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Russell's 1903–1905 Anticipation of the Lambda Calculus

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It is well known that the circumflex notation used by Russell and Whitehead to form complex function names in *Principia Mathematica* played a role in inspiring Alonzo Church's 'Lambda Calculus' for functional logic developed in the 1920s and 1930s. Interestingly, earlier unpublished manuscripts written by Russell between 1903 and 1905—surely unknown to Church—contain a more extensive anticipation of the essential details of the Lambda Calculus. Russell also anticipated Schönfinkel's Combinatory Logic approach of treating multi-argument functions as functions having other functions as value. Russell's work in this regard seems to have been largely inspired by Frege's theory of functions and 'value-ranges'. This system was discarded by Russell due to his abandonment of propositional functions as genuine entities as part of a new tack for solving Russell's paradox. In this article, I explore the genesis and demise of Russell's early anticipation of the Lambda Calculus.

1. Introduction

The Lambda Calculus, as we know it today, was initially developed by Alonzo Church in the late 1920s and 1930s (see, e.g. *Church 1932, 1940, 1941, Church and Rosser 1936*), although it built heavily on work done by others, perhaps most notably, the early works on Combinatory Logic by Moses Schönfinkel (see, e.g. his *1924*) and Haskell Curry (see, e.g. his *1930*). Church would have been the first to admit that his work was also largely influenced by Frege and Russell. However, in this article, my primary aim is not to examine Russell's influence on Church; rather, I discuss some early manuscripts of Russell's (from 1903 to 1905), in which he, inspired by Frege, had anticipated much more of the Lambda Calculus than Church himself probably ever imagined. That Russell did such work is interesting on its own, but what is perhaps more interesting is why it never surfaced during his lifetime. Russell undertook this work during a period in which he was trying desperately to find a solution to Russell's paradox and related antinomies. However, his standards were high; he did not want a formal dodge, he wanted a philosophically, even *metaphysically*, motivated explanation for the avoidance of the contradictions. The work he did anticipating the Lambda Calculus fitted well with some of his earlier philosophical views on the nature of functions and propositions, but did not fit so well with the views he developed after adopting the theory of descriptions in 1905. Therefore, Russell scrapped this early work and it never saw the light of day. A close examination of Russell's anticipation of Lambda Calculus sheds light on the development of his logical and metaphysical views, and may even prompt us to take another look at later developments in these areas of thought.

2. Church, *Principia Mathematica*, and the Lambda Calculi

My main goal in this paper is to examine the early work of Russell's that *anticipated* the Lambda Calculus, not work that actually influenced its development.

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However, let us for the moment return to Church and the development of his work, to serve as a basis for comparison. I shall also make brief note of those aspects of Russell's *later* work in *Principia Mathematica* that did actually influence Church's Lambda Calculus, again simply for purposes of contrast with the discussion of Russell's early manuscripts to follow.

There is not simply one formal system that goes by the name 'Lambda Calculus', but a whole family of related systems. What they have in common are the notions of λ-abstracts and λ-conversion.[1] A λ-abstract, according to Church's original design, was to be understood as a term standing for a function itself, rather than some unspecified value of that function. Thus, as a convenient name of the mathematical function f such that:

$$f(x) = x^2 + 4x + 4$$

we might use the λ-abstract '$\lambda x(x^2 + 4x + 4)$'. Within different subsystems, these could be interpreted as names either of functions-in-intension, or of functions-in-extension (see, e.g. *Church 1941*, pp. 2–3). More contemporary work on the 'pure theory' of the Lambda Calculus has moved away from this original intent, and has treated systems in which one cannot, without distortion, interpret the λ-abstracts as standing for functions, or arguably, for anything else. However, for present purposes, let us stick with this original interpretation.

A λ-abstract contains a bound variable. λ-conversion refers to the replacement of this bound variable with the argument to the function, or vice-versa. Therefore, the expressions:

$$\lambda x(x^2 + 4x + 4)a$$

$$a^2 + 4a + 4$$

can be regarded as 'λ-converts'. Within a formal system containing λ-abstracts, rules of inference that allow λ-conversion, i.e. the replacement of a subexpression with one of its λ-converts, are usually included.

In the Lambda Calculus, multi-place functions are usually treated as functions having functions as value. If the number 5 is taken as argument, the function $\lambda y(\lambda x\ (x + y))$ yields as value (by conversion), the function, $\lambda x(x + 5)$. This way, all functions can be treated, formally, as having a single argument-place, greatly simplifying the rules of the system. This approach is borrowed from Combinatory Logic, and was first suggested (in print) by Schönfinkel (*1924*).

Church's original work on the Lambda Calculus was part of an attempt to develop a type-free higher-order logical system for the foundations of mathematics. However, the untyped Lambda Calculus has proven itself to be a very shaky basis for any consistent logical system (see, e.g. *Kleene and Rosser 1935, Curry 1942*). The typed Lambda Calculus has done considerably better (see, e.g. *Church 1940*). Later on, the Lambda Calculus proved its worth in recursion theory and the study of computability, and even served as basis for certain types of programming languages. Nevertheless, Church's initial motivation for this work grew out of his interests in the foundations of mathematics, and more specifically, out of his desire to make up for

1 The notion Church called 'λ-conversion' has sometimes been renamed, or respecified, as 'β-conversion' by contemporary logicians, mathematicians and computer scientists (see, e.g. *Barendregt 1981, Hindley and Seldin 1986*). I use the term as Church originally did.

certain shortcomings, as he saw them, of Whitehead and Russell's *Principia Mathematica*.

The exact relationship between Church's innovations and *Principia* is somewhat complicated. Church claims his first publication on the Lambda Calculus (*Church 1932*, n. 346) was the end-product of work undertaken as a NEH National Research Fellow in 1928–1929, soon after finishing his dissertation under Oswald Veblen. *Principia*, with its new second edition, was still very much on the minds of most able-minded logicians. Church himself published a review of Books II and III of *Principia* in 1928. Church is known to have spent much of 1928 and 1929 at the University of Göttingen, where he interacted with Hilbert and Curry (see *Grattan-Guinness 2000*, p. 453, *Curry 1980*, p. 88). *Principia* was a often topic of conversation there too. In the introduction to the new edition, and earlier in his *1919 Introduction to Mathematical Philosophy* (p. 151), Russell admitted that they had erred in not listing substitution or replacement rules for free variables among the basic principles of the system, though he still does not formulate such rules in explicit terms. Apparently, it was in attempting to state substitution rules in simple terms that lead Curry to his early Combinatory Logic and to study Schönfinkel in the mid-1920s (see, e.g. *Scott 1980*, p. 224, *Grattan-Guinness 2000*, n. 453; *Curry 1980*, pp. 85–6).[2] Curry surely discussed this work with Church, and the latter apparently wrote up his first formulation of the Lambda Calculus near the time he was in Göttingen (see, e.g. *Curry 1980*, p. 88), no doubt influenced by Curry's still-developing work on combinators.[3]

The notation used by Church is said to have been derived, accidentally, from the circumflex notation used by Whitehead and Russell in *Principia Mathematica*. Therein they used the notation '$2\hat{x} + 4$' for the function f such that $f(x) = 2x + 4$ and the notation '$\hat{x}(x^2 = 4)$' for class abstracts. Church, who saw little need to distinguish classes from functions taken extensionally, originally meant to use the notation '$\hat{x}(\ldots x \ldots)$' instead of '$\lambda x(...x...)$' for his λ-abstracts, but due to typesetting difficulties, the notation was changed by one typesetter to '$\hat{}x(\ldots x \ldots)$' and by another, finally, to '$\lambda x(...x...)$' (see *Barendregt 1991*, p. 182). With the Lambda Calculus, Church hoped to make up for difficulties in the programme defended by *Principia Mathematica*, such as the overly complicated theory of types, and the lack of proper substitution rules. Indeed, the latter difficulty held Church's interest as well as Curry's. It was Church, guided to a large extent by what he had learned in stating conversion and abstraction rules for the Lambda Calculus, who, following incomplete attempts by Hilbert and Ackermann (*1928*, p. 53) and others, first published properly stated rules for replacement that would be adequate for a system as rich as *Principia Mathematica*, although he apparently relied heavily on notes from a lecture by Gödel taken by his students S. C. Kleene and J. B. Rosser (see *Church 1935, Kleene and Rosser 1965*).

However, the influence of *Principia* upon Church's Lambda Calculus seems mainly limited to his motivations and overall vision; most likely, it had little influence on the details. Certainly, not much of a case can be made for the suggestion that *Principia Mathematica* contains something like a proto-Lambda Calculus in its use of the circumflex notation. Some writers on Russell have claimed that the circumflex notation was not intended as an official part of the symbolism of *Principia Mathematica*, and that, rather, it was simply a device Whitehead and Russell used in their informal discussion when they wished to discuss functions as such (see, e.g.

2 For the results of Curry's efforts, see *1929*.

3 For more on the relation between Combinatory Logic and the Lambda Calculus, see *Hindley and Seldin 1986*.

Landini 1998, pp. 264–267).[4] One never encounters circumflex 'terms' as parts of the fundamental axioms, and no rules of 'concretion' or 'conversion' are listed in *Principia Mathematica* to govern their use (as pointed out in *Quine 1963*, p. 249). Indeed, the circumflex notation for functions used by Whitehead and Russell, as such, is inadequate to the task of expansion towards a Lambda Calculus; alterations would be necessary to avoid ambiguity (see *Hatcher 1982*, p. 109). Consider, for example, the two-place function abstract '$\hat{x} > \hat{y}$'; if this were written with two arguments, it would be unclear how the conversion should take place; for example is '$(\hat{x} > \hat{y})(a, b)$' to be converted to '$a > b$' or to '$b > a$'? (I will later suggest that Russell may have been aware, years earlier, of the limitations of this notation, which, given its perseverance in *Principia* lends credence to the view that it was not part of the 'official symbolism'.) Certainly, Church did not simply extract the rules of the Lambda Calculus by reading *Principia*; Curry, Schönfinkel and Hilbert surely provided more help when it came to the essential details of what was new in his innovation. If we wished to understand the actual historical development of the Lambda Calculus, a study of these figures would figure more prominently.

Unfortunately, the writings of Russell's that *would have* helped Church the most had he had the opportunity to see them, written between 1903 and 1905, lay buried among Russell's unpublished manuscripts until quite recently. To understand Russell's early anticipation of the Lambda Calculus more fully, however, we must take a step further back, to Frege.

3. Frege's *Wertverläufe*

The system of Frege's *Grundgesetze der Arithmetik* is usually taken as the first higher-order predicate calculus. However, strictly speaking, Frege's system is not a predicate calculus at all, but a function calculus. Instead of 'predicates', Frege insists on functions onto (reified) truth-values, the True and the False. Functions with one argument-place that always yield truth-values as value he calls 'concepts'; similar functions with multiple argument-places he calls 'relations'. For Frege, concepts, relations and other functions are thought to be 'incomplete' or 'unsaturated' entities. As such, he believes that a sign for a function should never occur without its argument place. So, even in informal discussion, Frege always writes '$f(\ )$' or '$f(\xi)$' and never just 'f' to name a function. When a function receives its argument, the incompleteness or unsaturatedness is removed, and the result, the *value*, is therefore not a function, but an 'object' (possibly a truth-value, possibly a number, etc.)

For these reasons, it is arguable that Frege's approach to logic is diametrically opposed to that taken in the Lambda Calculus,[5] in which functions are named by λ-abstracts that may appear *with or without* their arguments, and in which many functions have other functions as value. But the situation is not quite so easy as that. Besides the incompleteness-tracking notation for functions, Frege also included notation for what he called 'value-ranges' (*Wertverläufe*) of functions. This notation consists of a Greek vowel written with a smooth-breathing accent-mark preceding a formula in which that Greek vowel sits in the argument-spot(s) of a function name. So

4 In this way, their use of the notation '$2\hat{x} + 4$' would be analogous to Frege's use of notation such as '$2\xi + 4$', which would not occur in any of the formal derivations in the *Grundgesetze*, only in his informal discussion. See *Frege 1893*, §1.

5 For further discussion of this issue, see *Potts 1973*. I do not, however, agree with Potts on all points.

if '$H(\ )$' is the name of the function (concept) that takes all humans to the True, and all non-humans to the False, '$\grave{\varepsilon}H(\varepsilon)$' names the value-range of this function. But what is a value-range? What seems to stick in most readers' memories is that he equates the value-range of a concept with its *extension*. Therefore, '$\grave{\varepsilon}H(\varepsilon)$' is taken as a name of the *class of all humans*. The axioms and principles Frege introduces to deal with value-ranges are therefore usually taken as an axiomatization of set or class theory, indeed, of *naïve*, *inconsistent set theory*, yielding straightaway to Russell's paradox of sets that do not 'contain' themselves.

But this is an overly narrow interpretation of Frege's value-ranges. Not only concepts have value-ranges: all (first-level) functions do. One of Frege's own examples is $\grave{\varepsilon}(\varepsilon^2 - 4)$ (*Frege 1891*, p. 143). There is no way to understand this as a class. When Frege gives such mathematical examples, he seems to construe the value-range as the complete argument-value pairing generated by the function, its *graph*, considered as an abstract object. (Indeed, 'graph' is an alternative translation of 'Wertverlauf'.) Mathematical function $f(\ )$ and mathematical function $g(\ )$ are thought to have the same value-range if and only if have the same value for every argument, that is, they determine the same 'graph'. Generalized to all functions, this becomes Frege's ill-fated Basic Law V. However, it should not be thought that Frege meant to suggest that *two different* functions could determine the same value-range. Indeed, when $f(\ )$ and $g(\ )$ have the same value for every argument, Frege not only says that $f(\ )$ and $g(\ )$ have the same value-range, but that $f(\ )$ and $g(\ )$ coincide, that they are *the very same function* (*Frege 1892*, p. 120–21). What then, is the *real difference*, between the *value-range* and the *function* itself?—after all, they seem to have the same identity conditions.[6] Frege himself claimed that that value-ranges of concepts 'have their being in the concept' (*Frege 1906*, p. 183). Perhaps Frege really thought of value-ranges as nothing more *functions treated as logical subjects*.

Stressing these aspects of Frege's theory of value-ranges, Cocchiarella (*1987*, Ch. 2) suggests that Frege's logic could be treated as equivalent to a system with nominalized functions, and presents a reconstruction of Frege's logic using λ-abstracts as though they were equivalent to Frege's value-range notation. In a 1942 dictionary of philosophy article on 'abstraction', Church (*1942*) himself identified both Russell's circumflex notation and Frege's smooth-breathing notation as earlier analogues of his '$\lambda x(...x...)$'. If we regard Frege's work on value-ranges as a sort of proto-Lambda Calculus, we can find in his work certain anticipations of some of the technical results of the Lambda Calculus. For example, a slight anticipation of the Lambda Calculus's approach to multi-place functions can be found in Frege's realization (*Frege 1893*, §36) that it is unnecessary to posit a different kind of entity to serve as value-ranges of two-place functions as it is possible to simply make use of the value-ranges of one-place functions that have value-ranges as value. For example, the two-place function $\xi + \zeta$ corresponds to the value-range $\grave{\alpha}\grave{\varepsilon}(\varepsilon + \alpha)$. Strictly speaking, $\grave{\alpha}\grave{\varepsilon}(\varepsilon + \alpha)$ is the value-range of the function $\grave{\varepsilon}(\varepsilon + \xi)$, that is, the function whose value for 5 as argument is $\grave{\varepsilon}(\varepsilon + 5)$, and whose value for 7 as argument is $\grave{\varepsilon}(\varepsilon + 7)$, etc. Using higher-order quantification and a description operator, Frege defines an operator '$\frown$', which is often read simply as a membership sign akin to the usual '$\in$', but is in fact defined such that, for any function $f(\ )$, whether a concept or otherwise, $x \frown \grave{\varepsilon}f(\varepsilon) = f(x)$. Therefore, $(7 \frown \grave{\alpha}\grave{\varepsilon}(\varepsilon + \alpha)) = \grave{\varepsilon}(\varepsilon + 7)$, and hence:

6 After learning of Russell's paradox, Frege suggested a revision to Basic Law V which did allow for differing functions to have the same value-range. However, originally, he thought these to have the same identity conditions.

$$
\begin{aligned}
&(5\frown(7\frown\grave{\alpha}\grave{\varepsilon}(\varepsilon + \alpha))) = \\
&(5\frown\grave{\varepsilon}(\varepsilon + 7)) = \\
&5 + 7 = \\
&12
\end{aligned}
$$

This is exactly parallel to the following result in the Lambda Calculus (by λ-conversion, etc.):

$$
\begin{aligned}
&(\lambda y \lambda x(x + y)7)5 = \\
&\lambda x(x + 7)5 = \\
&5 + 7 = \\
&12
\end{aligned}
$$

Moreover, as Frege (*1893*, §§ 34–35) himself also realized, the use of value-ranges and the sign '$\frown$' allows one to proxy the use of any second-level concepts (concepts that are true or false of functions) with an equivalent first-level concept that is true or false of value-ranges. Thus, Frege's paradigm case of a second-level concept, *generality*, represented by the universal quantifier, '$\smile^{\mathfrak{a}}\!\!-\phi(\mathfrak{a})$', which takes function *f*( ) to the True just in case *f*( ) has the True as value for all arguments, could in a way be 'reduced in level' to a first-level function Π( ), which has the True as value just in case its argument is the *value-range* of a function yielding the True for all arguments. (Frege himself does not define such a sign, but it is clearly possible in his system.) In the Lambda Calculus, where there is no distinction between functions and 'value-ranges', and the λ-abstracts themselves stand for functions, such a 'reduction in level' is unnecessary; therein, the function Π is taken as a *primitive*. The more usual notation for quantification, '$\forall x(\ldots x\ldots)$' is simply *defined* as '$\Pi(\lambda x(\ldots))$', and no variable-binding operators besides 'λ' are needed.

So there are certain definite ways in which an anticipation of something at least very much like a Lambda Calculus was present in Frege's work. Church is known to have studied Frege in quite some detail, and here, more so than Russell, there was likely even some amount of historical influence on *the details of* Church's work. There are differences to be sure; Frege did continue to maintain a difference between functions and value-ranges, the former understood as 'unsaturated entities', the latter as objects,[7] and when it came to functions themselves, Frege never did rid himself wholly of multi-place functions. Interestingly, in a November 1904 letter to Russell, Frege briefly did consider a notation for 'function abstraction', written with a rough-breathing accent (rather than a smooth-breathing) over the initial Greek vowel, such that '$\acute{\varepsilon}(\varepsilon + 7)$' would stand for the function whose value-range is $\grave{\varepsilon}(\varepsilon + 7)$. This rough-breathing abstract could then be followed by an argument (e.g. in '$\acute{\varepsilon}(\varepsilon + 7)5 = 12$'), but would not *need* to be. However, he rejects this approach as violating the distinction between function signs, which must represent the unsaturated nature of functions, and proper names, which stand for objects (*Frege 1980*, pp. 161–62). Here it seems Frege comes even closer to anticipating all of the essential details of the Lambda Calculus; the proposal seems to differ only by arbitrary matters of notation. Frege's rejection of the approach tells us something about his philosophy: how seriously he took the distinction between functions and objects, and the corresponding distinction

7 Church himself labeled Frege's understanding of functions as 'unsaturated' (*ungesättigt*) as 'somewhat problematical' (*Church 1951*, p. 101), and so it is not difficult to see why he moved in the direction he did. (See also 1951 p. 9.)

between function names and proper names. Russell's much more extended consideration of a similar approach, and his own rejection thereof, similarly tells us quite a lot about his changing views during the period from 1903 to 1905.

4. Russell's initial response to Frege

As is well known, Russell first became interested in symbolic logic through the work of Giuseppe Peano and his associates. His early logical works, for the most part, followed the trends and conventions of that school of logicians: he used Peanist notation, followed Peanist methods, and worked on Peanist problems. This was especially true while Russell was composing the *Principles of Mathematics* from 1900 – 1902; Peano can certainly be seen as the single greatest influence on Russell during this period. However, after finishing the body of *Principles* in mid-1902, Frege's works also began to have quite a lot of influence over him. Although he had been aware of Frege's work prior to this, it is only at this time that he had a chance to study Frege's work in detail. He was so impressed by the great overlap between their views, that he felt the need to make certain last-minute changes in the body of the *Principles* to accommodate Frege's insights (for the details, see *Grattan-Guinness 1996*), as well as include a special appendix discussing Frege, primarily in order to justify his position on the points at which they diverged. There are several points worth mentioning in this context from this discussion.

First, Russell (*1903*, p. 503) rejected Frege's 'curious' claim that truth-values are to be taken as denoted by complete statements, and insisted on his own theory of propositions as superior to Frege's division between thoughts and truth-values. Therefore, he also took pains to distinguish his notion of a *propositional function* from the Fregean notion of a concept. Frege treats concepts as entirely on a par with other, mathematical-style, functions that take an object as argument and yield any other object as value; the function corresponding to the predicate '... is human' is philosophically no different than a function such as that which 'the father of ...' stands for or the square-root function; it simply happens to have truth-values rather than numbers or people as value. For Russell, however, *propositional functions*, whose values are not truth-values, but propositions, are taken as more primitive than mathematical-style functions. Indeed, he insisted that the latter must be seen as derived from the former (*1903*, p. 508). Thus the truth of any statement of the form '$y = f(x)$', such as 'James Mill = the father of (John Stuart Mill)' is claimed to be derivative from a relational statement of the form '$\phi(y, x)$' such as 'James Mill fathered John Stuart Mill'.

Russell also took issue with Frege's account of the unsaturatedness of functions. Frege speaks of functions as the denotations of incomplete expressions; as Russell interprets this, metaphysically, functions are thereby depicted as what remains when something is taken away from a propositional complex. However, Russell claims that an entity cannot simply be taken away from a proposition without destroying the unity of that proposition (*1903*, p. 509). On Russell's own understanding of (propositional) functions, they are not incomplete, nor do they contain a 'gap' or 'empty space'; instead, a propositional function is a proposition-like unity that contains a variable, rather than a definite term[8] like Socrates or Humanity. For the early Russell, a variable is not a letter or anything linguistic, but is in the same

8 Russell (*1903*, p. 43) uses 'term' in a non-linguistic way to signify any entity that can occur as part of a proposition.

metaphysical category as what he calls a 'denoting concept', something which, when it occurs in a proposition, makes the proposition not about it, but about one or more entities to which it bears a special relation. When a variable occurs in a proposition-like construction, it 'ambiguously denotes' its different values; and hence what one has is not a constant proposition, but a propositional function.

For Russell of the *Principles*, for some proposition of the form ϕa, the derived propositional function is not some incomplete or gappy ϕ, or as Frege would write it '$\phi(\ )$'. It is, rather, an entity just like the proposition ϕa, except containing a ('real', as opposed to an 'apparent') variable rather than a definite term a. Russell explicitly denies the existence of incomplete entities such as Frege's $\phi(\ )$, which elsewhere he calls 'assertions' (*1903*, pp. 83–4, 505). Indeed, he says, 'the ϕ in ϕx is not a separate and distinguishable entity: it lives in the propositions of the form ϕx, and cannot survive analysis' (*1903*, p. 88). Russell is sometimes read here, wrongly, as denying the existence of propositional functions as entities. Instead, he is claiming that the structured remainder of the proposition over and above the entity to be varied, or, what amounts to the same, *the remainder of the propositional function without the variable*, cannot be seen as a single entity. If one attempts to 'pull apart' a complete proposition such as *Socrates is mortal*; one cannot divide it into two entities: Socrates and an 'assertion' made about Socrates: *( ) is mortal*. The latter 'cannot survive analysis'. If one wants to understand what *Socrates is mortal* and *Plato is mortal* have in common, one must consider them both as values of the propositional function *x is mortal*, where the function has a *variable* in place of a definite term, rather than simply an empty spot or gap to be saturated or completed. The inclusion of this variable within the structured whole is necessary to maintain its unity (*1903*, p. 106).

For early Russell, because propositional functions differ from propositions only in containing variables, the study of propositional functions cannot be separated from the study of variables. As variables are simply a special sort of denoting concept, Russell's theory of propositional functions is deeply interwoven with his theory of meaning and denotation. For this reason, the later topic figures more prominently in his discussion of functions than one might otherwise expect.

In the appendix, Russell also expressed his disagreement with Frege's related thesis that functions and concepts cannot be denoted by proper names or phrases that can operate as logical subject. According to Russell, anything that *is* can be made into a logical subject, and to suggest otherwise for any entity *A* is to contradict oneself, for the very statement of the position, '*A* cannot be the subject of a proposition', involves doing precisely what it claims impossible. If concepts are indeed entities at all, they must also be able to serve as logical subjects.

Lastly, Russell criticizes many aspects of Frege's theory of value-ranges, although he does express some admiration for other aspects, particularly the use of double value-ranges to capture the extensions of relations. Besides the obvious complaint that Frege's theory leads to contradiction, Russell's primary complaint is that, although Frege makes clear what the identity conditions for value-ranges are, he never precisely explains what we are supposed to take them to *be* in and of themselves (*1903*, p. 511). Value-ranges of concepts are not to be understood as collections or aggregates of the objects falling under the concepts (as Russell himself understood classes); instead, value-ranges of concepts are supposed to 'have their being' in the concepts. However, as objects, they cannot simply be *identified* with the concepts as functions. What, then, are they? This last complaint is indicative of Russell's characteristically metaphysical and philosophical approach to logic. It would not be enough for him to have a logical system that was free of contradiction and worked well formally; one needed a philosophically robust understanding of its metaphysical presuppositions, and how the various signs in that system related to objective structures in the world (i.e. for him, *propositions*).

While Russell is largely critical of Frege in the Appendix to the *Principles*, it is quite clear that his close study of Frege's work during this period had a large impact on him. Indeed, over the next few years, Russell's manuscripts show a distinctly Fregean influence in both doctrine and topic. Fregean themes, such as the attempt to understand the nature of functions, and the distinction between 'meaning' and 'denotation', dominated Russell's work during this period. Russell was of course interested in these topics before reading Frege, but his encounter with Frege seems to have newly plunged Russell into them. Of course, his primary occupation during this period was an attempt to find a philosophically adequate way of reconstructing his logical principles that solved the contradictions and allowed the logicist programme to proceed. Naturally, Russell was interested in the appendix Frege dedicated to Russell's paradox and added to the second volume of the *Grundgesetze*, published in late 1902. Despite that Frege's proposed fix is somewhat arbitrary and logically flawed, Russell's reaction seems to have been quite deferential; indeed, in a last-minute footnote added to the *Principles*, Russell claims that it contains what is probably the correct solution to the paradox (*1903*, p. 522). This remark appears rather striking, having been placed at the end of an appendix containing harsh criticism of Frege's very notion of a function and a value-range. Russell soon discovered problems with Frege's proposal, but during this period he certainly took Frege's views very seriously.

5. Function abstracts in Russell's early work

This brings us to May 1903. Earlier we made note of a letter written by Frege to Russell in which he considered and rejected a rough-breathing notation for function abstraction. This letter was written in response to a letter Russell sent Frege in May 1903. In that letter, Russell described a new approach to solving some of the difficulties plaguing their shared logicist enterprise. Therein, Russell suggested that they try to do away with classes altogether, and make do, instead, simply with functions. Rather than, for example defining a cardinal-similarity relation between two classes whose members stand in a $1-1$ correspondence, Russell suggested defining a cardinality relation for two (propositional) functions that would hold of functions that held of the same number of things, and so on. Doing away with classes at least solved Russell's paradox in the classes-of-all-classes-not-members-of-themselves form. The versions involving *predicates* and *propositional functions* that do not apply to themselves remained, but at least there was no need for separating individuals, classes, classes of classes, and so on, into distinct logical types, as Russell had begrudgingly and unhappily suggested in Appendix B of the *Principles*.

In his response, Frege points out that such an approach would need a way of placing names of functions in subject position of relations, etc. and that to be able to treat more complicated functions, it would need a notation for forming function 'abstracts', leading to the discussion of the rough-breathing notation. The need for an abstraction operator, however, was not news to Russell. Had Frege read closer, he would have noticed that Russell's letter began:

> I received your letter [of 21 May 1903] this morning, and I am replying to it at once, for I believe I have discovered that classes are entirely superfluous. Your designation $\grave{\varepsilon}\,\phi(\varepsilon)$ can be used for ϕ itself, and $x \frown (\grave{\varepsilon}\,(\phi(\varepsilon)))$ for $\phi(x)$. (*Frege 1980*, 158)

Notice here that the suggestion is not to do away with Frege's notation for value-ranges or classes, '$\grave{\varepsilon}\,\phi(\varepsilon)$', but, instead, to use this notation *for the functions themselves*. Russell's suggestion was to replace class abstracts with function abstracts. This is much clearer from Russell's manuscripts starting the time near when this letter was written (and long before he received Frege's reply).

We noted earlier the great influence Peano and his school had on Russell, especially early on. It is possible that Russell first encountered function abstracts in Peano's work. At one passage in the second edition of the *Formulaire*, Peano comes to something like functional abstraction, seemingly understanding it, *obscurely*, as the *inverse* of the application of function to argument. Employing the overbar notation that was used generally in his logic for inversion, Peano writes: 'we indicate by $a\bar{x}$ the sign of the function f, such that $fx = a$. Thus one has $(fx)\bar{x} = f$' *Peano 1897*, p. 277. In a later work Peano has the notation '$ux|x$' for functions, where the '|' is again described as a 'the sign of inversion' *Peano 1900*, p. 356. However, the notation was not much used, and was never clearly distinguished from similar notation used for the instruction to replace one variable or sign for another. (For more on this, see *Grattan-Guinness 2000*, pp. 245, 255). There is no evidence that these passages had any effect on Russell's thought.

It is much more likely, especially given the smooth-breathing notation he uses, the philosophical questions he raises about functions (discussed below), and how eager he was to share his 'discovery' with Frege, that the larger inspiration for Russell's use of function abstraction in May 1903 was Frege's work on functions and value-ranges. Russell outlined the basis for the new notation at the opening of a manuscript from this period bearing the Frege-reminiscent title 'Functions and objects':

> When any expression contains a constituent x, it is possible to conceive of x being replaced by y or z or any other term, without any other change being made in the expression. When this is done, something remains constant while the term in question is changed. This something is to be called a *function*. The expression containing x is called the *value* of the function for the *argument* x; if we substitute y for x, the result is the value of the function for the argument y. If ϕ denotes the function, $\phi|x$ will be used to denote the value of the function for argument x; and conversely, if X denotes an expression containing x, $\grave{x}(X)$ will be used to denote the function involved. Thus $\grave{x}(X)|x$ will be another symbol for X; and if we denote by Y what X becomes when y is substituted for x, $\grave{x}(X)|y$ will be another symbol for Y. Also $\grave{x}(\phi|x)$ will be another symbol for ϕ; and thus $\grave{x}(\phi|x)\}|x$ will be another symbol for $\phi|x$. (*Russell 1994*, p. 50)

Unfortunately, here again, we see evidence of Russell's characteristic sloppiness about use and mention. However, overlooking this, with just a few changes of notation, a similar passage could easily occur in the introductory passages to a text on the Lambda Calculus. If we use X schematically for a formula containing variable 'x' and Y for the same formula only containing 'y' where X contains 'x', then $\ulcorner\lambda x(X)\urcorner$ will stand for a function, such that, by λ-conversion, it will hold that $\ulcorner\forall y[\lambda x(X)y = Y]\urcorner$. Similarly, if '$\phi$' is a name of a function, '$\lambda x(\phi x)$' is another name for that function, etc. Russell's use of the sign '|' to stand for the application of a function to its argument appears somewhat as a curiosity here, and was perhaps modified from Peano's notation for substitution. The smooth-breathing notation, of course, comes from Frege, but note that Russell has substituted normal Roman letters for Frege's Greek vowels.

As Russell continues, the parallel to Frege's work on value-ranges runs deeper. Probably building on Frege's slight anticipation of Schönfinkel's approach to multi-place functions, Russell completes the thought:

> It will be found unnecessary to regard functions of two or more variables as radically different from functions of one variable; we shall find it possible to treat all such functions as functions whose values are themselves functions. (*Russell 1994*, p. 51)

In a manuscript dated from the same period entitled 'Primitive propositions for functions', Russell tries out new primitive propositions such as the following (*1994*, pp. 53–55):

$$\vdash: \dot{x}(X) = \phi . \supset . \phi|x = X \qquad \text{Pp.}$$

$$\vdash: \dot{x}(\phi|x) = \phi \qquad \text{Pp.}$$

And definitions such as the following:

$$\varepsilon = \dot{\phi}\{\dot{\xi}(\phi|\xi)\} \qquad \text{Df.}$$

$$x\mathrm{R}y = (\mathrm{R}|y)|x \qquad \text{Df.}$$

From which one derives results such as:

$$\vdash x\,\varepsilon\, f = f|x$$

This result allows functions to proxy for classes. Applying this approach to logical operators, Russell notes that '$p \supset q$' is really shorthand for '$(\supset|q)|p$', and '$p. \supset .q \supset p$' shorthand for '$[\supset|\{(\supset|p)|q\}]|p$'.

Here we can see that Russell's form of functional abstraction is significant steps closer to later Lambda Calculi than Frege's smooth-breathing notation. Multi-place functions have been purged from the logic entirely. The abstracts stand for functions-proper, and can be directly fed arguments. Unlike in Frege's system, and like the Lambda Calculus, the abstraction operation can bind function variables (as in Russell's definition of 'ε'). The use of the approach is foundational to the entire system. One could simply remove value-ranges and the associated axioms from Frege's system and be left with a perfectly functioning, consistent, higher-order logic. Here, what Russell is suggesting is not just an add-on that could be simply removed while preserving the core. Unfortunately, even in this period, passages such as those above are the closest Russell comes to stating explicit conversion, concretion or replacement rules (which would have been much easier for this system than for *Principia Mathematica*), so the development of the system was far from complete. Of course, none of these works were intended for public reading, and it is possible that he would have made up for such lacunae had he prepared something for publication, though unlikely given his later omissions.

Over the next two years, nearly all of Russell's logical manuscripts contain function abstracts used as terms. However, the notation he used changed continuously over the period, in a way that almost seems calculated to frustrate the modern interpreter. As we have seen, Russell originally used the smooth-breathing borrowed from Frege notation for functions, and aimed to do without classes altogether. In a letter to Russell from Whitehead in late April 1904, Whitehead made use of the circumflex notation such as '$\phi`\hat{x}$' that appears later in *Principia* for functions, adding, 'note the circumflex over the $\hat{x}$ marks that the variable x on the right hand side is only apparent'. Whitehead also replaced Russell's use of the vertical bar '|' for the application of a function to an argument with a raised inverted comma. Uncomfortable with Russell's abandonment of classes, in a letter written a month later, Whitehead also suggested a way of reintroducing classes by means of a relation sign 'Kl' such that 'u Kl $\phi`\hat{x}$' would mean that $u$ is the class defined by propositional function $\phi`\hat{x}$. By means of this notation, one could define a *class* abstract written '$\grave{x}\phi`\hat{x}$' for those functions for which there is a $u$ such that $u$ Kl $\phi`\hat{x}$.

Russell tended to follow Whitehead's lead in matters of notation, and during this period he begins to use the smooth-breathing notation again for classes rather than functions. For a while, he adopts the circumflex notation for function abstracts, and indeed sees it as in some ways 'more philosophically correct' (see, e.g. *Russell 1994*, p. 272). As we saw, in the *Principles*, Russell had suggested that a function should be understood as a proposition-like complex containing a variable rather than definite term at one or more places. While he sometimes doubted this original understanding of a function during this period, he often returned to it. Notation such as the smooth-breathing notation suggests that there is something in addition to the variable, the greater-than relation, and two making up the function written '$\grave{x}(x > 2)$'. Rather, the function is just the *meaning* of '$x > 2$'. By itself '$x > 2$' *denotes* the various values of the function; however, if we want a name of the function itself, we simply place the circumflex over the variable to suggest we wish to speak of the *meaning*, not the various complexes the meaning 'ambiguously denotes'. As Russell puts it at one point:

> The circumflex has the same sort of effect as inverted commas have. E.g., we say
> Any man is a biped;
>
> "Any man" is a denoting concept.
> The difference between $p \supset q. \supset .q$ and $\hat{p} \supset \hat{q}. \supset .\hat{q}$ corresponds to the difference between any man and "any man". (*Russell 1994*, p. 129)

Circumflexion is thus one way to speak of meanings rather than denotations. Like the inverted commas, which cannot be understood as a function that maps the denotation onto the meaning, we do not have here some function that operates on a *value* of a function and yields the function itself (as Peano's inversion notation would seem to have it). The sign for the function should not be *more complex* than the sign for one of its various values.

However, towards the end of 1904, Russell begins using a different notation for functions, more like the smooth-breathing notation, except with an initial variable written with a circumflex followed by an expression containing that variable (i.e. the very notation Church, before typesetting problems, had intended). Its first occurrence appears on a page in his working notes entitled 'Notation for functions', and he begins (*Russell 1994*, p. 240, see also 265–72):

$$\hat{z}(\phi`z) \text{ always instead of } \phi`\hat{z}$$

The reason for the new notation is not discussed, but it may be that Russell had some awareness of the ambiguities generated with the simpler circumflex notation. For a short period, Russell used this notation for function abstraction, alongside smooth-breathing notation for classes. As this new circumflex notation is the same as the notation used later on for *classes* in *Principia Mathematica*, and given that he *initially* used the smooth-breathing notation for functions, a present-day reader of Russell's manuscripts must take great care to avoid confusion.

6. Russell's changing views on the nature of functions

That Russell's notation changed during this period is of relatively minor interest. What is of more interest is the evolution of his views on the subject matter for which the notation was designed. True to form, Russell was not simply interested in putting forth a formal system or new style of notation; for him, a logical system or style of notation would be regarded as adequate only to the extent that it reflected the true nature of the entities in question. His consideration of these logical apparati comes always alongside discussion of functions, their nature, and how they occur (or fail to occur) within complexes such as propositions.

For Russell, there were two questions that continued to come to the surface concerning the nature of functions. They were not at all new for him; indeed, he had addressed them already in the *Principles*, but he again and again found occasions for reconsidering them. The first had to with the relative fundamentality of complexes and functions. The view of the *Principles*, we have seen, is that functions are themselves those complexes containing variables. Since the same complexes could exist save containing a normal term instead of a variable, there would be no way of explaining or defining complexes in terms of functions. Here, the notion of a complex is more fundamental than that of a function. In reading Frege, he had encountered the claim that functions should be understood as *unsaturated* (*ungesättigt)* or *incomplete*, and that the unity of complex 'judgeable contents' is to be explained by the cohering of functions with their arguments.[9] Despite the criticisms of Frege's views leveled in Appendix A of the *Principles* and elsewhere, Russell for while was attracted to the possibility of understanding the application of functions to arguments as 'the logical genesis of all complexes' (see, e.g., 1994, p. 50). He goes on:

> Thus functions, except in the case of the function whose value for the argument x is x itself, which we may call the identical function, are simpler than their values: their values are complexes formed of themselves together with a term. The logically correct course, therefore, would be to begin with the function ϕ, and proceed to its value $\phi|x$, not to start with X and proceed to $\grave{x}(X)$ (1994, p. 51).

9 Here I am using Frege's early terminology of the *Begriffsschrift* (1879). After the sense/reference distinction is in place, Frege still speaks as though the unity of complexes is derived from the unsaturated or incomplete nature of functions, but exactly how this is consistent with the remainder of his views is actually rather unclear. As Frege himself points out later in his career, at the level of reference, the value of a function is not *composed* of the function and argument (*Frege 1979*, p. 255). And at the level of sense, it is not clear that the incomplete or unsaturated entities can properly be construed as functions, as I have discussed elsewhere (*Klement 2002b*, pp. 65–76). So in what sense can a function be said to cohere with, be completed by, or form a whole with, its value? Church may well have been right in calling Frege's doctrine of the incompleteness of functions 'problematical'.

On this view, the functions, *on their own*, do not contain variables or anything else 'in' their argument spots; they instead, bind together with their arguments to form complexes. At first blush, this view seems absurd with regard to certain functions, such as the square root function; the value of this function for four as argument is two; yet surely, two is not a *complex* formed from four and this function. In this case, Russell would appeal to his theory of denoting: when the function and argument come together, they form what he called a *denoting complex*; and when this complex appears in a proposition, the proposition is not about it, but about some entity to which it is related—in this case, the number two. Russell calls functions resulting in denoting complexes 'denoting functions' and distinguishes them from propositional or 'undenoting' functions (*Russell 1994*, pp. 331 – 2, 374).

However, despite the possibility of providing some explanation for the metaphysics of propositions and complex meanings, Russell was dissatisfied with it philosophically. The abstraction notation itself, he thought, suggested that functions ought to be derived from complexes, not vice-versa. As he put the point when discussing the same dilemma later on:

> Is the function derived from the complex, or the complex from the function? The notation $\hat{x}(\phi\text{‘}x)$ suggests the former.
>
> ...
>
> ... if the function is *not* derived from the complex, it ought to be possible to speak of it otherwise than as $\hat{x}(\phi\text{‘}x)$; yet this not possible with particular functions. Suppose, e.g., we wish to speak of the function $\hat{x}(x \supset x)$. In language, we may call this function 'self-implication'. But this name indicates substantially the same process as is indicated by $\hat{x}(x \supset x)$. The function is the manner in which, whatever x may be, the constituents of $x \supset x$ are combined. And we cannot describe a way of combining constituents except by presupposing constituents. (*Russell 1994*, p. 265)

The attempt to take complexes as derived from functions fails to make sense of certain functions. Let 'p' represent some constant proposition. The formula '$p \supset p$', given what was said above, is shorthand for '$(\supset |p)|p$'. If complexes are formed from functions, this complex seems to be generated from the following procedure: the function $\supset$ binds together with the proposition p to form the complex $\supset |p$; this complex is itself a function, and can bind together again with p to form the complex $(\supset |p)|p$, i.e., $p \supset p$. Yet, it would seem that $p \supset p$ is not just a value of the function $\supset |p$, but also the function $\hat{x}(x \supset x)$. But there is no way to regard this later function as a constituent of $(\supset |p)|p$. For at least certain functions, Russell suggests, we must begin already with a complex in mind, and arrive at the function by analysis of the complex, that is to regard parts of the complex as replaceable by other constituents, resulting in a similar complex differing from the original only in certain specified ways.

On this way of looking at a function, a function corresponds to 'mode of combining' entities to form a whole; but itself is not a constituent of the whole that is formed. On this understanding, $\hat{x}(x \supset x)$ gives us a recipe for combining entities to form propositions such as $p \supset p$. However, it is not a constituent of the propositions so formed. (It could be regarded as part of the *meaning* of '$\hat{x}(x \supset x)|p$', but it is not a part of what this denotes, viz., $p \supset p$.) He writes:

> The mode of combination of the constituents of a complex is not itself one of the constituents of the complex. For if it were, it would be combined with the other

constituents to form the form the complex; hence we should need to specify the mode of combination of the constituents with their mode of combination, thus what we supposed to be the mode of combination of the constituents would be only a mode of combination of *some* of the constituents. In short, in a complex, the combination is a combination of *all* the constituents, and cannot therefore be itself one of the constituents. (*Russell 1994*, p. 98)

This, Russell hopes, will help provide some direction for aiming to solve the contradictions involving classes and functions. If $\hat{x}(x \supset x)$ is not a detachable *component* of $p \supset p$, then one could similarly argue that the 'function' $\hat{x}\{(\phi) : x = \phi . \supset \sim \phi`x\}$ involved in Russell's paradox is not a separable component of a proposition such as $(\phi) :\ a = \phi . \supset \sim \phi`a$, and perhaps not an entity at all. Indeed, Russell spent a great deal of effort trying to delimit a certain class of complexes from which one could *not* abstract a function. However, within this approach, he never quite found an adequate way of delimiting such a class.

The other question that resurfaced again and again had to do with statements of the form '$y = f(x)$' such as 'James Mill = the father of (John Stuart Mill)'. As we saw, in the *Principles*, Russell thought these to be derivative from relational statements of the form '$\phi(x, y)$' or 'James Mill fathered John Stuart Mill'. In the years to follow, Russell was somewhat undecided about this issue as well. After all, in this period, he could not pretend that functions with complete propositions as value were somehow fundamental. On his new view of the nature of multi-place functions, the form $\phi(x, y)$ had been replaced with the form $(\phi|x)|y$; the function ϕ here has other functions as value, not propositions. Earlier, in discussing Russell's consideration of the possibility that functions may provide the basis for explaining complexes, we noted that Russell does countenance functions whose values may in no way contain their arguments, and explained such functions in terms of the functions yielding denoting complexes. The function involved in the example given earlier, which we might now write in Russell's abstraction notation as $\dot{x}$ (the father of x), could be understood as an example of such a 'denoting function'. While he was not, at least at first, adverse to allowing such functions, per his usual practice, he attempted to reduce them to as few as possible. Eventually he came to the conclusion that they could all be reduced to two forms, both of which he understood as functions that themselves take *propositional functions* as argument.

The first was written as the inverted iota, $\iota\!\!\!\rotatebox{180}{}$ ℩, and it was understood as a function mapping the propositional function it takes as argument onto the sole argument to that propositional function that yields a true proposition if there is such an argument, and a chosen object, otherwise.[10] Using this device, in a certain sense, Russell maintained the core of his earlier view that propositional functions were to be taken as fundamental in that he would analyse '$y =$ the father of x' by beginning with the relation R, a function whose value for every argument x is a propositional function that itself yields a true proposition only for fathers of x as argument. He would then analyse 'the father of x' as '℩'(R'x)'. The other denoting function Russell at times took as pri-

10 Russell borrowed the inverted iota from Peano, who used it for a function mapping a singleton class onto its single member, although Russell's understanding of it here also owes a lot to Frege's sign '\' from *Grundgesetze* §11, which stands for a function that has as value, if its argument is the value-range of a concept true of only one object, the sole object that falls under the concept, and a chosen object otherwise.

mitive was the function that maps a propositional function onto the class of entities satisfying that function. However, as we have seen, at times, Russell thought classes could be done away with entirely, and at other times, he considered the function to be in fact definable in a similar way to the father case using the inverted iota and Whitehead's relation Kl discussed above. However, Russell was never quite happy with the function ℩, and it bothered him to have only a single function that worked so differently than the others he had on the table. I have discussed his dislike of this use of the symbol ℩ elsewhere (*Klement 2002a*). Of course, he finally managed to purge his logic completely of denoting functions in 1905 when he came across the theory of 'incomplete symbols' of 'On Denoting' and the possibility of defining away both descriptive phrases and class-terms in context.

7. Abandonment of the approach

Russell's proto-Lambda Calculus never saw the light of day during his lifetime. The interesting question is why. We have just seen that Russell was not entirely satisfied with its treatment of what he called 'denoting functions'. However, in his treatment of *propositional* functions, the question remains as to why Russell did not continue to make use of an abstraction operator and treat multi-place functions as functions onto other functions. It is now well known that soon after developing the theory of descriptions in 1905, Russell began work on his so-called 'substitutional theory' in which neither classes, nor relations-in-extension, nor even propositional functions are taken as genuine entities, but are instead proxyed by means of an intensional logic of propositions and the notion of 'substitution' of entities within a proposition (see, e.g. *Russell 1905, 1906, 1973*). In many ways, the substitutional theory is almost diametrically opposed in metaphysical outlook to Russell's function-heavy theories of 1903 and 1904. One makes functions central; the other banishes them altogether. However, the development in Russell's mind from the one theory to the other is actually rather more direct and natural than one might suspect. Let us examine, then, how it happened.

We saw in the last section that although Russell took seriously the supposition that the application of a function to an argument might be the basis for the unity of propositions and denoting complexes, he eventually returned to the conclusion that the reverse was true: at least many functions could only be gotten at by analysis of complexes, and that functions, even propositional functions, are not constituents of their values. On Russell's view, this view was consistent with, suggested by, and perhaps even *demanded* by, the abstraction notation, $\hat{x}(\ldots x \ldots)$. Here one begins with a complex containing one (or more) variable as a constituent; that variable is then highlighted so that one does not consider any particular value of the variable, nor what the complex comes when the variable takes on a value, but the function itself. If complexes are prior to functions, then it is not the application of function to argument that generates complexes. Rather, it is through one or more entities, and specifically, one or more entities we would now call 'universals', occurring in a proposition 'as concept' (i.e. predicatively) rather than 'as term' (i.e. as logical subject) as he had put in the *Principles* (*Russell 1903*, pp. 44–46). However, one or more entities coming together with certain properties or relations to form a proposition is not now to be understood as the application of function to argument, on pain of circularity.

Russell then realized that there was something strange about the notation he had been using for functions. Here the difficulty is with the part of the symbol for an abstract that appears *after* the circumflected variable; that is the '$\ldots x \ldots$'. part of '$\hat{x}(\ldots x \ldots)$'. This should be the name of the complex from which the function is derived. It itself should not be understood as 'put together' from a function and its argument. However, in stating the general rules for the notation, Russell had hitherto often used abstracts such as '$\hat{x}(\phi\text{‘}x)$', but notice that the interior formula '$\phi\text{‘}x$' itself represents the application of a function to an argument. (Indeed, it would seem, the function ϕ would seem to be the very function the abstract is supposed to name.) Russell explained the oddity of the situation as follows:

> The notation $\hat{z}(\phi\text{‘}z)$, thought admirable when $\phi\text{‘}z$ is replaced by some particular complex, is open to grave objection when used to denote a general variable function. For it suggests that the function is derived from $\phi\text{‘}z$, whereas $\phi\text{‘}z$ already involves ϕ, which is the function. (*Russell 1994*, p. 269)

Russell began to feel that the very notation '$\phi\text{‘}x$' was in some sense problematic. If the 'ϕ' in '$\phi\text{‘}x$' already stands for the function, then it already contains the variable as such. A value of this function comes from replacing the variable with one of its values. In a period in which he vacillated between using Whitehead's simpler circumflex notation and the revised notation noted above, Russell commented on the difficulties as follows:

> According to what seems the best view, $\phi\text{‘}x$ is short for $(\phi\text{‘}\hat{x})\frac{x}{\hat{x}}$, and is not identical in *meaning* with the straightforward complex containing x that $\phi\text{’}x$ *stands for*.
>
> But this involves a new difficulty: we ought to write ϕ simply, instead of $\phi\text{‘}\hat{x}$, or else $\phi\text{‘}\hat{x}$ would have to be replaced by $(\phi\text{‘}\hat{x})\frac{\hat{x}}{\hat{x}}$, or ϕ will be meaningless. Thus in fact, when ϕ appears in the function-position, as in $\phi\text{‘}x$, it must stand for the function itself, *i.e.* for the complex with $\hat{x}$ in place of x. Thus, e.g., suppose $\phi\text{‘}x =_x . \; x = x$, then $\phi = (\hat{x} = \hat{x})$, *i.e.* $\hat{x}(x = x)$. Thus $\phi\text{‘}\hat{x}$ is wrong. We have $\phi\text{‘}x = \phi\frac{x}{\hat{x}}$. (*Russell 1994*, p. 256)

This is a difficult passage, one which Russell surely would have made clearer if he had intended it to be read by anyone except himself.

I interpret it as follows. If a function is to be understood as a complex containing a variable, written '$\phi\text{‘}\hat{x}$', then a certain value of that function for some argument x, hitherto written $\phi\text{‘}x$, is best understood as that which the complex becomes when the argument is substituted for the variable, and thus is best represented as $(\phi\text{‘}\hat{x})\frac{x}{\hat{x}}$. But if the application of a function to an argument is best understood by means of this sort of *substitution* of argument for variable, that is, if we should regard the notation '$\phi\text{’}x$' as shorthand for '$(\phi\text{‘}\hat{x})\frac{x}{\hat{x}}$', then that very notation should not form a part of the notation for the complexes from which we form the function names, for fear of generating an infinite regress. Because '$(\phi\text{‘}\hat{x})\frac{x}{\hat{x}}$' contains the very sort of notation it is supposed to explicate on the left side where the function abstract appears—recall that $\phi\text{‘}\hat{x}$' is a notation variant of '$\hat{x}(\phi\text{‘}x)$' —one would need to replace the '$\phi\text{‘}\hat{x}$' that appears in '$(\phi\text{‘}\hat{x})\frac{x}{\hat{x}}$' with '$(\phi\text{‘}\hat{x})\frac{\hat{x}}{\hat{x}}$', resulting in '$((\phi\text{‘}\hat{x})\frac{\hat{x}}{\hat{x}})\frac{x}{\hat{x}}$'. This, of course, gets us precisely nowhere.

At the end, then, Russell suggests a different way of looking at things. The function itself, rather than being written in abstract form '$\phi\text{‘}\hat{x}$' or '$\hat{x}(\phi\text{‘}x)$', should be written simply as 'ϕ'. Its notation should not presuppose application of function to argument. This is precisely what we should expect given the metaphysical picture: the function itself is a complex containing a variable, but the complex is not formed from composing the function and the variable, so that notation is inappropriate in forming a name of the function. The function itself then should just be represented in a simpler way, one that does not presuppose application of functions to arguments. Here Russell simply uses 'ϕ', although for a more specific example it would be appropriate to use a different sign such as $\hat{x} = \hat{x}$ ', which contains no occurrences of applying functions to arguments. (Identity here is understood as the *relation*, occurring as *concept*, and giving this complex its unity, not as a two-place function.) It is then just presupposed that the complex this stands for contains some variable. When we wish to proceed to one of the values of this function, abbreviated 'ϕ'x' we write '$\phi \frac{x}{\hat{x}}$'.

At this point Russell appears to be poised rather precariously between the notation he had been using involving functional abstracts and the later substitutional notation. Still, there is nothing in the above to rule out using abstracts completely. As Russell himself had put it, there does not seem to be anything wrong with the abstraction notation when one is treating some *particular* complex, as with '$\hat{x}(x = x)$'. The difficulty only seems to come in with a formula such as '$\hat{x}(\phi$'$x)$', in which the function itself already seems presupposed by the internal 'ϕ'. However this does not in itself seem to be an insurmountable obstacle to the notation. However, in Russell's eyes it seemed to be a slightly larger problem, as he thought something like '$\hat{x}(\phi$'$x)$' would be necessary in stating basic principles for the notation. To state a principle dealing with conversion, Russell's inclination would be to write in a form such as:

$$\vdash \hat{x}(\phi\text{`}x)y = \phi\text{`}y$$

But this of course involves the questionable sort of notation. To the modern reader, one cannot help but feel that Russell here would be greatly helped by the contemporary distinction between object-language function variables and schematic letters or metalinguistic variables. The above would be better stated in schematic form. What we want on the left half of the equation is any particular formula involving 'x' (standing for any complex containing a variable), and on the right, we want the same formula save having 'y' instead of 'x'. Yet without the concept of a schematic letter, Russell was forced to view the above 'ϕ' as a genuine functional variable, and began to doubt the very basis of the notation.

As these sorts of difficulties cropped up again and again for Russell during this period, we find him trying various sorts of approaches. At points we see him trying to make up for the above deficiency by using notations along the lines of '$\hat{x}(X)$' or '$\hat{x}(p)$', where he was forced to read the 'X' and 'p' as *object-language* variables ranging over complexes. More and more we see him trying out various substitution-style notations. Earlier we saw the notation $\phi \frac{x}{\hat{x}}$. for the value of function ϕ for argument x. Here the ϕ itself is supposed to represent the whole proposition-like complex containing the variable $\hat{x}$. Realizing that the use of the letter 'ϕ' seems to be most appropriate when following the traditional style of notation that would put this instead as '$\phi(x)$' or suchlike, Russell begins to start using notations such as the following:

> The indefinable $p \frac{x}{\text{Var}}$ has the following meaning: If p is a mode of combination containing the variable-as-such '$\hat{x}$', then $p \frac{x}{\text{Var}}$ is to mean the result of substituting x for $\hat{x}$, i.e., the value of p for the case of x; otherwise $p \frac{x}{\text{Var}}$ is to mean $(s). s$. (*Russell 1994*, p. 275)

Because the function, or more precisely, the 'mode of combination' is now understood as a propositional *complex* containing the variable, it is appropriate to use a propositional variable for it. Elsewhere, Russell uses a similar notation save with a 'C' instead of a 'p' to stand for the complex in which the substitution is to be effected (see *Russell 1994*, pp. 361 – 66). Russell has not abandoned completely his notation for abstracts, as the '$\hat{x}$'' in the quotation above indicates. Indeed, if we were to deal with some particular instance of this notation, the 'p' might be replaced by some particular proposition-like complex in which the 'variable-as-such' appears, e.g., '$\hat{x} = \hat{x}$'. (It is rather striking that in all this philosophical rumination about substitution of entities at the ontological level, Russell apparently still did not see the need to formally state rules of replacement or substitution for variables at the linguistic or notational level. This only serves to further indicate Russell's greater concern for metaphysics than for language.)

This approach, which Russell began in late 1904 and worked on through early 1905, was relatively short-lived, as it was wedded to certain aspects of his earlier theory of denoting which he abandoned upon discovery of the theory of descriptions. The above explication of '$p\frac{x}{\mathrm{Var}}$' presupposes that, in use, p will usually be a complex containing 'the variable-as-such, '$\hat{x}$' '.[11] Variables are therefore still taken as entities on par with denoting concepts. Second, even when the function we are treating in this way is a straightforward propositional function rather than a denoting function, the notation requires a meaning/denotation separation. In this notation, if 'a' is some constant, then '$(\hat{x} = a)\frac{a}{\hat{x}}$' will stand for the same proposition as '$(\hat{x} = \hat{x})\frac{a}{\hat{x}}$', but these two expressions nevertheless seem to differ in meaning.

After abandoning the meaning/denoting distinction and the theory of denoting concepts/denoting complexes, these aspects of the approach obviously needed to be changed as well. After 'On Denoting', Russell does not seem quite sure what to make of the metaphysical nature of variables nor how (or even if!) they occur in propositions, but he surely no longer regards them as operating like denoting concepts.[12] Certainly, then he cannot understand a 'function' or 'mode of combination' as a proposition-like complex containing one or more variables, where the variables are understood as before. However, Russell realized it was possible to do all the work of functions by beginning with determinate propositions and expanding the notion of substitution to allow substitution for constant elements, not just 'variables'. That is, rather than the form: $p\frac{x}{\mathrm{Var}}$, where, for the substitution to take effect in the normal way, p must be supposed to be a complex containing the 'variable-as-such' like $\hat{x} = \hat{x}$, Russell focused instead on a more general form $p\frac{x}{y}$ where p could be any proposition, and y could be *any* entity occurring within it.[13] Rather than beginning with the complex $\hat{x} = \hat{x}$, containing this mysterious thing, the 'variable-as-such',

11 Russell had, as early as 1903, considered a broader notion of substitution in complexes written $p\frac{x}{y}$ which stood for the result of substituting entity x for y in p; where y might be a particular entity in complex p; that is if p were the proposition that *Socrates is bald*, if we substitute Plato for Socrates, then we arrive at *Plato is bald*. However, problems were encountered when the complexes in which the substitution were to be effected were denoting complexes, since it was unclear whether the substitution should be effected in the *meaning* of these complexes or their denotations. Either way had its problems (see, e.g. *Russell 1994*, pp. 308 – 9). The approach discussed here is more narrow; it deals only with complexes containing the variable, and substitutions for the variable. Here it is obviously the *meanings* in which the substitutions are taking place, as indicated by Russell's suggestion that the combination contain the 'variable-as-such, $\hat{x}$ —recall that the circumflex is used to speak about the meaning as opposed to denotation.

12 For discussion of Russell's new theory of the variable after 'On Denoting', see *Landini 1998*, Ch. 3. However, Landini and I disagree about the ontological status of variables prior to 1905 (in informal discussion).

we begin with a perfectly respectable proposition like $b = b$; to get to another proposition in which a different entity is put together with this same 'mode of combination' we consider the substitution $(b = b)\frac{a}{b}$. Through a similar process, we can get any arbitrary proposition of this form, i.e., any proposition we previously would have described as a value of the same propositional function.

What, however, of the other way in which this approach seems at odds with the new theory of denoting? That would seem to remain. Consider the two substitutions: '$(p \supset q)\frac{p}{q}$', and '$(q \supset q)\frac{p}{q}$'. These two expressions stand for the same proposition, although they seem to differ in *meaning* somehow. Invoking the resources of his new theory of descriptions, Russell was able to avoid this difficulty as well. Rather than taking the results of such substitutions, and expressions of the form '$p\frac{x}{y}$' as primitive; Russell chose instead of begin with a complex *relation*, written '$p\frac{x}{y}!q$', which means that q results from the substitution of x for y in p. Simpler constructions of the form '$p\frac{x}{y}$' are then treated as 'incomplete symbols' of the form, '$(\imath q)(p\frac{x}{y}!q)$' which would need to be treated in context using the mechanics of the theory of descriptions (see *Russell 1905*, p. 7, *Russell 1973*, p. 169). The apparent differences in meaning between expressions with the same apparent denotation are explained away in the same way as with 'the author of *Waverly*' and 'the author of *Ivanhoe*'.

The basic idea of this new approach eschewing functions altogether was first described, as one of a number of alternative approaches, in Russell's paper 'On some difficulties in the theory of transfinite numbers and order types', probably written in late October and early November of 1905 and published the following year by the London Mathematical Society. By December, this new 'substitutional theory', as it has come to be called, was more fully developed as a logical system (see, e.g. *Russell 1905*). Russell continued to work on the substitutional theory, in various guises, for the next two years. It is in many ways a natural outgrowth of his struggles to come to terms with the nature of functions in 1903–1905.

The change must indeed have been a welcome one, because the mature theory of substitution, unlike that from late 1904 and early 1905, provided a ready-made solution to the paradox, because it has *done away* with functions as entities altogether. In the late pre-'On Denoting' theory, although one arrived at the *values* of functions through the notion of substitution, the functions themselves were understood as specific kinds of complexes, and although those complexes were not understood as *constituents* of their *values* in the typical cases; they could still be constituents of *some* complexes and constitute arguments to other functions or even to themselves. We take p as the 'mode of combination' $\hat{q}(\sim q\frac{q}{\mathrm{Var}})$, and ask whether the proposition that results when we substitute p for the variable it contains, viz., $p\frac{p}{\mathrm{Var}}$ is true, we arrive at the same familiar contradiction. On the new approach, however, there is no way to construct a *constant* proposition with the same features. There are no functions; only constant propositions and the new four-place relation. The work of functions is proxyed by what Russell calls a 'matrix'; a pair consisting of a proposition and an entity to-be-substituted-for in that proposition, but matrices are not themselves entities, but 'logical fictions', and are not themselves values of the variables of the logical system, and Russell's contradiction is avoided (see, e.g. *Russell 1973*, pp. 170–73).

13 This broader notion of substitution was not an entirely new idea, as discussed in note 11. The worries about denoting complexes ceased to be a problem after 'On Denoting', so it was natural to return to this approach at this time. After 'On Denoting', propositions were the only type of complex he needed to allow.

Of course, once Russell had started down a path in which functions were to be treated as non-entities, function abstracts disappeared from his logical notation, and the work he did anticipating the Lambda Calculus was abandoned. If, like Russell, we want our logical notation to reflect the structure of that which we're using it to represent, then it is inappropriate to use some sort of abstraction notation—whether that notation uses smooth-breathings, circumflexes or lambdas—to form names of functions unless functions are genuine entities. To Russell's mind, the notation, '$\hat{x}(\ldots x \ldots)$' or '$\ldots \hat{x} \ldots$'. suggests a complex containing a variable-entity; after abandoning such entities in his metaphysics, he similarly abandoned the notation. After December 1905, Russell's logical manuscripts no longer contain function abstracts as a part of what would seem to be the 'official symbolism'.

This is not to say that something of Russell's anticipation of Lambda Calculus-esque functional abstraction did not survive into Russell's substitutional theory (and beyond). Earlier we discussed, for example the Lambda Calculus approach of treating multi-place functions as functions onto other functions. Of course, in the substitutional theory, there are no functions at all, so certainly this view cannot be quite preserved. However, in the substitutional theory, multi-place functions are proxyed through the notion of taking the result of one substitution and performing another substitution within the result (for details, see *Landini 1998*, pp. 132–5), which, I suppose, is as close as one could come to a substitutional-analogue of the Lambda Calculus approach. Of course, the substitutional theory itself did not see the light of day until after Russell's death, so with the exception of the remnants of such an approach that appear with the circumflexion notation in *Principia*, Church and others more or less had to explore the notion of functional abstraction without tutelage from Russell.

8. Conclusion

To sum up: in 1903, after he had done a close study of Frege's logic and inspired by Frege's *Wertverläufe* notation, Russell hit upon a sort of logical system of functional abstraction operating very much like modern Lambda Calculi. However, in Russell's mind, this style of logical calculus and the notation it invoked went hand in hand with a certain understanding of the metaphysical structure of propositions and related complexes. However, he was never able to fully work out all the problems as he saw it with this metaphysical theory—not the least of which the persistence of the contradiction involving propositional functions not true of themselves—and in late 1905 he abandoned both the logical notation and metaphysical theory in favour of one that eschewed functions altogether.

Russell's persistent occupation with metaphysics stands out in strong juxtaposition from the work of almost all technical logicians to follow. Not only is virtually all of the recent work on the Lambda Calculus done completely independently of metaphysical inquiry, much of it does not even attempt to maintain the original intended interpretation of the abstracts as standing for functions in any straightforward sense. Even Church, the originator of the Lambda Calculus as we know it, who was no stranger to philosophy, did not fret overly much about the underlying metaphysics. Although, *like Russell before him*, he hoped to develop the system into a type-free foundations for arithmetic, when he realized that the type-free Lambda Calculus was formally flawed, rather that questioning the entire style of notation, he opted instead for adding a theory of types to dodge the formal difficulties in his foundational work (as in *Church 1940*), and also worked on the type-free Lambda Calculus as a system for treating functions independently from a system of logic (as in *Church 1941*). The early Russell, himself usually credited as the originator of type-theory, would have seen no value in a system invoking a system of types independently of a metaphysical theory explaining why the system of types was necessary and philosophically well-grounded. According to his own early metaphysical outlook, 'whatever is, is one', and there must therefore be a type that subsumes all entities (see, e.g. *Russell 1994*, p. 51). Some functions must allow any entity, including themselves, as argument (*1994*, p. 52), in violation of the sort of type-strictures imposed by Church. One cannot imagine Church having much patience for this sort of argument. However, that Russell refused to accept any theory until he found a theory that was consistent with his metaphysical scruples, is, I think, a testament to his philosophical integrity.[14]

Of course, Russell's metaphysical views are not inviolate. Indeed, many of the arguments that lead him to abandon function abstraction seem somewhat idiosyncratic and could certainly be called into question. Yet there is certainly something to be said for being wary of too easily positing functions as entities. I do not think it is pure coincidence that the most common forms of logic and set theory in use these days *do not* take functions as central or fundamental. For example, in most contemporary set theories, functions are reduced to sets of Wiener-Kuratowski ordered pairs. Indeed, contemporary Church-style Lambda Calculi seem almost unique in taking functions as more primitive than sets, classes and predicates, etc. I do not mean to suggest that no coherent metaphysical picture could be told to underwrite this approach. But it might indeed be quite difficult. When Church does wax philosophical, he seems more attracted to a Fregean metaphysics and semantic theory than to Russellian doctrines. This may provide some help, but perhaps not as much as one might think. Church himself has called Frege's theory of the incompleteness of functions 'somewhat problematical' (*1951*, p. 101), but it is this feature of functions that Frege himself used to justify his 'levels'-distinction and block the functions version Russell's paradox (*Frege 1980*, pp. 132–3). Does Church have any justification for his types of functions, apart from providing a formal dodge to the paradoxes? If not, as philosophically minded logicians, how seriously should we take his Lambda Calculus?

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14 For more on how the substitutional theory and later developments in Russell's logic are consistent, even motivated by, some of metaphysical doctrines Russell adopted as early as the *Principles*, see *Landini 1998*.

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