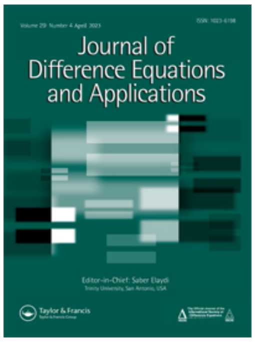

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A global picture for the planar Ricker map: convergence to fixed points and identification of the stable/unstable manifolds

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ABSTRACT

A quadratic Lyapunov function is demonstrated for the non-invertible planar Ricker map $(x, y) \mapsto (xe^{r-x-\alpha y}, ye^{s-y-\beta x})$ which shows that for $\alpha, \beta > 0$, and $0 < r, s \leq 2$ all orbits of the planar Ricker map converge to a fixed point. We establish that for $0 < r$, $s < 2$, whenever a positive equilibrium exists and is locally asymptotically stable, it is globally asymptotically stable (i.e. attracts all of $(0, \infty)^2$). Our approach bypasses and improves on methods that rely on monotonicity, which require $0 < r, s \leq 1$. We also use the Lyapunov function to identify the one-dimensional stable and unstable manifolds when the positive fixed point exists and is a hyperbolic saddle.

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1. Introduction

Consider the planar Ricker map on $C_+ := [0, \infty)^2$, which is the non-invertible map

$$(x, y) \in C_+ \mapsto \left(xe^{r-x-\alpha y}, ye^{s-y-\beta x}\right), \tag{1}$$

where we take r, $s > 0$ and $\alpha, \beta > 0$. The map occurs in the context of theoretical ecology and, along with its one-dimensional version [19], has been extensively studied (for an extensive bibliography, epecially in the ecology context, see the recent [13]). In this article we use a Lyapunov function to study the global dynamics of (1), an approach which enables us to prove global convergence to a fixed point when $0 < r, s \leq 2$, and global asymptotic stability of the positive fixed point when it exists for $0 < r$, $s < 2$, a range which improves significantly on existing published results. Moreover, again using the Lyapunov function, we identify the stable and unstable manifolds, and in particular for the case where there is a positive fixed point which is a hyperbolic saddle. As a result we are able to completely classify the dynamics of the Ricker map when $< r, s \leq 2$. We find that the map satisfies the classical exclusion principle for some parameters and coexistence of species for others,

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in exactly the same fashion as the planar competitive Lotka-Volterra differential equation model [2].

The corresponding one-dimensional map $h(x) = xe^{r-x}$ obtained by restricting trajectories of (1) to the forward invariant line $y = 0$ is known to be globally convergent on $(0, \infty)$ to $x = r$ when $r \leq 2$. At $r = 2$ a locally stable 2-cycle is born, and as r increases past 2 period-doubling takes place. On page 516 of [18] May shows that $(x - r)^2$ is a Lyapunov function for the one-dimensional Ricker map h when $0 < r < 2$. Our Lyapunov function is equivalent to May's (up to a constant) when the map (1) is restricted to either axis. In [18] May also shows that there is a stable 2-cycle for $2 < r < 2.526$ and this 2-cycle becomes unstable for $r > 2.526$ and a stable 4-cycle appears which is stable for $r < 2.656$. May shows that this period doubling then leads to chaos for $r > 3.102$.

The picture of global stability in the literature for the planar map (1) is less complete. It is known that the dynamics of the planar system (1) can be highly complicated; see for example, [13], who study bifurcations in (1) in parameter regions outside of those considered in the present article.

For parameter regions similar to those that we study here, Fisher and Goh [8] consider a scaled but equivalent system to (1) and put forward a logarithmic Lyapunov function for the positive fixed point when it exists, but do not provide a proof that their Lyapunov function is decreasing along trajectories. However, their proposed Lyapunov function does not apply in all cases where $0 < r, s < 2$. In fact, even for the map $T(x) = xe^{1.92(1-x)}$ (which is equivalent to $xe^{1.92-x}$ after rescaling x), with their Lyapunov function $V(x) = \frac{x^2}{2} - \log x$ we find that $V(T(x)) - V(x)$ changes sign at approximately $x = 1.51136$ and $x = 1.57895$, so that V is then not a Lyapunov function.

In [12] the authors consider the special case of symmetry in (1) when $T(x, y) = (xe^{a(1-x)-by}, ye^{a(1-y)-bx})$, where they focus their investigation on invariant regions and bifurcation analysis, but not global stability of fixed points. In [17], the authors provide a complete analysis of *local* stability of the positive fixed point of (1) when it exists (see Equation (10) in Remark 3.1). To show *global* asymptotic stability of the positive fixed point (i.e. show that it is asymptotically stable and that it attracts $(0, \infty)^2$) has proved more challenging. In fact it has been conjectured that the positive fixed point is globally asymptotically stable if and only if it is (locally) asymptotically stable (e.g. [6]). Our results here increase support of this conjecture, but fall short of proving it.

In [25], Smith used monotone systems theory for orderings induced by the 4th quadrant to establish convergence of trajectories of (1) when $r, s \leq 1$. An alternative viewpoint uses that the map is retrotone on a forward-invariant rectangle R (see [21]).

Definition 1.1: A map $T = (F, G) : C_+ \to C_+$ is retrotone (e.g. [10,21]) in a subset $Q \subset C_+$ if for $x, y \in Q$ such that $F(x) \geq F(y)$ and $G(x) \geq G(y)$ but $T(x) \neq T(y)$ we have $x_1 > y_1$ provided $y_1 > 0$ and $x_2 > y_2$ provided $y_2 > 0$.

A retrotone map is sometimes also called a competitive map. The terminology sometimes can cause confusion in an ecological context: A retrotone map must involve competitive interactions between species, but a map that models competitive interactions between species may not be a retrotone map. When the Ricker map is retrotone (see for example [10,11,21]), global attraction to fixed points can be obtained using the results for planar competitive or cooperative systems due to Smith [25]. However, the parameter values for retrotonicity (assuming $r, s > 0$):

$$r + s \le 1 + rs(1 - \alpha\beta) \le 2 \tag{2}$$

are in a proper subset of those given in Theorem 3.6, since in particular (2) implies that $r, s \le 1$. Another approach is to use the carrying simplex for retrotone systems (e.g. [3,9,11]). Roughly speaking retrotonicity of a map means that the inverse of the map (1) restricted to the rectangle R is monotone for the first quadrant ordering, so that there is a carrying simplex [9–11,21] consisting of an invariant curve that connects the axial fixed points (and includes the positive fixed point when it exists). Since the carrying simplex attracts all nonzero points in the non-negative orthant, global stability results are easily obtained by restriction of trajectories to the carrying simplex. Both approaches establish the conjecture that local asymptotic stability implies global asymptotic stability when $0 < r, s \le 1$. However, both the methods of Smith and the carrying simplex only apply to (1) when $0 < r, s \le 1$.

In the appendix of [25] Smith also used properties of planar competitive maps to identify the stable and unstable manifolds of the Ricker map when the map parameters are such that it is a competitive map (which requires $0 < r, s \le 1$ at least). Similar to Smith, we identify the stable and unstable manifolds, in particular for the case where there is a positive hyperbolic saddle fixed point, but for the larger parameter region $0 < r, s < 2$. Our proof relies strongly of the existence of a Lyapunov function, not monotonicty, which is not available to us, and is also contained in an appendix.

In attempts to extend Smith's results beyond $r, s \le 1$ progress has been made by several authors who establish the existence of a one-dimensional attracting invariant manifold akin to the carrying simplex, but when (1) is no longer retrotone [4,17,23,24]. For example, in [4] using singularity theory, the authors established conditions on the existence of the attracting invariant manifold by looking at intersections of critical curves and bounds on r, s that also depend on the competition parameters α, β. Recent work in [22–24], improved the previous results and found optimal bounds of the parameters, but do not cover all parameter cases when $0 < r, s \le 2$.

Here we are able to show global convergence of (1) to a fixed point of T (the actual fixed point may depend on the initial data) for all parameter values $\alpha, \beta > 0$ (and not just $\alpha\beta < 1$) and $0 < r, s \le 2$.

The presentation is split into two parts. The main part is a complete description of the dynamics of the Ricker map (1) for $0 < r, s < 2$. The second part, for which the proofs are given in the appendix, is devoted to identifying the stable and unstable manifolds when the positive fixed point of (1) exists and is a hyperbolic saddle point.

2. Some background on Lyapunov functions

Let X be a metric space with metric d and subsets $A, B \subset X$. $\overline{A}$ denotes closure of a set A and we set $\mathbb{N} = \{0, 1, 2, \ldots\}$.

For $G \subset X$ we say that a continuous function $V : G \to \mathbb{R}$ is a Lyapunov function for the (continuous) map $T : X \to X$ in G if $\Delta V = V \circ T - V$ satisfies $\Delta V(x) \le 0$ (or $\Delta V(x) \ge 0$) whenever $x, T(x) \in G$.

We recall that the omega limit set for $x \in X$ is the set $\omega(x)$ of points $y \in X$ such that there exists a sequence $\mathbb{N} \ni n_k \to \infty$ as $k \to \infty$ and $\lim_{k\to\infty} T^{n_k}(x) = y$. An omega limit set for x is non-empty if its orbit $\{T^k(x)\}_{k\in\mathbb{N}}$ is precompact (i.e. is contained in a compact set). Unlike the situation in continuous dynamical systems, the omega limit set $\omega(x)$ is, in general, not connected, but it is invariantly connected [14]; that is, it is not the union of two disjoint non-empty open (closed) invariant sets.

A set H is said to be stable (under T) relative to a set $S \subset X$, if given a neighbourhood W of H there exists a neighbourhood U of H such that $T^k(U \cap S) \subset W$ for all $k \in \mathbb{N}$. The basin of attraction of H in an open set $S \subset X$, $B_S(H)$, is the union of all (relatively open) sets $V \cap S$ with V open in X such that $\text{dist}(T^k(V \cap S), H) \to 0$ as $k \to \infty$; $B_S(H) = \{x \in S : \omega(x) \neq \emptyset \text{ and } \omega(x) \subset H\}$. We say that H is asymptotically stable relative to S if H is stable relative to S and $B_S(H)$ is nonempty. We say that H is globally asymptotically stable relative to S if $B_S(H) = S$.

H is a (uniform) repeller (for T) relative to S if there is some open set $U \subset X$ containing H and some N_0 such that $T^k(x) \notin U$ for all $k \geq N_0$ and $x \in (U \cap S) \setminus H$. For example, a fixed point $z \in X$ is a uniform repeller when all eigenvalues of $DT(z)$ exceed 1 in modulus.

The following Proposition applied to (1) will be useful for both establishing convergence to a fixed point of orbits, and also in identifying the stable and unstable manifolds of fixed points. It appears in various guises in the Mathematical Genetics literature in relation to the Fundamental Theorem of Natural Selection (e.g. [15,16]). We provide a proof of our statement of the result below for convenience.

Proposition 2.1: *Let X be a metric space and $M \subset X$. Let $S : M \to X$ be a continuous map with Lyapunov function $V : X \to \mathbb{R}$, and suppose that $V(S(x)) = V(x)$ if and only if x is a fixed point of S.*

Let $x_0 \in M$ be such that $S^k(x_0) \in M$ for all k. Then if $S^k(x_0)$ is relatively compact, the omega limit set $\omega(x_0)$ is compact, invariant, connected and consists only of fixed points of S.

Hence if the fixed points of S are finite in number, $S^k(x_0)$ converges to a fixed point as $k \to \infty$.

Proof: For $x_0 \in M$, $S^k(x_0) \in M$ and $V(S^k(x_0))$ is monotonic in k and bounded since $S^k(x_0)$ is relatively compact. As $S^k(x_0)$ is relatively compact, $\omega(x_0)$ is non-empty, compact and invariant (e.g. Theorem 3.1 in [20]). Hence $V(S^k(x_0)) \to c$ as $k \to \infty$ where c is a constant. Let $y \in \omega(x_0)$ so that there is some $k_n \to \infty$ as $n \to \infty$ such that $S^{k_n}(x_0) \to y$. Then $c = \lim_{n\to\infty} V(S^{k_n}(x_0)) = V(y)$. So V is constant on $\omega(x_0)$. Since $\omega(x_0)$ is forward invariant and for $y \in \omega(x_0)$, $V(S(y)) = V(y) = c$ and hence, by hypothesis, y is a fixed point of S. Thus $\omega(x_0)$ is compact, invariant and consists of only fixed points. It remains to show that $\omega(x_0)$ is connected, if not, then it is the union of two non-empty disjoint closed sets, A and B. Since both A and B consist of fixed points, they must be invariant. This contradicts the property that the omega limit set is invariantly connected. Hence $\omega(x_0)$ is connected. ■

3. A quadratic Lyapunov function for the planar Ricker map

To study (1) we will use the Lyapunov function

$$V(x,y) = V_R(x,y) := \beta x^2 + \alpha y^2 + 2\alpha\beta xy - 2r\beta x - 2s\alpha y. \tag{3}$$

For convenience, let us set $X = r - x - \alpha y$, $Y = s - y - \beta x$ so that (1) becomes $T(x,y) = (xe^X, ye^Y)$. We will find conditions on the Ricker map T for which V_R is a Lyapunov function on C_+: $\Delta V_R := V_R \circ T - V_R$ satisfies $\Delta V_R(x,y) \leq 0$ for all $(x,y) \in C_+$.

For (1) there are always fixed points at $(0,0)$, $(r,0)$, $(0,s)$, and there is a positive fixed point

$$(p,q) = \left(\frac{r-\alpha s}{1-\alpha\beta}, \frac{s-\beta r}{1-\alpha\beta}\right), \tag{4}$$

whenever $\alpha < \frac{r}{s} < \frac{1}{\beta}$ or $\frac{1}{\beta} < \frac{r}{s} < \alpha$. For convenience we use the notation $C_{++} = (0,\infty)^2$, so that a positive fixed point $(p,q) \in C_{++}$. If $\alpha = \frac{r}{s} = \frac{1}{\beta}$ then the lines $x + \alpha y = r$ and $y + \beta x = s$ coincide and all points in C_+ on these lines are fixed points. Hence unless $\alpha = \frac{r}{s} = \frac{1}{\beta}$ all fixed points are isolated.

The derivative of T is

$$DT(x,y) = \begin{pmatrix} (1-x)e^{r-x-\alpha y} & -\alpha x e^{r-x-\alpha y} \\ -\beta y e^{s-y-\beta x} & (1-y)e^{s-y-\beta x} \end{pmatrix}. \tag{5}$$

With σ denoting the spectrum of a matrix,

$$\sigma(DT(0,0)) = \{e^r, e^s\}, \quad \sigma(DT(r,0)) = \{1-r, e^{s-\beta r}\}, \quad \sigma(DT(0,s)) = \{1-s, e^{r-s\alpha}\}, \tag{6}$$

and when (p,q) exists,

$$\sigma(DT(p,q)) = \sigma\begin{pmatrix} (1-p) & -\alpha p \\ -\beta q & (1-q) \end{pmatrix}, \tag{7}$$

which we will analyse further later.

As noted in [25] we have

Lemma 3.1: *Let $T = (F,G)$ and $D = [0, e^{r-1}] \times [0, e^{s-1}]$, $r,s \geq 0$. Then $T(C_+) \subset D$ so that all trajectories of T in C_+ are bounded and the compact set D is forward invariant. Moreover, $[0,r] \times [0,s] \subset D$.*

Proof: For any $y \geq 0$, $F(x,y) \leq \phi_1(x) := xe^{r-x}$ and $\phi_1([0,\infty)) = [0, e^{r-1}]$. Similarly for any $x \geq 0$, $G(x,y) \leq ye^{s-y}$. Thus $T(C_+) \subset [0, e^{r-1}] \times [0, e^{s-1}]$. Since $e^a \geq 1 + a$ for $a \in \mathbb{R}$ we obtain $e^{r-1} \geq r$ and $e^{s-1} \geq s$, and so $[0,r] \times [0,s] \subset D$. ∎

Now we show that V_R is a Lyapunov function for (1). For $(x,y) \in C_+$ we have

$$\begin{aligned}\Delta V_R(x,y) &= \beta x^2(e^X-1)^2 + 2\alpha\beta xy(e^X-1)(e^Y-1) + \alpha y^2(e^Y-1)^2 \\ &\quad - 2\beta xX(e^X-1) - 2\alpha yY(e^Y-1) \\ &\leq \beta x^2(e^X-1)^2 + \alpha y^2(e^Y-1)^2 + \alpha\beta xy((e^X-1)^2 \\ &\quad + (e^Y-1)^2) - 2\beta xX(e^X-1) - 2\alpha yY(e^Y-1) \\ &= (\beta x^2 + \alpha\beta xy)(e^X-1)^2 + (\alpha y^2 + \alpha\beta xy)(e^Y-1)^2 \\ &\quad - 2\beta xX(e^X-1) - 2\alpha yY(e^Y-1) \\ &= -\beta x(X-r)(e^X-1)^2 - \alpha y(Y-s)(e^Y-1)^2 \\ &\quad - 2\beta xX(e^X-1) - 2\alpha yY(e^Y-1) \\ &= -\beta x(e^X-1)((X-r)(e^X-1)+2X) - \alpha y(e^Y-1)((Y-s)(e^Y-1)+2Y).\end{aligned} \tag{8}$$

Now define, for $u \in \mathbb{R}$,

$$\Phi(u,t) = (1-e^u)(t-u) + 2u$$

so that ΔV_R becomes

$$\Delta V_R(x,y) \leq -\beta x(e^X-1)\Phi(X,r) - \alpha y(e^Y-1)\Phi(Y,s). \tag{9}$$

Lemma 3.2: *Let $\Omega(u,t) = (e^u-1)\Phi(u,t) = (e^u-1)((1-e^u)(t-u)+2u)$ with $0 < t \leq 2$. Then $\Omega(u,t) \geq 0$ for all $u \in \mathbb{R}$ with equality if and only if $u = 0$.*

Proof: We compute

$$\begin{aligned}\Phi(u,t) &= (1-e^u)(t-u) + 2u, \\ \Phi_u(u,t) &= 1 + e^u(1-t+u), \\ \Phi_{uu}(u,t) &= e^u(2-t+u).\end{aligned}$$

As $t \in (0,2]$, $\Phi_{uu}(u,t) > 0$ for $u > 0$, so $\Phi_u(u,t)$ is strictly increasing in $u > 0$. As $\Phi_u(0,t) = 2-t \geq 0$, $\Phi(u,t)$ is increasing for $u > 0$. For $u < 0$, since $e^{-u} > 1-u$, we have $\Phi_u(u,t) = e^u(e^{-u}+1-t+u) > e^u(2-t) \geq 0$ so $\Phi_u(u,t) > 0$ if $u < 0$. Hence $\Phi(u,t)$ is strictly increasing for all $u \neq 0$. Then $\Phi(0,t) = 0$ implies that $\mathrm{sgn}(\Phi(u,t)) = \mathrm{sgn}(u)$ and that $\Phi(u,t) = 0$ if and only if $u = 0$. Since also $e^u - 1 = 0$ if and only if $u = 0$ and $\mathrm{sgn}(e^u-1) = \mathrm{sgn}(u)$ we see that $\Omega(u,t) \geq 0$ with equality if and only if $u = 0$. ■

For $r, s \in (0,2]$, Lemma 3.2 applied to (9) shows that $\Delta V_R(x,y) = 0$ if and only if $(x,y) = (0,0)$, $(x,Y) = (0,0)$, $(X,y) = (0,0)$ or $(X,Y) = (0,0)$, i.e. only at a fixed point of T (but note that there may be non-isolated fixed points).

Corollary 3.3: *For $0 < r, s \leq 2$, and for $(x,y) \in C_+$, $\Delta V(x,y) \leq 0$ with equality if and only if (x,y) is a fixed point of (1).*

Taking $S = T$ and $M = T(C_+)$ in Proposition 2.1 we obtain

Theorem 3.4: *When $\alpha, \beta > 0$ and $0 < r, s \leq 2$ every orbit of* (1) *converges to a fixed point.*

Proof: Without loss of generality we may take an initial point $(x_0, y_0) \in D$. By Corollary 3.3, $\omega((x_0, y_0))$ consists of fixed points of T. Unless $\alpha = \frac{r}{s} = \beta$ the fixed points of T are finite in number, and since $\omega((x_0, y_0))$ is connected, $\omega((x_0, y_0))$ must be a singleton and hence we obtain convergence.

If $\alpha = \frac{r}{s} = \frac{1}{\beta}$ then there is a line of fixed points $\{(x, y) : r - x - \alpha y = 0\}$. Moreover, $T(x, y) = (xe^{s\alpha - x - \alpha y}, ye^{s - y - \frac{x}{\alpha}})$. Then for $V_1(x, y) = xy^{-\alpha}$ $(y \neq 0)$ we have $V_1(T(x, y)) = V_1(x, y)$. Hence the non-empty $\omega((x_0, y_0))$ is a subset of the single point intersection of $\{(x, y) : xy^{-\alpha} = x_0 y_0^{-\alpha}\}$ and $\{(x, y) : r - x - \alpha y = 0\}$, so that $\omega((x_0, y_0)$ is again a single fixed point and we obtain convergence. ■

By a curve in the open set $U \in \mathbb{R}^2$ we mean a pair (I, γ) where $I \subset \mathbb{R}$ is an interval and $\gamma : I \to U$ is continuously differentiable. Note that we have not excluded the possibility of cusps where $\gamma' = 0$. Notice that curves can have self-intersections. We will call curves that do not self-intersect 'simple'.

A heteroclinic orbit is a sequence of points $\{x_k\}$ with $x_k \in T^{-1}(x_{k+1})$ such that $\lim_{k\to\infty} x_k = P^*$ and $\lim_{k\to\infty} x_{-k} = Q^*$ where P^*, Q^* are distinct fixed points of T. When $P^* = Q^*$ we say that the orbit is homoclinic.

If there is an invariant curve γ joining a fixed point $P^* \in C_+$ to a different fixed point $Q^* \in C_+$ we will call γ a heteroclinic curve. Similarly, when $P^* = Q^*$ we call the invariant curve a homoclinic blue curve.

Lemma 3.5: *There is no homoclinic orbit for* (1).

Proof: Let P^* be a fixed point of T and $\eta = \{z_k\}_{k=-\infty}^{\infty}$ a homoclinic orbit of P^* such that $\lim_{k\to\pm\infty} z_k = P^*$. Since V_R decreases along the orbit, $V_R(z_{-k}) > V_R(z_{-1}) > V_R(z_0) > V_R(z_1) > V_R(z_k)$ for all $k \geq 2$ when z_0 is not a fixed point. Now take $k \to \infty$ to obtain the contradiction $V_R(P^*) > V_R(P^*)$. ■

For hyperbolic fixed points $(x^*, y^*) \in C_+$ we are guaranteed an open neighbourhood $U \subset C_+$ such that the local stable and unstable manifolds (one of which may be empty)

$$W^s_{loc}(x^*, y^*) := \{(x, y) \in U : \lim_{t\to\infty} T^t(x, y) = (x^*, y^*) \text{ and } T^t(x, y) \in U \quad \forall\, t \in \mathbb{N}\}$$

$$W^u_{loc}(x^*, y^*) := \{(x, y) \in U : \lim_{t\to\infty} T^{-t}(x, y) = (x^*, y^*) \text{ and } T^{-t}(x, y) \in U \quad \forall\, t \in \mathbb{N}\}$$

are embeddings, and are as smooth as the map T (see, for example, [20]).

For a fixed point $(x^*, y^*) \in C_+$ we denote by $W^u(x^*, y^*) = \bigcup_{t\geq 0} T^t(W^u_{loc}(x^*, y^*))$ its global unstable set (we do not say 'unstable manifold' as for non-invertible maps, such as (1), $W^u(x^*, y^*)$ may self-intersect). Similarly the global stable set $W^s(x^*, y^*)$ of (x^*, y^*) is the forward invariant set $W^s(x^*, y^*) = \{(x, y) \in C_+ : T^t(x, y) \to (x^*, y^*) \text{ as } t \to \infty\}$, and it is obtained as the union $W^s(x^*, y^*) = \bigcup_{t\in\mathbb{N}} T^{-t}(W^s_{loc}(x^*, y^*))$ (see, for example, [7]). The stable set cannot self-intersect, but it may have several components when T is not invertible.

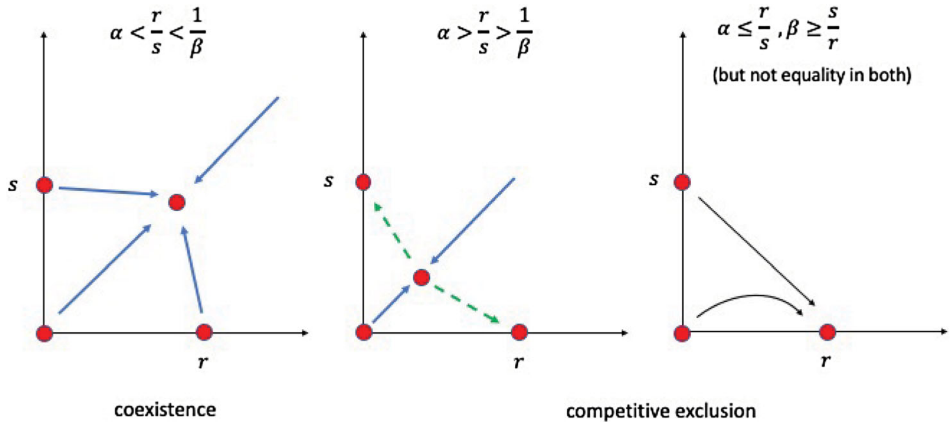

Figure 1. Global dynamics for the Ricker model (1) when $0 < r, s \leq 2$ showing stable and unstable sets for various parameter values. The solid dots are fixed point of (1). In the left and middle plots the green (dotted) arrows are the unstable set of the positive fixed point, and the blue (solid) arrows are the stable set of the positive fixed point. The figure shows that from an ecological perspective the (interior) dynamics results in either coexistence or competitive exclusion. Note that the depiction of the unstable set in not meant to imply it is a manifold, as we have not here ruled out potential self-crossings.

In the appendix we identify the global stable and unstable sets of a positive fixed point when it is a hyperbolic saddle. This case demonstrates the competitive exclusion principle of ecology (here competitive refers to the species-species interactions and not that the map is necessarily retrotone). For this positive fixed point there is a stable set which is a curve joining the origin to infinity, and a unstable set which joins one boundary fixed point to the other. This is illustrated in Figure 1. The stable set divides the first quadrant into two regions, and each correspond to the stable manifold of the boundary fixed point that they contain.

The main challenge for finding these stable and unstable manifolds lies in the fact that the Ricker map is not invertible, and the unions of iterates of local stable and unstable manifolds mentioned above must be shown to have no gaps. The existence of a Lyapunov function greatly simplifies the proofs given in the appendix of the existence of the stable and unstable sets: There can be no invariant closed curves, and orbits must evolve while respecting that the Lyapunov function is strictly decreasing away from fixed points. A further complication, due to noninvertibility of the Ricker map, is that for some parameter values that we consider, orbits can oscillate above and below the unstable set of the positive fixed point.

Now we state our main Theorem for global dynamics of the Ricker map when $0 < r, s \leq 2$.

Theorem 3.6: *For the Ricker map*

$$T(x, y) = (xe^{r-x-\alpha y}, ye^{s-y-\beta x}), \quad (x, y) \in C_+,$$

with $r, s \in (0, 2)$, *T has axial fixed points* $(r, 0)$ *and* $(0, s)$, *and the origin is always a repelling fixed point relative to* C_+. *There exists a unique positive fixed point* (p, q) *whenever either* $r > \alpha s$ *and* $s > \beta r$ *or* $r < \alpha s$ *and* $s < \beta r$. *Moreover,* *R1* *If $\alpha < \frac{r}{s} < \frac{1}{\beta}$, then the unique positive fixed point (p,q) is globally asymptotically stable relative to C_{++}.*

R2 *If $\alpha > \frac{r}{s} > \frac{1}{\beta}$, then the unique positive fixed point (p,q) is a saddle point and $W^s(p,q)$ comprises the union of a heteroclinic curve joining $(0,0)$ to (p,q), but excluding $(0,0)$ and (p,q), and a curve joining (p,q) to infinity. $W^s(p,q)$ divides C_+ into two open and disjoint regions R_1, R_2 with $C_+ \setminus \{0,0\} = R_1 \cup W^s(p,q) \cup R_2$ where $(0,s) \in R_1$ and $(r,0) \in R_2$. $(0,s)$ is globally asymptotically stable relative to R_1 and $(r,0)$ is globally asymptotically stable relative to R_2.*

R3 *If $\alpha s \le r, \beta r \ge s$ but not $\alpha = \frac{r}{s} = \frac{1}{\beta}$ then there is no positive fixed point. $(r,0)$ is globally asymptotically stable relative to $(0,\infty) \times [0,\infty)$. $(0,s)$ is a saddle point with $W^s(0,s) = \{0\} \times (0,\infty)$ and $W^u(0,s)$ is a heteroclinic curve joining $(0,s)$ to $(r,0)$.*

R4 *If $\alpha s \ge r, \beta r \le s$ but not $\alpha = \frac{r}{s} = \frac{1}{\beta}$ then there is no positive fixed point. $(0,s)$ is globally asymptotically stable relative to $[0,\infty) \times (0,\infty)$, and $(r,0)$ is a saddle point with $W^s(r,0) = (0,\infty) \times \{0\}$ and $W^u(r,0)$ is a heteroclinic curve joining $(r,0)$ to $(0,s)$.*

R5 *If $\alpha = \frac{r}{s} = \frac{1}{\beta}$ then, for every point $\rho \in C_+ \setminus \{(0,0)\}$, $\{T^n(\rho)\}$ converges to a fixed point on $L_0 = \{(x,y) \in C_+ : x + \alpha y = r\}$.*

See Figure 2 for an illustration of these possible outcomes.

Proof: The eigenvalues of $DT(0,0)$ are e^r, e^s which both exceed 1, since $r, s > 0$. Hence $(0,0)$ is always a uniform repeller relative to C_+. From May's work [18], when $r < 2$ all nonzero points on the x−axis converge to the axial fixed point $(r,0)$.

Now consider the items R1-R5.

R1: The spectrum of $DT(0,s)$ is $\{e^{r-\alpha s}, 1-s\}$ and the spectrum of $DT(r,0)$ is $\{1-r, e^{s-\beta r}\}$. Hence for $0 < r, s < 2$ and $\alpha < \frac{r}{s} < \frac{1}{\beta}$ the axial points are hyperbolic saddles and with stable manifolds subsets of the axes. Hence $(r,0), (0,s) \notin \omega((x,y))$ for any $(x,y) \in C_{++}$. Moreover $(0,0)$ is a uniform repeller, so $(0,0) \notin \omega((x,y))$ for any $(x,y) \in C_{++}$. Thus all points in C_{++} converge to (p,q) which is globally asymptotically stable since (p,q) is (locally) asymptotically stable.

R2: By Theorem A.3 in the appendix, (p,q) is a hyperbolic saddle and $W^s(p,q)$ consists of the union of a heteroclinic curve, minus $(0,0)$, joining $(0,0)$ to (p,q) and curve from (p,q) to infinity. $W^s(p,q)$ divides C_+ into two disjoint, open and forward invariant sets R_1 with $(0,s) \in R_1$ and R_2 with $(r,0) \in R_2$ such that $C_+ \setminus \{0,0\} = R_1 \cup W^s(p,q) \cup R_2$.

R3: If $\alpha s \le r, s \le \beta r$ but not $\alpha = \frac{r}{s} = \frac{1}{\beta}$. Set $z = y^r/x^s$, so that (1) gives

$$z' = ze^{x(s-r\beta)+y(s\alpha-r)}$$

for the next iterate z' of z in terms of z[1]. Under the stated conditions or r, s, α, β we see that $z' \le z$. Since iterates of z are non-increasing and bounded below by 0, the iterates of z tend to a fixed point z^*. Suppose that $z^* \ne 0$. We already know by Theorem 3.4 that all orbits converge to a fixed point; lets call it (x^*, y^*), so that $z^* = y^*/x^*$. Since the origin is repelling $(x^*, y^*) \ne (0,0)$, and $x^* > 0$ since orbits of z are bounded. If $z^* > 0$, $y^* > 0$, which together with $x^* > 0$ contradicts $x^*(s - r\beta) + y^*(s\alpha - r) = 0$. Hence $z^* = 0$ which gives $x^* = r$ and $y^* = 0$.

R4: This is just R3 with r and s swapped and α, β swapped.

R5: This is part of Theorem 3.4. ■

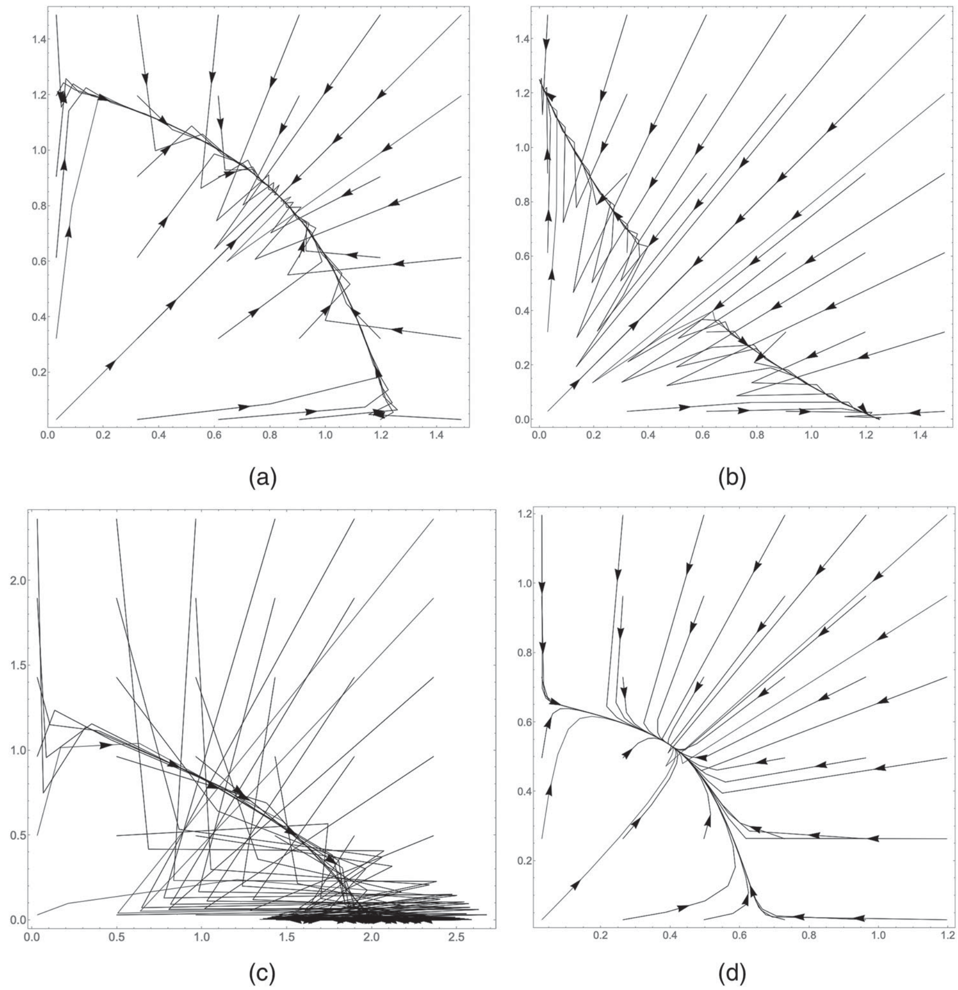

Figure 2. Examples of dynamics catalogued in Theorem 3.6. Each plot shows trajectories of points (hollow circles) starting on a uniform grid of initial points. (Some of these initial points lie close to fixed points (e.g. (0, 0)), but no initial point is exactly positioned at a fixed point.) (a) R1: $r = s = 5/4, \alpha = \beta = 1/2$, (b) R2: $r = s = 5/4, \alpha = \beta = 3/2$. (c) R3: $r = 2, s = 5/4, \alpha = 1/2, \beta = 5/4$, (d) $r = s = 2/3, \alpha = 1/2$, $\beta = 1/3$, so that T is retrotone (see Definition 1.1). In this case there is a carrying simplex which is $\Sigma = \overline{W^u(2/3, 0) \cup W^u(0, 2/3)}$.

Remark 3.1: The Lyapunov function method here does not prove that (local) asymptotic stability of the positive fixed point in the Ricker model implies global asymptotic stability (i.e. asymptotic stability in C_+) as our Theorem 3.6 is only for $0 < r,\ s < 2$. For example, when $\alpha = 0.4, \beta = 0.2, r = 2.2, s = 1.6$, (numerically generated) trajectories starting in $(0, \infty)^2$ globally converge to $(1.696, 1.261)$. It is known (e.g. [17]) by way of the Jury condition applied to the derivative of T at the fixed point,

$$DT(p, q) = \begin{pmatrix} \dfrac{\alpha\beta + r - \alpha s - 1}{\alpha\beta - 1} & \dfrac{\alpha(r - \alpha s)}{\alpha\beta - 1} \\ \dfrac{\beta(s - \beta r)}{\alpha\beta - 1} & \dfrac{\alpha\beta - \beta r + s - 1}{\alpha\beta - 1} \end{pmatrix},$$

that the Ricker model has an asymptotically stable positive fixed point if and only if

$$4(\alpha\beta - 1) + 2(1-\alpha)s + 2(1-\beta)r < (\alpha s - r)(\beta r - s) < (1-\alpha)s + (1-\beta)r \quad (10)$$

which allows for values $r, s \geq 2$ not covered by Theorem 3.6. When $0 < r, s \leq 2$ the Lyapunov function V_R satisfies the conditions of (i), (ii), but we have not been able to verify (iii) as some axial fixed points are no longer hyperbolic when $r = 2$ or $s = 2$ (or both).

It has been conjectured (e.g. [6]) that (10) is a necessary and sufficient condition for global stability of a positive fixed point, but this remains an open problem.

Notes

1. We thank one of the referees for pointing out this method of proof for R3 which improves upon our original approach that used centre manifold theory.

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No potential conflict of interest was reported by the author(s).

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Appendix

Appendix 1. Stable and unstable manifolds of a positive fixed point in the Ricker model

In [25], Smith shows (Proposition 6.3) that when $0 < r, s < 1$ and the positive fixed point is a saddle, there is a C^1 smooth separatrix curve Γ that contains the origin, the positive fixed point, and extends to infinity. The set $\Gamma \setminus \{0, 0\}$ is the stable manifold of the positive fixed point. We extend Smith's results to values r, s beyond $0 < r, s < 1$ with values $0 < r, s < 2$.

To establish that all orbits converge to a fixed point we did not need the map T to be invertible, but in order to find the stable manifold of P we will need to consider inverse images of points. The map T is not invertible, so the inverse image of a point may not be unique. Thus we now elaborate on the map T and its inverse images.

Following [12], we consider the image under T of each half-line $\ell_\theta := \{(t\cos\theta, t\sin\theta) : t \in [0, \infty)\}$, $\theta \in [0, \pi/2]$, so that $C_+ = \bigcup_{\theta\in[0,\pi/2]} \ell_\theta$ (see Figure A1). Then $\overline{T(\ell_\theta)}$ defines a loop in C_+ from O to itself. $T(C_+) = \bigcup_{\theta\in[0,\pi/2]} T(\ell_\theta)$ is a compact and simply-connected subset of C_+. Let LC_{-1} be the set of points $(x, y) \in C_+$ where $\det DT(x, y) = 0$, and let $LC_0 = T(LC_{-1})$.

For convenience we set $\Omega = T(C_+) \setminus LC_0$ (the interior of $T(C_+)$ in C_+). The boundary of $\overline{\Omega}$ in C_+ is the envelope of the curves $T(\ell_\theta)$, and so corresponds to the image of the set LC_{-1} under the map T. Since in the present case of a hyperbolic saddle $\alpha\beta > 1$, $LC_{-1} \subset C_+$ is the curve given by the graph of $y = \frac{1-x}{1+(\alpha\beta-1)x}$ for $x \in [0, 1]$. LC_{-1} is a decreasing curve in C_+ connecting $(0, 1)$ to $(1, 0)$. Let S_1 be the set of points below LC_{-1} in C_+ and S_2 the set of points above LC_{-1} in C_+. Then $T(S_1) = T(S_2) \cup \{O\} = \Omega$, $T(\overline{S_1}) = T(S_1) \cup T(LC_{-1}) = T(S_1) \cup LC_0 = \overline{\Omega}$, $T(\overline{S_2}) = T(S_2 \cup LC_{-1}) = T(S_2) \cup LC_0 = \overline{\Omega} \setminus \{O\}$.

Remark A.1: It is important to note that since $T(C_+) = \overline{\Omega}$, any positive fixed point P of T belongs to $\overline{\Omega} \cap C_{++}$, and moreover P belongs to $\Omega \cap C_{++}$ unless $P \in LC_0 \cap LC_{-1}$.

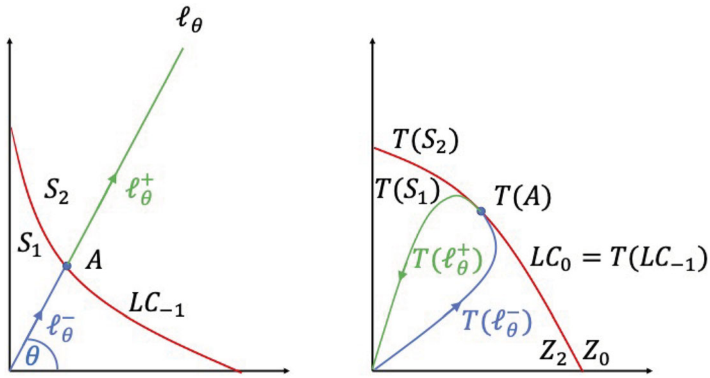

Figure A1. T maps the line ℓ_θ onto a simple curve $T(\ell_\theta) = T(\ell_\theta^+) \cup T(\ell_\theta^-)$ and $\overline{T(\ell_\theta)}$ is a closed curve joining O to itself. T maps S_1 onto $T(S_1)$ and S_2 onto $T(S_2)$ where $T(S_1) = T(S_2) \cup \{O\}$. The envelope of all curves $T(\ell_\theta)$ is $LC_0 = T(LC_{-1})$. $C_+ = Z_2 \cup LC_0 \cup Z_0$ where points in Z_0 have no preimage, and points in Z_2 have 2 preimages. That the curve $T(\ell_\theta)$ encloses a convex set can be shown fairly easily by showing that the curvature of $T(\ell_\theta)$ does not vanish.

We will find it convenient to view T restricted to each of S_1, S_2 as maps in their own right. For $i = 1, 2$, let $T_i : \overline{S_i} \to T(\overline{S_i})$ be the restriction of T to $\overline{S_i}$. As we will show below, both T_1, T_2 are homeomorphisms and we will denote their inverses by $T_i^{-1} : T(\overline{S_i}) \to \overline{S_i}$, $i = 1, 2$.

We recall that a map $f : X \to Y$, where X, Y are topological spaces, is called proper if whenever $K \subset Y$ is compact in Y, $f^{-1}(K)$ is compact in X. From [1] we have the following result:

Theorem A.1: *Let $f : X \to Y$ be a proper local homeomorphism from a connected metric space X to a simply-connected metric space Y. Then f is a homeomorphism from X onto Y.*

Lemma A.2: *$T_1 : \overline{S_1} \to T(\overline{S_1})$ is a homeomorphism and $T_2 : \overline{S_2} \to T(\overline{S_2})$ is also a homeomorphism with $T_1(\overline{S_1}) = \overline{\Omega}$ and $T_2(\overline{S_2}) = \overline{\Omega} \setminus \{O\}$.*

Proof: Let K be a compact subset of the bounded set $T_2(S_2) = \Omega \setminus \{O\}$. Since $O \notin T_2(S_2)$ and K is compact, $T_2^{-1}(K)$ is bounded. Since T is continuous, $T_2^{-1}(K)$ is closed and hence being also bounded is compact. Hence T_2 restricted to S_2 is a proper map. On the other hand, S_1 is bounded and the inverse image $T_1^{-1}(K)$ of a compact set $K \subset \Omega$ is closed. Since S_1 is also bounded, $T_1^{-1}(K)$ is compact. Hence T_1 restricted to S_1 is a proper map. ∎

T is a local homeomorphism and so by Theorem A.1, T_1 restricted to S_1 and T_2 restricted to S_2 are homeomorphisms.

$T_1 : \overline{S_1} \to T(\overline{S_1}) = \overline{T(S_1)}$ is continuous, bijective, $\overline{S_1}$ is compact, and hence T_1 is a homeomorphism of $\overline{S_1}$ onto $T(\overline{S_1})$ (e.g. Theorem 5.4 in [5]). $\overline{S_2}$ is not compact, so we need an alternative approach. LC_{-1} and LC_0 are homeomorphic to the interval $[0, 1]$, so T restricted to LC_{-1} is a homeomorphism from LC_{-1} to $LC_0 = T(LC_{-1})$. $T_2 : \overline{S_2} \to \overline{\Omega} \setminus \{O\}$ is also continuous and bijective. Let T_2^{-1} be defined on $\overline{\Omega} \setminus \{O\} = (\Omega \setminus \{O\}) \cup LC_0$ piecewise via the homeomorphisms T_2^{-1} restricted to $\Omega \setminus \{O\}$ and T_2^{-1} restricted to LC_0. Suppose $z_n \to z \in \overline{\Omega} \setminus \{O\}$ but $T_2^{-1}(z_n) \not\to T_2^{-1}(z)$. Then $z_n = T(T_2^{-1}(z_n)) \not\to z = T(T_2^{-1}(z))$ as T is injective on $\overline{S_2}$. Hence T_2^{-1} is continuous on $\overline{\Omega} \setminus \{O\}$ and is a homeomorphism.

Remark A.2: Note that if $P^* \in \Omega$ is a fixed point of T_1^{-1}, then $T_1^{-1}(P^*) = P^*$ so that $P^* = T(T_1^{-1}(P^*)) = T(P^*)$. Thus the fixed points of T_1^{-1}, T_2^{-1} are a subset of the fixed points of T.

Moreover, by Corollary 3.3, $V_R(T_1^{-1}(Q)) \geq V_R(Q)$ for $Q \in \overline{\Omega}$ since $Q = T(T_1^{-1}(Q))$. Furthermore, if $V_R(T_1^{-1}(Q)) = V_R(Q)$ then $V_R(T_1^{-1}(Q)) = V_R(T(T_1^{-1}(Q)))$ so $T_1^{-1}(Q)$ is a fixed point of T by Corollary 3.3. As $Q = T(T_1^{-1}(Q)) = T_1^{-1}(Q)$, Q is a fixed point of T. Hence $V_R(T_1^{-1}(Q)) \geq V_R(Q)$ with equality only if Q is a fixed point of T. Similarly for T_2^{-1}. Hence Proposition 2.1 can be used with T_1^{-1} and T_2^{-1}.

Theorem A.3: *Assume that* $\alpha > \frac{r}{s} > \frac{1}{\beta}$ *and* $0 < r, s < 2$. *Then the following conclusions hold for* (1).

(a) *The positive fixed point P of* (1) *given by* (4) *is a hyperbolic saddle point.*
(b) *Identification of $W^u(P)$: There is a continuous curve $\ell \subset C_{++}$ from $(r, 0)$ through P to $(0, s)$ such that $T(\ell) = \ell$ and ℓ is the unstable manifold of P, i.e. $W^u(P) = \ell$.*
(c) *Identification of $W^s(P)$: There is a simple continuous curve γ joining the origin O (but not including O) through P to infinity such that $T(\gamma) \subset \gamma$, $T^{-1}(\gamma) = \gamma$, and γ is the stable manifold of P in C_+, i.e. $W^s(P) = \gamma$. Moreover, $\overline{\gamma}$ divides C_+ into two regions, R_1 containing $(r, 0)$ and R_2 containing $(0, s)$, such that $C_+ = R_1 \cup \overline{\gamma} \cup R_2$, $T(R_1) \subset R_1$ and $T(R_2) \subset R_2$.*

Proof: (a) When $\alpha > \frac{r}{s} > \frac{1}{\beta}$, so that in particular $\alpha\beta > 1$, it is clear from (4) that $p > 0$ and $q > 0$. The Jacobian of the map T at (p, q) is

$$DT(p,q) = \begin{pmatrix} 1-p & -\alpha p \\ -\beta q & 1-q \end{pmatrix}.$$

$DT(p, q) - I_2$ has eigenvalues $\mu_\pm$ of opposite sign since $\det(DT(p, q) - I_2) = pq(1 - \alpha\beta) < 0$. Hence $DT(p, q)$ has at least one positive eigenvalue $\lambda_+ = \mu_+ + 1 > 1$. On the other hand the second eigenvalue is $\lambda_- = 1 + \mu_- < 1$, so to ensure $\lambda_- > -1$ we need $\mu_- > -2$, which is the case when $\det(DT(p, q) - (-I_2)) > 0$. Using (4) and simplifying we find that

$$\begin{aligned} \det(DT(p,q) + I_2) &= 4 - 2p - 2q + (1 - \alpha\beta)pq \\ &= \frac{1}{\alpha\beta - 1}\left(\left(\frac{4}{rs} - 1\right)(\alpha s - r)(\beta r - s) \right. \\ &\quad \left. + \left(\frac{4}{r} - 2\right)(\alpha s - r) + \left(\frac{4}{s} - 2\right)(\beta r - s)\right) \\ &\geq 0 \end{aligned}$$

when $\alpha > \frac{r}{s} > \frac{1}{\beta}$, $0 < r, s \leq 2$, with equality only if $r = s = 2$.

To prove parts (b) and (c), we observe that an eigenvector v of $DT(p, q)$ associated with λ_- (λ_+) is determined from $(DT(p, q) - I_2)v = \mu_- v$ ($\mu_+ v$). Since $DT(p, q) - I_2$ is a negative matrix, by the Perron-Frobenius theorem μ_- (λ_-) has an associated unique positive unit eigenvector which we call e_1. It is obvious to check that the two components of any eigenvector associated with μ_+ (λ_+) have opposite sign and we can choose a unit eigenvector e_0 with a positive first component. Note that

$$V_R(x, y) - V_R(p, q) = \beta[x - p + \alpha(y - q)]^2 - \alpha(\alpha\beta - 1)(y - q)^2.$$

As $\alpha\beta > 1$, the level curve of V_R determined by $V_R(x, y) = V_R(p, q)$ consists of two straight line segments in C_+ each with the two endpoints on the axes. The two line segments intersect at (p, q) and divide C_+ into four mutually disconnected regions: two triangles $E_1 \ni (r, 0)$ and $E_2 \ni (0, s)$, a quadrilateral $D_1 \ni O$ and the unbounded region D_2 above the two line segments. Also, $V_R(x, y) < V_R(p, q)$ for $(x, y) \in E_1 \cup E_2$ and $V_R(x, y) > V_R(p, q)$ for $(x, y) \in D_1 \cup D_2$.

(b) Identification of $W^u(P)$

Let $E = \overline{E_1 \cup E_2}$. Then $(x, y) \in E$ if and only of $V_R(x, y) \leq V_R(p, q)$. E is forward invariant by Corollary 3.3 and contains P, $(r, 0)$, $(0, s)$. (Note that parts of E may lie outside Ω).

Take a local unstable manifold of P, $\gamma_0 \subset E$, with endpoints P_0, Q_0. For each $k \geq 0$, $\gamma_k = T^k(\gamma_0)$ is a continuous curve in E and $\gamma_0 \subset \gamma_1 \subset \gamma_2 \cdots$, where P_k and Q_k are endpoints of γ_k.

Since the unstable eigenvalue is positive, by the Hartman-Grobman theorem and continuity of T, $P_k \in E_1$ and $Q_k \in E_2$ for all $k \geq 0$. P_k and Q_k converge to fixed points of T, but not O since $O \notin E$ and also not P by Lemma 3.5. Thus $P_k \to (r, 0)$ and $Q_k \to (0, s)$ as $k \to \infty$. Then $\ell = \bigcup_{k \geq 0} T^k(\gamma_0)$ is the unstable manifold of P and consists of a $T-$invariant curve in C_{++} from $(0, s)$ through P to $(r, 0)$ (but excluding $(r, 0), (0, s)$).

As mentioned earlier for non-invertible maps it is possible for the unstable manifold to self-intersect. Our numerical simulations suggest that no self-intersections of the unstable manifold can occur, but we have been unable to prove it.

(c) Identification of $W^s(P)$

We recall Remark A.1 that $P \in \Omega$ unless $P \in LC_{-1} \cap LC_0 \subset \partial\overline{\Omega}$ which is case (iii) below.

If $P \in S_1$ then $DT(P)$ has two positive eigenvalues, whereas if $P \in S_2$, $DT(P)$ has eigenvalues of opposite sign. Let γ_0 be a local stable manifold of P, so that $T(\gamma_0) \subset \gamma_0$ and $T^n(Q) \to P$ as $n \to \infty$ for all $Q \in \gamma_0$, with $V_R(T^n(Q)) \downarrow V_R(P)$ as $n \to \infty$, so by Corollary 3.3, $\gamma_0 \subset D_1 \cup D_2 \cup \{P\}$.

Since we have excluded $\alpha = \frac{r}{s} = \frac{1}{\beta}$, one eigenvalue has modulus greater than one, and the other has modulus less than one, so local stable and unstable manifolds are one-dimensional. By the Hartman-Grobman theorem for maps, since P is hyperbolic, in a small enough neighbourhood of P orbits of (1) are topologically conjugate to those of the linearization of (1) at P. Hence, since T is continuous, if $P \in S_1$, orbits on γ_0 either stay in D_1 or stay in D_2, whereas if $P \in S_2$ orbits on γ_0 alternate in D_1 and in D_2.

We split part (c) into 3 distinct possibilities: (i) $P \in S_2 \cap \Omega$ (so that P lies above LC_{-1} but below LC_0, $-1 < \lambda_- < 0$), (ii) $P \in S_1 \cap \Omega$ (so that P lies below both LC_0 and LC_{-1}, $0 < \lambda_- < 1$), and (iii) $P \in LC_0 \cap LC_{-1}$ (so $\lambda_- = 0$).

In (i) we use the homeomorphism $T_2 : \overline{S_2} \to T(\overline{S_2})$ first and then $T_1 : \overline{S_1} \to T(\overline{S_1})$ since $P \in S_2$ is a fixed point of T_2, and in (ii) we use the homeomorphism $T_1 : \overline{S_1} \to T(\overline{S_1})$ first since $P \in S_1$ is a fixed point of T_1.

(i) $P \in S_2 \cap \Omega$

The first step for (i) is to show that without loss of generality γ_0 can be chosen to be a simple curve with endpoints $Q_0 \in LC_{-1}$ and $P_0 \in LC_0$.

Case (ia): Suppose $\gamma_0 \cap LC_{-1} = \{Q_0\}$.

Since $P \in \Omega$ (and so $P \notin LC_0$), the local stable manifold γ_0 can be chosen so that $\gamma_0 \subset \overline{\Omega}$ and $Q_0 \in LC_{-1}$ as an endpoint of γ_0. Clearly, γ_0 is a simple curve in $\overline{\Omega} \cap (D_1 \cup D_2 \cup \{P\})$. Let $Q_1 := T_2(Q_0) \in LC_0$, and γ_1 be the section of γ_0 between Q_0 and P. Then $\gamma_2 := T_2(\gamma_1) = T(\gamma_1) \subset \gamma_0$ with endpoints P, Q_1. Let $\gamma = \gamma_1 \cup \gamma_2$, which is a simple curve from $Q_0 \in LC_{-1}$ through P to $Q_1 \in LC_0$. Then $T_2^k(\gamma) = T_2^k(\gamma_1) \cup T_2^k(\gamma_2) \subset T_2^k(\gamma_0) \cup T_2^{k+1}(\gamma_0) \to \{P\}$ as $k \to \infty$. Hence γ is a local stable manifold of P that contains $Q_0 \in LC_{-1}$ and $Q_1 \in LC_0$ as endpoints and $\gamma \subset \overline{\Omega} \cap (D_1 \cup D_2 \cup \{P\})$.

Case (ib): Suppose $LC_{-1} \cap \gamma_0 = \emptyset$ but $\gamma_0 \cap LC_0 = \{P_0\}$.

γ_0 can be chosen so that $P_0 \in LC_0$ is an endpoint of γ_0 and in $\overline{\Omega} \cap (D_1 \cup D_2 \cup \{P\})$. Let $P_{-1} := T_2^{-1}(P_0) \in LC_{-1}$, and γ_1 be the section of γ_0 with endpoints P, P_0. Then $\gamma_2 := T_2^{-1}(\gamma_1)$ is a simple curve in $\overline{S_2}$ joining $P_{-1} \in LC_{-1}$ and P, and $\gamma := \gamma_2 \cup \gamma_1$ is a simple curve joining $P_{-1} \in LC_{-1}$ and $P_0 \in LC_0$. Moreover, γ is a local stable manifold for P, since if $z \in \gamma$, either $z \in \gamma_1$ or $z \in \gamma_2$; if $z \in \gamma_1$ then $z \in \gamma_0$, whereas if $z \in \gamma_2$, $T(z) \in \gamma_1 \subset \gamma_0$. Clearly, $\gamma \subset \overline{\Omega} \cap (D_1 \cup D_2 \cup \{P\})$.

Case (ic): Suppose $(LC_0 \cup LC_{-1}) \cap \gamma_0 = \emptyset$.

We suppose that $(LC_0 \cup LC_{-1}) \cap T_2^{-k}(\gamma_0) = \emptyset$ for all $k \geq 0$ and seek a contradiction. Notice that for each $k \geq 0$, $T_2^{-k}(\gamma_0) \subset T_2^{-k-1}(\gamma_0) \subset (D_1 \cup D_2 \cup \{P\})$ and $T^{-k}(\gamma_0)$ is between LC_{-1} and LC_0. Let P_0 be the endpoint of γ_0 in D_2. The iterates of P_0 under T_2^{-1} alternate in D_1 and in D_2 (as in Case (i) $P \in S_2$), so $T_2^{-2k}(P_0) \in D_2 \cap \Omega$ and $T_2^{-2k-1}(P_0) \in D_1 \cap \Omega$. As T_2 is a homeomorphism, each $T_2^{-k}(\gamma_0)$ is a simple curve in the compact region of $\overline{D_1 \cup D_2}$ between LC_{-1} and LC_0. Thus, there are $Q \in \overline{D_1}$ and $Q' \in \overline{D_2}$ such that $T_2^{-2k}(P_0) \to Q'$, $T_2^{-2k-1}(P_0) \to Q$ as $k \to \infty$. Hence, $T_2^{-2k+2}(P_0) = T^2(T_2^{-2k}(P_0)) \to T^2(Q') = Q'$, $T_2^{-2k+1}(P_0) = T(T_2^{-2k}(P_0)) \to T(Q') = Q$ as $k \to \infty$. If $Q \neq Q'$ then Q' is not a fixed point of T so, by Corollary 3.3, $V_R(Q') = V_R(T^2(Q')) < V_R(T(Q')) < V_R(Q')$, a contradiction. This shows that $Q = Q'$ is a fixed point of T in the compact region of $\overline{D_1 \cup D_2}$ between LC_{-1} and LC_0. As P is the only fixed point of T in this region, we have $T_2^{-k}(P_0) \to$

P, $T_2^k(P_0) \to P$ as $k \to \infty$. This shows that $\{T^n(P_0)\}$ for integers n is a homoclinic orbit, which is against Lemma 3.5. These contradictions show that for some $K \geq 0$, $(LC_0 \cup LC_{-1}) \cap T_2^{-K}(\gamma_0) \neq \emptyset$.

We conclude that in case (i), without loss of generality γ_0 can be chosen to be a local stable manifold that lies between LC_{-1} and LC_0.

Now we show how to use γ_0 and T_1^{-1} to construct $W^s(P)$ for Case (*i*).

Without loss of generality we have $\gamma_0 \subset \overline{\Omega}$ as the local stable manifold with end points $P_{-1} \in LC_{-1}$ in $\Omega \cap D_1$ and $P_0 \in LC_0$ in $\overline{\Omega} \cap D_2$. Set $P_{-2} := T_1^{-1}(P_{-1})$, which is defined since $P_{-1} \in \Omega$. Then $T_1^{-1}(\gamma_0) \subset \overline{S_1}$ is a simple curve with end points $P_{-2} \in S_1$ and $P_{-1} \in LC_{-1} \cap \Omega \cap D_1$. Since $V_R(T_1^{-1}(Q)) \geq V_R(Q)$ with equality if and inly if Q is a fixed point of T_1, for all $Q \in \gamma_0$ and inter $n > 0$ we have $V_R(T_1^{-1}(Q)) > V_R(Q) \geq V_R(T^n(Q)) \geq V_R(P)$. Thus, $T_1^{-1}(\gamma_0) \subset (D_1 \cup D_2)$. As $T_1^{-1}(\gamma_0)$ is a simple continuous curve, $P_{-1} \in D_1$ and D_1 is not connected to D_2, we must have $T_1^{-1}(\gamma_0) \subset D_1$. As T maps $\overline{D_1} \setminus (D_1 \cup \{P\})$ into $E_1 \cup E_2$ due to $V_R(T(Q)) < V_R(Q)$ for non-fixed point Q, by continuity of T and O being a uniform repeller, $D_1 \subset T(D_1) \subset \overline{\Omega}$. Then $T_1^{-1}(\gamma_0) \subset \overline{S_1} \cap D_1 \subset \overline{\Omega}$.

Continuing to apply T_1^{-1}, for each n we obtain a simple curve $\bigcup_{k\geq 0}^{n} T_1^{-k}(\gamma_0)$ which joins $P_{-n}, P_{1-n}, \ldots, P_{-1}, P$ and P_0, where $P_{-k} := T_1^{-k}(P_0)$. Moreover, $V_R(P_{-k})$ is increasing with k and $V_R(T_1^{-1}(Q)) = V_R(Q)$ if and only if Q is a fixed point of T_1^{-1} and thus also of T (see Remark A.2). Since O is the only fixed point of T_1^{-1} in $\overline{S_1} \cap D_1$, and is stable for T_1^{-1}, by Proposition 2.1, $P_{-k} \to O$ as $k \to \infty$. Let $\gamma = \cup_{k\geq 0} T_1^{-k}(\gamma_0)$. Then γ is a simple curve joining P_0 to (but not including) O.

Now set $\ell = T^{-1}(\gamma)$. $T_1^{-1}(\gamma)$ is a simple curve between P_{-1} and O and $T_2^{-1}(\gamma)$ is a simple curve from P_{-1} to infinity. Thus ℓ is a simple curve from (but not including) O through P to infinity.

If $z \in \ell$, $T(z) \in \gamma = \cup_{k\geq 0} T_1^{-k}(\gamma_0)$, so for some K, $T_1^K(z) \in \gamma_0 \subset W^s(P)$. Hence $\ell \subset W^s(P)$. On the other hand, if $z \in W^s(P)$ then there is a smallest integer $k \geq 0$ such that $T^k(z) \in \gamma_0$. If $k = 0$ then $z \in \gamma_0 \in \ell$. Suppose $k > 0$. If $z \in \overline{S_1} \cap W^s(P)$ then $z = T_1^{-k}(T^k(z)) \in T_1^{-k}(\gamma_0) \subset \ell$. If $z \in S_2 \cap W^s(P)$ then $z = T_2^{-1}(T_1^{-k+1}(T^k(z))) \subset T_2^{-1}(T_1^{-k+1}(\gamma_0)) \subset \ell$. Therefore, $W^s(P) \subset \ell$ and hence, since we showed the reverse inclusion above, $\ell = W^s(P)$.

Case (ii) $P \in S_1 \cap \Omega$

Case (iia): Suppose $\gamma_0 \cap LC_0 = \{P_0\}$.

γ_0 lies in $\overline{\Omega}$ and can be chosen so that $P_0 \in LC_0$ is an endpoint of γ_0 in D_2. Let Q_0 be the endpoint of γ_0 in D_1. Let γ_1 be the section of γ_0 in $D_1 \cap \Omega$. Then $T_1^{-1}(\gamma_1)$ is a simple curve in $S_1 \cap D_1$ connecting $Q_{-1} := T_1^{-1}(Q_0)$ and P. Moreover $\gamma_1 \subset T_1^{-1}(\gamma_1)$. From case (i) above we know that $(S_1 \cap D_1) \subset \overline{\Omega}$. So we may continue to apply T_1^{-1} to obtain an increasing sequence of simple curves $T_1^{-k}(\gamma_1)$ in $D_1 \cap \overline{\Omega}$, with one endpoint $Q_{-k} = T_1^{-k}(Q_0)$. By Proposition 2.1, $Q_{-k} \to Q^* \in \overline{D_1}$ where Q^* is a fixed point of T_1^{-1} and hence also of T. Since P and O are the only two fixed points of T in $\overline{D_1}$ and $Q^* \neq P$ due to no homoclinic orbits by Lemma 3.5, we must have $Q^* = O$. Then $\gamma := \gamma_0 \cup \bigcup_{k\geq 0} T_1^{-k}(\gamma_1)$ is a local stable manifold of P that joins P_0 through P to (but excluding) O. Now define $\ell = T^{-1}(\gamma)$, which is a simple curve that joins O through P to infinity. Using almost the same argument as in Case (i), we find $\ell = W^s(P)$.

Case (iib): Suppose $\gamma_0 \cap LC_0 = \emptyset$.

Then we may choose $\gamma_0 \subset \Omega$ and $T_1^{-1}(\gamma_0) \subset S_1$ is a simple curve (since T_1 is a homeomorphism on $\overline{S_1}$). Either $T_1^{-1}(\gamma_0) \cap LC_0 \neq \emptyset$ or $T_1^{-1}(\gamma_0) \subset \Omega$ and $T_1^{-2}(\gamma_0) \subset S_1$. We continue in this way so that either there is some K such that $T_1^{-K+1}(\gamma_0) \subset \Omega$ but $T_1^{-K}(\gamma_0) \cap LC_0 \neq \emptyset$, which is Case (iia), or $T_1^{-k}(\gamma_0) \subset \Omega$ for all $k \geq 0$.

In this latter case, let P_0 be the endpoint of γ_0 in D_2. Then $P_{-k} := T_1^{-k}(P_0) \in \Omega \cap D_2$ for all $k \geq 0$ is bounded, $V_R(P_{-k})$ is increasing with $k \geq 0$, so by Proposition 2.1, $T_1^{-k}(P_0) \to P^* \in \overline{D_2}$, where P^* is a fixed point of T_1^{-1} and hence also of T. Since P is the only fixed point in $\overline{D_2}$, we must have $P^* = P$ and a contradiction follows since there are no homoclinic orbits by Lemma 3.5, and we arrive at Case (iia) again.

Case (iii): $P \in LC_{-1} \cap LC_0$.

Let γ_0 be a local stable manifold of P chosen such that $\gamma_0 \cap LC_{-1} = \{P\}$ (this is valid since γ_0 is locally the graph of an increasing function, whereas LC_{-1} is the graph of a decreasing function) and $(\gamma_0 \setminus \{P\}) \subset D_1$.

By the same way as in Case (ii), $\gamma = \bigcup_{k\geq 0} T_1^{-k}(\gamma_0)$, γ is a simple curve from O (excluding O) to P such that $T(\gamma) = \gamma \subset \overline{\Omega}$, $\gamma_0 \subset \gamma$, and $\lim_{n\to\infty} T^n(Q) = P$ for any $Q \in \gamma$. Now set $\ell = T^{-1}(\gamma) \subset S_1 \cup \overline{S_2} = C_+$, which is a simple curve from O through P to infinity. Finally, $\ell = W^s(P)$ is established as in Case (i). ■