Gödel's "Slingshot" Argument and His Onto-Theological System

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1 Introduction

"Slingshot" arguments are usually formulated to demonstrate some unexpected simplicity of semantical or ontological structures lying behind certain philosophical ideas about meaning and reference. Church's reasoning from 1943 [6] is often mentioned as the first argumentation of this kind, where Church criticizes Carnap's idea of treating a denotation of a sentence as a relation between the sentence and a proposition. The aim of Church's critique was to show that the theory proposed by Carnap, which reduced meanings of terms and sentences to extensions, actually led to some kind of collapse: it emerged that there are only two denotations of all sentences – Fregean logical values the True and the False. The same conclusion (although justified in a different way) was again formulated by Church in 1956 [7, §04], and the similar arguments for the paradoxical statement that any two sentences with the same truth value refer to only one fact (proposition) were considered by Gödel (1944[1] [16]) and Davidson (1967 [9]). Two approaches by Quine

[1]It is an interesting detail that until now it is not clear if Gödel formulated his argument perhaps before Church's formulation of the argument in 1943. According to the Gödel – Schilpp correspondence ([19, p. 217–232], see also Parsons [36]), Gödel was invited by Schilpp on November 18, 1942 to write a paper on Russell's logical work. As stated in the correspondence, Church read the manuscript for the purpose of linguistic correction sometimes between June/July 1943 and September 27, 1943, when the manuscript was finally sent to the editor. At the end of the text Gödel expressed his thankfulness to Church for linguistic corrections. However it is not evident

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(in 1953 [38], and 1960 [37, §41]) are also regarded as “slingshot” argumentations. They intend to undermine the sense of constructing modal logic in view of the fact that necessity is reducible to factuality.

The common and crucial feature of all the mentioned arguments is to use abstract terms, definite descriptions or/and modal contexts together with some generally acceptable logical tools: the way of understanding logical equivalence and the rule of composition according to which the substitution of any expression by a co-referential expression does not change the reference of an expression where this substitution has been carried out. The relatively simple construction of these arguments is considered on the one hand as a source of their high pragmatic power to undermine the validity of the so called *situational/factualistic/propositionalistic* (and also modal) semantics or ontologies, and on the other as a reason to speak of them as of rather non-interesting logical puzzles.[2] We refrain from general statements about the mentioned arguments and focus our analysis on a version of Gödel’s “slingshot” argument reconstructed in a specific Gödelian onto-theological context.

Our motivation comes at first from Gödel’s original text: the commentary to Russell’s work on mathematical logic [16], which contains Gödel’s “slingshot” argument, is considered in general as being the essay that "marks the transition of Gödel’s main attention from the quest for definite mathematical results in logic to investigations of more distinctly philosophical (and historical) character” [52, p. 305], and as being “the first and the most extended philosophical statement” of Gödel [36, p. 103]. We claim that some of these philosophical references are clearly present also in the fragment containing the “slingshot” argument. In the frame of our philosophical interpretation we link (at least some version of) Gödel’s “slingshot” to his theological views in order

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that Church had at that time already finished his review of Carnap’s book. Although Church’s review is published in the *Philosophical Review* for May 1943 (nr. 3 of vol. 52), it is possible, as mentioned by Neale [30], that this number was actually published later than May, and thus only after Church had already read Gödel’s paper. The content of Church’s paper does not help in solving this question. Church does not refer to Gödel (nor Gödel to Church’s review), and Church’s 1943 argumentation is to some extent different from that which is present in Gödel’s approach. Church uses λ abstractions instead of $\imath$ descriptions, and gives more details about meaning (*Sinn*) semantics (similarities are emphasized in [45]).

[2] Cf. discussion between Neale and Oppy [34].

to get a more clear and complete picture.

The idea to link Gödel's "slingshot" argument with Gödel's ontological argument and his theological views was put forward by Sobel.[3] We elaborate upon it in the context of Gödelian theory of the most positive being and formulate a certain kind of Gödel's "slingshot" which does not trivialize this theory but rather makes the vision of its universe richer, where everything that happens (and everything that does not happen) is essentially dependent on a necessarily existent God. In this respect, our standpoint also follows the interpretation of modal collapse from [27]. The reduction of necessity to factuality (again originally noticed by Sobel in [43, 42], and proved in an interesting way with the use of abstraction operator and a sort of substitution into modal contexts) is not understood here as a negative effect of Gödel's onto-theological theory of reality but as an intended realization of the "rise of modalities to the perfect being" [27].

2 The problem and the argument

Gödel formulated his "slingshot" argument in order to comment on the way in which the Russellian concept of using expressions with operator "the" avoids, as he calls it, "Frege's puzzling" conclusion according to which all true propositions signify the only one object – the True, and false propositions signify the False [16, p. 122]. Gödel's commentary shows at first the compatibility of two components of Russell's philosophy: his theory of reference and his realism, being reduced over time to logical atomism.[4] Simultaneously, it exposes the essential difference between Russell's approach and Frege's idea to treat sentences as denoting expressions. Contrary to Frege, Russell claims that sentences play essentially a different semantical role than names: we say that names *denote* (or *signify*) objects, while sentences *indicate* facts (situations). To be more precise, only true sentences refer to facts, and false ones do not indicate anything, or simply: they indicate *nothing*, as expressed by

[3]It is expressed in the correspondence with a co-author of this paper (S. Kovač) on February 5, 2004, as well as in [45, p. 135].

[4]Gödel mentions Russell's "pronouncedly realistic attitude", which "has been gradually decreasing in the course of time" [16, p. 120–121]. Even in Russell's theory of definite descriptions Gödel recognizes a realistic attitude of finding the "right" solution (unless Russell meant it merely in a psychological sense) [16, p. 123].

Gödel in an analogy to Frege's truth-value the False [16, p. 122]. Therefore, in the context of Russell's proposal, as Gödel notes, the problem of a possible collapsing effect may actually concern only a plurality of facts corresponding to true sentences, and this effect is blocked by Russell precisely by treating descriptions as contextual expressions, not as proper names. Russell eliminates definite descriptions by definitions according to which expressions containing syncathegorematic phrases with "the" abbreviate some complex sentences built with quantifiers and variables (this is the realization of the idea that every simple sentence with a definite description is a statement about concepts). After all, in Gödel's opinion, the formal advantage of Russellian theory does not lead to a general and convincing solution. Russellian approach is not satisfactory, especially from Gödel's Platonic perspective: there are still definitions to be considered that introduce names referring to investigated objects. So, according to Gödel, the way of blocking the Fregean puzzle as proposed by Russell rather "evades" the problem than offers the real solution [16, p. 123].

All Gödel's argumentation is expressed in a rather sketchy form. We thus start by extracting its main steps.

Gödel begins with a consideration about the semantical status of definite descriptions and assumes, in contrast to Russell, that

> **P1** a descriptive phrase denotes the object described [16, p. 123].

This assumption together with the second "apparently obvious axiom" according to which

> **P2** the signification of a composite expression, containing constituents which have themselves a signification, depends only on the signification of these constituents (not on the manner in which this signification is expressed) [16, p. 122]

leads to the paradoxical statement that

> (*) the sentence "Scott is the author of *Waverley*" signifies the same thing as "Scott is Scott" [16, p. 122].

Gödel sketches the extension of this unwanted effect to any pair of sentences. To this aim, he formulates further assumptions:

Assm1 "$\varphi(a)$" and the proposition "a is the object which has the property φ and is identical with a" means the same thing [16, p. 122, ftn. 5]

Assm2 every proposition "speaks about something", i.e., can be brought to the form $\varphi(a)$ [16, p. 122, ftn. 5]

and infers the final conclusion that

SG all true sentences have the same signification (as well as all false ones) [16, p. 122].

Gödel points out that the crucial nerve of this collapse actually comes from the conjunction of mentioned postulates **P1** and **P2**, and hence the obvious way to avoid the collapse would be to reject either **P1** or **P2**. Russell rejects **P1**, accepting that in contrast to it, "a descriptive phrase denotes nothing at all" [16, p. 123], and probably does not drop (or restrict) the principle expressed by **P2**. As already said, Gödel finds that the way chosen by Russell is not in general satisfactory. Before we sketch a few other possible solutions consisting of restrictions of **P2**, we want to show the main reasoning in some precise and clear logical form.

3 Formal reconstruction of Gödel's "slingshot" argument. Free non-Fregean proposal

There are quite extensive reconstructions of Gödel's argumentation. Most of them are expressed in metalanguage to keep the semantical character of the argument. This style – in some aspects very different one from each other – is taken e.g. by Neale in [30] and also by Sobel in [45]. Since proposals offer in part a very informative analysis of Gödel's approach, we refer to some of their results. However, in our presentation we use the idea initiated by Wójtowicz [53, 54], developed by Shramko and Wansing [40, 41], and express Gödel's "slingshot" argument in an object language with identity connective $\equiv$ of non-Fregean logics. The motivation for choosing this tool for an analysis of a sort of "Fregean collapse" follows the philosophical background of non-Fregean logics pointed by their founder Suszko [46, 47]: let us remind ourselves that the essential idea lying behind this approach was to distinguish the equivalence of propositions (facts) from their identity, and so to describe the situational ontology, in which the following implication is not valid:

$$\textbf{FA} \quad (A \leftrightarrow B) \rightarrow (A \equiv B).$$

Non-Fregean logics are actually intended to describe some possible connections between equivalence and identity that are weaker than those expressed by **FA**. "Slingshot" arguments in general revolve around certain conditions just leading to **FA**.

We proceed in a way similar to the approach from [41] but with some modifications and new remarks.

We take the minimal non-Fregean logic PCI as a starting point.

The logic considered is formulated in a so called W *language*.[5] Its vocabulary consists of: (i) individual variables: $x, y, z, x_1, \ldots$; (ii) predicates: $X^1, Y^1, Z^1, X^1_1, \ldots; X^2, \ldots$; identity predicate $=$; (iii) connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \equiv$; (iv) quantifier symbols $\forall, \exists$; (v) parentheses.

Individual terms are individual variables, while formulas are defined in the following way:

$$\phi ::= X^n x_1 \ldots x_n \mid x_1 = x_2 \mid \neg\phi \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \vee \phi_2)$$
$$\mid (\phi_1 \rightarrow \phi_2) \mid (\phi_1 \leftrightarrow \phi_2) \mid (\phi_1 \equiv \phi_2) \mid \forall x\phi \mid \exists x\phi$$

Expression $x_1 \neq x_2$ will be used as the abbreviation for $\neg x_1 = x_2$.

Further, we use α and β as metavariables representing terms or formulas, and metasymbol $\doteq$ so that $(\alpha \doteq \beta)$ is understood as $\alpha = \beta$ or $(\alpha \equiv \beta)$, dependently on the category of α and β.

By a *universal closure of* a formula ϕ we understand any formula obtained from ϕ by prefixing to it any sequence of universal quantifiers with possible length 0.

Following Bloom [5], we characterize the considered logic by using the above notion of universal closure. We accept the axiomatics given by Omyła in [31].

[5]The general characterization of W *languages* is given e.g. in [5] ('W' for Wittgenstein).

Definition 1 (PCI)**.** *Logic* PCI *is defined by all universal closures of*

CPC *classical propositional axiom schemes,*

and of the formulas of the following shapes:

Q1 **∀1a** $\forall x\phi \rightarrow \phi(y/x)$, *$y$ is free for x in ϕ*

∀1b $\forall x(\phi \rightarrow \psi) \rightarrow (\forall x\phi \rightarrow \forall x\psi)$

∀1c $\phi \rightarrow \forall x\phi$, *where* $x \notin \mathsf{free}(\phi)$

∃/∀ $\exists x\phi \leftrightarrow \neg\forall x\neg\phi$

ID **i1** $\alpha \doteq \beta$, *where α and β are different at most in case of bound variables*

i2 $(\phi \equiv \psi) \rightarrow (\phi \rightarrow \psi)$

iSubs $(\alpha_i \doteq \beta_j \wedge \ldots \wedge \alpha_{i+k} \doteq \beta_{j+k})$

$$\rightarrow (F(\alpha_i, \ldots, \alpha_{i+k}) \equiv F(\beta_j/\alpha_i, \ldots, \beta_{j+k}/\alpha_{i+k})),$$

for every $k \geq 0$ and every F being a predicate (then α, β are individual variables) or every F being a connective (then α, β are formulas; in case of two-place connectives, the prefix notation is changed to the infix one)

Q1/ID **i∀** $\forall x(\phi \equiv \psi) \rightarrow (\forall x\phi \equiv \forall x\psi)$

i∃ $\forall x(\phi \equiv \psi) \rightarrow (\exists x\phi \equiv \exists x\psi)$.

The primitive rule is **MP**: $\vdash \phi \rightarrow \psi, \phi \implies \vdash \psi$.

Let us note that both identities are reflexive, symmetric and transitive:

Proposition 1. *All universal closures of the following schemes are* PCI *derivable:*

PCI	$\vdash \alpha \doteq \alpha$	(ref $\doteq$)
PCI	$\vdash (\alpha \doteq \beta) \rightarrow (\beta \doteq \alpha)$	(sym $\doteq$)
PCI	$\vdash (\alpha \doteq \beta) \rightarrow ((\beta \doteq \gamma) \rightarrow (\alpha \doteq \gamma))$	(trans $\doteq$).

Scheme (ref $\doteq$) is a case of scheme (i1); (sym $\doteq$) and (trans $\doteq$) easily follow from the axioms.

Let us now investigate possible extensions of PCI by definite descriptions. The first proposals to use definite descriptions in certain non-Fregean systems come from Bloom [5] and Omyła (first published in [33]). Both authors follow the style of Lewandowski and Suszko [28] and refer to the approach by Bernays [4] in order to retain the Fregean idea that grammatical rules for the use of such expressions, treated as individual terms, should be independent of any extralinguistic conditions.[6]

The symbol $\imath$ is used in [4] in context $\imath x(\phi, t)$, which reads: *the only one x which is ϕ or is otherwise t*, where t is an individual variable or constant.

The same notation is adopted in [28], together with the following schemes:

D1* $\exists_1 x\phi(x) \rightarrow \phi(\imath x(\phi(x), t)/x)$

D2* $\neg\exists_1 x\phi(x) \rightarrow \imath x(\phi(x), t) = t,$

where

M∃₁ $\exists_1 x\phi =_{def} \exists x\forall y(\phi \leftrightarrow x = y).$

D1* and **D2*** may be added to the first order predicate logic with identity as axioms introducing definite descriptions. As it is shown in [28], this addition is non-creative and equivalent to the second way of introducing $\imath$ terms by assuming the equivalence

$$(\imath/\leftrightarrow^*) \quad t' = \imath x(\phi(x), t) \leftrightarrow (\forall x(\phi(x) \leftrightarrow x = t') \vee (\neg\exists_1 x\phi(x) \wedge t' = t)).$$

The fact that the extension of any W language by $(\imath/\leftrightarrow^*)$ is non-creative was shown by Bloom in [5]. Omyła uses $(\imath/\leftrightarrow^*)$ to introduce $\imath$ terms in [32].

The described solution could be modified to keep the usual use of $\imath$ operator and take expressions $\imath x\phi$ as terms. After the appropriate enlargement of the language, all universal closures of the following schemes would be accepted:

D1 $\exists_1 x\phi(x) \rightarrow \phi(\imath x\phi(x)/x)$

[6] Another approach comes from Hilbert and Bernays [22, p. 383]. In this case any $\imath$ term may be introduced into a language only if the unique existence of the object which fulfills the condition given by this term is provable. The use of descriptions for which this condition is not fulfilled is incorrect already on the level of the formation of expressions.

D2 $\neg\exists_1 x\phi(x) \rightarrow \imath x\phi(x) = \imath y \neg y = y$,

coming from **D1*** and **D2*** for $t =: \imath y \neg y = y$.[7] Equivalently, all universal closures of the following equivalence could be adopted:

$$(\imath/\leftrightarrow) \quad t = \imath x\phi(x) \leftrightarrow (\forall x(\phi(x) \leftrightarrow x = t) \vee (\neg\exists_1 x\phi(x) \wedge t = \imath y \neg y = y)).$$

The same way of introducing $\imath$ terms into classical predicate logic with identity is proposed by Kalish and Montague in [23].[8] Sobel, too, bases his analysis of Gödel's "slingshot" argument on this approach [45]. However Kalish and Montague block the derivability of expressions like $\exists x(x = \imath y \neg y = y)$ ($\ominus$) by not introducing $t = t$ instead of $x = x$. They modify the unwanted use of $\imath$ terms (occuring in the same position as free variables) in connection with identity by adopting the primitive rule $\forall x(x = t \rightarrow \phi) \implies \phi(t/x)$ into their system. After all, we do not want to weaken the sense of existential quantifier to the sense connected with the provability of ($\ominus$), but choose another solution of this problem offered by the free logic approach.[9] The use of the existence predicate regulated by free logic will enable us to articulate certain connections between the "slingshot" effect and the necessary existence of some individuals.[10]

We are going to base our analysis on a free version of PCI. To this end we consider the minimal positive free logic PFL expressed in first-order predicate language with $\imath$ terms and predicate =. We use symbols t and t' for individual variables or $\imath$ terms.

Definition 2 (PFL)**.** PFL *is characterized by*

CPC *classical propositional axiom schemes,*

formulas of the shapes: $\forall$**1b**, $\forall$**1c**, $\exists/\forall$*, and*

[7] Bernays introduces primitive constant 0 to the vocabulary, and then uses the definition $\imath x\phi(x) =_{def} \imath x(\phi(x), 0)$ (cf. [4, p. 54–55]).

[8] They consider a natural deduction system and so, instead of **D1** and **D2**, they introduce two primitive rules for "proper" and "improper" descriptions, respectively [23, p. 318–319].

[9] This is obvious in connection with the onto-theological system of Gödel considered below, in which the existence of God is expressed just by $\exists$ (cf. Theorem 1 below).

[10] In analysis of Gödelian "slingshot" argument from [40], PCI is used (and some strengthening of it), conservatively extended with $\imath$ terms. The derivability of ($\ominus$) is redundant with respect to the results presented there.

∀1aE	$\forall x\phi \to (Et \to \phi(t/x))$	*t is free for x in* ϕ
E	$\forall x Ex$	
i1*	$t = t$	
Subs	$t_1 = t_2 \to (\phi(t_1/x) \to \phi(t_2/x))$	t_1, t_2 *are free for x in* ϕ
Eid	$Et \leftrightarrow \exists x x = t$	

The basic rules are **MP** *and* **U1**: $\vdash \phi \implies \vdash \forall x\phi$.

Now we combine PCI and PFL. We take our W language with $\imath$ terms of the form $\imath x\phi$, and consider the following system:

Definition 3 ($\mathsf{fPCI}_\imath$)**.** $\mathsf{fPCI}_\imath$ *is defined by all universal closures of schemata* **CPC**, **∀1aE**, **E**, **∀1b**, **∀1c**, **∃/∀**, **Eid**, **i1**, **i2**, **iSubs** *and rule* **MP**.

(Now metavariables α and β represent descriptive terms, too.)

It is rather obvious that $\mathsf{fPCI}_\imath$ is consistent. Let us add to PFL, conservatively extended with connective ≡, all formulas of the shape

$$\textbf{FAA} \quad (\psi \leftrightarrow \chi) \leftrightarrow (\psi \equiv \chi).$$

This enrichment, which we name $\mathsf{PFL}_\equiv$+**FAA**, is consistent. Otherwise, a derivation of some formula A as well as of $\neg A$ could be formulated in $\mathsf{PFL}_\equiv$ + **FAA**. Every step of this derivation would be obtained by means of $\mathsf{PFL}_\equiv$ or, if the transformed formula contains ≡, it could be eliminated by means of **FAA** and **MP**, and thus, we could proceed using only axioms and rules of $\mathsf{PFL}_\equiv$. In this way, the derivation of A and $\neg A$ would yield inconsistency of $\mathsf{PFL}_\equiv$. The $\mathsf{fPCI}_\imath$ is consistent because it is a subsystem of $\mathsf{FD}_\equiv$ + **FAA**.

To start with our formulation of the "slingshot" argument, we say that the assumption **P1** is accepted by the introduction of definite descriptions as individual terms into our language.

The principle of composition **P2** is expressed by scheme **iSubs**.

We use the notation $\mathsf{L}[\mathsf{T}^\Phi]$ to speak about any consistent theory T based on logic L and such that Φ is provable in T.

Let us come to Gödel's argumentation concerning the unexpected statement (*). Sentences "Scott is Scott" and "Scott is the author of *Waverley*" are meant to "signify the same thing", which follows directly from the following fact:

Proposition 2 (**SG1**). $\mathsf{fPCI}_{\imath}[\mathsf{T}^{\Phi}] \vdash t = t \equiv t = \imath x\phi(x)$, *where* $\Phi =: t = \imath x\phi(x)$ *for some* t *and* ϕ.

Proof.

1	$t = \imath x\phi(x)$	Φ
2	$(t = t \wedge t = \imath x\phi(x)) \rightarrow (t = t \equiv t = \imath x\phi(x))$	iSubs
3	$t = t \equiv t = \imath x\phi(x)$	1, 2, i1

□

The next step is to prove a more general result. To this end we consider Gödel's assumption **Assm1**. Let us take, first, its cautious formulation using the following scheme:

($\circ$) $P(t) \equiv t = \imath x(x = t \wedge P(x))$, for any atomic formula $P(t)$

We name $\mathsf{fPCI}_{\imath}[\mathsf{T}^{\circ}]$ any consistent theory T based on $\mathsf{fPCI}_{\imath}$, such that $\mathsf{T} \vdash (\circ)$.

Now we can state the following:

Proposition 3 (**SG2**). $\mathsf{fPCI}_{\imath}[\mathsf{T}^{\circ}] \vdash (Xt \wedge Yt) \rightarrow (Xt \equiv Yt)$

Proof.

1	$Xt \equiv t = \imath x(x = t \wedge Xx)$	($\circ$)
2	$Yt \equiv t = \imath x(x = t \wedge Yx)$	($\circ$)
3	$Xt \rightarrow t = \imath x(x = t \wedge Xx)$	1, i2
4	$Yt \rightarrow t = \imath x(x = t \wedge Yx)$	2, i2
5	$t = \imath x(x = t \wedge Xx) \rightarrow (t = t \equiv t = \imath x(x = t \wedge Xx))$	i1, iSubs
6	$t = \imath x(x = t \wedge Yx) \rightarrow (t = t \equiv t = \imath x(x = t \wedge Yx))$	i1, iSubs
7	$Xt \rightarrow (t = t \equiv t = \imath x(x = t \wedge Xx))$	3, 5
8	$Yt \rightarrow (t = t \equiv t = \imath x(x = t \wedge Yx))$	4, 6
9	$Xt \rightarrow (t = t \equiv Xt)$	7, 1, trans $\equiv$
10	$Yt \rightarrow (t = t \equiv Yt)$	8, 2, trans $\equiv$
11	$(Xt \wedge Yt) \rightarrow (Xt \equiv Yt)$	9, 10, trans $\equiv$

□

To obtain a result more general than **SG2**, we assume the scheme:

($\bullet$) $\phi \equiv t = \iota x(x = t \wedge \phi(/t))$, ϕ possibly containing t.

Identity ($\circ$) is a special case of ($\bullet$), and in this sense ($\bullet$) gives a more extended meaning to **Assm1** than ($\circ$).

Now we can notice that in any consistent theory $\mathsf{fPCI}_\iota[\mathsf{T}^\bullet]$ the conjunction of any two formulas ψ and χ implies their identity:

Proposition 4 (SG3). $\mathsf{fPCI}_\iota[\mathsf{T}^\bullet] \vdash (\psi \wedge \chi) \to (\psi \equiv \chi)$

Proof.

1	$\psi \equiv t = \iota x(x = t \wedge \psi)$	($\bullet$)
2	$\chi \equiv t' = \iota x(x = t' \wedge \chi)$	($\bullet$)
3	$\psi \to t = \iota x(x = t \wedge \psi)$	1, i2
4	$\chi \to t' = \iota x(x = t' \wedge \chi)$	2, i2
5	$t = \iota x(x = t \wedge \psi))$	i1, iSubs
	$\to (t = t \equiv t = \iota x(x = t \wedge \psi)$	
6	$\psi \to (t = t \equiv \psi)$	3, 5, 1, trans $\equiv$
7	$t = t' \to (\chi \equiv t = \iota x(x = t \wedge \chi))$	2
8	$t = t' \to (\chi \to t = \iota x(x = t \wedge \chi))$	7, i2
9	$t = \iota x(x = t \wedge \chi)$	i1, iSubs
	$\to (t = t \equiv t = \iota x(x = t \wedge \chi))$	
10	$t = t' \to (\chi \to (t = t \equiv t = \iota x(x = t \wedge \chi))$	8, 9
11	$t = t' \to (\chi \to (t = t \equiv \chi))$	7, 10, trans $\equiv$
12	$t = t' \to (\psi \wedge \chi \to (\psi \equiv \chi))$	11, 6, trans $\equiv$
13	$t \neq t' \equiv t = \iota x(x = t \wedge x \neq t')$	($\bullet$)
14	$t \neq t' \equiv t' = \iota x(x = t' \wedge x \neq t)$	($\bullet$)
15	$t \neq t' \wedge \psi \to \iota x(x = t \wedge x \neq t')$	13, 3, i2
	$= \iota x(x = t \wedge \psi)$	
16	$\iota x(x = t \wedge x \neq t') = \iota x(x = t \wedge \psi) \to$	iSubs
	$(t = \iota x(x = t \wedge x \neq t') \equiv t = \iota x(x = t \wedge \psi))$	
17	$(t \neq t' \wedge \psi) \to (t = \iota x(x = t \wedge x \neq t')$	15, 16
	$\equiv t = \iota x(x = t \wedge \psi))$	
18	$t \neq t' \wedge \psi \to (t \neq t' \equiv \psi)$	17, 13, 1 trans $\equiv$
19	$t \neq t' \wedge \chi \to (t \neq t' \equiv \chi)$	like 18, by 14, i2, iSubs, 2
20	$t \neq t' \to (\psi \wedge \chi \to \psi \equiv \chi)$	18, 19, trans $\equiv$
21	$(\psi \wedge \chi) \to (\psi \equiv \chi)$	12, 20

□

The derivation is essentially the same as in case of $\mathsf{PCI}_\imath$ extended by (•), as presented in [41].

In connection with **SG3**, let us notice that, according to [41], in no $\mathsf{PCI}_\imath[\mathsf{T}^\bullet]$ we could prove

$$(**) \quad (\neg\psi \wedge \neg\chi) \rightarrow (\psi \equiv \chi).$$

Otherwise, we would already reach the collapse of $\leftrightarrow$ and $\equiv$.

Perhaps **SG3** could justify the abovementioned Gödel's remark about non-symmetry of Frege's and Russell's theories: although Frege and Russell have essentially different opinions about reference (*denotation* or *signification*) of true sentences, they keep similar view in case of false ones

> [...] according to Russell's terminology and view, true sentences "indicate" facts and, correspondingly, false ones indicate nothing. Hence Frege's theory would in a sense apply [in the context of Russell's construction] to false sentences, since they all indicate the same thing, namely nothing. But different true sentences may indicate many different things. [16, p. 122]

After all, the authors of [41] say that the acceptance of (•) allows the derivation of **FA** on the ground of the next known non-Fregean logic $\mathsf{WBQ}_\imath$.

Definition 4. *The logic* $\mathsf{WBQ}_\imath$ *is the conservative* $\imath$ *extension of* WBQ *which results from* PCI *by the addition of the following primitive rule:*

RB $A \leftrightarrow B \implies A \equiv B$, *for any* $A \leftrightarrow B$ *which is a law of classical predicate logic.*[11]

WBQ (or already its sentential fragment WB) weakens the difference between identity and equivalence in case of classical laws. In particular, the formulas of the following shape are WBQ derivable:

$$(\neg|^{\equiv}) \quad (\neg\psi \equiv \neg\chi) \rightarrow (\psi \equiv \chi)$$

[11] A, B represent any formulas of any given W language.

A proof can be established directly from **iSubs** and **RB**.

For our aim it is sufficient to introduce only the extension of fPCI_7 by $(\neg|^{\equiv})$, named $\mathsf{fPCI}_7 + (\neg|^{\equiv})$. The addition of (•) leads us to **FA**:

Proposition 5 (SG4). $\mathsf{fPCI}_7 + (\neg|^{\equiv})\ [\mathsf{T}^\bullet] \vdash (\psi \leftrightarrow \chi) \to (\psi \equiv \chi)$

The proof consists, first, of the derivation of $(\psi \wedge \chi) \to (\psi \equiv \chi)$ as it is for **SG3** (Proposition 4), and then, in the same way, of the derivation of $(\neg\psi \wedge \neg\chi) \to (\neg\psi \equiv \neg\chi)$. Finally, we use $(\neg|^{\equiv})$.

We generalize the above result to logics stronger than $\mathsf{fPCI}_7 + (\neg|^{\equiv})$:

Proposition 6 (SG5). *For any* $\mathsf{L} \supseteq \mathsf{fPCI}_7 + (\neg|^{\equiv})$, $\mathsf{L}[\mathsf{T}^\bullet] \vdash (\psi \leftrightarrow \chi) \to (\psi \equiv \chi)$.

This shows that a deductive minimum to achieve **FA** is even weaker than that which is described in Theorem 2.2 [41, p. 28].

As a comment to stronger and stronger versions of the "slingshot" argument, let us say that they are always derivable thanks to a special combination of logical and extralogical components. In our approach, the first essential step is the decision to treat definite descriptions as individual terms. The next one is the application of substitution in the contexts with ≡. The extralogical assumptions become stronger and stronger: in the weakest version, **SG1**, we have to assume some non-trivial identity of individuals: $t = \imath x\phi(x)$; in the strongest version, **SG5**, we need to assume the identity of some kind of propositions/facts (situations) (•). Actually, the assumption (•) is mentioned as a generalization of **Assm1** justified by **Assm2**. In Gödel's opinion, this generalization validates the general form of his "slingshot" argument. Actually, **SG5** may be considered as a proposition realizing this effect. However, we want to check the effect of disturbing the balance between logical and extralogical assumptions by looking also for a justification of (•).[12]

[12] **Assm1** and its formal counterparts would be questioned on the ground of Quine's *principle of shallow analysis*, according to which an adequate formalization of any sentence should not be more complex than the formalized sentence. Following this idea, the sentence "Scott is the author of Waverley" would be adequately formalized by $\phi(t)$ and not by $t = \imath x\phi(x)$, because the second formula has higher level of complexity than the considered sentence. The same applies to any sentence "*a* is the object which has the property φ and is identical with *a*" – the *shallow* formalization would be $\varphi(a)$ and not $a = \imath x(x = a \wedge \phi(a/x))$. Baumgartner [3] shows that the acceptance of Quine's principle blocks the discussed argumentation. Quine's principle leads to the restriction of the substitution in ≡ contexts to the expressions with the same level of complexity. Thus this principle is not applicable to (•).

To this aim, we consider the weakest theory FD of descriptions, based on minimal free logic PFL (proposed in [51]).

FD is obtained from PFL by the addition of the scheme

AFD $\forall y(y = \imath x\phi \leftrightarrow (\forall x(\phi \rightarrow x = y) \wedge \phi(y/x)))$, *with different x and y.*

So we consider the following system:

Definition 5 (FD)**.** FD = PFL + **AFD**[13]

We note that

Proposition 7. $\mathsf{FD} \vdash \forall y\, (\phi \leftrightarrow y = \imath x(x = y \wedge \phi(x/y)))$, $x \notin \mathsf{free}(\phi)$.

Proof.

1	$\phi \leftrightarrow \phi$	CPC
2	$y = y$	i1*
3	$\forall x((x = y \wedge \phi(x/y)) \rightarrow x = y)$	CPC, U1
4	$\phi \leftrightarrow (\forall x((x{=}y \wedge \phi(x/y)) \rightarrow x = y) \wedge y{=}y \wedge \phi)$	1, 2, 3
5	$Ey \rightarrow (y = \imath x(x = y \wedge \phi(x/y)) \leftrightarrow (\forall x((x = y \wedge \phi(x/y)) \rightarrow x = y) \wedge y = y \wedge \phi))$	AFD, $\forall$1aE
6	$Ey \rightarrow (\phi \leftrightarrow y = \imath x(x = y \wedge \phi(x/y)))$	4, 5
7	$\forall y\,(Ey \rightarrow (\phi \leftrightarrow y = \imath x(x = y \wedge \phi(x/y))))$	6 U1
8	$\forall y(\phi \leftrightarrow y = \imath x(x = y \wedge \phi(x/y)))$	7 $\forall$1b, E

□

[13] The next logic – FD_1 – results from FD by adding $t = \imath x\, x = t$ (**AFD1**). In this frame we can speak also about non-existent objects using descriptive terms: in FD_1 we say that even contradictory objects are self-identical: $\imath y \neg y = y = \imath x\, x = \imath y \neg y = y$. FD_1 still seems to be a rather intuitive theory of descriptions in comparison with its strengthening to FD_2. In FD_1 there still does not appear the questionable solution with the existence of the special outer domain individual *null*: $\imath y \neg y = y$. This problematic situation appears in FD_2 obtained from FD by adding all universal closures of $t = \imath x\phi(x) \leftrightarrow \forall y(t = y \leftrightarrow (\phi(y) \wedge \forall x(\phi \rightarrow y = x)))$ (**AFD2**, x and y being different variables).

Van Fraassen and Lambert [51] consider FD_2 as a sort of marriage of Russelian and Fregean ideas on descriptions. Indeed, we can derive, in FD_2, $E!\imath x\phi \leftrightarrow \exists y\forall x(\phi \leftrightarrow x = y)$ (with the condition from **AFD**), which expresses the Russellian way of using descriptive phrases as well as: $\exists y\forall x(\phi \leftrightarrow x = y) \rightarrow \phi(\imath x\phi(x)/x)$, $\neg\exists y\forall x(\phi \leftrightarrow x = y) \rightarrow \imath x\phi(x) = \imath y \neg y = y$. These implications realize precisely Fregean style of treating $\imath$ terms (cf. **D1**, **D2**).

Let us now combine FD with the next strengthening of PCI.

This time we refer to logic WTQ, which extends WBQ. More precisely, we extend our W language by sentential constant 1 and modal operator $\Box$ for necessity, and add to WBQ new axioms of the following shapes:

$$({}_1|^{\equiv})\quad 1 \equiv (A \to A)$$
$$({}_\Box|^{\equiv})\quad \Box A \equiv (A \equiv 1),$$

as well as the following specific rule:

$$\textbf{RT}\quad \vdash A \leftrightarrow B \Longrightarrow \vdash A \equiv B, \quad \textit{for any } A \leftrightarrow B \in \mathsf{WBQ}.$$

It is already known that $\Box$ has $S4$ properties in the logic thus obtained, called here $\mathsf{WTQ}_{1\Box}$ (cf.[31, p.108]). We use only the following facts:

Proposition 8.

$$\mathsf{WTQ}_{1\Box} \vdash \Box(A \to B) \to (\Box A \to \Box B) \qquad (\mathbf{K})$$
$$\mathsf{WTQ}_{1\Box} \vdash A \equiv B \leftrightarrow \Box(A \leftrightarrow B) \qquad ({}_{\equiv}|^{\leftrightarrow}_{\Box})$$
the rule **Nec**: $\vdash A \Longrightarrow \vdash \Box A$ *is derivable in* $\mathsf{WTQ}_{1\Box}$.

By a $\Box$ *closure of* a formula ϕ we understand any formula obtained from ϕ by prefixing it with any sequence of $\Box$ with possible length 0.

We consider the conservative $\Box$ extension of PCI named $\mathsf{PCI}_\Box$. All $\Box$ closures of $\mathsf{PCI}_\Box$ theorems, together with $\Box$ closures of the formulas of the shapes ${}_\neg|^{\equiv}, \mathbf{K}, {}_{\equiv}|^{\leftrightarrow}_{\Box}$ form a subsystem of $\mathsf{WTQ}_{1\Box}$. We consider a free version of the subsystem with $\imath$ terms and $\Box$, named $\Box\big(\mathsf{fPCI}_{\imath\Box} + ({}_\neg|^{\equiv}, \mathbf{K}, {}_{\equiv}|^{\leftrightarrow}_{\Box})\big)$. (The obtained logic is consistent: it is a subsystem of the free version of $\mathsf{WTQ}_{1\Box}$, defined analogously to $\mathsf{fPCI}_\imath$ – cf. Def. 3.)

To meet the intended description of $\imath$ terms originating from FD, we add the following $\Box$ closure of **AFD**:

$$\Box\textbf{AFD}\quad \Box\forall y(y = \imath x\phi \leftrightarrow (\forall x(\phi \to x = y) \land \phi(y/x))),$$
with different x and y.

The addition of $\Box$**AFD** to our system gives the following effect:

Proposition 9. $\mathsf{fPCI}_{\imath\Box} + ({}_\neg|^{\equiv}, \mathbf{K}, {}_{\equiv}|^{\leftrightarrow}_{\Box}) + \Box\mathbf{AFD} \vdash \Box Et \to (\phi \equiv t = \imath x(x = t \land \phi(x/t)))$, *with different x and t.*

Proof.

1	$\Box\forall y\,(\phi \leftrightarrow y = \imath x(x = y \wedge \phi(x/y)))$	$\Box$AFD, cf. Prop.7 *via* AFD
2	$\Box(Et \rightarrow (\phi \leftrightarrow t = \imath x(x = t \wedge \phi(x/t))))$	$\Box 1\forall aE$, K, 1
3	$\Box Et \rightarrow \Box(\phi \leftrightarrow t = \imath x(x = t \wedge \phi(x/t)))$	K, 2
4	$\Box Et \rightarrow (\phi \equiv t = \imath x(x = t \wedge \phi(x/t)))$	${}_{\equiv}|^{\leftrightarrow}_{\Box}$, 3

$\Box$

Thus, we may say that in any theory T based on $\Box\big(\mathsf{fPCI}_{\imath\Box} + ({}_{\neg}|^{\equiv}, \mathbf{K}, {}_{\equiv}|^{\leftrightarrow}_{\Box})\big) + \Box\mathbf{AFD}$, and such that $\mathsf{T} \vdash \Box Et$, we can prove our general formulation of **Assm1**. Moreover:

Proposition 10. *For any* $\mathsf{L} \supseteq \Box\big(\mathsf{fPCI}_{\imath\Box} + ({}_{\neg}|^{\equiv}, \mathbf{K}, {}_{\equiv}|^{\leftrightarrow}_{\Box})\big) + \Box\mathbf{AFD}$, $\mathsf{L}[\mathsf{T}^{\Box Et}] \vdash (\bullet)$.

Finally, we reach the "slingshot" effect in the following sense:

Proposition 11 (SG6). *For any* $\mathsf{L} \supseteq \Box\big(\mathsf{fPCI}_{\imath\Box} + ({}_{\neg}|^{\equiv}, \mathbf{K}, {}_{\equiv}|^{\leftrightarrow}_{\Box})\big) + \Box\mathbf{AFD}$, $\mathsf{L}[\mathsf{T}^{\Box Et}] \vdash (\psi \leftrightarrow \chi) \rightarrow (\psi \equiv \chi)$.

The proof is based on Proposition 10 – we proceed in the same way as in the case of **SG4** (Proposition 5).

Interestingly, Propositions 10 and 11 may be considered in connection with the already mentioned Sobel's suggestion that the validity of Gödel's argumentation is dependent on the necessary existence of some individuals.

Let us try to investigate Gödel's possible way of reasoning, starting from the analysis sketched by Neale [30], who tries to formulate a satisfactory justification for **Assm1**.

Following Neale, we would say that if every sentential expression F would have a reference, or could be reformulated so as to have a reference, then F would have some logical value. Thus, we would have the equivalence (or perhaps even necessary equivalence) of F and the statement about the identity of an individual y with the object a given by the condition of being identical with y and fulfilling F (even if this fulfilment would not depend on a). A possible "procedure" of reformulating every sentential expression into some refering subject-predicate sentence is pictured by Neale:

> [...] he [Gödel] would say that "Socrates snored and Plato snored" can be rendered as "Socrates is *an x such that x snored and Plato snored*", and that "all men snore" can be rendered as somethig like "Socrates is *an x such that all men snore*" (harmlessly assuming a non empty universe) [30, p. 130].

Actually, the main nerve of the described way of finding a referential counterpart of any sentential expression is to have at disposal some existing individual (like Socrates) which would 'guarantee' this reference. But what would happen if this referential 'guarantor' would cease to exist? Would we not obtain the situation in which some sentential expressions have lost their reference? These questions lead Sobel to propose the following reinterpretation of Gödelian text:

> A response available to Gödel to this problem with his further 'assumption' [**Assm2**] would be to change it to say that every proposition 'speaks about something that exists necessarily'. I believe that Gödel thought that numbers are necessary existents, and I know that he was inclined to think that God is a necessary existent. [45, p. 135]

The acceptance of the necessary existence of some individual would guarantee a fixed reference of every sentence – as mentioned by Gödel in **Assm2** – and, in effect, also the validity of the general form of **Assm2**.

The above assumption about the necessary existence of something has, of course, extralogical provenience, and thus the obtained collapse has not to be associated only with logical tools. However, this logical background should still be discussed: especially the counterpart of Fregean rule of composition **P2**, as formulated by Gödel. As it may be already seen in case of the proof of **G1**, Axiom **iSubs**, which expresses **P2**, seems to lose its obviousness when applied to $\imath$ terms. The meaning of $\equiv$ in PCI is very strong: in connection with (**i1**), the formulas referring to the same situation may be different at most with respect to bound variables, and **iSubs**, applied to $\imath$ terms, changes the basic structure of formulas. The problem remains even in connection with the weaker meaning of $\equiv$ as determined in **SG6**. This weaker meaning seems to follow the original Gödel's intention – such an interpretation is suggested by Neale and Sobel. The symbol $\equiv$ is (extensionally) understood as necessary equivalence, but the application of substitution in referentially non-transparent contexts with $\Box$ is still problematic. A possible solution for this difficulty would be to introduce some restriction on **iSubs**. In this case we could apply Føllesdall's proposal to redefine a class of expressions treated as singular terms, and to restrict the instantiation in modal non-transparent contexts only to "genuine singular terms", which necessarily apply to their objects and are in this sense *rigid* [11, p. 95].

Let us accept the following meaning of rigid terms:

Definition 6 (**Rigid term**). *Term t is rigid in* T *iff* $\mathsf{T} \vdash x = t \rightarrow \Box\, x = t$.

We name $\mathsf{L}^*[\mathsf{T}]$ any consistent theory T based on L with **iSubs** restricted to the terms rigid in T.

Now our restriction blocks all "slingshot" derivations in the sense of **SG1**–**SG6** but, interestingly, it is still possible to reduce identity to equivalence if to the assumption of the necessary existence of some individual we add the condition that every description if the individual is a rigid term, i.e.:

Proposition 12 (**SG6**). *Let us take any* $\mathsf{L} \supseteq \Box\Big(\mathsf{fPCI}_{\imath\Box} + ({}_{\neg}|^{\equiv}, K, {}_{\equiv}|^{\leftrightarrow}_{\Box})\Big) + \Box$**AFD** *and any theory* T *based on* L*, such that* (Φ)*: if (i)* $\mathsf{T} \vdash \Box Et$*, and (ii) for every formula* ϕ*,* $\imath x(x = t \wedge \phi(x/t))$ *is rigid, then*

$$\mathsf{L}^*[\mathsf{T}^{\Phi}] \vdash (\psi \leftrightarrow \chi) \rightarrow (\psi \equiv \chi).$$

The proof is obtained in a similar way as for **SG3** (Proposition 4 – terms $\imath x(x = t \wedge \psi)$ and $\imath x(x = t \wedge \chi)$ are rigid). After that, we proceed like in the proof of **SG5** (Proposition 11).

In the following section we will meet all the assumptions expressed in Proposition 12 leading to the modal collapse. However, this will happen in a specific perspective of Gödel's onto-theological context with a necessarily existent God-like being and with all its descriptions rigid.

4 Onto-theological "slingshot"

The aim of this section is to show that a sort of "slingshot" argument can be established on the ground of a Gödelian onto-theological system.

We recall that Gödel in his 1944 discussion concludes that there are two ways to defend the view that "different true sentences may indicate many different things" [16, pp. 122–123]) instead to allow the collapse of the signification of all true sentences into one and the same fact (see here Section 2): either to reject prinicple **P1** (a definite description denotes the object described) or to reject principle **P2** (compositionality of the signification, with the irrelevance of the "manner" of expression).

Since Gödel was in [16] not convinced by Russell's eliminative theory of definite descriptions, we will in the following formalization of a Gödelian ontology retain definite descriptions as individual terms, but will axiomatically restrict the principle of the compositionality – the principle of substitution will be restricted to non-modal context. In addition, the first-order instantiation of universal formulas will be restricted to rigid terms (with existence predicated). Gödel's assumptions **Assm1** and **Assm2** will not be axiomatically presupposed, although they will be provable in the system. Thus, we will propose a second-order onto-theological system $\mathsf{fGO}_\imath$, where no existent object will be axiomatically assumed, as if the system is "universally free" ("inclusive"). However, since in $\mathsf{fGO}_\imath$ (like in GO[14]) the necessary existence of an object (God) is provable, the system turns out not to be universally free, but "exclusive", and that in a "constructive" way, namely, proving which object(s) exist(s).

We now first define the language of $\mathsf{fGO}_\imath$ and thereafter describe the axiomatic system.

The language of $\mathsf{fGO}_\imath$

Vocabulary: first-order variables $x, y, z, x_1, \ldots$; second-order variables $X^1, Y^1, Z^1, X^1_1, \ldots; X^2, \ldots$; third-order term $\mathcal{P}^1$; operators $\neg, \rightarrow, \Box, \forall, \imath, \lambda$; parentheses. A first-order term (t), a second-order term (T), and a formula (ϕ) are defined in the following way:

$$t ::= x \mid \imath x \phi$$

$$T ::= X^n_m \mid (\lambda x_1 \ldots x_n . \phi)$$

$$\phi ::= T^n t_1 \ldots t_n \mid t_1 = t_2 \mid \mathcal{P}T^1 \mid \neg\phi \mid (\phi_1 \rightarrow \phi_2)$$

[14]For the definition of GO see [21]. It is TG of [48]. Cf. Gödel's outline from 1970 in [17] as well as [42, 43] (including Scott's emendation).

$\mid \Box\phi \mid \forall x\phi \mid \forall X\phi\})$

$\vee, \wedge, \leftrightarrow, \exists, \Diamond, \neq, \top$, and $\bot$ are defined in a familiar way.

Besides, we will use the notion of substitutability, in the sense that a term t is substitutable for x in ϕ if t or a free variable occurring in t does not become bound by $\forall$, λ, or $\imath$ operator when t is substituted for x in ϕ. Similarly for second-order terms.

Abbreviations

$Et =_{def} \exists x x = t$ (see **Eid** in Definition 2),

$\overline{X} =_{def} (\lambda x.\neg Xx)$,

$Gt =_{def} \forall X(\mathcal{P}X \to Xt)$,

$\mathcal{E}\int\int(X, t) =_{def} Xt \wedge \forall Y(Yt \to \Box\forall y(Xy \to Yy))$,

$Nt =_{def} \forall Y(\mathcal{E}\int\int(Y, t) \to \Box\exists x Yx)$

Axiomatic system $\mathsf{fGO}_\imath$

Axiom schemes:

CPC classical propositional axiom schemes

K $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

T $\Box\phi \to \phi$

5 $\Diamond\phi \to \Box\Diamond\phi$

∀1a $\forall x\phi \to (Et \to \phi(t/x))$, t is rigid and substitutable for x in ϕ (for "rigid" see Definition 6 as well as Axiom **=R**)[15]

∀1b $\forall x(\phi \to \psi) \to (\forall x\phi \to \forall x\psi)$

∀1c $\phi \to \forall x\phi$, $x \notin \mathsf{free}(\phi)$

∀2a $\forall X\phi \to \phi(T/X)$, T is substitutable for X in ϕ

∀2b $\forall X(\phi \to \psi) \to (\forall X\phi \to \forall X\psi)$

∀2c $\phi \to \forall X\phi$, $X \notin \mathsf{free}(\phi)$

=1 $t = t$

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[15] Without the rigidity condition, $Et \to \exists x\Box x = t$ would be provable for any t (from Axiom **=1** below); see Garson in [13, p. 282–3, 284–5]. See also [12, pp. 626, 628].

=R $x = y \to \Box x = y$

E $\forall x E x$

λConv $\phi(t/x) \leftrightarrow (\lambda x.\phi)(t)$, t is substitutable for x in ϕ,

D $\forall y(y = \imath x\phi \leftrightarrow (\forall x(\phi \to x = y) \land \phi(y/x)))$, x and y are different variables (cf. **AFD** above)

Sub $t_1 = t_2 \to (\phi(t_2/x) \to \phi(t_1/x))$, x does not occur in the scope of $\Box$ in ϕ, and t_1 and t_2 are substitutable for x in ϕ.

We add special Gödel's axioms about "positivity" of properties:

GA1 $\mathcal{P}\overline{X} \leftrightarrow \neg\mathcal{P}X$

GA2 $(\mathcal{P}X \land \Box\forall x(Xx \to Yx)) \to \mathcal{P}Y$

GA3 $\mathcal{P}G$

GA4 $\mathcal{P}X \to \Box\mathcal{P}X$

GA5 $\mathcal{P}N$

Note that by axiom **λConv** the unrestricted comprehension scheme is expressed: for each formula there is a corresponding second-order term.

Rules **MP, U1, Nec**: $\vdash \phi \implies \vdash \Box\phi$, and **U2**: $\vdash \phi \implies \vdash \forall X\phi$. (For **MP** and **U1**, see PCI and PFL above. See also Proposition 8 for **Nec**.)

Rigid term in $\mathsf{fGO}_\imath$ is defined as for T in Definition 6. We note that Deduction Theorem holds for $\mathsf{fGO}_\imath$ with standard restrictions for $\forall$ and $\Box$. (For convenience, in justifications of proofs we will disregard the distinction between normal and bold letters).

In the following we will recall some propositions and theorems of GO which also hold for $\mathsf{fGO}_\imath$, and add some new propositions in order to show that the slingshot theorem is provable in $\mathsf{fGO}_\imath$. This theorem will be conceived in the sense of the provable necessary biconditional between any two sentences supposed to hold. Afterwards, we will add some notes on a possible strengthened sense of the slingshot theorem.

4.1 Necessary existence, uniqueness, and rigidity of God

Theorem 1 (Neccessary existence of God)**.** $\mathsf{fGO}_\imath \vdash \Box\exists x G x$

Proof. Similar to the proof in GO [21, 48, 17, 43], which can be reproduced in fGO_7, without dependence on Axioms $\exists x Ex$ and $\forall x\phi \to \phi$ ($x \notin \mathsf{free}(\phi)$). We give some examples where first-order quantification is involved. – (a) In the first part, in Gödel's version of the proof, $\mathcal{P}X \to \Diamond\exists Xx$ is proved on the ground of the previously proved $\mathcal{P}(\lambda x.x = x)$. From $\forall x(\neg Xx \to (Xx \to \neg x = x))$ we deduce $\forall x\neg Xx \to \forall x(Xx \to \neg x = x)$ by $\forall$**1b**. By **Nec** we obtain $\Box(\forall x\neg Xx \to \forall x(Xx \to \neg x = x))$, and by **K**, $\Box\forall x\neg Xx \to \Box\forall x(Xx \to \neg x = x)$. From Axiom $(\mathcal{P}X \wedge \Box\forall x(Xx \to \neg x = x)) \to \mathcal{P}(\lambda x.\neg x = x)$ and $\mathcal{P}X$ as assumption, $\Box\neg Xx \to \mathcal{P}(\lambda x.\neg x = x)$ follows, from where (by contraposition) $\neg\mathcal{P}(\lambda x .\neg x = x) \to \neg\Box\forall x\neg Xx$ is derivable. From the positivity of $(\lambda x.x = x)$ and Axiom **GA1** (together with the definition of $\exists$), the proposition $\mathcal{P}X \to \Diamond\exists xXx$ is proved. – (b) Near the end of the proof, the proposition $\exists xGx \to \Box\exists xGx$ is proved. Starting point is the previously proved proposition $Gx \to \Box\exists xGx$. From there, by contraposition, **U1**, and $\forall$**1b**, $\forall x\neg\Box\exists xGx \to \forall x\neg Gx$ is deducible, and by $\forall$**1c**, $\neg\Box\exists xGx \to \forall x\neg Gx$ follows. Therefore, by contraposition and the definition of $\exists$, we obtain $\exists xGx \to \Box\exists xGx$. □

The following two propositions with their proofs are also part of the proof of Theorem 1.

Proposition 13. $\mathsf{fGO}_7 \vdash Gx \to \forall X(Xx \to \mathcal{P}X)$

Proof. Can be proved in quite a similar way as in GO. □

Proposition 14. $\mathsf{fGO}_7 \vdash Gx \to \mathcal{E}\!\int\!\!\int(G, x)$

Proof. Similarly as in GO. From the definition of G we have $\forall y(Gy \to (\mathcal{P}X \to Xy))$, and thus, by propositional logic, $\forall y(\mathcal{P}X \to (Gy \to Xy))$. From there, $\forall y\mathcal{P}X \to \forall y(Gy \to Xy)$ follows by $\forall$**1b**. By $\forall$**1c** (and propositional logic), we have $\mathcal{P}X \to \forall y(Gy \to Xy)$. By **Nec** and **K**, $\Box\mathcal{P}X \to \Box\forall y(Gy \to Xy))$ follows. Now, by Axiom **GA4** and Proposition 13, we deduce $Gx \to (Xx \to \Box\forall y(Gy \to Xy))$, and by propositional logic we obtain $Gx \to (Gx \wedge (Xx \to \Box\forall y(Gy \to Xy)))$. Hence, $Gx \to (Gx \wedge \forall X(Xx \to \Box\forall y(Gy \to Xy)))$ is derivable by **U2**, and $\forall$**2a**–**c**, which is equivalent with $Gx \to \mathcal{E}\!\int\!\!\int(G, x)$ (by the definition of $\mathcal{E}\!\int\!\!\int$). □

We now prove that there is at most one God.

Proposition 15 (At most one God)**.** $\mathsf{fGO}_{\eta} \vdash Gx \to (Gy \to x = y)$

Proof.

1	Gx	assumption
2	Gy	assumption
3	$\forall X(Xy \to \mathcal{P}X)$	2 Prop. 13
4	$(\lambda z.z = y)(y) \to \mathcal{P}(\lambda z.z = y)$	3 $\forall$2a
5	$y = y \to \mathcal{P}(\lambda z.z = y)$	4 λConv.
6	$\mathcal{P}(\lambda z.z = y)$	5 =1, MP
7	$\forall X(\mathcal{P}X \to Xx)$	1 def. G
8	$\mathcal{P}(\lambda z.z = y) \to (\lambda z.z = y)(x)$	7 $\forall 2a$
9	$(\lambda z.z = y)(x)$	6, 8 MP
10	$x = y$	9 λConv.
11	$Gx \to (Gy \to x = y)$	1(2)–10 Ded. Theor.

For line 5, see the note on the unrestricted "comprehension scheme" after the list of axioms above. – Alternatively, this proposition can be proved on the ground of $(\mathcal{E}\int\int(G,x) \wedge \mathcal{E}\int\int(G,y)) \to \forall X(Xx \leftrightarrow Xy)$ $(**)$ as following from Gödel's definition of $\mathcal{E}\int\int$ [49]. Assume Gx and Gy. From $(**)$ and from Proposition 14, $\forall X(Xx \leftrightarrow Xy)$ follows. Hence we obtain $x = y$. – For another version of a proof (in a weakened GO), see [24]. A similar idea can be found in [2, p. 296] for Anderson's revised version of Gödel's ontological system.[16] □

In the reconstruction of the "slingshot" argument, the definite description of God, $\imath xGx$, will have a crucial role. We first prove some related propositions.

Proposition 16 (Rigidity of God)**.** $\mathsf{fGO}_{\eta} \vdash y = \imath xGx \to \Box y = \imath xGx$

Proof. We assume $y = \imath xGx$. By **D** we derive Gy, and by **Sub**, $G\imath xGx$, from where $\forall X(\mathcal{P}X \leftrightarrow X\imath xGx)$ follows by the definition of G. From Gy we obtain $\forall X(Xy \to \mathcal{P}X)$ by Proposition 13. Now, a similar reasoning is used like in the proof of Proposition 15, so that the following formulas are being derived after one another: $(\lambda z.\Box y = z)(y) \to$

[16] See [49] for a discussion and possible ways to supplement the axioms explicitly mentioned in Scott's version of Gödel's ontological proof [43] in order to prove the unity and the necessary identity of God (i.e. $(Gx \wedge Gy) \to x = y$ and $Gx \to \Box\forall y(Gy \to x = y)$). For the latter proposition in particular, see [20].

$\mathcal{P}(\lambda z.\Box y = z)$, $\Box y = y \to \mathcal{P}(\lambda z.\Box y = z)$, $\mathcal{P}(\lambda z.\Box y = z)$, $\mathcal{P}(\lambda z.\Box y = z)$, $\mathcal{P}(\lambda z.\Box y = z) \to (\lambda z.\Box y = z)(\imath x Gx)$ (from $\forall X(\mathcal{P}X \leftrightarrow X\imath x Gx)$ above), $(\lambda z.\Box y = z)(\imath x Gx)$, and $\Box y = \imath x Gx$. By Deduction Theorem, the proposition follows. □

In general, we note that $Gx \to \Box Gx$ almost immediately follows from Hájek's theorem $\forall x(Gx \leftrightarrow \forall Y(\mathcal{P}Y \leftrightarrow \Box Yx))$, proven in GO [21].[17]

Proposition 17 (The unique existent God)**.** $\mathsf{fGO}_\imath \vdash \exists x x = \imath x Gx$ *(i.e. $E\imath x Gx$)*

Proof.

1	$Gx \wedge Ex$	assumption
2	$\forall y(Gx \to (Gy \to x = y))$	Prop. 15, U1
3	$Gx \to \forall y(Gy \to x = y)$	2 ∀1b–c
4	$\forall y(Gy \to x = y)$	1, 3 MP
5	$Gx \wedge \forall y(Gy \to x = y)$	1, 4 CPC
6	$x = \imath x Gx$	5 D, MP
7	$x = \imath x Gx \to (Ex \to E\imath x Gx)$	Sub
8	$\exists x x = \imath x Gx$	1, 6, 7 MP, def. *E*
9	$(Gx \wedge Ex) \to \exists x x = \imath x Gx$	1–8 Ded. Theor.
10	$\forall x((Gx \wedge Ex) \to \exists x x = \imath x Gx)$	9 U1
11	$\forall x \neg \exists x x = \imath x Gx \to \forall x \neg (Gx \wedge Ex)$	10 contrp., ∀1b
12	$\neg \exists x x = \imath x Gx \to \forall x \neg (Gx \wedge Ex)$	11 ∀1c, CPC
13	$\exists x(Gx \wedge Ex) \to \exists x x = \imath x Gx$	12 contrp., def.*E*
14	$\exists x x = \imath x Gx$	13 Theor. 1, T, MP

□

Hence, $\mathsf{fGO}_\imath \vdash \exists x Ex$.

Corollary 1. $\mathsf{fGO}_\imath \vdash G\imath x Gx$

Proof. From Axiom **D** (substitution $\imath x Gx/y$) and Propositions 16 and 17. □

[17] Besides, in a system including Axioms $\forall x \exists X \Box \forall y(Xy \leftrightarrow x = y)$ or $\exists X \forall y(Xy \leftrightarrow x = y)$ (see [20] and [49], respectively), which ensure the existence of a uniquely descriptive property for each first-order object, Proposition $Gx \to \Box Gx$ is easily deducible via $Gx \to \Box \forall y(Gy \to x = y)$ (mentioned in footnote 16) by standard means.

Further, the proved existence of God makes vacuous instantiation possible:

Corollary 2. $\mathsf{fGO}_\imath \vdash \forall x\phi \rightarrow \phi$, $x \notin \mathsf{free}(\phi)$

Proof. From $\forall x\phi \rightarrow (E\imath xGx \rightarrow \phi(\imath xGx/x))$ (Axiom $\forall$**1a**), and **=R** with Propositions 16 and 17. □

The following proposition will also be needed in the course of the proofs:

Proposition 18 (Positivity of facts)**.** $\mathsf{fGO}_\imath \vdash \phi \rightarrow \mathcal{P}(\lambda x.\phi)$, $x \notin \mathsf{free}(\phi)$.

Proof. Assume Gx. If we also assume ϕ (a fact), we can derive $(\lambda x.\phi)(x)$ by a vacuous λ-conversion with x/x, x not occurring free in ϕ. By applying Proposition 13 we obtain $\mathcal{P}(\lambda x.\phi)$. Thus, by the deduction theorem and a bit propositional logic $Gx \rightarrow (\phi \rightarrow \mathcal{P}(\lambda x.\phi))$ follows. From there and from $\forall x(Gx \rightarrow (\phi \rightarrow \mathcal{P}(\lambda x.\phi)))$ (by **U1**), we deduce $\exists xGx \rightarrow (\phi \rightarrow \mathcal{P}(\lambda x.\phi))$ (by first-order reasoning using $\forall$**1b** and $\forall$**1c**). Finally, referring to Theorem 1 and **T**, we derive the proposition. – A proof can be also arranged as a branch to the modal collapse proof if already available (cf. [27]). □

Proposition 19. $\mathsf{fGO}_\imath \vdash X\imath xGx \rightarrow \Box X\imath xGx$

Proof.

1	$\forall x(Gx \rightarrow \forall X(Xx \rightarrow \mathcal{P}X))$	Prop. 13
2	$E\imath xGx \rightarrow (G\imath xGx \rightarrow \forall X(X\imath xGx \rightarrow \mathcal{P}X))$	1 $\forall$1a, Prop.16
3	$\forall X(X\imath xGx \rightarrow \mathcal{P}X)$	2 Cor. 1, Prop. 17
4	$X\imath xGx \rightarrow \mathcal{P}X$	3 $\forall$2a
5	$X\imath xGx \rightarrow \Box\mathcal{P}X$	4 GA4, CPC
6	$\Box(\mathcal{P}X \rightarrow X\imath xGx)$	Prop. 17, def. G, Nec
7	$X\imath xGx \rightarrow \Box X\imath xGx$	5, 6 K

□

4.2 Slingshot by G

On the grounds of the previously proved propositions we can devise a modal "slingshot" argument in $\mathsf{fGO}_\imath$, where the identity of the "signification" of ϕ and ψ is reduced, at first, to necessary biconditional $\Box(\phi \leftrightarrow \psi)$. The proof is essentially connected with G and $\imath xGx$, since the being of which we provably know, within $\mathsf{fGO}_\imath$, that it exists, is x which is G (possibly nothing else exists).[18]

In $\mathsf{fGO}_\imath$, the following proposition obviously follows from Proposition 18 and **GA4** by propositional modal logic:

$$\phi \rightarrow (\psi \rightarrow \Box(\mathcal{P}(\lambda x.\phi) \leftrightarrow \mathcal{P}(\lambda x.\psi))) \quad x \notin \mathsf{free}(\phi, \psi) \tag{1}$$

(Of course, we could also prove the similar proposition with $\wedge$ instead of $\rightarrow$.)

Further, from the definition of G and Proposition 13 it follows:

$$\phi \rightarrow (\psi \rightarrow \Box((\lambda x.\phi)(\imath xGx) \leftrightarrow (\lambda x.\psi)(\imath xGx))) \tag{2}$$

From (2) we obtain:

$$\phi \rightarrow (\psi \rightarrow \Box(\phi \leftrightarrow \psi)) \tag{3}$$

by λ conversion. We can now establish the "slingshot" argument for negated ("false") propositions:

$$\neg\phi \rightarrow (\neg\psi \rightarrow \Box(\phi \leftrightarrow \psi)) \tag{4}$$

Proof. Analogous to Sobel [45]: from $\Box(\neg\neg\phi \leftrightarrow \neg\neg\phi)$ and from $\neg\phi \rightarrow (\neg\psi \rightarrow \Box(\neg\phi \leftrightarrow \neg\psi))$ (see (3)), we get $\neg\phi \rightarrow (\neg\psi \rightarrow \Box(\neg\neg\phi \leftrightarrow \neg\neg\psi))$, by the replacement of $\neg\psi$ for $\neg\phi$ in the tautology. Proposition $\neg\phi \rightarrow (\neg\psi \rightarrow \Box(\phi \leftrightarrow \psi))$ follows. □

The modal collapse, too, is provable on the ground of the available propositions:

$$\phi \rightarrow \Box\phi \tag{5}$$

Proof. From Propositions 18 and 19 it is provable that $\phi \rightarrow \Box(\lambda x.\phi)(\imath xGx)$, with $x \notin \mathsf{free}(\phi)$, from where $\phi \rightarrow \Box\phi$ follows. □

[18]Cf. Sobel remark in [45, p. 135].

The “slingshot” argument can be easily derived from the modal collapse.[19]

In still another way, the modal “slingshot” theorem can be proved from **Assm1** (page 5), which is deducible by means of Axiom **D**:

Proposition 20. $\mathsf{fGO}_{\imath} \vdash \forall y(\phi \leftrightarrow y = \imath x(x = y \wedge \phi(x/y)))$, $x \notin \mathsf{free}(\phi)$.

Proof. Analogous to the proof of Proposition 7. □

Corollary 3. $\mathsf{fGO}_{\imath} \vdash \phi \leftrightarrow \imath xGx = \imath x(x = \imath xGx \wedge \phi(x/\imath xGx))$, $x \notin \mathsf{free}(\phi)$ *($\imath xGx$ possibly not occurring in ϕ).*

Proof. From Proposition 20 with $\imath xGx$ for y, and from Propositions 17 and 16 ($\imath xGx = \imath xGx$ from **=1**). □

As we can see, the proof of Corollary 3 requires the provable existence of a rigidly determined being (because of the application of Axiom **∀1a**), and this has been proved of the most positive being ($\imath xGx$) in the Propositions 16 and 17.

Using Corollary 3 and Proposition 19, we can now prove the “slingshot” theorem in the sense of a necessary conditional:

Theorem 2. $\mathsf{fGO}_{\imath} \vdash \phi \rightarrow (\psi \rightarrow \Box(\phi \leftrightarrow \psi))$

Proof.

1	$\phi \leftrightarrow \imath xGx = \imath x(x = \imath xGx \wedge \phi(x/\imath xGx))$ $x \notin \mathsf{free}(\phi)$	Corollary 3
2	$\Box(\phi \leftrightarrow \imath xGx = \imath x(x = \imath xGx \wedge \phi(x/\imath xGx))$	1 Nec
3	$\psi \leftrightarrow \imath xGx = \imath x(x = \imath xGx \wedge \psi(x/\imath xGx))$ $x \notin \mathsf{free}(\psi)$	Corollary 3
4	$\Box(\psi \leftrightarrow \imath xGx = \imath x(x = \imath xGx \wedge \psi(x/\imath xGx)))$	2 Nec

[19]Cf. Sobel’s remark on this in [45, p. 147]. As to the possible acceptability of the modal collapse, see Orilia [35, p. 130–131]; for the possibility of Gödel’s acceptance of the modal collapse, see Adams [1]; in favor of Gödel’s real acceptance of the modal collapse, see Co-author [24, p. 582] and [25], Sobel [44], and Co-author [26, 27].

5	ϕ	assumption
6	ψ	assumption
7	$\iota xGx = \iota x(x = \iota xGx \wedge \phi(x/\iota xGx))$	1, 5 MP
8	$(\lambda y.x = \iota x(x = y \wedge \phi(x/\iota xGx))(\iota xGx)$	7 λConv
9	$\Box \iota xGx = \iota x(x = \iota xGx \wedge \phi(x/\iota xGx))$	8, Prop. 19
10	$\iota xGx = \iota x(x = \iota xGx \wedge \psi(x/\iota xGx))$	3, 6 MP
11	$(\lambda y.x = \iota x(x = y \wedge \psi(x/\iota xGx))(\iota xGx)$	11 λConv
12	$\Box \iota xGx = \iota x(x = \iota xGx \wedge \psi(x/\iota xGx))$	11, Prop. 19
13	$\iota xGx = \iota x(x = \iota xGx \wedge \psi(x/\iota xGx)) \leftrightarrow \iota x(x = \iota xGx \wedge \phi(x/\iota xGx)) = \iota x(x = \iota xGx \wedge \psi(x/\iota xGx))$	7 Sub, CPC
14	$\Box(\iota xGx = \iota x(x = \iota xGx \wedge \psi(x/\iota xGx)) \leftrightarrow \iota x(x = \iota xGx \wedge \phi(x/\iota xGx)) = \iota x(x = \iota xGx \wedge \psi(x/\iota xGx)))$	13 λConv, Prop. 19
15	$\Box \iota x(x = \iota xGx \wedge \phi(x/\iota xGx)) = \iota x(x = \iota xGx \wedge \psi(x/\iota xGx))$	12, 14 K, MP
16	$\Box(\iota x(x = \iota xGx \wedge \phi(x/\iota xGx)) = \iota x(x = \iota xGx \wedge \psi(x/\iota xGx)) \rightarrow (\iota xGx = \iota x(x = \iota xGx \wedge \phi(x/\iota xGx)) \leftrightarrow \iota xGx = \iota x(x = \iota xGx \wedge \psi(x/\iota xGx))))$	Sub, Nec, CPC
17	$\Box(\iota xGx = \iota x(x = \iota xGx \wedge \phi(x/\iota xGx)) \leftrightarrow \iota xGx = \iota x(x = \iota xGx \wedge \psi(x/\iota xGx)))$	15, 16 K, MP
18	$\Box(\phi \leftrightarrow (\iota xGx = \iota x(x = \iota xGx \wedge \psi(x/\iota xGx)))$	2, 17 CPC, K, MP
19	$\Box(\phi \leftrightarrow \psi)$	4, 18 CPC, K
20	$\phi \rightarrow (\psi \rightarrow \Box(\phi \leftrightarrow \psi))$	5(6)–19 Ded. Th.

□

If there would be no existing object whatsoever, the slingshot argument would only conditionally hold (on the condition that the objects considered exist). If we have a proof of the necessary existence of God, then the (non-conditional) slingshot argument should hold "vacuously" unless sentences are about God. The argument is almost the same as if it would be for any (necessarily) existing object. Another kind of proof, more specific for G, would be dependent on the essence of God and on the necessity of each property of God (similarly as in the modal collapse proof, see [43, 10]).

Remark 1 (Models)**.** *Let us briefly consider a model and assignment for* $\mathsf{fGO}_{\imath}$ *defined as* $\langle W, R, D, q, D(n), I, a\rangle$*, where* W *is a set of worlds,* R *reflexive and euclidean relation on* W*,* D *a first-order domain of the model,* q *a function from* W *to* $\wp D$*,* $D(n) \subseteq \wp(D^n)^W$ *a set of world relative second-order domains for each predicate arity (n) (*$D(n)$ *should be chosen in the right way – in accordance with the satisfaction of formulas on the ground of a previously defined frame),* I *an interpretation function, and a some valuation of variables. Assume that the interpretation of the third-order term* $\mathcal{P}$ *corresponds to the axioms of positivity. Let a definite description* $\imath x\phi$ *denote at* w *a unique member of* $q(w)$ *that satisfies* ϕ*, and a member of* D *outside* $q(w')$ *for any* $w' \in W$ *otherwise. – We take as an example a most simple, one-world model* $\mathfrak{M}$ *with* $W = \{w\}$*,* $R = \{\langle w, w\rangle\}$*,* $D = \{g, 0\}$*,* $q(w) = \{g\}$*, a properly defined* $D(n)$*, with* $I(\mathcal{P})$ *being a (principal) ultrafilter of properties generated by g. It can be shown that all the axioms and rules of* $\mathsf{fGO}_{\imath}$ *as well as Theorem 2 and the modal collapse are satisfied in this model. As an example, we take Axiom* **D***. Let* $d \in q(w)$ *be the value of y for w. If* $[\![\imath x\phi]\!]_a^{\mathfrak{M},w}$ *denotes (the unique)* $d \in q(w)$ *that satisfies* ϕ*, then both sides of* **D** *hold. If* ϕ *does not uniquely denote d, then* $\phi(y)$ *on the right side does not hold for d, and the left side of* **D** *(*$y = \imath x\phi(x)$*) does not hold either – it remains that* $[\![\imath x\phi]\!]_a^{\mathfrak{M},w} = 0$*. – In the chapter "Causal interpretation of Gödel's ontological proof" (this book), we more extensively describe the semantics for the case of an extended and slightly modified logic* QCGO *[27], containing causal terms instead of modal operators.*

4.3 The sameness of facts

A considerable and natural restriction on the strict biconditional ($\Box(\phi \leftrightarrow \psi)$), as well as on necessary identity ($\Box t_1 = t_2$) can be applied to define anew the "slingshot" collapse. Let us introduce $\equiv$ ("sameness of facts") as a sentence connective (in analogy with Section 3) and the following axioms:

AI $\phi \equiv \psi$, with ψ having the following shapes: ϕ, $\neg\neg\phi$, $\phi \wedge \phi$, $\phi \wedge \top$, $\phi \vee \phi$, $\phi \vee \bot$, $(\lambda y.\phi(y/t))(t)$ ($y \notin \mathsf{free}(\phi)$), $\forall y \phi'(y/x)$ if $\phi = \forall x \phi'$ ($y \notin \mathsf{free}(\phi')$, y is substitutable for x in ϕ'),

AIe $(e_1 \doteq e_2) \to (e(e_1/e_0) \doteq e(e_2/e_0))$, where $\doteq$ stands for $=$ or $\equiv$, and e, e_i are expressions (terms or formulas), with the restriction as for **Sub**,

DI $\forall y(y = \imath x\phi \equiv (\forall x(\phi \to x = y) \wedge \phi(y/x)))$, x and y are different variables,

I2 $(\phi \equiv \psi) \to \Box(\phi \to \psi)$.

Definition 7 ($\mathsf{fGO}_{\imath\equiv}$). *Let* $\mathsf{fGO}_{\imath\equiv}$ *be like* $\mathsf{fGO}_{\imath}$*, with* **AIe** *and* **DI** *instead of* **Sub** *and* **D***, respectively, and extended by* **AI**, **I2**, **i∀**, *and* **i∃** *(see Definition 1 above).*

In $\mathsf{fGO}_{\imath\equiv}$, the $\equiv$-slingshot theorems for the positive ("truth") and the negative ("falsehood") case are provable.

Proposition 21. $\mathsf{fGO}_{\imath}\equiv \vdash \forall y(\phi \equiv (y = \imath x(x = y \wedge (\lambda y.\phi)(x))), x \notin \mathsf{free}(\phi)$.

Proof. Like the proof of Proposition 7, replacing $\leftrightarrow$ with $\equiv$ throughout the proof, using **AI** in line 4, **AIe** in line 6, and **DI** instead of **D**. □

Theorem 3. $\mathsf{fGO}_{\imath}\equiv \vdash \phi \to (\psi \to (\phi \equiv \psi))$

Proof.

1	ϕ	assumption
2	ψ	assumption
3	$\phi \equiv (\imath xGx = \imath x(x = \imath xGx \wedge \phi(x/\imath xGx)))$, $x \notin \mathsf{free}(\phi)$	Prop. 21
4	$\psi \equiv (\imath xGx = \imath x(x = \imath xGx \wedge \psi(x/\imath xGx)))$, $x \notin \mathsf{free}(\psi)$	Prop. 21
5	$\Box(\phi \leftrightarrow (\imath xGx = \imath x(x = \imath xGx \wedge \phi(x/\imath xGx))))$	3 I2
6	$\phi \leftrightarrow (\imath xGx = \imath x(x = \imath xGx \wedge \phi(x/\imath xGx)))$	5 T
7	$\imath xGx = \imath x(x = \imath xGx \wedge \phi(x/\imath xGx))$	1, 6 MP
8	$\Box(\psi \leftrightarrow (\imath xGx = \imath x(x = \imath xGx \wedge \psi(x/\imath xGx))))$	4 I2
9	$\psi \leftrightarrow (\imath xGx = \imath x(x = \imath xGx \wedge \psi(x/\imath xGx))$	8 T
10	$\imath xGx = \imath x(x = \imath xGx \wedge \psi(x/\imath xGx))$	2, 9 MP
11	$\imath x(x = \imath xGx \wedge \phi(x/\imath xGx))$ $\equiv \imath x(x = \imath xGx \wedge \psi(x/\imath xGx))$	7, 10 AIe

12	$\phi \equiv (\imath x Gx = \imath x(x = \imath x Gx \wedge \psi(x/\imath x Gx)))$	3, 11 AIe
13	$\phi \equiv \psi$	4, 12 AIe
14	$\phi \to (\psi \to (\phi \equiv \psi))$	1(2)–13 Ded. Th.

□

Theorem 4. $\mathsf{fGO}_{\imath\equiv} \vdash \neg\phi \to (\neg\psi \to (\phi \equiv \psi))$

Proof. Analogous to the proof of (4) above, starting from $\neg\neg\phi \equiv \neg\neg\phi$ (**AI3**) and $\neg\phi \to (\neg\psi \to (\neg\phi \equiv \neg\psi))$ (Theorem 3). □

An analogous modification of AO (in the nomenclature by Hájek [21]), that is, Anderson's emendation of GO [2], to $\mathsf{fAO}_{\imath\equiv}$, makes possible to prove the "slingshot" collapse. We will, first, define $\mathsf{fAO}_{\imath}$, starting from AO.

As is known, **GA1** is replaced in AO by

AA1 $\mathcal{P}\overline{X} \to \neg\mathcal{P}X$,

and the abbreviations (definitions) of G, $\mathcal{E}\!\int\!\!\int$, and N are as follows:

$$Gt =_{def} \forall X(\mathcal{P}X \leftrightarrow \Box Xt)$$
$$\mathcal{E}\!\int\!\!\int(X, t) =_{def} \forall Y(\Box Yt \leftrightarrow \Box\forall x(Xx \to Yx)).$$

Axioms **GA2–5** as well as the definition of N remain nominally the same as in GO. The axioms on positivity are named **AA1–5**.

Let us note that in AO the "slingshot" argument does not hold. The countermodel for modal collapse described in [2] is also a countermodel for the "slingshot". In this countermodel there is a world w with one object in its domain, and a world w' with two objects its domain. In w' both sentences "There is at least one object" (1) and "There are at least two objects" (2) are true, whereas $\Box((1) \leftrightarrow (2))$ does not hold in w' since in w (accessible to w') (1) is true but (2) is not.[20]

[20]Let us also note that in the modifications of GO by the restrictions of the comprehension scheme (λ abstraction) as in [8] and [20] by Czermak and Hájek, respectively, the "slingshot" argument does not hold. For example, in Czermak's countermodel for the modal collapse, there is a world w with an object d satisfying ϕ, and a world w' where no object satisfies ϕ. Thus (analogously to Anderson's countermodel), in w' sentences "At least one object is ϕ" (3) and "At least two objects are ϕ" (4) are both false,

Let us define fAO_ι like fGO_ι with the only difference consisting in the replacement of axioms and definitions as mentioned above in connection with AO. It can be shown that $\Box\exists xGx$, $Gy \to \Box y = \imath xGx$, and $y = \imath xGx \to \Box y = \imath xGx$ are provable in fAO_ι.

Proposition 22. $\mathsf{fAO}_\iota \vdash \Box\exists xGx$

Proof. Anderson's proof [2] can be routinely accommodated to "universally free" basis. □

Proposition 23. $\mathsf{fAO}_\iota \vdash Gy \to \Box y = \imath xGx$

Proof. From $\Box(\lambda y.y = x)(x)$, and assuming Gx, $\mathcal{P}(\lambda y.y = x))$ follows (Anderson's definition of G), and thus, assuming Gy, we derive $\Box(\lambda y.y = x)(y)$ (the same definition). Therefore, $Gx \to (Gy \to y = x)$ is deducible (cf. also the alternative proof of Proposition 15 above and [2]). From there,

1	$Gy \to \forall y(Gy \to y = x))$	U1, $\forall 1b$, $\forall 1c$
2	$Gy \to (\forall y(Gy \to y = x) \wedge Gx)$	1 CPC
3	$Gx \to x = \imath xGx$	2 D
4	$\Box\forall x(Gx \to x = \imath xGx)$	3 U1, Nec
5	$\mathcal{P}(\lambda x.x = \imath xGx)$	4 AA3, AA2
6	$Gy \to \Box(\lambda x.x = \imath xGx)(y)$	5 def. of G
7	$Gy \to \Box y = \imath xGx$	6 λConv

□

Proposition 24. $\mathsf{fAO}_\iota \vdash y = \imath xGx \to \Box y = \imath xGx$

Proof. From $y = \imath xGx$, according to Axiom **D**, and from Proposition 23, $y = \imath xGx \to \Box y = \imath xGx$ follows. □

If we modify fAO_ι to $\mathsf{fAO}_{\iota\equiv}$ as mentioned, i.e. in analogous way as fGO_ι to $\mathsf{fGO}_{\iota\equiv}$, the ≡-"slingshot" and, by **I2**, □-"slingshot" are deducible similarly as by the proof of Theorem 3.

but in w (accessible to w') (3) is true but (4) is false and hence $\Box((3) \leftrightarrow (4))$ is not true at w. In Hájek's Example 2, there are at least two worlds (w, w') and two objects (g, h) so that, extending the example a bit, we ccould have g and h satisfying property X at w, and g satisfying X while h not satisfying X at w'. Hence, although both "g is X" and "h is X" are true at w, "$\Box(g$ is $X \leftrightarrow h$ is $X)$" is not true at w.

A concluding remark

The reconstruction of a "slingshot" argument within fGO_7 may perhaps throw some light on Gödel's statement (regarding the collapsing conclusion of the argument) that "there is something behind it which is not yet completely understood" [16, p. 123]. What seems to be obvious ("apparently obvious" "slingshot" axioms) has in general a hardly acceptable consequence (the collapse of the signification of sentences). Russell's elimination of definite descriptions can prevent the "slingshot" argument, but is as such for Gödel not quite plausible (Gödel has a "feeling that . . . the problem . . . has only been evaded" by Russell's solution [16, p. 123]). In Gödelian onto-theological systems like the ones considered, the mentioned presuppositions of the "slingshot" argument (**P1**–**2**, **Assm1**–**2**) are partly embedded in the axioms and partly derived from them. What seems to be obvious, especially the assumptions that something exists, and that there is anything at all the terms (including descriptions) could denote and the propositions could speak about, are in fact only (not from the start obvious) consequences of onto-theology. In addition, what does not seem to be obvious nor acceptable (like the collapsing "slingshot" argument) should be in fact accepted on the basis of Gödelian onto-theological presuppositions considered (including an extension of AO by factual identity, $\equiv$). From the 1940s to 1970s, Gödel's attempts are documented to establish an ontological system containing an ontological proof. The question arises why Gödel would not have been aware of such consequences of his onto-theology, and found the "slingshot" (similarly as "modal collapse") acceptable – not as an obvious common sense truth, but as a consequence of a deeper, ontologically analysed theistic world-view.

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