Aversion to the variability of pay and optimal incentive contracts

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Aversion to the variability of pay and optimal incentive contracts[*]

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Abstract

In a moral hazard setting with a performance additive in effort and a symmetrically distributed noise term, I show that compensation contracts which are convex in performance are suboptimal when the agent has mean-variance preferences. With step contracts, I show that sticks are more efficient than carrots: an exogenously given lower bound on payments is binding at the optimum. Intuitively, the variance of the agent's pay conditional on a high effort should be as low as possible, while it should be as high as possible conditional on a low effort. From an ex ante perspective, which is relevant for effort inducement, this maximizes the rewards associated to high effort, and the punishments associated to low effort. These results call into question the widespread use of stock-options and contracts with rewards-like features to provide incentives to risk averse executives.

JEL classification: D86; J33; M52.

Keywords: compensation; contract theory; incentives; moral hazard; optimal contracts.

Although it is possible to derive the optimal compensation contract in some particular settings,[1] in general we do not know whether concave contracts or convex contracts, punishments or rewards, are more efficient when incentive provision implies a deviation from first-best risk sharing. This paper contributes to answering this question. The main finding runs counter to the convention wisdom: even though making the agent's pay convex in his performance protects him against severe losses but still provides incentives in the form of upside participation, we show that convex compensation contracts are suboptimal for agents who are averse to the variability of their pay.

We study a simple principal-agent model of moral hazard in which the agent's effort results in a rightward translation of the symmetric probability distribution of his performance measure. In order to focus on the effect of risk aversion on contract design, we assume that the agent has mean-variance preferences. We then show that it is optimal for the equilibrium variance of his pay to be a decreasing function of his effort. In particular, this necessary condition for optimality rules out convex contracts, including stock-options.[2]

Intuitively, each agent can be viewed as having a type, namely the probability distribution of performances associated to the level of effort he exerted. The objective of giving adequate incentives while minimizing deviations from the first-best risk sharing rule for the type who exerts high effort is notably achieved by exposing him to a lower variance of pay than the type who exerts low effort. As a result, not only do agents exert higher effort to reduce the variance of their pay, but the type who exerts high effort may also be approximately insured in equilibrium, which minimizes the agency cost of incentive provision.

These results imply that measuring the strength of incentives using observed pay-performance

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[1]For instance, for an exponential distribution and an agent with square root utility (Holmstrom 1979), for a lognormal distribution and an agent with CRRA utility (Dittmann and Maug 2007), for a gamma distribution and an agent with HARA utility (Hemmer Kim and Verrecchia 2000).

[2]The managerial compensation literature, which generally uses CRRA-lognormal models, tends to reach similar conclusions. For example, the optimal strike (or exercise price) of stock-options is approximately zero in a CRRA-lognormal model when stock-options enter the risk averse agent's participation constraint – as they should in a model of efficient contracting (Hall and Murphy 2002). This implies that granting restricted stocks, whose payoff is a linear function of the stock price, is preferable to granting stock-options, whose payoff is a convex function of the stock price. This is consistent with our results. In a calibration of a CRRA-lognormal model to a sample of U.S. CEOs, optimal unconstrained contracts are concave for the vast majority of CEOs (Dittmann and Maug 2007). This is also consistent with our results.

sensitivities may be misleading. The managerial compensation literature (Jensen and Murphy 1990, Murphy 1999) has tended to measure incentives using the sensitivity of the manager's pay to his firm's performance. However, we show that making the variance of pay a decreasing function of effort in equilibrium (this can be achieved by making compensation a concave function of performance) reduces the average pay-performance sensitivity required in equilibrium to induce a given level of effort. This may contribute to explaining low observed pay-performance sensitivities (Jensen and Murphy 1990). Indeed, to the extent that a higher effort increases the likelihood of good performances, (ex ante) incentives can be strong even if pay-performance sensitivities are low when measured ex post. For example, if incentives are primarily provided through punishments for poor performances, incentives can be strong even if punishments are rarely administered in equilibrium. In this regard, the widespread use of stock-options and the associated rise in pay-performance sensitivities throughout the 1980s and the 1990s (Hall and Liebman 1998, Frydman and Saks 2008) may indicate a deterioration in the efficiency of incentive provision rather than an increase in the amount of explicit incentives provided.

The paper starts by deriving the aforementioned results by not restricting attention to a particular class of contracts. Then it studies options-based contracts, and step contracts. With options, contracts based on a short position in a put option are more efficient than contracts based on a long position in a call option (i.e., stock-options). We show that a contract taking the form of a short position in a put option with an arbitrarily small exercise price is approximately optimal when the performance measure is normally distributed. With step contracts characterized by two levels of payments and with an exogenously given lower bound on payments, we find that the low payment is equal to the lower bound at the optimum. In the limit, with an arbitrarily low lower bound on payments, we show that the optimal contract is the Mirrlees approximation of the optimal contract (Mirrlees 1975). The fact that punishments make the variance of pay a decreasing function of effort makes them more efficient than rewards. However, for a sufficiently high lower bound on payments, we also find that it is impossible to use punishments for incentive purposes without giving a rent to the agent. In this case, it may be optimal to use rewards instead. Contracts with rewards features may thus

be constrained-efficient.

Some papers have already recognized that punishments are more efficient than rewards to incentivize risk averse agents. The superiority of punishments over rewards in a monitoring problem is driven by properties of the utility function (Jewitt 1988). A contract based on extreme punishments approximates the first-best in a moral hazard problem when the agent is risk averse (Mirrlees 1975). The contribution of this paper is to isolate and highlight the impact of the agent's aversion to the variability of his pay on the optimal structure of his incentives in a simple but general setting.

Other papers argue that rewards are more efficient than punishments: it is optimal to offer a convex, call-option like contract to a risk-neutral agent protected by limited liability (Innes 1990, Tirole 2006). The risk aversion models of moral hazard and the risk-neutrality-limited liability models of moral hazard thus tend to generate different predictions regarding the form of the optimal contract. Punishments can be optimal with the former, while rewards are generally optimal with the latter. Whereas it is well-understood why rewards are optimal with a risk neutral agent protected by limited liability, this paper contributes to explaining why punishments may be optimal for a risk averse agent. While some of the reasons were implicit in Mirrlees (1975), we make them more explicit.

Section 1 presents the model. Section 2 derives the main results in the general case. Section 3 compares calls and puts-based compensation. Section 4 studies step contracts. Section 5 concludes.

1   The model

We consider a standard principal-agent relationship. At time 0, the agent exerts some costly effort $e \in [0, \infty)$. Effort shifts the mean of the contractible performance measure $\tilde{\pi}$, which is realized at time 1. At time -1, the risk-neutral principal offers to the agent a compensation contract $W(\pi)$, which specifies the agent's payment at time 1 as a function of π. For simplicity, the discount rate is zero.

We assume that the performance measure writes as

$$\tilde{\pi} = e + \tilde{\epsilon}$$

The random variable $\tilde{\epsilon}$ is realized at time 1, but is unobservable. It is distributed according to the continuous probability density function (p.d.f.) φ, which is symmetric around zero, and the associated cumulative distribution function (c.d.f.) Φ. The mean of $\tilde{\epsilon}$ is normalized at zero and its variance is σ^2. Denote the p.d.f. of $\tilde{\pi}$ by ψ.

In order to isolate the effect of risk aversion on contract design, we only consider the first and second-order moments of the distribution of payments received by the agent. Our reduced-form model assumes that the agent likes the mean and dislikes the variance of payments. In addition to being very tractable, this approach has the merit of focusing on risk aversion, to the exclusion of other factors. Formally, for any future random payment $\tilde{W}$ and effort e, the agent's objective function takes the form of a mean-variance criterion:

$$E[\tilde{W}|e] - \omega var[\tilde{W}|e] - \frac{c}{2}e^2 \tag{1}$$

where ω is the weight attributed to variance, and the third term if the cost of effort. The parameter ω is therefore a measure of risk aversion. The agent has nonnegative reservation utility $\bar{U}$, defined as the minimum value of (1) in equilibrium such that he accepts the contract $W(\pi)$. At time 0, given a compensation contract $W(\pi)$, the agent chooses e to maximize his objective function.

For any given w, the compensation contract $W(\pi)$ may be decomposed as $W(\pi) = w + f(\pi)$. We assume that the probability distribution is such that any optimal compensation profile is nondecreasing in the performance measure. The problem of the principal is to design $W(\pi)$ to minimize the expected cost of compensation $E[W(\tilde{\pi})]$, subject to the constraints that the agent accepts the contract (the participation constraint) and exerts a given level of effort $e^\star$ (the incentive constraint).

A given contract $\{w, f\}$ satisfies the participation constraint if and only if

$$w + E[f(\tilde{\pi})|e^\star] - \omega var[f(\tilde{\pi})|e^\star] - c\frac{e^{\star 2}}{2} \geq \bar{U} \qquad (2)$$

For a given f, define $\underline{w}$ as the value of w which satisfies (2) as an equality. For (2) to be satisfied, we must have $w \geq \underline{w}$.

Assume that the effort cost c is large enough for the first-order approach to be valid for all given compensation contracts under consideration (this imposes that we consider a bounded set of contracts). This is simply a sufficient condition for the effort choice problem to be always concave when required, even when the compensation profile is convex. This technical requirement guarantees an interior solution to the effort choice problem. For a given f, the necessary and sufficient first-order condition with respect to effort is then

$$E[f'(\tilde{\pi})|e] - \omega\frac{\partial}{\partial e}var[f(\tilde{\pi})|e] - ce = 0 \qquad (3)$$

The contract f must be such that this equation is verified at $e = e^\star$, so that the equilibrium level of effort is $e^\star$. We call this condition the incentive constraint.

We define the first-best cost of eliciting effort $e^\star$, $C^{FB}(e^\star)$, and the first-best wage $w^\star$, as

$$C^{FB}(e^\star) \equiv \bar{U} + c\frac{e^{\star 2}}{2} \equiv w^\star \qquad (4)$$

For a given f which satisfies the incentive constraint in (3) for $e = e^\star$ and a given $w \geq \underline{w}$ which satisfies the participation constraint in (2) for this f, the second-best cost of eliciting effort $e^\star$ is

$$C^{SB}(e^\star) \equiv E[W(\tilde{\pi})|e^\star] = w + E[f(\tilde{\pi})|e^\star]$$

Using (2) and the definition of $\underline{w}$ yields

$$C^{SB}(e^\star) = \bar{U} + c\frac{e^{\star 2}}{2} + \omega var[f(\tilde{\pi})|e^\star] + (w - \underline{w})$$

Given f and w, we define the agency cost of eliciting effort $e^\star$ as

$$AC_{f,w}(e^\star) \equiv C^{SB}(e^\star) - C^{FB}(e^\star) = \omega var[f(\tilde{\pi})|e^\star] + (w - \underline{w}) \tag{5}$$

Since $C^{FB}(e^\star)$ is given, the principal's objective of minimizing the expected cost of compensation is equivalent to minimizing the agency cost $AC_{f,w}(e^\star)$ of eliciting a given effort $e^\star$. First, for any given f, this implies setting $w = \underline{w}$, given (5) and the fact that we must have $w \geq \underline{w}$ for the participation constraint (2) to be satisfied. For any given f which satisfies the incentive constraint in (3) for $e = e^\star$, the agency cost of eliciting effort $e^\star$ at the optimum is therefore

$$AC_f(e^\star) = \omega var[f(\tilde{\pi})|e^\star] \tag{6}$$

Second, since the agency cost is proportional to the variance of f at the optimum, the problem of the principal may be rewritten as

$$\min_f var[f(\tilde{\pi})|e^\star] \qquad \text{s.t.} \qquad (3) \text{ for } e = e^\star \tag{7}$$

We say that a given contract f is dominated if and only if the agency cost of implementing $e^\star$ can be reduced while leaving $e^\star$ unchanged. In view of (6), a contract f is dominated if and only if there exists another contract which also satisfies the incentive constraint (3) for $e = e^\star$ but which is characterized by a lower variance.

2   Convex contracts are dominated

We begin by deriving the central results of the paper, by not restricting attention to a particular class of compensation contracts. The first result below is proven in the Appendix. It implies that, at the equilibrium level of effort, any optimal contract is such that a marginal increase in effort reduces the variance of the agent's pay.

Lemma 1: Any contract such that

$$\frac{\partial}{\partial e} var[W(\tilde{\pi})|e^\star] > 0$$

is dominated.

Any contract f for which the variance of compensation at the equilibrium effort is locally strictly increasing in effort is suboptimal. Indeed, a symmetrical contract h with the same average slope and the same variance as f can be constructed. However, this new contract provides more incentives, since a marginal increase in effort at the equilibrium level of effort *reduces* the variance of compensation – whereas a marginal increase in effort has the opposite effect with the initial contract f. The new contract h, which necessarily provides excessive incentives, may therefore be flattened. This diminishes the equilibrium variance of pay, and therefore reduces agency costs. The approach is illustrated in figure 1, where the initial contract f is convex, the symmetrical contract h is concave, and its flattened transformation $\frac{1}{\theta}h$, with $\theta > 1$, is the dotted line. Finally, the fixed wage w is adjusted in order to satisfy the participation constraint. With mean-variance preferences, this vertical translation of payments does not affect either incentives or the variance of pay.

Interestingly, the average pay-performance sensitivity of the dominated contract f (the convex contract on figure 1) is *higher* than the pay-performance sensitivity of the new contract $\frac{1}{\theta}h$ (the dotted line on figure 1) which induces the same effort. Indeed, we show in Proof 1 that, in equilibrium,

$$E[f'(\tilde{\pi})|e^\star] = E[h'(\tilde{\pi})|e^\star]$$

So that, since $\theta > 1$,

$$E[f'(\tilde{\pi})|e^\star] > E[\frac{1}{\theta}h'(\tilde{\pi})|e^\star]$$

Even though both contracts elicit the same level of effort $e^\star$, an empirical (ex-post) measure of the pay-performance sensitivity with type-f contracts will be larger than with type-h contracts. This implies that the level of incentives provided by different contracts may only be ranked
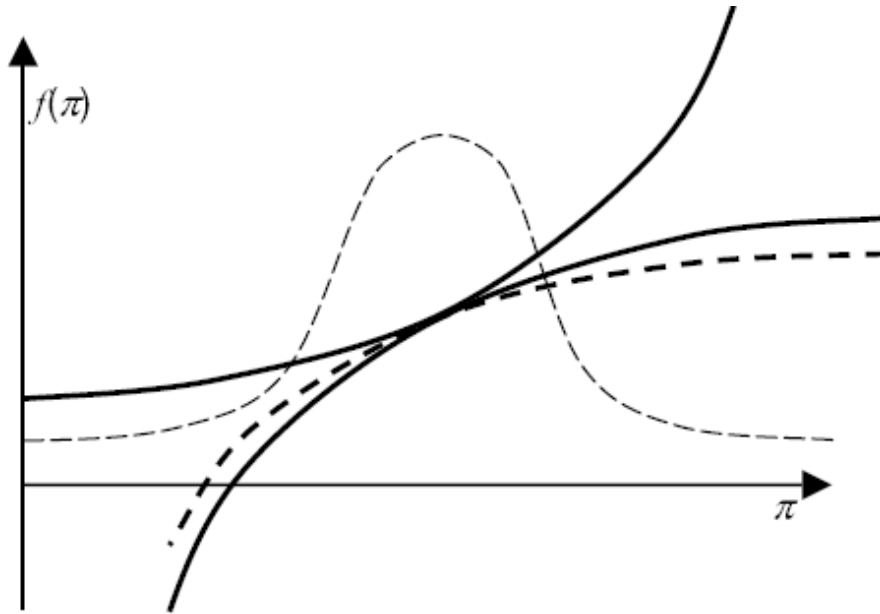
Figure 1: The convex contract is given. The concave contract with a solid line is the transformation of the convex contract with the same average slope and the same variance described in the Proof of Lemma 1. The concave contract with a dotted line is the flattened version of the aforementioned concave contract, which induces the same level of effort as the convex contract.

according to the pay-performance sensitivity criterion if these contracts all share the same form.

Unsurprisingly, Lemma 1 renders convex contracts suboptimal. The result below only applies to contracts which are differentiable twice.

Theorem 1: It is suboptimal for $W(\pi)$ to be convex in π.

This is due to the simple fact that the variance of compensation at the equilibrium level of effort is locally strictly increasing in effort with a convex contract, which is suboptimal. These results suggest that agents who are averse to the variance of their pay are more efficiently incentivized when the sensitivity of their pay to their performance is higher for low performances than for high performances.

The Mirrlees (1975) result that no optimal contract exists in a certain setting is theoretically

powerful but leaves us with little guidance on how to design compensation contracts in practice, all the more that a Mirrlees-type step contract based on extreme punishments for extremely low performance may not be feasible. We not only show that a class of contracts, namely convex contracts, is dominated, but we also show how to construct a contract which improves on any given convex contract. Crucially, such a contract would be concave. We also provide a clear economic intuition to explain why it is suboptimal to make the variance of the pay of a risk averse agent increasing in his effort in equilibrium.

3   Options-based compensation

In this section, we restrict attention to options-based contracts, which are the simplest piece-wise linear contracts. We start by extending the result of Theorem 1 to any contract which takes the form of a long position in call options, which is convex but not differentiable twice.

Corollary 1: Any contract of the form

$$W(\pi) = w + a \max\{\pi - S, 0\} \tag{8}$$

where $a > 0$, is suboptimal.

We show in the proof of Corollary 1 that for a contract of the form (8), then $\frac{\partial}{\partial e} var[f(\tilde{\pi})|e] > 0$. A direct application of Lemma 1 then gives us the result that any contract of the form (8) is dominated. In addition, an application of the proof of Lemma 1 shows that any contract of the form (8) is dominated by a contract taking the form of a short position in put options. This is not surprising. With a long position in call options, increasing effort increases the variance of pay. With a short position in put options, increasing effort decreases the variance of pay. With a linear contract, a change in effort does not affect the variance of pay.[3]

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[3]A long call contract is convex, while a short put contract is concave. In addition, we know from the proof of Theorem 1 that $\frac{\partial}{\partial e} var[f(\tilde{\pi})|e^\star] > 0$ if f is convex. We know from Proof 1 that $\frac{\partial}{\partial e} var[f(\tilde{\pi})|e^\star] < 0$ if f is concave. Finally, if f linear, then its mirror image h as defined in the Proof of Lemma 1 is defined by $h(\pi) = f(\pi)$ for any π. Since we must also have $\frac{\partial}{\partial e} var[h(\tilde{\pi})|e^\star] = -\frac{\partial}{\partial e} var[f(\tilde{\pi})|e^\star]$, we necessarily have

The next result shows that a short put contract with a very low strike is quasi-optimal when the performance measure is normally distributed.

Proposition 1: If $\tilde{\epsilon}$ is normally distributed, then a contract of the form

$$W(\pi) = w^\star + \varepsilon + a \min\{\pi - \underline{S}, 0\}$$

where $w^\star$ is the first-best cost of eliciting effort $e^\star$, ε is positive but arbitrarily small, $\underline{S}$ is negative and arbitrarily low, and a is set to satisfy the incentive constraint, is approximately optimal.

With this contract, the agent receives the first-best wage $w^\star$ (as defined in (4)) corresponding to effort $e^\star$ almost everywhere in equilibrium, but he is very severely punished for extremely low performances, which are extremely unlikely if adequate effort is exerted. Intuitively, with a distribution whose likelihood ratio $\frac{\psi_e}{\psi}$ is arbitrarily small at the left tail of the distribution, it is almost certain that a very low performance is the result of low effort. This means that estimating effort at the left tail of the distribution allows to discriminate almost perfectly between an agent who exerted effort $e^\star$ and an agent who exerted a lower effort. The former type is approximately insured and receives the first-best wage almost everywhere (in the sense that he receives the first-best wage with a probability which tends to one as $\underline{S}$ approaches minus infinity), whereas the latter type is exposed to a very harsh punishment with a much higher probability. This mechanism almost achieves first-best risk sharing, while also providing adequate incentives. Finally, note that it differs from the Mirrlees approximation of the optimal contract (Mirrlees 1975), which is a step contract.

All the aforementioned results rely on the symmetry of the probability distribution. This hypothesis was adopted as a normalization: with a linear contract and a symmetrically distributed performance measure, a change in effort leaves the variance of pay unchanged. On

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$\frac{\partial}{\partial e} var[f(\tilde{\pi})|e^\star] = 0$ if f is linear.

the contrary, with a linear contract and a non-symmetrically distributed performance measure, a change in effort typically changes the variance of pay. This matters because performance measures tend not to be symmetrically distributed. For example, the long-run distribution of stock prices is approximately lognormal. However, a lognormal transformation of a lognormally distributed random variable is normally distributed. The curvature of any contract may then be reinterpreted with the tools developed in this paper. In particular, with a lognormal distribution, the contract which is such that the variance of pay is invariant to effort (this role is played by the linear contract when the performance measure is symmetrically distributed) is an affine transformation of the log function, which is concave. A fortiori, giving a CEO a compensation contract convex in a lognormally distributed stock price (with restricted stocks and stock-options, say) makes the variance of his pay an increasing function of his effort. Indeed, such a contract could equivalently be represented as a function of a normal (not lognormal) variable, and it would then be even more convex than when plotted against the stock price. Using Theorem 1, we know that this contract, which is convex as a function of a symmetrically distributed random variable, is dominated. To conclude, not only are the results robust to the case of the lognormal distribution, they even argue more strongly for the suboptimality of convex contracts in this case.

4 Step contracts

To take into account a lower bound on payments, potentially due to limited liability, and determine at which level of performance it is preferable to concentrate incentives, we now study step contracts. A step contract pays off m for $\pi < e^\star + p$, and M for $\pi \geq e^\star + p$, with $M > m$ and a given p. If a step contract is characterized by $p < 0$, then we say that it uses "sticks". Indeed, with such a contract, the absolute value of $m - E[W(\tilde{\pi})]$ is larger than that of $M - E[W(\tilde{\pi})]$, by construction.[4] Otherwise we say that the contract uses "carrots". The next result shows that the "lower bound constraint" is always binding at the optimum.

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[4]We demonstrate this in passing in the proof of Proposition 2.

Proposition 2: With the constraint that $W(\pi) \geq L$ for any π, the optimal step contract is characterized by $m = L$.

For step contracts which induce the same given level of effort and for which the participation constraint binds, we show in the proof of Proposition 2 that both $\frac{d\Phi(p)}{dm}$ and $\frac{dM}{dm}$ are strictly positive for any value of m: both p and M are strictly increasing in m at the margin, for any value of m. This in turn implies that the parameters p and M of contracts which satisfy both the participation constraint as an equality and the incentive constraint are strictly increasing in m. Denote the parameters of the contract characterized at the optimum (when $m = L$) by $p = 0$ by $\{0, m_0, M_0\}$.

For a given $L < m_0$, denote the parameters of the optimal contract which satisfies both the participation constraint as an equality and the incentive constraint by $\{p_{-1}, m_{-1}, M_{-1}\}$. We know from Proposition 2 that $m_{-1} = L < m_0$, so that $p_{-1} < 0$ (by definition of m_0, and because p is increasing in m) and $M_{-1} < M_0$ (by definition of M_0, and because M is increasing in m). In this case, the optimal step contract involves a "punishment" m_{-1} for poor performances, which are relatively unlikely in equilibrium, since $p < 0$. The optimal contract uses sticks.

An intuition for Proposition 2 is that contracts with sticks provide more incentives to agents averse to the variability of their pay. The incentive constraint with step contracts, derived in the proof of Proposition 2 in (42), may be rewritten as

$$(M - m)[\varphi(p) + \omega\varphi(p)(M - m)(1 - 2\Phi(p))] = ce^\star \tag{9}$$

We know that $\Phi(p) < 0.5$ if and only if $p < 0$, since the probability distribution of $\tilde{\epsilon}$ is symmetric around the mean. In view of (9), for given payments m and M, the left-hand side of the incentive constraint is increasing in ω if and only if $p < 0$. That is, the amount of incentives provided is an increasing function of the agent's aversion to the variability of his pay if and only if the contract uses sticks. Whenever the agent is risk averse, this effect is maximized by equating the low payment m to the lower bound L.

In the limit, the optimal contract when L is arbitrarily small is the Mirrlees approximation

of the optimal contract (which is obtained in a setting without any lower bound on payments): $m = L$, p tends to minus infinity, and M tends to the first-best wage $w^\star$. In addition, the agency cost is approximately zero with this contract. We prove these two claims when c, $e^\star$, and ω are bounded away from zero and $ce^\star$ is bounded from above, in the Appendix.

When $L > m_0$, since $m = L$ at the optimum, and both p and M are increasing in m for contracts which satisfy the participation constraint as an equality and the incentive constraint, we know that either the optimal contract uses carrots (i.e., it is characterized by $p > 0$), or it does not satisfy the participation constraint as an equality. We make this argument formal, and show that using sticks when $L > m_0$ involves giving a rent to the agent:

Corollary 2: If $L > m_0$, a contract with $m = L$ and a given $p = \underline{p} < 0$ which satisfies the incentive constraint does not bind the participation constraint.

If $L > m_0$, then using any incentive-compatible contract with sticks rather than carrots results in the agent being paid more than would be needed to keep him at his reservation utility. To avoid this, it may be in the principal's interests to use carrots rather than sticks.

In short, punishments are optimal with unconstrained contracting (with $L \to -\infty$) or with a sufficiently low lower bound on payments, whereas rewards may be optimal with a sufficiently high lower bound on payments. This suggests that carrots or rewards are only used when downward deviations from the first-best wage can only be small, i.e., when the agent cannot be sufficiently punished.[5]

5   Conclusion

This paper shows that it is inefficient to use either carrots or convex contracts to provide monetary incentives to agents who are averse to the variability of their pay. This is because of the general principle that the variance of pay of agents who exert low effort should be higher

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[5]For example, the agent may not own much which can be at stake (wealth, a reputation, a potential future career, etc.), and he may have good outside opportunities, so that dismissal is not very costly from his perspective.

than for agents who exert high effort. However, it may be optimal to use carrots to incentivize an agent protected by limited liability, since in this case using sticks could involve giving him a rent.

The model explains the paucity of piece-rate or bonus-type incentive schemes in most firms (Medoff and Abraham 1980, Baker Jensen and Murphy 1988). It also underlines that pay-performance sensitivities as measured in equilibrium will be low if incentive schemes rely more on punishments than on rewards. This is simply because the performances of agents who exert high effort will tend to be good, so that punishments are rarely administered ex post, even though their existence provides adequate incentives ex ante.

This being said, the model cannot be reconciled with the use of stock-options for CEOs. The most plausible explanation for their widespread use may be the preferential fiscal and accounting treatment they are granted (Hall and Murphy 2003). In other words, the public authorities and the regulators might have skewed pay practices in a suboptimal direction. Alternatively, boards and compensation committees may still be primarily concerned with pay-performance sensitivities, and disregard the incentive effects of the shape of compensation profiles which we have emphasized. This also suggests that there is scope for improvements. At the very least, our results show that risk aversion (in the sense of an aversion for the variability of payments) alone cannot justify the use of contracts with rewards features.

To conclude, we have highlighted the implications of an aversion towards the variance of payments for the optimal design of incentive contracts. More applied work would design mechanisms which take into account this principle, whether in the particular context of an employment relationship, or in other areas where the provision of incentives matters. Future theoretical work could determine the implications for contract design of higher-order derivatives of the utility function, notably the preference for a positive skewness and the aversion to kurtosis.

6   Appendix

Proof of Lemma 1:

For any contract such that $\frac{\partial}{\partial e} var[W(\tilde{\pi})|e^{\star}] > 0$, i.e. such that $\frac{\partial}{\partial e} var[f(\tilde{\pi})|e^{\star}] > 0$, we need to show that it is possible to design another contract that induces the same effort $e^{\star}$ at a lower variance. Indeed, we know from section 1 that, given the new contract, w adjusts to satisfy the participation constraint: $w = \underline{w}$, where $\underline{w}$ satisfies the participation constraint (2) as an equality (note that a change in w neither affects the incentive constraint in (3) nor the variance of compensation). Finally, we know from (6) that the new contract with a lower variance is associated with a lower agency cost, and therefore dominates the former.

To this end, in the (π, W) space, consider the contract symmetrical to f with respect to the point $(e^{\star}, f(e^{\star}))$: the function defining the new compensation profile is obtained by taking a function symmetrical to f with respect to the horizontal line going through the point $(e^{\star}, f(e^{\star}))$, then taking a function symmetrical to this new function with respect to the vertical line going through the same point. Denote the function thus obtained by h.

To start with, assume that effort does not change. Then, we show in Proof 1 later in the Appendix that

$$var[h(\tilde{\pi})|e^{\star}] = var[f(\tilde{\pi})|e^{\star}]$$

$$E[h'(\tilde{\pi})|e^{\star}] = E[f'(\tilde{\pi})|e^{\star}]$$

$$\frac{\partial}{\partial e} var[h(\tilde{\pi})|e^{\star}] < 0$$

We also show later in the Appendix that a sufficient condition for the first-order approach to hold for the contract h if it holds for the contract f is that $E[f''(\pi)] > 0$.

However, effort, as defined in (3) by the first-order condition of the agent's problem, will increase under h. Indeed, $e^{\star}$ solves

$$e^{\star} = \frac{1}{c}\Big(E[f'(\tilde{\pi})|e^{\star}] - \omega \frac{\partial}{\partial e} var[f(\tilde{\pi})|e^{\star}]\Big) \tag{10}$$

with $\frac{\partial}{\partial e} var[f(\tilde{\pi})|e^{\star}] > 0$. The effort $\hat{e}$ induced by h solves

$$\hat{e} = \frac{1}{c}\Big(E[h'(\tilde{\pi})|\hat{e}] - \omega \frac{\partial}{\partial e} var[h(\tilde{\pi})|\hat{e}]\Big) \tag{11}$$

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But, since $E[h'(\tilde{\pi})|\hat{e}] = E[f'(\tilde{\pi})|e^\star]$ and $\frac{\partial}{\partial e}var[h(\tilde{\pi})|\hat{e}] < 0$, we have $\hat{e} > e^\star$.

For any e and any $\theta > 0$,

$$E\left[\frac{1}{\theta}h'(\tilde{\pi})|e\right] = \frac{1}{\theta}E\left[h'(\tilde{\pi})|e\right] \tag{12}$$

$$\frac{\partial}{\partial e}var\left[\frac{1}{\theta}h(\tilde{\pi})|e\right] = \frac{1}{\theta^2}\frac{\partial}{\partial e}var\left[h(\tilde{\pi})|e\right] \tag{13}$$

It follows from these two equations that the terms $E\left[\frac{1}{\theta}h'(\tilde{\pi})|e\right]$ and $\frac{\partial}{\partial e}var\left[\frac{1}{\theta}h(\tilde{\pi})|e\right]$ are monotonically decreasing in θ. Furthermore, their first-order derivatives with respect to θ exist and are finite, for $\theta \in (0, \infty)$. Lastly, for $\theta = 1$, effort $\hat{e}$ is induced. Therefore, there exists a $\theta > 1$ such that

$$e^\star = \frac{1}{c}\left(E\left[\frac{1}{\theta}h'(\tilde{\pi})|e^\star\right] - \omega\frac{\partial}{\partial e}var\left[\frac{1}{\theta}h(\tilde{\pi})|e^\star\right]\right) \tag{14}$$

where $e^\star$ solves (10).

The compensation profile $\frac{1}{\theta}h$ induces the same effort as f, but at a lower agency cost: the variance of h is equal to the variance of f, so that the variance of $\frac{1}{\theta}h$ is lower than the variance of f. The proof is complete.

Proof 1:

Consider the function g, symmetrical to f with respect to the horizontal line going through the point $(e^\star, f(e^\star))$. Up to an additive constant equal to $2f(e^\star)$, g is the opposite of f: $g(\pi) = -f(\pi) + 2f(e^\star)$. Its variance writes as

$$var[g(\tilde{\pi})] \equiv E\left[\left(g(\tilde{\pi}) - E[g(\tilde{\pi})]\right)^2\right] = E\left[\left(-f(\tilde{\pi}) + 2f(e^\star) + E[f(\tilde{\pi})] - 2f(e^\star)\right)^2\right]$$

$$= E\left[\left(f(\tilde{\pi}) - E[f(\tilde{\pi})]\right)^2\right] \equiv var[f(\tilde{\pi})] \tag{15}$$

Consider the function h, symmetrical to g with respect to the vertical line going through the point $(e^\star, f(e^\star))$. By definition, it writes as

$$h(\pi) = g(\pi - 2(\pi - e^\star)) = g(-\pi + 2e^\star)$$

where $\varphi(x)$ is the p.d.f. for a symmetrically distributed variable x of mean 0 and variance σ^2. Because the p.d.f. of $\tilde{\pi}$ is centred around the mean $e^\star$ of $\tilde{\pi}$, we have $\psi(-\pi + 2e^\star) = \psi(\pi)$. The expectation of $h(\tilde{\pi})$ is

$$E[h(\tilde{\pi})] \equiv \int_{-\infty}^{\infty} h(\pi)\psi(\pi)d\pi = \int_{-\infty}^{\infty} g(-\pi + 2e^\star)\psi(\pi)d\pi$$

$$= \int_{-\infty}^{\infty} g(\pi)\psi(-\pi + 2e^\star)d\pi = \int_{-\infty}^{\infty} g(\pi)\psi(\pi)d\pi \equiv E[g(\tilde{\pi})] \tag{16}$$

where the second equality uses the definition of g, the third involves a change of variable, and the fourth uses the symmetry of φ unveiled above. Equalities below involve the same steps, plus the fact that $E[g(\tilde{\pi})]$ is a constant.

$$var[h(\tilde{\pi})] \equiv \int_{-\infty}^{\infty} \Big(h(\pi) - E[h(\tilde{\pi})]\Big)^2 \psi(\pi)d\pi = \int_{-\infty}^{\infty} \Big(g(-\pi + 2e^\star) - E[g(\tilde{\pi})]\Big)^2 \psi(\pi)d\pi$$

$$= \int_{-\infty}^{\infty} \Big(g(\pi) - E[g(\tilde{\pi})]\Big)^2 \psi(-\pi + 2e^\star)d\pi = \int_{-\infty}^{\infty} \Big(g(\pi) - E[g(\tilde{\pi})]\Big)^2 \psi(\pi)d\pi \equiv var[g(\tilde{\pi})] \tag{17}$$

Eventually, combining (15) with (17),

$$var[h(\tilde{\pi})] = var[g(\tilde{\pi})] = var[f(\tilde{\pi})]$$

The second part of the proof compares $E[h'(\tilde{\pi})]$ to $E[f'(\tilde{\pi})]$. Using the definition of g and remembering that $f(e^\star)$ is a constant, we get $g'(\pi) = -f'(\pi)$, and $E[g'(\tilde{\pi})] = -E[f'(\tilde{\pi})]$. Besides, the derivative of h with respect to π is $h'(\pi) = -g'(-\pi + 2e^\star)$. Going once again through the same steps,

$$E[h'(\tilde{\pi})] \equiv \int_{-\infty}^{\infty} h'(\pi)\psi(\pi)d\pi = \int_{-\infty}^{\infty} -g'(-\pi + 2e^\star)\psi(\pi)d\pi$$

$$= \int_{-\infty}^{\infty} -g'(\pi)\psi(-\pi + 2e^\star)d\pi = \int_{-\infty}^{\infty} -g'(\pi)\psi(\pi)d\pi \equiv -E[g'(\tilde{\pi})] \tag{18}$$

Combining these two results, $E[h'(\tilde{\pi})] = -E[g'(\tilde{\pi})] = E[f'(\tilde{\pi})]$.

The third part of the proof shows that the derivatives with respect to effort of the conditional

variances of $f(\tilde{\pi})$ and $h(\tilde{\pi})$ have opposite signs. On the one hand, using the definition of g,

$$\frac{\partial}{\partial e}var[g(\tilde{\pi})|e] = \frac{\partial}{\partial e}var[-f(\tilde{\pi}) + 2f(e^\star)|e] = \frac{\partial}{\partial e}var[-f(\tilde{\pi})|e] = -\frac{\partial}{\partial e}var[f(\tilde{\pi})|e] \qquad (19)$$

On the other hand,

$$\frac{\partial}{\partial e}var[g(\tilde{\pi})|e] = \frac{\partial}{\partial e}\int_{-\infty}^{\infty}\Big(g(\pi) - E[g(\tilde{\pi})]\Big)^2\varphi(\epsilon)d\epsilon$$

$$= \int_{-\infty}^{\infty}2\Big[\frac{\partial}{\partial\pi}g(\pi) - \frac{\partial}{\partial\pi}E[g(\tilde{\pi})]\Big]\Big(g(\pi) - E[g(\tilde{\pi})]\Big)\varphi(\epsilon)d\epsilon$$

Using the definition of h,

$$\frac{\partial}{\partial e}var[h(\tilde{\pi})] = \frac{\partial}{\partial e}\int_{-\infty}^{\infty}\Big(h(\pi) - E[h(\tilde{\pi})]\Big)^2\varphi(\epsilon)d\epsilon$$

$$= \int_{-\infty}^{\infty}-2\Big[\frac{\partial}{\partial\pi}g(-\pi + 2e^\star) - \frac{\partial}{\partial\pi}E[g(-\tilde{\pi} + 2e^\star)]\Big]\Big(g(-\pi + 2e^\star) - E[g(-\tilde{\pi} + 2e^\star)]\Big)\varphi(\epsilon)d\epsilon$$

$$= \int_{-\infty}^{\infty}-2\Big[\frac{\partial}{\partial\pi}g(\pi) - \frac{\partial}{\partial\pi}E[g(\tilde{\pi})]\Big]\Big(g(\pi) - E[g(\tilde{\pi})]\Big)\varphi(-\epsilon)d\epsilon$$

$$= \int_{-\infty}^{\infty}-2\Big[\frac{\partial}{\partial\pi}g(\pi) - \frac{\partial}{\partial\pi}E[g(\tilde{\pi})]\Big]\Big(g(\pi) - E[g(\tilde{\pi})]\Big)\varphi(\epsilon)d\epsilon$$

$$= -\frac{\partial}{\partial e}var[g(\tilde{\pi})|e] \qquad (20)$$

Combining (19) with (20), we have $\frac{\partial}{\partial e}var[h(\tilde{\pi})|e] = -\frac{\partial}{\partial e}var[f(\tilde{\pi})|e]$ for every e.

The validity of the first-order approach

Suppose that, for any contract f characterized by $E[f''(\pi)] > 0$,

$$E[f''(\tilde{\pi})|e] + \frac{\partial^2}{\partial e^2}var[f(\tilde{\pi})|e] - c < 0 \qquad (21)$$

for any nonnegative e. Then the first-order approach is valid for the contract f, since the problem of the agent is concave in effort. In this case, we will show that, for the corresponding

contract h defined in the proof of Lemma 1,

$$E[h''(\tilde{\pi})|e] + \frac{\partial^2}{\partial e^2} var[h(\tilde{\pi})|e] - c < 0 \tag{22}$$

for any nonnegative e, so that the first-order approach is valid for the contract h.

First, since $h''(\pi) = -f''(-\pi + 2e)$ for any given e,

$$E[h''(\tilde{\pi})|e] \equiv \int_{-\infty}^{\infty} h''(\pi)\psi(\pi|e)d\pi = \int_{-\infty}^{\infty} -f''(-\pi + 2e)\psi(\pi|e)d\pi$$

$$= \int_{-\infty}^{\infty} -f''(\pi)\psi(-\pi + 2e|e)d\pi = -\int_{-\infty}^{\infty} f''(\pi)\psi(\pi|e)d\pi \equiv -E[f''(\tilde{\pi})|e] \tag{23}$$

Since $E[f''(\tilde{\pi})|e] > 0$, $E[h''(\tilde{\pi})|e] < 0$.

Second, we now show that, for any given e,

$$\frac{\partial^2}{\partial e^2} var[h(\tilde{\pi})|e] = \frac{\partial^2}{\partial e^2} var[f(\tilde{\pi})|e]$$

We know from the proof of Theorem 1 that, for any h,

$$\frac{\partial}{\partial e} var[h(\tilde{\pi})] = 2cov(h'(\tilde{\pi}), h(\tilde{\pi})) \tag{24}$$

Taking the second derivative,

$$\frac{\partial}{\partial e} cov(h'(\tilde{\pi}), h(\tilde{\pi})) = \int_{-\infty}^{\infty} \frac{\partial}{\partial e} \Big\{ \Big(h'(\pi) - E[h'(\pi)] \Big) \Big(h(\pi) - E[h(\tilde{\pi})] \Big) \Big\} \varphi(\epsilon)d\epsilon$$

$$= \int_{-\infty}^{\infty} \Big(\Big(h''(\pi) - E[h''(\tilde{\pi})] \Big) \Big(h(\pi) - E[h(\tilde{\pi})] \Big) + \Big(h'(\pi) - E[h'(\tilde{\pi})] \Big)^2 \Big) \varphi(\epsilon)d\epsilon \tag{25}$$

The second term in the integral above is the same for h as for f, as shown in Proof 1. Substituting for the expression that defines h, the first term in the integral above rewrites as

$$\int_{-\infty}^{\infty} \Big(-f''(-\pi + 2e) - E[-f''(-\tilde{\pi} + 2e)] \Big) \Big(-f(-\pi + 2e) + 2f(e) - E[-f(-\tilde{\pi} + 2e) + 2f(e)] \Big) \varphi(\epsilon)d\epsilon$$

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After some changes of variables, this becomes

$$\int_{-\infty}^{\infty} \Big(- f''(\pi) - E[-f''(\tilde{\pi})]\Big)\Big(- f(\pi) + 2f(e) - E[-f(\tilde{\pi})] - 2f(e)\Big)\varphi(\epsilon)d\epsilon$$

Rearranging, the first term in the integral in (25) is equal to

$$\int_{-\infty}^{\infty} \Big(f''(\pi) - E[f''(\tilde{\pi})]\Big)\Big(f(\pi) - E[f(\tilde{\pi})]\Big)\varphi(\epsilon)d\epsilon$$

All in all,
$$\frac{\partial}{\partial e}cov(h'(\tilde{\pi}), h(\tilde{\pi})) = \frac{\partial}{\partial e}cov(f'(\tilde{\pi}), f(\tilde{\pi}))$$

Using (24), we have
$$\frac{\partial^2}{\partial e^2}var[h(\tilde{\pi})|e] = \frac{\partial^2}{\partial e^2}var[f(\tilde{\pi})|e] \tag{26}$$

Because of (23) and (26), for any nonnegative e,

$$E[h''(\tilde{\pi})|e] + \frac{\partial^2}{\partial e^2}var[h(\tilde{\pi})|e] - c < E[f''(\tilde{\pi})|e] + \frac{\partial^2}{\partial e^2}var[f(\tilde{\pi})|e] - c$$

We have assumed in (21) that the right-hand side of this inequality is negative for any non-negative e, so that the left-hand-side is negative as well: (22) holds for any nonnegative e, and the first-order approach is valid for h.

Proof of Theorem 1:

It is sufficient to show that if f is convex in π, then $\frac{\partial}{\partial e}var[f(\tilde{\pi})|e^\star] > 0$. Lemma 1 then gives us the desired result.

To start with,

$$\frac{\partial}{\partial e}var[f(\tilde{\pi})|e^\star] = \int_{-\infty}^{\infty} \frac{\partial}{\partial e}\Big(f(e^\star + \epsilon) - E[f(e^\star + \tilde{\epsilon})]\Big)^2\varphi(\epsilon)d\epsilon$$

$$= \int_{-\infty}^{\infty} 2\Big(f(e^\star + \epsilon) - E[f(e^\star + \tilde{\epsilon})]\Big)\Big(f'(\pi) - E[f'(\tilde{\pi})]\Big)\varphi(\epsilon)d\epsilon \tag{27}$$

The function f being increasing and convex in the performance measure,

$$\frac{\partial f(\pi)}{\partial \epsilon} = \frac{\partial f(\pi)}{\partial \pi} \frac{\partial \pi}{\partial \epsilon} = f'(\pi) > 0$$

$$\frac{\partial f'(\pi)}{\partial \epsilon} = \frac{\partial f'(\pi)}{\partial \pi} \frac{\partial \pi}{\partial \epsilon} = f''(\pi) > 0$$

Therefore,

$$cov(f(\tilde{\pi}), f'(\tilde{\pi})) > 0 \tag{28}$$

Additionally,

$$cov(f(\tilde{\pi}), f'(\tilde{\pi})) = \int_{-\infty}^{\infty} \Big(f(\pi) - E[f(\tilde{\pi})] \Big) \Big(f'(\pi) - E[f'(\tilde{\pi})] \Big) \varphi(\epsilon) d\epsilon$$

Which is positive because of (28). Applying these two results to (27) completes the proof.

Proof of Corollary 1:

It is sufficient to show that a contract of the form described in (8) is characterized by $\frac{\partial}{\partial e} var[f(\tilde{\pi})|e^\star] > 0$. Lemma 1 then gives us the desired result. For a contract of the form described in (8),

$$f(\pi) = a \max\{\pi - S, 0\} \tag{29}$$

As in the proof of Theorem 1,

$$\frac{\partial}{\partial e} var[f(\tilde{\pi})|e^\star] = \int_{-\infty}^{\infty} 2(f(e^\star + \epsilon) - E[f(e^\star + \tilde{\epsilon})])(f'(\pi) - E[f'(\tilde{\pi})])\psi(\pi)d\pi = 2cov(f(\tilde{\pi}), f'(\tilde{\pi})) \tag{30}$$

where $f'(\pi) = 0$ for $\pi < S$, and $f'(\pi) = a > 0$ for $\pi > S$, so that $E[f'(\tilde{\pi})] > 0$, and $f'(\pi) > E[f'(\tilde{\pi})]$ for $\pi > S$.

Additionally, by definition of the covariance, and for the contract defined in (8),

$$cov(f(\tilde{\pi}), f'(\tilde{\pi})) = \int_{-\infty}^{S} (f(\pi) - E[f(\tilde{\pi})])(f'(\pi) - E[f'(\tilde{\pi})])\psi(\pi)d\pi$$

$$+ \int_S^\infty (f(\pi) - E[f(\tilde{\pi})])(f'(\pi) - E[f'(\tilde{\pi})])\psi(\pi)d\pi \qquad (31)$$

For f of the form (29), $f(\pi) = 0$ for $\pi < S$, and $f(\pi) > 0$ for $\pi > S$, so that $E[f(\tilde{\pi})] > 0$. It follows that $f(\pi) < E[f(\tilde{\pi})]$ for $\pi < S$. We are now going to show that both integrals in (31) are positive.

First, $f'(\pi) = 0 < E[f'(\tilde{\pi})]$ for $\pi < S$. It follows from this inequality and $f(\pi) < E[f(\tilde{\pi})]$ for $\pi < S$ that the first integral in (31) is positive. Second,

$$\int_S^\infty f(\pi)\psi(\pi)d\pi = E[f(\tilde{\pi})]$$

So that

$$\int_S^\infty f(\pi)\psi(\pi)d\pi > (1 - \Psi(S))E[f(\tilde{\pi})]$$

which implies that

$$\int_S^\infty f(\pi)(f'(\pi) - E[f'(\tilde{\pi})])\psi(\pi)d\pi > \int_S^\infty E[f(\tilde{\pi})](f'(\pi) - E[f'(\tilde{\pi})])\psi(\pi)d\pi$$

since $f'(\pi) - E[f'(\tilde{\pi})] = a - E[f'(\tilde{\pi})]$ for $\pi > S$, which is a constant. This shows that the second integral in (31) is positive.

Applying these two results to (31) yields

$$cov(f(\tilde{\pi}), f'(\tilde{\pi})) > 0 \qquad (32)$$

Equation (30) then implies that $\frac{\partial}{\partial e}var[f(\tilde{\pi})|e^\star] > 0$ for a contract of the form described in (8), and Lemma 1 gives the desired result.

Proof of Proposition 1:

The mean-variance criterion of an agent exerting effort $e^\star$ is

$$\int_{-\infty}^S (w^\star + \varepsilon + a(\pi - S))\psi(\pi|e^\star)d\pi + \int_S^\infty (w^\star + \varepsilon)\psi(\pi|e^\star)d\pi$$

$$-\omega\left[\int_{-\infty}^{S}\Big(w^{\star}+\varepsilon+a(\pi-S)-E[f(\tilde{\pi})]\Big)^{2}\psi(\pi|e^{\star})d\pi+\int_{S}^{\infty}\Big(w^{\star}+\varepsilon-E[f(\tilde{\pi})]\Big)^{2}\psi(\pi|e^{\star})d\pi\right]$$

For a given S, the slope a is set to satisfy the incentive constraint, which is

$$\int_{-\infty}^{S}(w^{\star}+\varepsilon+a(\pi-S))\frac{\partial}{\partial e}\psi(\pi|e^{\star})d\pi+\int_{S}^{\infty}(w^{\star}+\varepsilon)\frac{\partial}{\partial e}\psi(\pi|e^{\star})d\pi$$

$$-\omega\left[\int_{-\infty}^{S}\Big(w^{\star}+\varepsilon+a(\pi-S)-E[f(\tilde{\pi})]\Big)^{2}\frac{\partial}{\partial e}\psi(\pi|e^{\star})d\pi+\int_{S}^{\infty}\Big(w^{\star}+\varepsilon-E[f(\tilde{\pi})]\Big)^{2}\frac{\partial}{\partial e}\psi(\pi|e^{\star})d\pi\right]=ce^{\star}\tag{33}$$

We show in Proof 2 later in the Appendix that, for any given w and S, there is one and only one positive a which satisfies the incentive constraint.

With $\varepsilon=0$, the mean-variance criterion is lower than $w^{\star}$ by the following positive amount (which is, by the definition of $w^{\star}$ in (4), the shortfall to meet the participation constraint):

$$\int_{-\infty}^{S}-a(\pi-S)\psi(\pi|e^{\star})d\pi$$

$$+\omega\left[\int_{-\infty}^{S}\Big(w^{\star}+a(\pi-S)-E[f(\tilde{\pi})]\Big)^{2}\psi(\pi|e^{\star})d\pi+\int_{S}^{\infty}\Big(w^{\star}-E[f(\tilde{\pi})]\Big)^{2}\psi(\pi|e^{\star})d\pi\right]\tag{34}$$

Given the likelihood ratio of the normal distribution, for any arbitrarily large M there exists a S low enough such that

$$\frac{\frac{\partial}{\partial e}\psi(\pi|e^{\star})}{\psi(\pi|e^{\star})}<-M\tag{35}$$

for all $\pi<S$. Or

$$\psi(\pi|e^{\star})<-\frac{1}{M}\frac{\partial}{\partial e}\psi(\pi|e^{\star})$$

Substituting, (34) is smaller than the positive amount

$$-\frac{1}{M}\int_{-\infty}^{S}-a(\pi-S)\frac{\partial}{\partial e}\psi(\pi|e^{\star})d\pi$$

$$-\frac{\omega}{M}\left[\int_{-\infty}^{S}\Big(w^{\star}+a(\pi-S)-E[f(\tilde{\pi})]\Big)^{2}\frac{\partial}{\partial e}\psi(\pi|e^{\star})d\pi+\int_{S}^{\infty}\Big(w^{\star}-E[f(\tilde{\pi})]\Big)^{2}\frac{\partial}{\partial e}\psi(\pi|e^{\star})d\pi\right]\tag{36}$$

Using the incentive constraint (33), (36) is equal to

$$-\frac{1}{M}\int_{-\infty}^{\infty}w^{\star}\frac{\partial}{\partial e}\psi(\pi|e^{\star})d\pi - \frac{1}{M}ce^{\star} \tag{37}$$

For a normal distribution,

$$\frac{\partial}{\partial e}\psi(\pi|e^{\star}) = \frac{\pi - e^{\star}}{\sigma^2}\psi(\pi|e^{\star})$$

The expression in (37) therefore writes as

$$-\frac{1}{M}\int_{-\infty}^{\infty}w^{\star}\frac{\pi - e^{\star}}{\sigma^2}\psi(\pi|e^{\star})d\pi - \frac{1}{M}ce^{\star} = -\frac{1}{M}\frac{w^{\star}}{\sigma^2}[E[\tilde{\pi} - e^{\star}]] - \frac{1}{M}ce^{\star} = -\frac{1}{M}ce^{\star}$$

Which is given and finite, so that this expression tends to zero as M approaches infinity. Since $w = w^{\star} + \varepsilon$, and the derivative of the mean-variance criterion with respect to w is equal to one, ε must be strictly positive but may be arbitrarily small for the participation constraint to be satisfied:

$$\varepsilon = \frac{1}{M}ce^{\star} \longrightarrow_{M\to\infty} 0$$

With the contract described in Proposition 1, the second-best cost of eliciting effort $e^{\star}$ is

$$w^{\star} + \varepsilon + aE\min\{0, \pi - \underline{S}\} < w^{\star} + \varepsilon \longrightarrow_{M\to\infty} w^{\star}$$

where $w^{\star}$ is by definition the first-best cost of eliciting effort $e^{\star}$. It follows from (5) that the agency cost is approximately zero, and the proof is complete.

Proof 2:

We show that for any given w and S, there is one and only one slope a of a put option contract that satisfies the incentive constraint.

The incentive constraint is

$$E[f'(\tilde{\pi})|e^{\star}] - \omega\frac{\partial}{\partial e}var[f(\tilde{\pi})|e^{\star}] - ce^{\star} = 0 \tag{38}$$

where the third term is a constant.

For a short put option with strike S, the first term on the left-hand-side of (38) is equal to

$$E[f'(\tilde{\pi})|e^\star] = \int_{-\infty}^{S-e^\star} a\varphi(\epsilon)d\epsilon = a\Phi(S - e^\star)$$

Given a finite S and a finite $e^\star$, this term is equal to zero for $a = 0$, and tends to infinity as a approaches infinity. Furthermore, this term is increasing in a:

$$\frac{\partial}{\partial a}E[f'(\tilde{\pi})|e^\star] = \Phi(S - e^\star) > 0$$

We now show that the second term on the left-hand-side of (38) has the same properties with respect to a. For a short put option with strike S,

$$-\omega\frac{\partial}{\partial e}var[f(\tilde{\pi})|e^\star] = -\omega\left[\int_{-\infty}^{S-e}\left(a(\epsilon-S+e)+w-E[f(\tilde{\pi})]\right)^2\varphi(\epsilon)d\epsilon+\int_{S-e}^{\infty}\left(w-E[f(\tilde{\pi})]\right)^2\varphi(\epsilon)d\epsilon\right]$$

$$= -\omega\left[-\left(a(S-e-S+e)+w-E[f(\tilde{\pi})]\right)^2\varphi(S-e)+\left(w-E[f(\tilde{\pi})]\right)^2\varphi(S-e)\right.$$

$$+2a\int_{-\infty}^{S-e}\left(a(\epsilon+e-S)+w-E[f(\tilde{\pi})]\right)\varphi(\epsilon)d\epsilon$$

$$-\int_{-\infty}^{S-e}2\frac{\partial}{\partial e}E[f(\tilde{\pi})]\left(a(\epsilon+e-S)+w-E[f(\tilde{\pi})]\right)\varphi(\epsilon)d\epsilon$$

$$\left.-\int_{S-e}^{\infty}2\frac{\partial}{\partial e}E[f(\tilde{\pi})]\left(w-E[f(\tilde{\pi})]\right)\varphi(\epsilon)d\epsilon\right]$$

The first line in the expression above is zero. The last two lines cancel each other out, since

$$\int_{-\infty}^{S-e}\left(a(\epsilon+e-S)+w\right)\varphi(\epsilon)d\epsilon + \int_{S-e}^{\infty}w\varphi(\epsilon)d\epsilon = E[f(\tilde{\pi})] \tag{39}$$

Eventually,

$$-\omega\frac{\partial}{\partial e}var[f(\tilde{\pi})|e^\star] = -2\omega a\int_{-\infty}^{S-e}\left(a(\epsilon+e-S)+w-E[f(\tilde{\pi})]\right)\varphi(\epsilon)d\epsilon \tag{40}$$

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This expression is equal to zero for $a = 0$, and tends to infinity as a approaches infinity. Furthermore, it is monotonically increasing in a. Indeed, differentiating (40) with respect to a,

$$\frac{\partial}{\partial a}\left\{-\omega\frac{\partial}{\partial e}var[f(\tilde{\pi})|e^\star]\right\} = -2\omega\int_{-\infty}^{S-e}(\epsilon+e-S)\varphi(\epsilon)d\epsilon - 2\omega\int_{-\infty}^{S-e}\left(a(\epsilon+e-S)+w-E[f(\tilde{\pi})]\right)\varphi(\epsilon)d\epsilon > 0$$

Both integrals in this expression are negative, so that both terms are positive.

To summarize, with a short put of strike S, the expression

$$E[f'(\tilde{\pi})|e^\star] - \omega\frac{\partial}{\partial e}var[f(\tilde{\pi})|e^\star]$$

is zero for $a = 0$, tends to infinity as a approaches infinity, and is monotonically increasing in a. Therefore, for any value of w and S, there exists one and only one a such that the incentive constraint in (38) is satisfied.

Proof of Proposition 2:

A step contract is defined by its floor m, its cap M, and the cutoff p. Bearing in mind that

$$E[W(\tilde{\pi})] = m\Phi(p) + M(1 - \Phi(p)) \tag{41}$$

$$var[W(\tilde{\pi})] = (m - E[W(\tilde{\pi})])^2\Phi(p) + (M - E[W(\tilde{\pi})])^2(1 - \Phi(p))$$

We compute derivatives with step contracts:

$$E[W'(\tilde{\pi})] = \lim_{a\to 0}\int_{p-a}^{p+a}\frac{M-m}{2a}d\Phi(\epsilon) = (M-m)\lim_{a\to 0}\frac{1}{2a}\int_{p-a}^{p+a}d\Phi(\epsilon)$$

$$= (M-m)\lim_{a\to 0}\frac{\Phi(p+a) - \Phi(p-a)}{2a} = (M-m)\varphi(p)$$

$$\frac{\partial}{\partial e}var[W(\tilde{\pi})] = \frac{\partial}{\partial e}\left\{\int_{-\infty}^{p-e+e^\star}(m - E[W(\tilde{\pi})])^2d\Phi(\epsilon) + \int_{p-e+e^\star}^{\infty}(M - E[W(\tilde{\pi})])^2d\Phi(\epsilon)\right\}$$

$$= -\varphi(p)(m - E[W(\tilde{\pi})])^2 + \varphi(p)(M - E[W(\tilde{\pi})])^2$$

This expression is negative if and only if

$$|m - E[W(\tilde{\pi})]| > |M - E[W(\tilde{\pi})]|$$

which is the case if and only if p is lower than zero (because of (41) and the fact that, for a symmetrically distributed random variable, $\Phi(0) = 0.5$).

The incentive constraint is

$$(M - m)\varphi(p) - \omega\varphi(p)\Big[M^2 - m^2 - 2(M - m)E[W(\tilde{\pi})]\Big] = ce^\star$$

Or

$$(M - m)\Big[\varphi(p) - \omega\varphi(p)\big(M + m - 2m\Phi(p) - 2M(1 - \Phi(p))\big)\Big] = ce^\star \tag{42}$$

We rewrite the incentive constraint in (42) as

$$\eta(m, M(m)) \equiv ce^\star$$

Suppose that a given step contract satisfies the incentive constraint. Following a change in m, the adjustment in M required for the incentive constraint to be satisfied, holding p constant, is

$$\frac{dM}{dm} = -\frac{\frac{\partial}{\partial m}\eta(m, M(m))}{\frac{\partial}{\partial M}\eta(m, M(m))} \tag{43}$$

The participation constraint is

$$m\Phi(p) + M(1 - \Phi(p)) - \omega\Big[(m - E[W(\tilde{\pi})])^2\Phi(p) + (M - E[W(\tilde{\pi})])^2(1 - \Phi(p))\Big] = \bar{U} + c\frac{e^\star}{2} \tag{44}$$

We rewrite (44) as

$$\mu(m, [\Phi(p)](m)) \equiv \bar{U} + c\frac{e^\star}{2}$$

Suppose that a given step contract satisfies the participation constraint. Following a change in m, the adjustment in p required for the participation constraint to be satisfied, holding M

constant, is

$$\frac{d\Phi(p)}{dm} = -\frac{\frac{\partial}{\partial m}\mu(m,[\Phi(p)](m))}{\frac{\partial}{\partial p}\mu(m,[\Phi(p)](m))} \tag{45}$$

A given contract which satisfies both the participation constraint and the incentive constraint is dominated if marginally changing m, and adjusting M and p accordingly to satisfy the incentive and participation constraints (so that the agent still participates and still selects the same level of effort $e^\star$), diminishes the expected payments made to the agent (the second-best cost of eliciting effort $e^\star$). We now calculate the sign of the total derivative of expected payments to the agent with respect to m

$$\frac{d}{dm}E[W(\tilde{\pi})] = \Phi(p) + m\frac{d\Phi(p)}{dm} + \frac{dM}{dm}(1-\Phi(p)) - M\frac{d\Phi(p)}{dm}$$

$$= \Phi(p) + (M-m)\left(-\frac{d\Phi(p)}{dm}\right) + \frac{dM}{dm}(1-\Phi(p)) \tag{46}$$

Start with the $\frac{d\Phi(p)}{dm}$ term. On the one hand,

$$\frac{\partial}{\partial m}\mu(m,[\Phi(p)](m)) = \Phi(p)-\omega\Big[2(1+\Phi(p))(m-E[W(\tilde{\pi})])\Phi(p)+2(1+\Phi(p))(M-E[W(\tilde{\pi})])(1-\Phi(p))\Big]$$

$$= \Phi(p) - 2\omega(1+\Phi(p))\Big[m\Phi(p) - m(\Phi(p))^2 - M(1-\Phi(p))\Phi(p) + M(1-\Phi(p))$$

$$-m\Phi(p)(1-\Phi(p)) - M(1-\Phi(p))^2\Big] = \Phi(p)$$

On the other hand,

$$\frac{\partial}{\partial\Phi(p)}\mu(m,[\Phi(p)](m)) = m-M-2\omega\Big[(M-m)(m-E[W(\tilde{\pi})])\Phi(p)+\big(m-m\Phi(p)-M(1-\Phi(p))\big)^2$$

$$+(M-m)(M-E[W(\tilde{\pi})])(1-\Phi(p)) - \big(m-m\Phi(p)-M(1-\Phi(p))\big)^2\Big]$$

$$= (M-m)\Big[-1-2\omega\big[m\Phi(p)+M(1-\Phi(p))-E[W(\tilde{\pi})]\big]\Big]$$

$$= -(M-m)$$

So that

$$\frac{d\Phi(p)}{dm} = \frac{\Phi(p)}{M-m} > 0 \tag{47}$$

Next, consider the $\frac{dM}{dm}$ term On the one hand,

$$\frac{\partial}{\partial m}\eta(m, M(m)) = -\varphi(p) + \omega\varphi(p)\Big[M - m - 2m\Phi(p) - 2M(1 - \Phi(p)) - (M - m)(1 - 2\Phi(p))\Big]$$

$$= -\varphi(p) + \omega\varphi(p)\Big[2m - 2M - 4m\Phi(p) + 4M\Phi(p)\Big]$$

On the other hand,

$$\frac{\partial}{\partial M}\eta(m, M(m)) = \varphi(p) - \omega\varphi(p)\Big[M - m - 2m\Phi(p) - 2M(1 - \Phi(p)) + (M - m)(1 - 2(1 - \Phi(p)))\Big]$$

$$= -\varphi(p) + \omega\varphi(p)\Big[2m - 2M - 4m\Phi(p) + 4M\Phi(p)\Big]$$

So that

$$\frac{dM}{dm} = 1 > 0 \tag{48}$$

Putting these results together, the change in the second-best cost of a contract following a marginal change in m is

$$\frac{d}{dm}E[W(\tilde{\pi})] = \Phi(p) - (M - m)\frac{\Phi(p)}{M-m} + \frac{dM}{dm}(1 - \Phi(p)) = 1 - \Phi(p) > 0 \tag{49}$$

which is positive, so any contract with $m > L$ is dominated. Therefore, at the optimum, $m = L$.

The Mirrlees approximation of the optimal contract :

We will demonstrate that if L is negative and arbitrarily small, if c, $e^\star$ and ω are bounded away from zero, and if $ce^\star$ is bounded from above, then the Mirrlees approximation of the optimal contract (which is obtained in a setting without any lower bound on payments) is the optimal contract, and the agency cost is approximately zero.

In the first part of the proof, we show in four steps that $m = L$, $M \longrightarrow_{L\to-\infty} w^\star$, $p \longrightarrow_{L\to-\infty} -\infty$.

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First, we know from Proposition 2 that the optimal contract is characterized by $m = L$.

Second, since the variance of $W(\tilde{\pi})$ is positive,

$$-\omega[(m - E[W(\tilde{\pi})])^2\Phi(p) + (M - E[W(\tilde{\pi})])^2(1 - \Phi(p))] < 0$$

For the participation constraint already written in (44) to be satisfied, we therefore need to have

$$m\Phi(p) + M(1 - \Phi(p)) > \bar{U} + c\frac{e^\star}{2} = w^\star$$

Since m is arbitrarily small, $0 < \Phi(p) < 1$, and $0 < 1 - \Phi(p) < 1$, this inequality requires that $M > w^\star$. For any given M, let ε be implicitly defined by $M \equiv w^\star + \varepsilon$.

Third, for a negative and arbitrarily small L, given that ω is bounded away from zero and that $M > w^\star$, all terms on the left-hand side of the incentive constraint (42) are negligible next to $\varphi(p)L^2\omega$. We must therefore have $\varphi(p)L^2\omega \approx ce^\star$ for the incentive constraint to be satisfied. This rewrites as

$$\varphi(p) \longrightarrow_{L\to-\infty} \frac{ce^\star}{\omega}\frac{1}{L^2} \tag{50}$$

Because L is arbitrarily small, and ω is bounded away from zero, the right-hand side of (50) tends to zero as L approaches minus infinity. Since $\varphi(p)$ tends to zero as p approaches minus infinity, (50) is verified if

$$p \longrightarrow_{L\to-\infty} -\infty \tag{51}$$

Fourth, the normal distribution is characterized by

$$\frac{\Phi(p)}{\varphi(p)} \longrightarrow_{p\to-\infty} 0 \tag{52}$$

Rewrite the participation constraint in (44) as

$$L\Phi(p) + (w^\star + \varepsilon)(1 - \Phi(p)) - \omega\Big[(L - E[W(\tilde{\pi})])^2\Phi(p) + (w^\star + \varepsilon - E[W(\tilde{\pi})])^2(1 - \Phi(p))\Big] = w^\star \tag{53}$$

Consider the first two terms on the left-hand side of (53): Substituting from (50) into (52):

$$\frac{\Phi(p)}{ce^\star}\omega L^2 \longrightarrow_{p\to-\infty} 0 \tag{54}$$

Since ω is bounded away from zero and $ce^\star$ is bounded from above, this implies that

$$E[W(\tilde{\pi})] = L\Phi(p) + (w^\star + \varepsilon)(1 - \Phi(p)) \longrightarrow_{p\to-\infty} w^\star + \varepsilon \tag{55}$$

Consider the third term on the left-hand side of (53). Since ω is bounded away from zero and $ce^\star$ is bounded from above, (54) also implies that

$$\Phi(p)L^2 \longrightarrow_{p\to-\infty} 0$$

So that, using (55),

$$(L - E[W(\tilde{\pi})])^2\Phi(p) + (w^\star + \varepsilon - E[W(\tilde{\pi})])^2(1 - \Phi(p))$$

$$= (L - w^\star - \varepsilon)^2\Phi(p) + (w^\star + \varepsilon - w^\star - \varepsilon)^2(1 - \Phi(p)) \longrightarrow_{p\to-\infty} 0 \tag{56}$$

Combining (55) and (56), the participation constraint in (53) rewrites as

$$w^\star + \varepsilon \longrightarrow_{p\to-\infty} w^\star \tag{57}$$

So that, given (51),

$$\varepsilon \longrightarrow_{L\to-\infty} 0 \tag{58}$$

and

$$M \longrightarrow_{L\to-\infty} w^\star \tag{59}$$

The first part of proof is complete.

Using the definition of the agency cost in (5), we know from (55) that the agency cost with

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the proposed contract is equal to

$$L\Phi(p) + (w^\star + \varepsilon)(1 - \Phi(p)) - w^\star \longrightarrow_{L \to -\infty} \varepsilon$$

Using (58), the agency cost is approximately zero for an arbitrarily small L, and the proof is complete.

Proof of Corollary 2:

The optimal contract is characterized by $m = L$, Φ is strictly increasing, and both $\frac{d\Phi(p)}{dm}$ and $\frac{dM}{dm}$ are strictly positive for contracts which satisfy the participation constraint as an equality and the incentive constraint (as shown in the proof of Proposition 2). An optimal contract which satisfies the participation constraint as an equality and the incentive constraint will therefore be characterized by a given $p = \underline{p} < 0$ if and only if $L < m_0$. In this case, denote the L and M associated to a given $\underline{p} < 0$ by $\underline{L}$ and $\underline{M}$, respectively. Such a contract can be denoted by $\{\underline{p}, \underline{L}, \underline{M}\}$. Since it satisfies the participation constraint as an equality, we have:

$$\Phi(\underline{p})\underline{L} + (1 - \Phi(\underline{p}))\underline{M} - \omega\Big[\Phi(\underline{p})(\underline{L} - E[W(\tilde{\pi})])^2 + (1 - \Phi(\underline{p}))(\underline{M} - E[W(\tilde{\pi})])^2\Big] - c\frac{e^\star}{2} = \bar{U} \quad (60)$$

where $E[W(\tilde{\pi})] = \Phi(\underline{p})\underline{L} + (1 - \Phi(\underline{p}))\underline{M}$.

Now suppose that $L = \bar{L} > m_0$. Consider the contract with $p = \underline{p}$ and $m = \bar{L}$. We determine the value of M such that it satisfies the incentive constraint, and we denote it by $\bar{M}$. The incentive constraint (42), derived in the proof of Proposition 2, may be rewritten as

$$(M - m)[\varphi(p) + \omega\varphi(p)(M - m)(1 - 2\Phi(p))] = ce^\star \quad (61)$$

In order for two given contracts characterized by the same value of p, namely $\underline{p}$, to satisfy (61), they need to be characterized by the same value of $M - m$, so that we must have

$$\bar{M} = \bar{L} + (\underline{M} - \underline{L}) \quad (62)$$

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For $L = \bar{L}$, the left-hand-side of the participation constraint for the contract with $p = \underline{p}$ which satisfies the incentive constraint writes as

$$\Phi(\underline{p})\bar{L} + (1 - \Phi(\underline{p}))\bar{M} - \omega\left[\Phi(\underline{p})(\bar{L} - E[W(\tilde{\pi})])^2 + (1 - \Phi(\underline{p}))(\bar{M} - E[W(\tilde{\pi})])^2\right] - c\frac{e^\star}{2} \quad (63)$$

where $E[W(\tilde{\pi})] = \Phi(\underline{p})\bar{L} + (1 - \Phi(\underline{p}))\bar{M}$.

Because of (62), both contracts, $\{\underline{p}, \underline{L}, \underline{M}\}$ and $\{\underline{p}, \bar{L}, \bar{M}\}$, have the same variance:

$$\Phi(\underline{p})(\underline{L} - E[W(\tilde{\pi})])^2 + (1 - \Phi(\underline{p}))(\underline{M} - E[W(\tilde{\pi})])^2 = \Phi(\underline{p})(\bar{L} - E[W(\tilde{\pi})])^2 + (1 - \Phi(\underline{p}))(\bar{M} - E[W(\tilde{\pi})])^2$$

(64)

However, since using successively (62) and $\underline{L} < m_0 < \bar{L}$ gives

$$\Phi(\underline{p})\bar{L} + (1 - \Phi(\underline{p}))\bar{M} = \Phi(\underline{p})\bar{L} + (1 - \Phi(\underline{p}))(\bar{L} + (\underline{M} - \underline{L}))$$

$$= \Phi(\underline{p})\underline{L} + (1 - \Phi(\underline{p}))\underline{M} + \bar{L} - \underline{L} > \Phi(\underline{p})\underline{L} + (1 - \Phi(\underline{p}))\underline{M} \quad (65)$$

it follows that the left-hand side of the participation constraint in (63) (for the $\{\underline{p}, \bar{L}, \bar{M}\}$ contract) is strictly larger than the left-hand side of the participation constraint in (60) (for the $\{\underline{p}, \underline{L}, \underline{M}\}$ contract). But since the latter satisfies the participation constraint as an equality, the former does not: (63) is not binding, and the proof is complete.

7 References

Baker, G.P., Jensen, M.C., Murphy, K.J.: Compensation and incentives: practice vs. theory. *J. Finance*, 43 (3), 593-616 (1988).

Dittmann, I., Maug, E.: Lower salaries and no options? On the optimal structure of executive pay. *J. Finance*, 62 (1), 303-343 (2007).

Grossman, S.J., Hart, O.D.: An analysis of the principal-agent problem. *Econometrica*, 51 (1), 7-45 (1983).

Frydman, C., Saks, R.E.: Executive compensation: a new view from a long-term perspective. 1936-2005, NBER working paper no. W14145 (2008).

Hall, B.J., Liebman, J.B.: Are CEOs really paid like bureaucrats? *Q. J. Econ.*, 113 (3), 653-691 (1998)

Hall, B.J., Murphy, K.J.: Stock options for undiversified executives. *J. Accounting Econ.*, 33, 3-42 (2002).

Hall, B.J., Murphy, K.J.: The trouble with stock-options. *J. Econ. Perspect.*, 17 (3), 49-70 (2003).

Hemmer, T., Kim, O., Verrecchia, R.E.: Introducing convexity into optimal compensation contracts. *J. Accounting Econ.*, 28, 307-327 (2000).

Holmstrom, B.: Moral hazard and observability. *Bell J. Econ.*, 10 (1), 74-91 (1979).

Innes, R.D.: Limited liability and incentive contracting with ex-ante action choices. *J. Econ. Theory*, 52, 45-67 (1990).

Jensen, M.C., Murphy, K.J.: Performance pay and top-management incentives. *J. Political Econ.*, 98(2), 225-64 (1990).

Jewitt, I.: Justifying the first-order approach to principal-agent problems. *Econometrica*, 56 (5), 1177-90 (1988).

Medoff, J.L., Abraham, K.G.: Experience, performance, and earnings. *Q. J. Econ.*, 95 (4), 703-736 (1980).

Mirrlees, J.A.: The theory of moral hazard and unobservable behavior, Part 1. Mimeo, Oxford (1975).

Murphy, K.J.: Executive compensation, In: Ashenfelter, O, Card, D.E., *Handbook of Labor Econ.*, pp 2485-2563. Elsevier (1999).

Tirole, J.: *The Theory of Corporate Finance.* Princeton University Press, 115-118 (2006).