > 0$, we have*

$$|\mathsf{N}_0^{(2)}(v)| \lesssim \|v\|_{H^\sigma}^4. \tag{5.15}$$

Proof. For notational simplicity, we drop the superscript (1) in the frequencies $n^{(1)} = n_r$ and $n_j^{(1)}$. From (5.10), we have

$$\sup_{n \in \mathbb{Z}} \sum_{\Gamma(n)} \frac{n_{\max}^{4s-8\sigma}}{|\phi_1|^2} \lesssim \sup_{n \in \mathbb{Z}} \sum_{\Gamma(n)} \frac{1}{|(n-n_1)(n-n_3)|^2 n_{\max}^{4-4s+8\sigma}} \lesssim 1, \tag{5.16}$$

provided that $4-4s+8\sigma>0$, namely, $s>0$. Then, by Cauchy-Schwarz inequality with $|\mathfrak{BT}(\mathcal{T}_1)|=1$ and (5.16), we have

$$\begin{aligned}|\mathsf{N}_0^{(2)}(v)| &\lesssim \sum_{\mathcal{T}_1\in\mathfrak{BT}(1)}\sum_{n\in\mathbb{Z}}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_1)\\ n_r=n}}\frac{n_{\max}^{2s-4\sigma}}{|\phi_1|}\prod_{a\in\mathcal{T}_1^\infty}\langle n_a\rangle^\sigma v_{n_a}\\ &\le \|v\|_{H^\sigma}\bigg\{\bigg(\sup_{n\in\mathbb{Z}}\sum_{\Gamma(n)}\frac{n_{\max}^{4s-8\sigma}}{|\phi_1|^2}\bigg)\cdot\bigg(\sum_{n\in\mathbb{Z}}\sum_{\Gamma(n)}\prod_{i=1}^3\langle n_i\rangle^{2\sigma}|v_{n_i}|^2\bigg)\bigg\}^{\frac12}\\ &\lesssim \|v\|_{H^\sigma}^4.\end{aligned}$$

This proves (5.15) □

Proceeding in an analogous manner, we obtain the following estimate on $\mathsf{R}^{(2)}$.

Lemma 5.10. *Let* $\mathsf{R}^{(2)}$ *be as in* (5.14). *Then, for* $s>\frac14$, *we have*

$$|\mathsf{R}^{(2)}(v)|\lesssim\|v\|_{H^\sigma}^6.$$

Proof. This lemma follows from the proof of Lemma 5.9 and $\ell^2\hookrightarrow\ell^6$, once we observe that

$$\sup_{n\in\mathbb{Z}}\sum_{\Gamma(n)}\frac{n_{\max}^{4s-12\sigma}}{|\phi_1|^2}\lesssim\sup_{n\in\mathbb{Z}}\sum_{\Gamma(n)}\frac{1}{|(n-n_1)(n-n_3)|^2 n_{\max}^{4-4s+12\sigma}}\lesssim 1,$$

provided that $4-4s+12\sigma>0$, namely, $s>\frac14$. □

As it is, we cannot estimate $\mathsf{N}^{(2)}$ in (5.14). By dividing the frequency space into A_1 defined in (5.11) and its complement A_1^c, we split $\mathsf{N}^{(2)}$ as

$$\mathsf{N}^{(2)}=\mathsf{N}_1^{(2)}+\mathsf{N}_2^{(2)},\tag{5.17}$$

where $\mathsf{N}_1^{(2)}$ is the restriction of $\mathsf{N}^{(2)}$ onto A_1 and $\mathsf{N}_2^{(2)}:=\mathsf{N}^{(2)}-\mathsf{N}_1^{(2)}$. Thanks to the frequency restriction A_1, we can estimate the first term $\mathsf{N}_1^{(2)}$.

Lemma 5.11. *Let* $\mathsf{N}_1^{(2)}$ *be as in* (5.17). *Then, for* $s>\frac{3}{10}$, *we have*

$$|\mathsf{N}_1^{(2)}(v)|\lesssim\|v\|_{H^\sigma}^6.$$

Proof. On A_1, we have $|\phi_2|\lesssim|\phi_1|$. Then, from (5.10), we have

$$\begin{aligned}&\sup_{n\in\mathbb{Z}}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n\\ |\phi_2|\lesssim|\phi_1|}}\frac{(n_{\max}^{(1)})^{4s-6\sigma}(n_{\max}^{(2)})^{-6\sigma}}{|\phi_1|^2}\\ &\lesssim\sup_{n\in\mathbb{Z}}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n}}\frac{1}{|\mu_1|^\alpha|\mu_2|^{2-\alpha}(n_{\max}^{(1)})^{-4s+6\sigma+2\alpha}(n_{\max}^{(2)})^{6\sigma+4-2\alpha}}\end{aligned}\tag{5.18}$$

for any $0\le\alpha\le2$. We impose $-4s+6\sigma+2\alpha>0$ and $6\sigma+4-2\alpha>0$, namely,

$$s>-\alpha+\frac32\qquad\text{and}\qquad s>\frac{\alpha}{3}-\frac16.\tag{5.19}$$

In view of the powers of $n_{\max}^{(1)}$ and $n_{\max}^{(2)}$ on the left-hand side of (5.18), we may assume that $1 \le \alpha \le 2$. From $|\mu_2| \lesssim (n_{\max}^{(2)})^2$, we have

$$|\mu_2|^{2-\alpha}(n_{\max}^{(2)})^{6\sigma+4-2\alpha} \gtrsim |\mu_2|^{3s-2\alpha+\frac{5}{2}-3\varepsilon}.$$

Now, we impose $3s - 2\alpha + \frac{5}{2} > 1$, namely,

$$s > \frac{2}{3}\alpha - \frac{1}{2}. \tag{5.20}$$

Under the conditions (5.19) and (5.20), it follows from (5.18) that there exists $\delta > 0$ such that

$$\sup_{n\in\mathbb{Z}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n\\ |\phi_2|\lesssim|\phi_1|}} \frac{(n_{\max}^{(1)})^{4s-6\sigma}(n_{\max}^{(2)})^{-6\sigma}}{|\phi_1|^2} \lesssim \sup_{n\in\mathbb{Z}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n}} \frac{1}{|\mu_1|^{1+\delta}|\mu_2|^{1+\delta}} \lesssim 1. \tag{5.21}$$

By optimizing the conditions (5.19) and (5.20) with $\alpha = \frac{6}{5}$, we obtain the restriction $s > \frac{3}{10}$.

• **Case 1:** We first consider the case $\Pi_2(\mathcal{T}_2) = \{r_2\}$. Namely, the second root node r_2 is a terminal node. By Cauchy-Schwarz inequality with (5.21), we have

$$\begin{aligned}
|\mathsf{N}_1^{(2)}(v)| &\lesssim \sum_{n\in\mathbb{Z}} \sum_{\substack{\mathcal{T}_2\in\mathfrak{BT}(2)\\ \Pi_2(\mathcal{T}_2)=\{r_2\}}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n}} \mathbf{1}_{A_1} \frac{\langle n_r\rangle^{2s}}{|\phi_1|} \prod_{a\in\mathcal{T}_2^\infty} v_{n_a}\\
&\lesssim \|v\|_{H^\sigma} \Bigg\{ \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{\mathcal{T}_2\in\mathfrak{BT}(2)\\ \Pi_2(\mathcal{T}_2)=\{r_2\}}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n}} \mathbf{1}_{A_1} \frac{\langle n\rangle^{2s-\sigma}}{|\phi_1|} \prod_{a\in\mathcal{T}_2^\infty\setminus\{r_2\}} v_{n_a} \Bigg)^2 \Bigg\}^{\frac{1}{2}}\\
&\lesssim \|v\|_{H^\sigma} \sup_{\substack{\mathcal{T}_2\in\mathfrak{BT}(2)\\ \Pi_2(\mathcal{T}_2)=\{r_2\}}} \Bigg(\sup_{n\in\mathbb{Z}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n}} \mathbf{1}_{A_1} \frac{(n_{\max}^{(1)})^{4s-6\sigma}(n_{\max}^{(2)})^{-6\sigma}}{|\phi_1|^2} \Bigg)^{\frac{1}{2}}\\
&\qquad \times \Bigg(\sum_{n\in\mathbb{Z}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n}} \prod_{a\in\mathcal{T}_2^\infty\setminus\{r_2\}} \langle n_a\rangle^{2\sigma}|v_{n_a}|^2 \Bigg)^{\frac{1}{2}}\\
&\lesssim \|v\|_{H^\sigma}^6.
\end{aligned}$$

• **Case 2:** Next, we consider the case $\Pi_2(\mathcal{T}_2) \neq \{r_2\}$. In this case, we need to modify the argument above since the frequency $n_r = n$ does not correspond to a terminal node. Noting that $\mathcal{T}_2^\infty = \Pi_1(\mathcal{T}_2)^\infty \cup \Pi_2(\mathcal{T}_2)^\infty$, we have

$$\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n}} \prod_{a\in\mathcal{T}_2^\infty} |v_{n_a}|^2 = \prod_{j=1}^{2} \Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\Pi_j(\mathcal{T}_2))\\ n_{r_j}=n}} \prod_{a_j\in\Pi_j(\mathcal{T}_2)^\infty} |v_{n_{a_j}}|^2 \Bigg). \tag{5.22}$$

Then, from (5.21) and (5.22), we have

$$\begin{aligned}
|\mathsf{N}_1^{(2)}(v)| &\lesssim \sum_{n\in\mathbb{Z}} \sum_{\substack{\mathcal{T}_2\in\mathfrak{BT}(2)\\ \Pi_2(\mathcal{T}_2)\neq\{r_2\}}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n}} \mathbf{1}_{A_1} \frac{\langle n_r\rangle^{2s}}{|\phi_1|} \prod_{a\in\mathcal{T}_2^\infty} v_{n_a} \\
&\lesssim \sup_{\substack{\mathcal{T}_2\in\mathfrak{BT}(2)\\ \Pi_2(\mathcal{T}_2)\neq\{r_2\}}} \sum_{n\in\mathbb{Z}} \bigg(\sum_{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)} \mathbf{1}_{A_1} \frac{(n_{\max}^{(1)})^{4s-6\sigma}(n_{\max}^{(2)})^{-6\sigma}}{|\phi_1|^2} \bigg)^{\frac{1}{2}} \\
&\qquad\qquad \times \bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n}} \prod_{a\in\mathcal{T}_2^\infty} \langle n_a\rangle^{2\sigma}|v_{n_a}|^2 \bigg)^{\frac{1}{2}} \\
&\lesssim \sup_{\substack{\mathcal{T}_2\in\mathfrak{BT}(2)\\ \Pi_2(\mathcal{T}_2)\neq\{r_2\}}} \sum_{n\in\mathbb{Z}} \bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ n_r=n}} \prod_{a\in\mathcal{T}_2^\infty} \langle n_a\rangle^{2\sigma}|v_{n_a}|^2 \bigg)^{\frac{1}{2}} \\
&\lesssim \sup_{\substack{\mathcal{T}_2\in\mathfrak{BT}(2)\\ \Pi_2(\mathcal{T}_2)\neq\{r_2\}}} \sum_{n\in\mathbb{Z}} \prod_{j=1}^{2} \bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\Pi_j(\mathcal{T}_2))\\ n_{r_j}=n}} \prod_{a_j\in\Pi_j(\mathcal{T}_2)^\infty} \langle n_{a_j}\rangle^{2\sigma}|v_{n_{a_j}}|^2 \bigg)^{\frac{1}{2}} \\
&\lesssim \sup_{\substack{\mathcal{T}_2\in\mathfrak{BT}(2)\\ \Pi_2(\mathcal{T}_2)\neq\{r_2\}}} \prod_{j=1}^{2} \bigg(\sum_{n\in\mathbb{Z}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\Pi_j(\mathcal{T}_2))\\ n_{r_j}=n}} \prod_{a_j\in\Pi_j(\mathcal{T}_2)^\infty} \langle n_{a_j}\rangle^{2\sigma}|v_{n_{a_j}}|^2 \bigg)^{\frac{1}{2}} \\
&\lesssim \|v\|_{H^\sigma}^6.
\end{aligned}$$

This completes the proof of Lemma 5.11. □

Before moving onto the next subsection, let us briefly describe how to handle the highly non-resonant part $\mathsf{N}_2^{(2)}$ in (5.17). On the support of $\mathsf{N}_2^{(2)}$, i.e. on A_1^c, we have

$$|\phi_1+\phi_2| \gg 6^3|\phi_1| \tag{5.23}$$

Namely, the phase function $\phi_1+\phi_2$ is "large" in this case and hence we can exploit this fast oscillation by applying the second step of the normal form reduction:

$$\begin{aligned}
\mathsf{N}_2^{(2)}(v) &= \partial_t\bigg[\sum_{\mathcal{T}_2\in\mathfrak{BT}(2)} \sum_{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)} \mathbf{1}_{A_1^c} \frac{\langle n_r\rangle^{2s} e^{-i(\phi_1+\phi_2)t}}{\phi_1(\phi_1+\phi_2)} \prod_{a\in\mathcal{T}_2^\infty} v_{n_a} \bigg] \\
&\quad - \sum_{\mathcal{T}_2\in\mathfrak{BT}(2)} \sum_{b\in\mathcal{T}_2^\infty} \sum_{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)} \mathbf{1}_{A_1^c} \frac{\langle n_r\rangle^{2s} e^{-i(\phi_1+\phi_2)t}}{\phi_1(\phi_1+\phi_2)} \mathcal{R}(v)_{n_b} \prod_{a\in\mathcal{T}_2^\infty\setminus\{b\}} v_{n_a} \\
&\quad - \sum_{\mathcal{T}_3\in\mathfrak{BT}(3)} \sum_{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_3)} \mathbf{1}_{A_1^c} \frac{\langle n_r\rangle^{2s} e^{-i(\phi_1+\phi_2+\phi_3)t}}{\phi_1(\phi_1+\phi_2)} \prod_{a\in\mathcal{T}_3^\infty} v_{n_a} \\
&=: \partial_t \mathsf{N}_0^{(3)}(v) + \mathsf{R}^{(3)}(v) + \mathsf{N}^{(3)}(v).
\end{aligned}$$

Using (5.23), we can estimate the first two terms $\mathsf{N}_0^{(3)}$ and $\mathsf{R}^{(3)}$ on the right-hand side in a straightforward manner. See Lemmas 5.12 and 5.14 below. As for the last term $\mathsf{N}^{(3)}$, we split it as $\mathsf{N}^{(3)} = \mathsf{N}_1^{(3)} + \mathsf{N}_2^{(3)}$, where $\mathsf{N}_1^{(3)}$ and $\mathsf{N}_2^{(3)}$ are the restrictions onto A_2 and its complement A_2^c, respectively. By exploiting the frequency restriction on $A_1^c \cap A_2$, we can estimate the first term

$\mathsf{N}_1^{(3)}$ (see Lemma 5.15 below). As for the second term $\mathsf{N}_2^{(3)}$, we apply the third step of the normal form reductions. In this way, we iterate normal form reductions in an indefinite manner.

5.4. **General step.** After the Jth step, we have

$$\begin{aligned}\mathsf{N}_2^{(J)}(v) &= \partial_t\bigg[\sum_{\mathcal{T}_J\in\mathfrak{BT}(J)}\sum_{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)}\mathbf{1}_{\bigcap_{j=1}^{J-1}A_j^c}\frac{\langle n_r\rangle^{2s}e^{-i\widetilde{\phi}_J t}}{\prod_{j=1}^J\widetilde{\phi}_j}\prod_{a\in\mathcal{T}_J^\infty}v_{n_a}\bigg]\\&\quad-\sum_{\mathcal{T}_J\in\mathfrak{BT}(J)}\sum_{b\in\mathcal{T}_J^\infty}\sum_{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)}\mathbf{1}_{\bigcap_{j=1}^{J-1}A_j^c}\frac{\langle n_r\rangle^{2s}e^{-i\widetilde{\phi}_J t}}{\prod_{j=1}^J\widetilde{\phi}_j}\mathcal{R}(v)_{n_b}\prod_{a\in\mathcal{T}_J^\infty\setminus\{b\}}v_{n_a}\\&\quad-\sum_{\mathcal{T}_{J+1}\in\mathfrak{BT}(J+1)}\sum_{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{J+1})}\mathbf{1}_{\bigcap_{j=1}^{J-1}A_j^c}\frac{\langle n_r\rangle^{2s}e^{-i\widetilde{\phi}_{J+1} t}}{\prod_{j=1}^J\widetilde{\phi}_j}\prod_{a\in\mathcal{T}_{J+1}^\infty}v_{n_a}\\&=:\partial_t\mathsf{N}_0^{(J+1)}(v)+\mathsf{R}^{(J+1)}(v)+\mathsf{N}^{(J+1)}(v).\end{aligned}\tag{5.24}$$

On $\bigcap_{j=1}^{J-1}A_j^c$, we have $|\phi_1|\geq 1$ and

$$|\widetilde{\phi}_j|\gg(2j+2)^3\max\big(|\widetilde{\phi}_{j-1}|,|\phi_1|\big)\geq(2j+2)^3\tag{5.25}$$

for $j=2,\dots,J$. As in [18, 27, 33], we control the rapidly growing cardinality $c_J=|\mathfrak{BT}(J)|$ defined in (5.9) by the growing constant $(2j+2)^3$ appearing in (5.25).

First, we estimate $\mathsf{N}_0^{(J+1)}$ and $\mathsf{R}^{(J+1)}$.

Lemma 5.12. *Let $\mathsf{N}_0^{(J+1)}$ be as in (5.24). Then, for any $s>\frac{1}{6}$, we have*

$$|\mathsf{N}_0^{(J+1)}(v)|\lesssim\frac{1}{\prod_{j=2}^J(2j+2)^{\frac{1}{3}}}\|v\|_{H^\sigma}^{2J+2}.\tag{5.26}$$

Here, the implicit constant is independent of J.

Proof. From (5.12), we have

$$|\phi_j|\lesssim\max\big(|\widetilde{\phi}_{j-1}|,|\widetilde{\phi}_j|\big).$$

Then, in view of (5.25), we have

$$(2j)^3|\phi_j|\ll|\widetilde{\phi}_{j-1}||\widetilde{\phi}_j|.\tag{5.27}$$

Hence, from (5.27) and then (5.25), we have

$$\prod_{j=1}^J|\widetilde{\phi}_j|^2\gg|\phi_1||\widetilde{\phi}_J|\prod_{j=2}^J\Big((2j)^3|\phi_j|\Big)\gg|\phi_1|^2\prod_{j=2}^J\Big((2j+2)^3|\phi_j|\Big).\tag{5.28}$$

We only discuss the case $\Pi_2(\mathcal{T}_J)=\{r_2\}$ since the modification is straightforward if $\Pi_2(\mathcal{T}_J)\neq\{r_2\}$. Given $s>\frac{1}{6}$, there exists small $\delta>0$ such that

$$\frac{(n_{\max}^{(j)})^{-6\sigma}}{|\phi_j|}\sim\frac{(n_{\max}^{(j)})^{-6\sigma}}{|\mu_j|(n_{\max}^{(j)})^2}\lesssim\frac{1}{|\mu_j|^{1+\delta}}.\tag{5.29}$$

Similarly, we have

$$\frac{(n_{\max}^{(1)})^{4s-6\sigma}}{|\phi_1|^2}\sim\frac{(n_{\max}^{(1)})^{4s-6\sigma}}{|\mu_1|^2(n_{\max}^{(1)})^4}\lesssim\frac{1}{|\mu_1|^2}.\tag{5.30}$$

Then, from (5.28), (5.29), and (5.30), we have

$$
\begin{aligned}
&\sup_{n\in\mathbb{Z}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)\\ n_r=n}} \mathbf{1}_{\bigcap_{j=1}^{J-1} A_j^c} \cdot (n_{\max}^{(1)})^{4s} \prod_{j=1}^{J} \frac{(n_{\max}^{(j)})^{-6\sigma}}{|\widetilde{\phi}_j|^2} \\
&\ll \frac{1}{\prod_{j=1}^{J}(2j+2)^3} \cdot \sup_{n\in\mathbb{Z}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)\\ n_r=n\\ \phi_j\neq 0\\ j=1,\dots,J}} \mathbf{1}_{\bigcap_{j=1}^{J-1} A_j^c} \frac{(n_{\max}^{(1)})^{4s-6\sigma}}{|\phi_1|^2} \prod_{j=2}^{J} \frac{(n_{\max}^{(j)})^{-6s}}{|\phi_j|} \\
&\lesssim \frac{1}{\prod_{j=1}^{J}(2j+2)^3} \cdot \sup_{n\in\mathbb{Z}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)\\ n_r=n\\ \mu_j\neq 0\\ j=1,\dots,J}} \frac{1}{|\mu_1|^2} \prod_{j=2}^{J} \frac{1}{|\mu_j|^{1+\delta}} \\
&\le \frac{C^J}{\prod_{j=1}^{J}(2j+2)^3}.
\end{aligned}
\tag{5.31}
$$

Hence, by Cauchy-Schwarz inequality and (5.31), we have

$$
\begin{aligned}
|\mathsf{N}_0^{(J+1)}(v)| &\lesssim \|v\|_{H^\sigma} \sum_{\substack{\mathcal{T}_J\in\mathfrak{BT}(J)\\ \Pi_2(\mathcal{T}_J)=\{r_2\}}} \Bigg\{ \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)\\ n_r=n}} \mathbf{1}_{\bigcap_{j=1}^{J-1} A_j^c} \frac{\langle n_{\max}^{(1)}\rangle^{4s-6\sigma}}{|\phi_1|^2} \prod_{j=2}^{J} \frac{\langle n_{\max}^{(j)}\rangle^{-6\sigma}}{|\widetilde{\phi}_j|^2} \Bigg) \\
&\qquad \times \Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)\\ n_r=n}} \prod_{a\in\mathcal{T}_J^\infty\setminus\{r_2\}} |v_{n_a}|^2 \Bigg) \Bigg\}^{\frac12} \\
&\lesssim \frac{c_J\cdot C^{\frac{J}{2}}}{\prod_{j=2}^{J}(2j+2)^{\frac32}} \|v\|_{H^\sigma}^{2J+2}.
\end{aligned}
\tag{5.32}
$$

Then, the desired bound (5.26) follows from (5.9). □

Remark 5.13. At the first inequality in (5.32), we needed the full power $\langle n_{\max}^{(j)}\rangle^{-6\sigma}$ only for those j's such that the three terminal nodes of the tree added in the $(j-1)$th step are also in $\mathcal{T}_J^\infty$. For example, $j = J$ satisfies this condition. For other values of j, a smaller power may suffice. Note, however, that we need to use (5.29) at least for $j = J$, thus requiring the regularity restriction $s > \frac16$. We therefore simply used the maximum power $\langle n_{\max}^{(j)}\rangle^{-6\sigma}$ for all $j = 1, \dots, J$ at the first inequality in (5.32). The same comments applies to Lemmas 5.14 and 5.15.

Lemma 5.14. *Let $\mathsf{R}^{(J+1)}$ be as in* (5.24)*. Then, for any $\frac16 < s \le \frac12$, we have*

$$
|\mathsf{R}^{(J+1)}(v)| \lesssim \frac{1}{\prod_{j=2}^{J}(2j+2)^{\frac13}} \|v\|_{H^\sigma}^{2J+4}. \tag{5.33}
$$

Here, the implicit constant is independent of J.

Proof. We consider two cases: (i) $|\widetilde{\phi}_J| \gtrsim |\phi_J|$ and (ii) $|\widetilde{\phi}_J| \ll |\phi_J|$.

- **Case 1:** $|\widetilde{\phi}_J| \gtrsim |\phi_J|$. From (5.10), we have

$$
(n_{\max}^{(J)})^{-4\sigma} \lesssim |\phi_J|^{-2\sigma} \tag{5.34}
$$

for $\sigma \le 0$, namely, $s \le \frac{1}{2}$. From (5.25), we have

$$|\widetilde{\phi}_J| = |\widetilde{\phi}_J|^{-2\sigma}|\widetilde{\phi}_J|^{1+2\sigma} \gg |\widetilde{\phi}_J|^{-2\sigma}|\phi_1|^{1+2\sigma} \gtrsim |\phi_J|^{-2\sigma}|\phi_1|^{1+2\sigma}, \tag{5.35}$$

provided that $1 + 2\sigma \ge 0$, namely, $s > 0$. We also observe that

$$\frac{(n_{\max}^{(1)})^{4s-6\sigma}}{|\phi_1|^{2+2\sigma}} \sim \frac{1}{|\mu_1|^{2+2\sigma}(n_{\max}^{(1)})^{4-4s+10\sigma}}. \tag{5.36}$$

Note that $4 - 4s + 10\sigma > 0$ and $2 + 2\sigma > 1$, provided that $s > \frac{1}{6}$. Then, by applying (5.28) and (5.35) followed by (5.29), (5.34), and (5.36), we have

$$\begin{aligned}
&\sup_{n\in\mathbb{Z}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)\\ n_r=n}} \mathbf{1}_{\bigcap_{j=1}^{J-1} A_j^c} \cdot (n_{\max}^{(J)})^{-4\sigma}(n_{\max}^{(1)})^{4s} \prod_{j=1}^{J} \frac{(n_{\max}^{(j)})^{-6\sigma}}{|\widetilde{\phi}_j|^2}\\
&\ll \frac{1}{\prod_{j=2}^{J}(2j)^3} \cdot \sup_{n\in\mathbb{Z}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)\\ n_r=n\\ \phi_j\neq 0\\ j=1,\dots,J}} \mathbf{1}_{\bigcap_{j=1}^{J-1} A_j^c} \frac{(n_{\max}^{(J)})^{-4\sigma}}{|\widetilde{\phi}_J|} \frac{(n_{\max}^{(1)})^{4s-6\sigma}}{|\phi_1|} \prod_{j=2}^{J} \frac{(n_{\max}^{(j)})^{-6\sigma}}{|\phi_j|}\\
&\ll \frac{1}{\prod_{j=2}^{J}(2j)^3} \cdot \sup_{n\in\mathbb{Z}} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)\\ n_r=n\\ \phi_j\neq 0\\ j=1,\dots,J}} \frac{(n_{\max}^{(J)})^{-4\sigma}}{|\phi_J|^{-2\sigma}} \frac{(n_{\max}^{(1)})^{4s-6\sigma}}{|\phi_1|^{2+2\sigma}} \prod_{j=2}^{J} \frac{(n_{\max}^{(j)})^{-6\sigma}}{|\phi_j|}\\
&\lesssim \frac{C^J}{\prod_{j=2}^{J}(2j)^3}.
\end{aligned} \tag{5.37}$$

Hence, proceeding as in (5.32) with (5.37) and (5.9), we obtain (5.33) in this case.

- **Case 2:** $|\widetilde{\phi}_J| \ll |\phi_J|$. In this case, we have $|\phi_J| \sim |\widetilde{\phi}_{J-1}|$. From (5.27), we have

$$\prod_{j=1}^{J} |\widetilde{\phi}_j|^2 \gg |\phi_1||\widetilde{\phi}_{J-1}||\widetilde{\phi}_J|^2 \prod_{j=2}^{J-1} \Big((2j)^3|\phi_j|\Big). \tag{5.38}$$

From (5.25), we also have

$$\begin{aligned}
|\widetilde{\phi}_J| &\gg (2J+2)^3|\widetilde{\phi}_{J-1}| \sim (2J+2)^3|\phi_J|,\\
|\widetilde{\phi}_{J-1}| &\gg (2J)^3|\phi_1|.
\end{aligned} \tag{5.39}$$

Thus, from (5.38) and (5.39), we have

$$\prod_{j=1}^{J} |\widetilde{\phi}_j|^2 \gg |\phi_1||\phi_J| \prod_{j=1}^{J} \Big((2j+2)^3|\phi_j|\Big). \tag{5.40}$$

Note that

$$\frac{(n_{\max}^{(J)})^{-10\sigma}}{|\phi_J|^2} \sim \frac{(n_{\max}^{(J)})^{-10\sigma}}{|\mu_J|^2(n_{\max}^{(J)})^4} \le \frac{1}{|\mu_J|^2}, \tag{5.41}$$

provided that $4+10\sigma>0$, namely, $s>\frac{1}{10}$. Then, from (5.40), (5.29), (5.30), and (5.41), we have

$$\begin{aligned}
&\sup_{n\in\mathbb{Z}}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)\\ n_r=n}}\mathbf{1}_{\bigcap_{j=1}^{J-1}A_j^c}\cdot(n_{\max}^{(J)})^{-4\sigma}(n_{\max}^{(1)})^{4s}\prod_{j=1}^{J}\frac{(n_{\max}^{(j)})^{-6\sigma}}{|\widetilde{\phi}_j|^2}\\
&\quad\ll\frac{1}{\prod_{j=1}^{J}(2j+2)^3}\cdot\sup_{n\in\mathbb{Z}}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)\\ n_r=n\\ \phi_j\neq0\\ j=1,\dots,J}}\frac{(n_{\max}^{(J)})^{-10\sigma}}{|\phi_J|^2}\frac{(n_{\max}^{(1)})^{4s-6\sigma}}{|\phi_1|^2}\prod_{j=2}^{J-1}\frac{(n_{\max}^{(j)})^{-6\sigma}}{|\phi_j|}\\
&\quad\lesssim\frac{C^J}{\prod_{j=1}^{J}(2j+2)^3}.
\end{aligned}\tag{5.42}$$

Hence, proceeding as in (5.32) with (5.42) and (5.9), we obtain (5.33) in this case. □

Finally, we consider $\mathsf{N}^{(J+1)}$. As before, we write

$$\mathsf{N}^{(J+1)}=\mathsf{N}_1^{(J+1)}+\mathsf{N}_2^{(J+1)},\tag{5.43}$$

where $\mathsf{N}_1^{(J+1)}$ is the restriction of $\mathsf{N}^{(J+1)}$ onto A_J defined in (5.11) and $\mathsf{N}_2^{(J+1)}:=\mathsf{N}^{(J+1)}-\mathsf{N}_1^{(J+1)}$. In the following lemma, we estimate the first term $\mathsf{N}_1^{(J+1)}$. Then, we apply a normal form reduction once again to the second term $\mathsf{N}_2^{(J+1)}$ as in (5.24) and repeat this process indefinitely. Lemma 5.16 below shows that, for a smooth function v, this error term $\mathsf{N}_2^{(J+1)}$ tends to 0 as $J\to\infty$.

Lemma 5.15. *Let $\mathsf{N}_1^{(J+1)}$ be as in (5.24). Then, for any $s>\frac{3}{10}$, we have*

$$|\mathsf{N}_1^{(J+1)}(v)|\lesssim\frac{1}{\prod_{j=2}^{J}(2j+2)^{\frac{1}{3}}}\|v\|_{H^\sigma}^{2J+4}.\tag{5.44}$$

Here, the implicit constant is independent of J.

Proof. On $A_J\cap A_{J-1}^c$, we have $|\widetilde{\phi}_{J+1}|\lesssim(2J+4)^3|\widetilde{\phi}_J|$ and thus

$$|\phi_{J+1}|\lesssim|\widetilde{\phi}_{J+1}|+|\widetilde{\phi}_J|\lesssim J^3|\widetilde{\phi}_J|.\tag{5.45}$$

Then, from (5.27), (5.45), and (5.25), we have

$$\begin{aligned}
J^3\prod_{j=1}^{J}|\widetilde{\phi}_j|^2&\gg|\phi_1|(J^3|\widetilde{\phi}_J|)^{1-\alpha}(J^3|\widetilde{\phi}_J|)^{\alpha}\prod_{j=2}^{J}\Big((2j)^3|\phi_j|\Big)\\
&\gtrsim|\phi_1|^{2-\alpha}|\phi_{J+1}|^{\alpha}\prod_{j=2}^{J}\Big((2j)^3|\phi_j|\Big)
\end{aligned}\tag{5.46}$$

for $0\le\alpha\le1$.

Writing

$$\frac{(n_{\max}^{(1)})^{4s-6\sigma}}{|\phi_1|^{2-\alpha}}\sim\frac{1}{|\mu_1|^{2-\alpha}(n_{\max}^{(1)})^{-4s+6\sigma-2\alpha+4}}\tag{5.47}$$

and

$$\frac{(n_{\max}^{(J+1)})^{-6\sigma}}{|\phi_{J+1}|^{\alpha}}\sim\frac{1}{|\mu_{J+1}|^{\alpha}(n_{\max}^{(J+1)})^{6\sigma+2\alpha}},\tag{5.48}$$

we impose $-4s+6\sigma-2\alpha+4>0$ and $6\sigma+2\alpha>0$, namely,

$$s>\alpha-\frac{1}{2} \qquad \text{and} \qquad s>-\frac{\alpha}{3}+\frac{1}{2}. \tag{5.49}$$

From $|\mu_{J+1}|\lesssim (n_{\max}^{(J+1)})^2$, we have

$$|\mu_{J+1}|^{\alpha}(n_{\max}^{(J+1)})^{6\sigma+2\alpha}\gtrsim |\mu_2|^{3s+2\alpha-\frac{3}{2}-3\varepsilon}.$$

We now impose $3s+2\alpha-\frac{3}{2}>1$, namely,

$$s>-\frac{2}{3}\alpha+\frac{5}{6}. \tag{5.50}$$

By optimizing the conditions (5.49) and (5.50) with $\alpha=\frac{4}{5}$, we obtain the restriction $s>\frac{3}{10}$. Hence, for $s>\frac{3}{10}$, it follows from (5.46), (5.47), (5.48), and (5.29) that

$$\begin{aligned}
&\sup_{n\in\mathbb{Z}}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{J+1})\\ n_r=n}}\mathbf{1}_{A_J\cap(\bigcap_{j=1}^{J-1}A_j^c)}\cdot(n_{\max}^{(J+1)})^{-6\sigma}(n_{\max}^{(1)})^{4s}\prod_{j=1}^{J}\frac{(n_{\max}^{(j)})^{-6\sigma}}{|\widetilde{\phi}_j|^2}\\
&\quad\ll\frac{J^3}{\prod_{j=2}^{J}(2j)^3}\cdot\sup_{n\in\mathbb{Z}}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{J+1})\\ n_r=n\\ |\phi_j|\neq 0\\ j=1,\dots,J+1}}\frac{(n_{\max}^{(J+1)})^{-6\sigma}}{|\phi_{J+1}|^{\alpha}}\frac{(n_{\max}^{(1)})^{4s-6\sigma}}{|\phi_1|^{2-\alpha}}\prod_{j=2}^{J}\frac{(n_{\max}^{(j)})^{-6\sigma}}{|\phi_j|}\\
&\quad\lesssim\frac{J^3}{\prod_{j=2}^{J}(2j)^3}\cdot\sup_{n\in Z}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{\mathbf{J+1}})\\ n_r=n}}\prod_{j=1}^{J+1}\frac{1}{|\mu_j|^{1+\delta}}\\
&\quad\lesssim\frac{C^{J+1}J^3}{\prod_{j=2}^{J}(2j)^3}
\end{aligned} \tag{5.51}$$

for some small $\delta>0$. Then, the desired bound (5.44) follows from the Cauchy-Schwarz argument with (5.51). □

We conclude this subsection by showing that the error term $\mathsf{N}_2^{(J+1)}$ in (5.43) tends to 0 as $J\to\infty$ under some regularity assumption on v. From (5.24), we have

$$\mathsf{N}_2^{(J+1)}(v)=-\sum_{\mathcal{T}_{J+1}\in\mathfrak{BT}(J+1)}\sum_{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{J+1})}\mathbf{1}_{\bigcap_{j=1}^{J}A_j^c}\frac{\langle n_r\rangle^{2s}e^{-i\widetilde{\phi}_{J+1}t}}{\prod_{j=1}^{J}\widetilde{\phi}_j}\prod_{a\in\mathcal{T}_{J+1}^{\infty}}v_{n_a}. \tag{5.52}$$

Lemma 5.16. *Let $\sigma>\frac{1}{2}$. Then, given any $v\in H^{\sigma}(\mathbb{T})$, we have*

$$|\mathsf{N}_2^{(J+1)}(v)|\longrightarrow 0,$$

as $J\to\infty$.

Proof. By the algebra property of $H^s(\mathbb{T})$, $s>\frac{1}{2}$, we can easily bound (5.52) by $o_{J\to\infty}(1)\|v\|_{H^s}^{2J+4}$, where the decay in J comes from (5.25) for $j=2,\dots J+1$. See also [27, Subsection 4.5]. □

Remark 5.17. We point out that one can actually prove Lemma 5.16 under a weaker regularity assumption $\sigma\geq\frac{1}{6}$. See [33, Lemma 8.15].

5.5. **Proof of Proposition 5.1.** We briefly discuss the proof of Proposition 5.1. Let v be a smooth global solution to (5.1). Then, by applying the normal form reduction J times, we obtain[17]

$$\frac{d}{dt}\bigg(\frac{1}{2}\|v(t)\|_{H^s}^2\bigg) = \frac{d}{dt}\bigg(\sum_{j=2}^{J+1}\mathsf{N}_0^{(j)}(v)(t)\bigg) + \sum_{j=2}^{J+1}\mathsf{N}_1^{(j)}(v)(t) + \sum_{j=2}^{J+1}\mathsf{R}^{(j)}(v)(t) + \mathsf{N}_2^{(J+1)}(v)(t).$$

For a smooth solution v, Lemma 5.16 allows us to take a limit as $J \to \infty$, yielding

$$\frac{d}{dt}\bigg(\frac{1}{2}\|v(t)\|_{H^s}^2\bigg) = \frac{d}{dt}\bigg(\sum_{j=2}^{\infty}\mathsf{N}_0^{(j)}(v)(t)\bigg) + \sum_{j=2}^{\infty}\mathsf{N}_1^{(j)}(v)(t) + \sum_{j=2}^{\infty}\mathsf{R}^{(j)}(v)(t).$$

Therefore, we obtain (5.3) for a smooth solution v to (5.1). For a rough solution $v \in C(\mathbb{R}; H^\sigma(\mathbb{T}))$, $-\frac{1}{5} < \sigma \le 0$, we can obtain the identity (5.3) by a limiting argument. This argument is standard and thus we omit details. See, for example, Subsection 8.5 in [33].

The bounds (5.4), (5.5), and (5.6) follow from Lemmas 5.9, 5.10, 5.11, 5.12, 5.14, and 5.15. This proves Proposition 5.1.

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