

FREGE AND THE LOGIC OF SENSE
AND REFERENCE

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Preface

This work is a light revision of my Ph.D. dissertation, originally entitled, “Redressing Frege’s Failure to Develop a Logical Calculus for the Theory of *Sinn* and *Bedeutung*,” written at the University of Iowa in 1999 and 2000. Sanity has prompted me to change the title.

The idea for the work first began to emerge while taking a graduate seminar taught by Gregory Landini on the development of Bertrand Russell’s philosophy in the years between *The Principles of Mathematics* and *Principia Mathematica*. Landini’s work and that of others is beginning to generate renewed interest in this brilliant period of Russell’s career. It certainly got my attention. It seemed to me then (and still does) that Russell’s writings during these years, most of them unpublished during his lifetime, represent such an adroit and engaged effort to develop a workable theory of the nature of intensional entities (“propositions,” “propositional functions,” “universals,” etc.) that it has taken mainstream analytic philosophy a full century of development to come close to matching Russell’s profound insights into the issues involved. That Russell changed his mind so often during this period, and that he held off putting his multitude of ideas into print until he could solve those difficulties with his views that in all likelihood he alone was subtle enough to discover or appreciate, is a testament to his intellectual integrity.

Despite this appreciation for Russell, I was at the time more sympathetic in general to a broadly Fregean theory of meaning. Naturally it occurred to me to ask to what extent the same concerns and insights Russell had in attempting to work out the details of a theory regarding the nature and make-up of his “propositions” would be mirrored in an attempt to work out the details of a theory regarding the nature and make-up of Frege’s “*Gedanken*” (and *Sinne* generally). This proved to be a very interesting direction of study. I was of

course delighted to find that some of these issues arose in their correspondence following Russell's discovery of the contradiction in Frege's logical system, but disappointed that neither philosopher was able to make his views entirely clear. Their discussion of the paradox from Appendix B of the *Principles of Mathematics* particularly fascinated me, because, though they were arguing for contrary positions, both seemed right. Frege seemed right that Russell's formulation of the paradox made use of potentially questionable aspects of Russell's own doctrines that are not shared within Frege's doctrine of *Sinn* and *Bedeutung*. But Russell seemed right the Cantorian construction lying at the heart of the paradox remained. (My ultimate conclusions on this issue can be found in Chapter 6.) It turned out that a full resolution of this impasse requires developing quite a number of aspects of Frege's theories beyond what Frege himself tells us. Given the questions to be answered, and the nature of Frege's overall philosophy, these concerns naturally evolved to take on the form of thinking about how Frege would have developed a system of intensional logic for the theory of *Sinn* and *Bedeutung*.

Hence, the present volume and both its old, and its new, titles. But make no mistake, the issues that prompted the present work are not solely or even fundamentally issues for logicians. They are metaphysical issues: *What is the nature of intensional entities (if there are any)? How are they constituted? Do they have parts? What are their parts? How many are there? What are their identity conditions? On what does their existence depend? What is their relation to the mind? to language?*

I have made some small changes to the text since the dissertation was defended last summer, but only those I could make without substantial rewriting. There are still quite a few ways in which the present text could be improved, and doubtless even a few outright flaws. If I spent another six months revising, I could probably correct the ones I have noticed so far. By then, however, I will surely have discovered others, and the process would continue à la one of Zeno's paradoxes. However, let me briefly mention a few areas of concern.

First, I fear that Chapter 3's discussion of certain controversial areas of exegesis with regard to Frege's understanding of *Sinne* (e.g., the nature of incomplete *Sinne*, the compatibility of the context principle with compositionality, etc.) may appear somewhat unsophisticated, especially compared to some of the more recent and detailed studies of these issues that have appeared in the recent secondary literature. Although I maintain all the conclusions reached in the chapter, certainly many of the arguments given could be developed further and could be made to track better the changes in Frege's views at different stages in his career. Any of the difficult interpretative issues broached in that chapter could easily make a book unto itself. My main aim for

the chapter was actually quite modest; I simply wanted to provide a limited defense of certain features of the logical system developed in Chapter 5 that might otherwise be thought to be based on a questionable reading of Frege. However, I urge the reader not to make up his or her mind about my interpretative stances taken in Chapter 3 without considering the ramifications a contrary stance would have for the attempt to develop a workable logical system. I made attempts to develop alternative expansions of the system of the *Grundgesetze* that adopted, e.g., the sense-function view of the senses of incomplete expressions, or a strong reading of the context principle, but found such expansions either nearly impossible to formulate or clearly unsuited for such indispensable tasks as solving the morning star/evening star belief puzzle.

Second, the task of Chapter 4 was to discuss previous attempts at formalizing the *Sinn/Bedeutung* distinction within a logical calculus. My attention was fixed narrowly on Church, and I unduly neglected some very important work by Pavel Tichý. While I was aware of his work, and even cite it from time to time in the present text, I should have examined it more carefully for a number of reasons. Firstly, Tichý discusses a number of problematic features of Church's Logic of Sense and Denotation, and argues for a very different form of "transparent intensional logic." Some of Tichý's arguments could equally be directed against my own system from Chapter 5. While I do not believe Tichý's arguments are conclusive, they deserved an explicit response. Secondly, Tichý himself argues that some sort of ramification is needed within the theory of *Sinn* and *Bedeutung*, though for reasons largely independent of those considered in this book. Tichý does not seem as pessimistic when it comes to providing philosophical justification for ramification, though it may be that his notion of a *rigid presentation* would provide some help in this regard. (Compare, e.g., my discussion of the non-rigidity of *Sinne* and its relevance for the justifiability of ramification on pp. 224-6.) The reader is strongly encouraged to consult his work and compare it to mine.

Similarly, the choice of Searle and Bergmann as the philosophers discussed in addition to Russell in Chapter 6 may appear somewhat odd. Surely, there are other philosophers whose work would be as deserving, if not more deserving, of mention. In retrospect, I can see that my choice was brought about by the peculiarities of the intellectual environment I found myself in at the time. Nevertheless, I hope the discussion of these philosophers suffices to demonstrate that the issues discussed with regard to Frege's views in this book have a bearing for quite a variety of different theories with regard to the nature of meaning. I invite the reader to consider for him or herself how such concerns might be raised for other philosophical positions neglected here.

Lastly, the logical system developed in Chapter 5 takes the form of an axiomatic system, and the axioms are simply stated and justified informally. No

attempt is made to develop a system of formal semantics to support or justify the axioms. This may seem somewhat anachronistic given the fruitful development of formal semantics in logic in the past century. To some extent, that I was working within the context of Frege's philosophy and the logical system of *Grundgesetze* made my approach more natural given its similarity to Frege's own. But surely, this response might be thought inadequate; I have access to many resources that were unavailable to Frege in 1893. However, in my defense, let me say the following, (1) I knew the system I was developing was inconsistent; therefore, I knew that there could be no completely satisfactory semantics for it, and (2) I know of no very *natural* way even to go about, e.g., developing "models" for this sort of logical system; those I can imagine are largely artificial and seem to involve problems as difficult and controversial as any problems they might aid in solving. Perhaps an astute reader can do better.

There are also some aspects of the notation and symbols I adopt that may seem strange, even humorous. (How often does one see the spades playing card symbol used as part of a logical derivation?) Many such oddities are due to the turns and twists of the history of the work. I have let them remain.

The vast majority of the research and writing of the dissertation was made possible due to the support of a Seashore-Ballard Dissertation Fellowship granted by the University of Iowa Graduate College for the 1999-2000 academic year, for which I am very grateful. I must also express my deep gratitude to the many people both at Iowa and elsewhere who gave me valuable advice, in terms of both the content and the presentation of this work, including Richard Fumerton, Laird Addis, David G. Stern, Gregg Oden, C.A. Anderson, Pieranna Garavaso, Thomas Williams, Evan Fales, Diane Jeske, and others I should not have forgotten to mention. My chief debt is, of course, to my thesis advisor, Gregory Landini, for endless help and inspiration during a very busy year of his professional life. My new colleagues at the University of Massachusetts have also been very supportive, and while their influence came only after the major theses of the work were conceived and developed, they have certainly helped make it possible to dot the i's and cross the t's. The strengths of this book are due largely to those mentioned; the faults are certainly entirely my own.

This book is dedicated to the memory of my mother, Anthea Klement, who, I'm sure, would have loved this book whether she understood it or not, and whom I miss very much.

K.C.K.
Amherst, MA
April 2001

Abbreviations

References to Gottlob Frege's works are cited in the text with the abbreviations listed here. Because three of the works named below are collections, the reader should refer to the bibliography to determine which piece in the relevant collection is being cited.

- BL:* *Basic Laws of Arithmetic: Exposition of the System*, trans. and ed. Montgomery Furth (Berkeley: University of California Press, 1964).
- BS:* *Begriffsschrift, a Formula Language, Modeled upon that of Arithmetic, for Pure Thought*, in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, ed. Jean van Heijenoort (Cambridge, Mass.: Harvard University Press, 1967), 1-82.
- CP:* *Collected Papers on Mathematics, Logic and Philosophy*, ed. Brian McGuinness (New York: Basil Blackwell, 1984).
- FA:* *Foundations of Arithmetic*, 2d ed., trans. J. L. Austin (Evanston: Northwestern University Press, 1980).
- GG:* *Grundgesetze der Arithmetik*, 2 vols. (Jena: H. Pohle, 1893-1903).
- PMC:* *Philosophical and Mathematical Correspondence*, ed. Gottfried Gabriel et al. (Chicago: University of Chicago Press, 1980).
- PW:* *Posthumous Writings*, ed. Hans Hermes, Friedrich Kambartel and Freidrich Kaulbach (Chicago: University of Chicago Press, 1979).

FREGE AND THE LOGIC OF SENSE
AND REFERENCE

The Need for a Logical Calculus for the Theory of *Sinn* and *Bedeutung*

INTRODUCTION

A full century after the publication of his *Grundgesetze der Arithmetik*, Gottlob Frege (1848-1925) remains among the most discussed, articulated, praised and criticized philosophical thinkers. However, his work has yet to be fully explicated or evaluated. This is in part because a full assessment of his logical, semantic and philosophical views could only be possible if those views were developed in greater detail than Frege himself managed in his lifetime. In particular, he failed to develop his logical system sufficiently in order to capture fully within it the complexity of his notable theories regarding semantics, meaning and language. Filling in this gap in the interpretation of Frege's philosophical position is necessary in part simply because doing so will give us a better understanding of what his logical and semantic views actually were. Moreover, there are certain prominent criticisms and potential problems with his philosophy that can only be properly recognized and evaluated when this gap is filled in.

The primary purpose of this work is to redress this void in Frege's work. In later chapters, I shall expand upon his logical system so that it does better justice to his core tenets in the philosophy of language, namely, the theory of *Sinn* and *Bedeutung*. Before doing so, it will be necessary to discuss, develop and explicate certain aspects of his logical and semantic views, which will involve engaging in controversial matters of interpretation. It will also require examining previous attempts at capturing broadly Fregean semantics within systems of logic. Once a truly Fregean logical calculus is in place, we shall then be in a better place to evaluate Frege's philosophical views. It will be discovered that eliminating this gap in his work provides him with the ability to respond to

certain objections to his views that appeared threatening only because certain aspects of those views had not been fully explicated. However, perhaps of greater interest, it will also be discovered that his views contain internal problems and inconsistencies that have hitherto gone unnoticed and unaddressed for the very same reason. In particular, I show that if Frege's logical system is developed to include his semantic views, it becomes possible to demonstrate certain Cantorian and semantical paradoxes within it, including paradoxes that do not rely on the questionable aspects of his extant system responsible for previously identified paradoxes. These findings are then further discussed in the attempt to understand the source of the difficulties within Frege's views, in general, and vis-à-vis the views of others.

Before embarking on this project, however, it is worth taking a closer look at precisely why it is necessary to develop a logical calculus for the theory of *Sinn* and *Bedeutung*, and how such a development would fit within Frege's scholarly project and philosophical system. We must examine in more detail what is missing from Frege's extant logical system, what sorts of additions are necessary, and why they are important. We first turn to an overall look at Frege's project and philosophical interests.

FREGE'S PROJECT: LOGICISM AND THE NOTION OF BEGRIFFSSCHRIFT

Frege is known for making contributions to many areas of philosophy, including logic, philosophy of mathematics, philosophy of language, philosophy of mind and metaphysics. In no way, however, can these be understood as representing distinct philosophical interests on the part of Frege himself. Although he had a certain amount of philosophical training, he was first and foremost a mathematician and his primary lifelong interest was in the foundations of arithmetic. His interests in other philosophical areas were not only all related to his interests in the philosophy of arithmetic, they grew out of them. In thinking through the nature of arithmetic, he came to the view now known as logicism, the thesis that mathematical truths are truths of pure logic, rather than being grounded in empirical observations, or, as Kant has maintained, "pure intuition." In some ways, Frege's logicism was a limited one. For example, unlike Russell, Frege restricted his logicism to arithmetic, and explicitly denied that geometry was a branch of logic (*CP* 112, *PMC* 100). However, in other ways, Frege's logicism took a naively ambitious form, in holding that the truths of arithmetic could all be formally deductively proven from a small number of purely logical axioms.¹

¹Gödel's incompleteness proof showed the impossibility of this aspiration. See Kurt Gödel, "On Formally Undecidable Propositions of *Principia Mathematica* and Related

It is this conviction, unsurprisingly, that engrossed Frege in the area in which he made his greatest philosophical achievements, the area of formal logic. He found the existent logical systems of his time, still based mostly on the Aristotelian syllogistic, unsuitable for realizing his logicist aspirations. Thus, he took it upon himself to rethink the basic truths of logic. In so doing, Frege created an approach to logic so original, fruitful and innovative that a century after its invention, the Aristotelian approach, dominant in academia for over two thousand years, has been almost wholly supplanted by the Fregean. We need not dwell here on the many details of the Fregean revolution in logic. The core of the advances he brought about is well known, and many details of his views will be discussed in the next chapter. One aspect of Frege's logical doctrines worth examining further in this context, however, is his advocacy of the development of a distinct logical syntax, or as he called it, a "Begriffsschrift," translated alternatively as "conceptual notation," or "concept-script."²

Frege probably took the term "Begriffsschrift" from a paper by Adolf Trendelenburg on Leibniz's idea of a logically ideal language.³ To use the Leibnizian terminology, Frege wanted his notation to perform the tasks of both a *calculus ratiocinator* and a *lingua characterica*.⁴ Frege's Begriffsschrift was to serve, firstly, as a more logically perspicuous language, one that eliminated wholly the ambiguities and unclarities present in ordinary language. It would capture clearly everything about a proposition relevant to inference, and leave out anything irrelevant (such as author's attitude, etc.) Every sign to be used in the Begriffsschrift was to be rigorously defined, so that it would be understood precisely the same by everyone. In this way, it would serve as a *lingua characterica*, a universal language perfectly suitable for scientific (*wissenschaftlich*) work. As a *calculus ratiocinator*, however, it would be further furnished with a small set of axioms taken as fundamental, and inferences in the Begriffsschrift would be limited to what could be drawn from these axioms by means of precisely specified rules of inference. In Frege's eyes,

Systems I," in *From Frege to Gödel*, ed. Jean van Heijenoort (Cambridge, Mass.: Harvard University Press, 1967), 596-616.

²When speaking of logical notation as conceived by Frege, I shall use the word Begriffsschrift, leaving it untranslated. Despite its foreign origin, I leave the word unitalicized, save when referring to Frege's early work of that name.

³The paper in question is "Über Leibnizes Entwurf einer allgemeinen Charakteristik," *Historische Beiträge zur Philosophie* 3 (1867): 51-65. On this point, see Hans Sluga, *Gottlob Frege* (London: Routledge and Kegan Paul, 1980), 48-52.

⁴For more on this distinction, see Jean van Heijenoort, "Logic as Calculus and Logic as Language," *Synthese* 17 (1967): 324-30. However, for reasons I discuss in the next chapter, I think van Heijenoort in some ways overstates matters.

it was unacceptable that in the demonstration of logical or mathematical truths that any underlying assumption or step in the deduction be left implicit. This was thought to be the only way of eliminating the possibility of an unseen error creeping in. In carrying out his strong logicist program, it was crucial that it be clear that every truth demonstrated be grounded entirely on purely logical principles, and that no appeal to intuition or observation be involved. The only way to see the logicist program carried out correctly, therefore, was the creation of just such an ideal scientific language.

However, Frege did not think that one could create an ideal scientific language overnight. He recommended proceeding in small steps. His personal goal in his logical works was not to create an entire *lingua characterica*, but rather to develop only its purely logical core. Thus, Frege did not consider his Begriffsschrift to be complete. In fact, every sign he himself introduced is a logical constant, and as such, the Begriffsschrift he himself developed is only capable of expressing thoughts dealing with pure logic, including, if logicism is true, mathematical truths. However, it must be remembered that Frege saw his Begriffsschrift as merely a first step towards the creation of an ideal, universal, scientific language. In his early work, *Begriffsschrift, eine der arithmetische nachgebildete Formelsprache des reinen Denkens*, Frege claims that his Begriffsschrift is but one small step in the creation of an ideal Leibnizian language, but a very important step, for the signs of logic represent the core of an ideal language. He continues:

If we take our departure from there, we can with the greatest expectation of success, proceed to fill in the gaps in the existing formula languages, connect their hitherto separated fields into a single domain, and extend this domain to include fields that up to now have lacked such a language. (BS 7)

Frege's own Begriffsschrift was aimed only at creating the logical core of an ideal language, and he hoped that his efforts would be combined with those of others, working in other fields, and gradually, symbols and axioms would be added to it in order to capture the truths of chemistry, psychology, astronomy and other sciences (*Wissenschaften*).

The work entitled *Begriffsschrift* of 1879 represents Frege's first attempt at developing his Begriffsschrift. Therein, he laid out the basics of his logical notation, but only hinted at how it would be used to demonstrate the truths of arithmetic. Although he went on to explain his logicist project informally in his 1884 *Die Grundlagen der Arithmetik* ("The Foundations of Arithmetic"), further development of his formal logical work was put off until the publication of the first volume of his magnum opus, *Grundgesetze der Arithmetik* ("Basic Laws of Arithmetic") in 1893. However, in the intervening time, Frege's views had

matured considerably. He found it necessary to make changes to the Begriffsschrift reflecting these changed views. On the surface, the two logical systems appear to be quite similar. With one exception, all of the primitive signs of the *Begriffsschrift* appear also in the *Grundgesetze*, the latter adding two new signs. However, the small changes to the syntax of Frege's Begriffsschrift belie the much more serious changes to its semantics, i.e., how the signs were to be understood. These changes were forced by the maturation of Frege's views on functions, concepts, *Sinn* and *Bedeutung*, which had come about largely in the early 1890s. Indeed, the changes were so profound that Frege was forced to abandon entirely an almost completed manuscript version of the *Grundgesetze* (BL 5-6). While the early Begriffsschrift of the *Begriffsschrift* introduced most of his important technical innovations in logic, only the logical system of the *Grundgesetze* represents Frege's mature logical theory, and only it is fully consistent with his other notable philosophical views. For this reason, in what follows, the focus will be on Frege's mature views, and the logical system of the *Grundgesetze*.

The Begriffsschrift of the *Grundgesetze* was both innovative and powerful. Indeed, by the time he had almost completed the second volume in 1902, it must have seemed to Frege that he had been completely successful in creating a logical calculus that was fully capable of carrying out his logicist project to its completion. Volume I had expounded the Begriffsschrift itself, and proceeded to use it to define the natural numbers and demonstrate their properties. In Volume II, he had moved on to the real numbers, and was well on his way to demonstrating their properties logically as well—a project he had planned to finish in the third volume. However, the Begriffsschrift of the *Grundgesetze* turned out to be *too* powerful. In June of 1902, while the second volume of the *Grundgesetze* was already in the publication process, Bertrand Russell wrote to Frege, praising his work but noting that it is subject to the paradox of classes bearing Russell's name.⁵ Thus, Frege's Begriffsschrift of the *Grundgesetze* is inconsistent, and given its truth-functional nature, every proposition formulable in the logical language is also demonstrable in the system. While Frege had succeeded in creating a logical calculus in which the truths of the natural and real numbers could be proven, it could also be used to prove the negations of these very truths. Frege was devastated. While he continued to do some interesting work for another five years or so, he never devised an adequate solution to the paradox, and the proposed third volume of the *Grundgesetze* was

⁵Russell to Frege, Haslemere, 16 June 1902, *Philosophical and Mathematical Correspondence*, by Gottlob Frege (Chicago: University of Chicago Press, 1980), 130-1.

never written. By the end of 1906, he seems to have completely lost his intellectual steam.⁶

While Russell's paradox may have proven to be an intellectual difficulty Frege never found it possible to overcome, along the way he *did* face and *overcome*—to his own satisfaction, at least—smaller intellectual puzzles and doubts that challenged his logicist program and the success of the creation of his Begriffsschrift. In doing so, Frege was often forced to take sides on fundamental and thoroughly theoretical intellectual issues, and it is from his attempts to meet these challenges that most of his notable philosophical views emerged. For example, Frege's theory of judgment, and to a lesser extent, his ontology, were conceived largely in defending his logicist thesis against the criticisms of those in rival schools, such as psychologism and formalism. However, more importantly in this context, Frege's theory of meaning ostensibly derived from the attempt to solve some purely logical puzzles that would have been of great relevance to the construction of the Begriffsschrift. To this I now turn.

THE THEORY OF *SINN* AND *BEDEUTUNG*

There have been many suggestions made as to what Frege's ultimate motivation or rationale was for his theory of meaning. Perhaps the best approach, however, is to look carefully at the works in which he introduced the theory and the uses to which it is put therein. The theory of *Sinn* and *Bedeutung* was first outlined, albeit briefly, in his article, "Funktion und Begriff" of 1891 (CP 137-56), and was expanded and explained in greater detail in perhaps his most famous work, "Über Sinn und Bedeutung" of 1892 (CP 157-77). In "Funktion und Begriff," the distinction between the *Sinn* and *Bedeutung* of a sign in language is first made with regard to mathematical equations. During Frege's time, there was a widespread dispute among mathematicians as to how the sign "=" should be understood. If we consider an equation such as, " $4 \cdot 2 = 11 - 3$," a number of Frege's contemporaries, for a variety of reasons, were wary of viewing this as an expression of an *identity* proper, or, in this case, as the claim that $4 \cdot 2$ and $11 - 3$ are one and the same thing. Instead, they posited some weaker form of "equality" such that the numbers $4 \cdot 2$ and $11 - 3$ would be said to be equal in number or equal in magnitude without thereby constituting the same thing. In opposition to the view that "=" signifies identity, such thinkers would point out

⁶Although Frege did return to writing and publishing in the late 1910s, most notably the "Logical Investigations" series of "Thoughts," "Negation," and "Compound Thoughts," these works are merely later drafts of works he had begun as early as 1896. For example, virtually the same views published in these works appear in almost the same order and structure in the work in his *Nachlaß* entitled simply "Logic," thought to have been written in 1897. Compare CP 351-89 to PW 126-51.

that $4 \cdot 2$ and $11 - 3$ cannot in all ways be thought to be the same. The former is a product, the latter a difference, etc.

In his mature period, however, Frege was an ardent opponent of this view, and argued in favor of understanding “=” as identity proper. He suggests that the expressions “ $4 \cdot 2$ ” and “ $11 - 3$ ” can be understood as standing for one and the same thing, the number eight, but that this single entity is determined or presented differently by the two expressions. Thus, he makes a distinction between the actual *number* a mathematical expression such as “ $4 \cdot 2$ ” stands for, and the way in which that number is determined or picked out. The former he called the “*Bedeutung*” (denotation or reference) of the expression, and the latter he called the “*Sinn*” (sense) of the expression.⁷ In Fregean terminology, an expression is said to *express* its *Sinn*, and *denote* or *refer to* its *Bedeutung*. The distinction between *Sinn* and *Bedeutung* was vitally important for the logicist project, in part because it helped uphold his stance on the nature of mathematical equations, and allowed him, in the *Begriffsschrift* of the *Grundgesetze*, to safely use “=” as a sign for identity proper. (Frege saw this a great improvement over the earlier approach he had taken in his *Begriffsschrift*, in which he was forced to introduce two signs, “=” for mathematical equality, and “≡” for “identity of content.”)

The distinction between *Bedeutung* and *Sinn* was expanded, primarily in “Über Sinn und Bedeutung,” as holding not only for mathematical expressions, but for all linguistic expressions (whether the language in question is a natural language or a formal language). One of his primary examples therein involves the expressions “the morning star” and “the evening star.” Both of these expressions refer to the planet Venus, yet they obviously denote Venus in virtue of different properties that it has. Thus, Frege claims that these two expressions have the same *Bedeutung* but different *Sinne*. The *Bedeutung* of an expression is the actual thing corresponding to it, in the case of “the morning star,” the *Bedeutung* is the planet Venus itself. The *Sinn* of an expression, however, is the

⁷The translations of the German terms “*Bedeutung*” and “*Sinn*” are problematic and controversial. In translations of and writings about Frege alone, “*Sinn*” has been translated both as “sense” and as “meaning”, and “*Bedeutung*” has had the misfortune of having been variously translated as “reference,” “denotation,” “meaning,” “significance,” “indication” and “nominatum.” In order to avoid embroiling myself in the controversy, and in keeping with recent trends, I shall leave the terms untranslated. For more on the translation controversy, see Michael Beaney, introduction to *The Frege Reader*, by Gottlob Frege (Oxford: Blackwell, 1997), 36-46; David Bell, “On the Translation of Frege’s *Bedeutung*,” *Analysis* 40 (1980): 191-5; Peter Long and Roger White, preface to *Posthumous Writings*, by Gottlob Frege, vi-viii, “On the Translation of Frege’s *Bedeutung*: A Reply to Dr. Bell,” *Analysis* 40 (1980): 196-202.

“mode of presentation” (CP 158) or cognitive content associated with the expression in virtue of which the *Bedeutung* is picked out.

Frege puts the distinction to work in solving a puzzle concerning identity claims. If we consider the two claims:

(1) the morning star = the morning star

(2) the morning star = the evening star

The first appears to be a trivial case of the law of self-identity, knowable *a priori*, while the second seems to be something that was discovered *a posteriori* by astronomers. However, if “the morning star” means the same thing as “the evening star,” then the two statements would also seem to have the same meaning, both involving a thing’s relation of identity to itself. However, it then becomes difficult to explain why (2) seems informative while (1) does not. Frege’s response to this puzzle, given the distinction between *Sinn* and *Bedeutung*, should be apparent. Because the *Bedeutung* of “the evening star” and “the morning star” is the same, both statements are true in virtue of this object’s relation of identity to itself. However, because the *Sinne* of these expressions are different—in (1) the object is presented the same way twice, and in (2) it is presented in two different ways—it is informative to learn of (2). While the *truth* of an identity statement involves only the *Bedeutungen* of the component expressions, the *informativity* of such a statement involves additionally the way in which those *Bedeutungen* are determined, i.e., the *Sinne* of the component expressions.

So far we have only considered the distinction as it applies to expressions that name some object (including abstract objects, such as numbers). For Frege, the distinction applies also to other sorts of expressions and even whole sentences or propositions.⁸ If the *Sinn/Bedeutung* distinction can be applied to whole propositions, it stands to reason that the *Bedeutung* of a whole proposition depends on the *Bedeutungen* of the parts and the *Sinn* of the proposition depends on the *Sinne* of the parts. In the example considered in the previous paragraph, it

⁸A note is in order concerning the word “proposition” in Frege’s philosophy. In English-speaking philosophical traditions, this word is used in a variety of ways, some involving significant ontological assumptions. These ontological assumptions must not be read into Frege. In English versions of Frege’s works, the word is usually used to translate Frege’s word “Satz,” which in other contexts would simply be translated as “sentence” or “statement.” In Frege’s terminology, a *Satz* is simply a bit of language, a complete declarative sentence, and does not carry with it any stronger ontological status. Frege himself is explicit about this. See, e.g., CP 242n, 332 PW 167-8, 206, PMC 149.

was seen that the truth-value of the identity claim depended on the *Bedeutungen* of the component expressions, while the informativity of what was understood by the identity claim depended on the *Sinne*. For this and other reasons, Frege concluded that the *Bedeutung* of an entire proposition is its truth-value, either the True or the False.⁹ The *Sinn* of a complete proposition is what we understand when we understand a proposition, which Frege calls a *Gedanke* (thought).¹⁰ Just as the *Sinn* of a name of an object determines how that object is presented, the *Sinn* of a proposition determines a method of determination for a truth-value. The propositions, “ $2 + 4 = 6$ ” and “the Earth rotates,” both have the True as their *Bedeutung*. However, this is in virtue of very different conditions holding, just as “the morning star” and “the evening star” refer to Venus in virtue of different properties.

In “Über Sinn und Bedeutung,” Frege limits his discussion of the *Sinn/Bedeutung* distinction to “complete expressions,” such as names purporting to pick out some object as well as whole propositions.¹¹ In other works, Frege makes it quite clear that the distinction can also be applied to “incomplete expressions,” which include functional expressions and grammatical predicates (*PW* 118-25, *PMC* 61-4). These expressions are incomplete in the sense that they contain an “empty space,” which, when filled, yields either a complex name referring to an object or a complete proposition. The incomplete expression “the square root of ()” contains a blank spot, which, when completed by a expression referring to a number, yields a complex expression also referring to a number, e.g., “the square root of sixteen.” The incomplete expression, “() is a planet” contains an empty place, which, when filled with a name, yields a complete proposition. According to Frege, the *Bedeutungen* of these incomplete expressions are not objects but functions. Objects (*Gegenstände*), in Frege’s terminology, are self-standing, complete entities, while functions are essentially incomplete, or as Frege says, “unsaturated” (*ungesättigt*) in that they must take something else as argument in order to yield a value (*CP* 182-94). The *Bedeutung* of the expression “square root of ()” is a

⁹This has struck many as a surprising and counterintuitive suggestion. However, it would take us too far afield here to discuss or debate the matter in detail. Frege’s primary arguments, in addition to the consideration noted here, are given at *CP* 163-5.

¹⁰Although the German word “*Gedanke*” translates unproblematically into English as “thought,” here too I shall use the German expression, simply to help the reader bear in mind the very technical way in which Frege uses this word.

¹¹There is some controversy among interpreters of Frege as to whether the claim that truth-values are the *Bedeutungen* of propositions is equivalent to the claim that propositions are “names” of truth-values, as some have thought. This is discussed in the next chapter.

function that takes numbers as arguments and yields numbers as values. The situation may appear somewhat different in the case of grammatical predicates. However, because Frege holds that complete propositions, like names, have objects as their *Bedeutungen*, and in particular, the truth-values the True or the False, he is able to treat predicates also as having functions as their *Bedeutungen*. In particular, they are functions mapping objects onto truth-values. The expression, “() is a planet” has as its *Bedeutung* a function that yields as value the True when saturated by an object such as Saturn or Venus, but the False when saturated by a person or the number three. Frege calls functions with one argument place that yield the True or False for every possible argument “concepts” (*Begriffe*), and calls similar functions with more than one argument place (such as that denoted by “() > ()”, which is doubly in need of saturation), “relations”. In Frege’s way of speaking, an object for which a concept has the True as value is said to “fall under” the concept (CP 190).

It is clear that functions are to be understood as the *Bedeutungen* of incomplete expressions, but what of the *Sinne* of such expressions? This is a difficult matter, and one which will receive much fuller treatment in Chapter 3. For the moment, however, it suffices to note that just as the same object (e.g., the planet Venus), can be presented in different ways, so also can functions be presented in different ways. While identity, as Frege uses the term, is a relation that holds only between objects, Frege believes that there is a relation similar to identity that holds between functions just in case they always share the same value for every argument (CP 200, PW 120-2). If we allow ourselves the common philosophical supposition that all and only creatures with hearts have kidneys, then, strictly speaking, the concepts denoted by the expressions “() has a heart” and “() has a kidney” are one and the same. Clearly, however, these expressions do not present that concept in the same way. For Frege, these expressions would have different *Sinne* but the same *Bedeutung*.

These aspects of Frege’s theory have a tremendous influence on his Begriffsschrift. The conclusion that predicates can be treated as standing for functions greatly simplifies the kinds of symbols used in his logical language, because it is unnecessary to use a different style of notation for concepts as for mathematical functions, for instance. Moreover, the treatment of complete propositions as having truth-values as their *Bedeutungen* also aids the construction of the Begriffsschrift insofar as it allows Frege to transcribe sentential connectives such as “and” and “or,” etc., as truth functions in the strictest sense—functions that take truth-values as argument and yield truth-values as value. More will be said about this, however, in the next chapter.

For the moment, it is worth turning to another motivation lying behind the development of the theory of *Sinn* and *Bedeutung*. Frege also uses the distinction to solve what appears to be a difficulty with Leibniz’s law. This law

was stated by Leibniz as, “those things are the same of which one can be substituted for another without loss of truth,” a sentiment with which Frege was in full agreement (*FA* §65, *CP* 164, 200). As Frege understands this, it means that if two expressions have the same *Bedeutung*, they should be able to replace each other within any proposition without changing the truth-value of that proposition. Normally, this poses no problem. The inference from:

(3) The morning star is a planet.

to the conclusion:

(4) The evening star is a planet.

in virtue of (2) above and Leibniz’s law is unproblematically valid. However, there seem to be some serious counterexamples to this principle. We know for example that “the morning star” and “the evening star” have the same customary *Bedeutung*. However, it is not always true that they can replace one another without changing the truth of a sentence. For example, if we consider the propositions:

(5) Gottlob believes that the morning star is a planet.

(6) Gottlob believes that the evening star is a planet.

If we assume that Gottlob does not know that the morning star and the evening star are the same thing, (5) may be true while (6) false or vice versa.

Frege meets this challenge to Leibniz’s law by making a distinction between what he calls the primary and secondary *Bedeutungen* of expressions. He suggests that when expressions appear in certain unusual contexts, they have as their *Bedeutungen* what are customarily their *Sinne*. In such cases, the expressions are said to have their *secondary Bedeutungen*. Typically, such cases involve what Frege calls “indirect speech” or “*oratio obliqua*,” as in the case of statements of beliefs, thoughts, desires and other so-called “propositional attitudes,” such as the examples of (5) and (6). However, expressions also have their *secondary Bedeutungen* (for reasons which should already be apparent) in contexts such as “it is informative that . . .” or “. . . is analytically true.”

Let us consider the examples of (5) and (6) more closely. To Frege’s mind, these statements do not deal directly with the morning star or evening star itself. Rather, they involve a relation between a believer and a *Gedanke* believed. *Gedanken*, as we have seen, are the *Sinne* of complete propositions. Beliefs depend for their make-up on how certain objects and concepts are presented, not

only the objects and concepts themselves. The truth of a belief claim, therefore, will depend not on the customary *Bedeutungen* of the component expressions of the stated belief, but on their *Sinne*. Since the truth-value of the whole belief claim is the *Bedeutung* of that belief claim, and the *Bedeutung* of any proposition, for Frege, depends on the *Bedeutungen* of its component expressions, we are led to the conclusion that the typical *Sinne* of expressions appearing in *oratio obliqua* are in fact the *Bedeutungen* of those expressions when they appear in that context. Such contexts can be referred to as “oblique contexts,” contexts in which the *Bedeutung* of an expression is shifted from its customary *Bedeutung* to its customary *Sinn*.

In this way, Frege is able to retain his commitment in Leibniz’s law. The expressions “the morning star” and “the evening star” have the same primary *Bedeutung*, and in any non-oblique context, they can replace each other without changing the truth-value of the proposition. However, since the *Sinne* of these expressions are not the same, they cannot replace each other in oblique contexts, because in such contexts their *Bedeutungen* are non-identical. The conclusion that Leibniz’s law can be retained despite these apparent exceptions was very significant for the development of Frege’s Begriffsschrift, for it allowed him to introduce an axiom capturing Leibniz’s law,¹² which was quite essential if his logicist program was to be a success.

THE LIMITATIONS OF THE BEGRIFFSSCHRIFT

As we have seen, the theory of *Sinn* and *Bedeutung* was largely motivated to solve certain questions directly germane to logical theory and the construction of Frege’s Begriffsschrift. It allowed him to maintain Leibniz’s law, to safely utilize the identity sign for mathematical equations, and so forth. Frege even included a summary of the theory in the *Grundgesetze* itself (*BL* §2). However, despite the importance of the theory of *Sinn* and *Bedeutung* to the development of the Begriffsschrift, Frege did not develop the Begriffsschrift sufficiently in order to fully capture within it all of the core tenets of the semantic theory. This requires more careful scrutiny.

Given the way in which the Begriffsschrift is constructed, every symbol that appears within it always has the same *Bedeutung*, its primary *Bedeutung*. There are no oblique contexts in the Begriffsschrift. Of course, this is not surprising. Frege lamented the ambiguities present in ordinary language, and facets of ordinary language, such as that the expression “the morning star” in some contexts refers to a planet and in other contexts refers to a *Sinn*, appeared to him as defects. The Begriffsschrift was deliberately designed to be a language in which every sign has a clear and univocal *Bedeutung* in all cases. However, if

¹²The axiom in question is Frege’s Basic Law III (*BL* §20).

oblique contexts are disallowed from the Begriffsschrift, the question arises as to whether it could be able to express everything expressed in ordinary language. Of course, the Begriffsschrift Frege developed was admitted to be incomplete, it presented only its logical core. However, if we imagine that this prohibition on obliquity in the Begriffsschrift is to stand even in its expansion, the question becomes whether the Begriffsschrift, even ideally expanded, could be able to capture statements of propositional attitudes and other statements that, in ordinary language, give rise to indirect contexts.

Here, there seems to be a problem. Let us suppose that “ $\mathcal{P}(\)$ ” is the Begriffsschrift sign for a function that yields the True if its argument is a planet, and yields the False otherwise, and “ \mathfrak{m} ” is its sign for the morning star.¹³ If so then:

$$(7) \vdash \mathcal{P}(\mathfrak{m})$$

might be taken to be the Begriffsschrift proposition expressing the *Gedanke* that the morning star is a planet. (The sign “ \vdash ”, called *the judgment stroke*, will be explained in the next chapter.) However, how do we express the *Gedanke* that Gottlob believes that the morning star is a planet? According to Frege, belief is a relation between a believer and a *Gedanke* believed. Let us suppose that “ \mathfrak{g} ” is a sign for Gottlob, and “ $\mathcal{B}(\ ,)$ ” is a sign for a relation whose truth-value is the True if the second argument is a *Gedanke* serving as the content of a belief held by the first argument, and the False otherwise. It might be tempting to suppose that the correct translation of “Gottlob believes that the morning star is a planet,” is:

$$(8) \vdash \mathcal{B}(\mathfrak{g}, \mathcal{P}(\mathfrak{m}))$$

However, this is not the case. The expression “ $\mathcal{P}(\mathfrak{m})$ ” does not *denote* the *Gedanke* that the morning star is a planet, it only expresses it. The *Bedeutung* of “ $\mathcal{P}(\mathfrak{m})$ ” is the True, and the True is not a *Gedanke* at all, and *a fortiori* not a *Gedanke* held by Gottlob. Belief involves an indirect context in ordinary language, and if we are barred from treating it as an oblique context in the Begriffsschrift, we are faced with a problem.

Frege might have attempted to respond to this problem by noting that while it is true that we cannot use the same signs in Begriffsschrift at times to refer to truth-values and at times to refer to *Gedanken*, nothing prevents us from

¹³Because on the reading given in the next chapter, normal Roman letters are used in the Begriffsschrift to express generalities, I here use lowercase Cyrillic characters for object constants, and uppercase script characters for function constants.



introducing new primitive constants that denote *Gedanken*, rather than express them. For example, we might introduce the sign “p” to have as its *Bedeutung* the *Gedanke* that the morning star is a planet. Then, instead of (8), we obtain:

$$(9) \vdash \mathcal{B}(g, p)$$

(9) then expresses the same as the ordinary language sentence (5), that Gottlob believes the morning star is a planet. Thus, it might be argued, it is not impossible for a language that itself has no oblique contexts to capture *Gedanken* that, if expressed in ordinary language, do give rise to oblique contexts. Therefore, the prohibition in the Begriffsschrift against ambiguity (and thus obliquity) need not spell disaster.

The problem, however, is that if we simply employ different signs for the primary and secondary *Bedeutungen* of ordinary language expressions, then, unlike ordinary language, the obvious connection between these signs becomes lost. In the example just considered, the *Sinn* of the expression “ $\mathcal{P}(m)$ ” is the *Bedeutung* of the sign “p”. The former expresses what the latter denotes. However, the connection between the two is not at all evident in the symbolism. In natural language, the connection between the expression “the morning star is a planet” appearing in a direct context, and *that same expression* appearing in an oblique context is, of course, obvious. Thus, in natural language, one would be able to infer from (3) and (5) the conclusion:

$$(10) \text{ Gottlob believes something true.}$$

However, no corresponding inference seems to be possible with (7) and (9) as premises. Although (7) and (9) are the Begriffsschrift translations of (3) and (5), unlike (3) and (5), they contain no common constituents, and no inference rule in Frege’s system can be applied to them to reach the desired conclusion.

This very point is sometimes raised as an objection against Frege’s theory of *Sinn* and *Bedeutung* and the doctrine of secondary *Bedeutung*. Simon Blackburn and Alan Code, for example, find implicit in Russell’s famous criticism in “On Denoting”¹⁴ of Frege’s doctrine of meaning precisely the objection that by holding that an expression such as “the morning star” does not refer to the same thing in (3) as it does in (5), Frege bars any possible inference to something such as (10).¹⁵ The interpretation of Russell’s criticism of Frege is

¹⁴Bertrand Russell, “On Denoting,” in *Essays in Analysis*, ed. Douglas Lackey (London: George Allen & Unwin, 1973), 103-19.

¹⁵Blackburn and Code use a slightly different example, but the point is the same. See “The Power of Russell’s Criticism of Frege: ‘On Denoting’ pp. 48-50,” *Analysis* 38

notoriously controversial, but in the end it does not matter whether this objection has its origins in Russell's work or whether it is more recent. Either way, it is an objection to which Frege must respond.

I do not think this objection to Frege is, in the end, convincing. What it does show, however, is that Frege's Begriffsschrift is in need of supplementation, even in its "purely logical" portion. As we have seen, Frege's theory of *Sinn* and *Bedeutung* is of a piece with his logical theory. However, it is not fully reflected in the Begriffsschrift he develops as such. As it stands, the Begriffsschrift has no symbols capable of expressing the relationship between a *Sinn* and its corresponding *Bedeutung*, including that between a *Gedanke* and its corresponding truth-value. Nor does the Begriffsschrift contain axioms stating the identity conditions of *Gedanken* or *Sinne* more generally. By adding logical symbols and axioms to Frege's Begriffsschrift dealing directly with the theory of *Sinn* and *Bedeutung*, the theory would be much more powerful and more capable of being put to use in the ways Frege himself intended. Alonzo Church, the first to attempt the expansion of a logical system roughly in line with Frege's semantic theories, dubs such an expansion the creation of "the Logic of Sense and Denotation,"¹⁶ or leaving the Fregean terms untranslated, "the Logic of *Sinn* and *Bedeutung*."

Frege himself never attempted to expand his Begriffsschrift in this way, but if we imagine that he had, it is easy to see how he might have responded to Blackburn and Code's objection. One could introduce a two-place function sign " $\Delta(,)$,"¹⁷ standing for a function whose value is the True just in case the first argument is a *Sinn* picking out the second argument as *Bedeutung*.¹⁸ In that case,

(1978): 65-77. See also the response given in Michael Dummett, *The Interpretation of Frege's Philosophy* (Cambridge, Mass.: Harvard University Press, 1981), 87-9.

¹⁶Alonzo Church, "A Formulation of the Logic of Sense and Denotation (Abstract)," *Journal of Symbolic Logic* 11 (1946): 31, "A Formulation of the Logic of Sense and Denotation," in *Structure, Method and Meaning: Essays in Honor of Henry M. Sheffer*, ed. P. Henle, H. Kallen and S. Langer (New York: Liberal Arts Press, 1951), 3-24, "Outline of a Revised Formulation of the Logic of Sense and Denotation," parts 1 and 2, *Noûs* 7 (1973): 24-33; 8 (1974): 135-56, "A Revised Formulation of the Logic of Sense and Denotation, Alternative (1)," *Noûs* 27 (1993): 141-57.

¹⁷I borrow the sign " Δ " from Church, although it is worth noting, that, following Kaplan, I reverse the position of the arguments. See David Kaplan, "Quantifying In," in *The Philosophy of Language*, 3d ed., ed. A.P. Martinich (New York: Oxford University Press, 1996), 353.

¹⁸According to Fregean terminology, an expression *expresses* its *Sinn* and *denotes* or *refers to* (bedeutet) its *Bedeutung*. However, he has no verb to express the relation between a *Sinn* and its corresponding *Bedeutung*. This is odd, considering that on his

one could capture the relation between “ $\mathcal{P}(m)$ ” and “ p ” with the following proposition:

$$(11) \vdash \Delta(p, \mathcal{P}(m))$$

Given (11), as well as certain axioms governing the use of “ $\Delta(,)$ ”, one could then proceed to infer on the basis of (7) and (9) the Begriffsschrift correlate of (10). (The details remain to be worked out anon.)

The inference mentioned above is only one sort of inference that is currently impossible in Frege’s Begriffsschrift but could be possible if it were expanded to include the logic of *Sinn* and *Bedeutung*. Once there were some sort of treatment within the Begriffsschrift of the indirect contexts of ordinary language, along with this treatment would come a certain methodology for “quantifying in” to such contexts.¹⁹ An example of an inference that involves “quantifying in” would be, to infer, on the basis of (5):

$$(12) \text{ There is something Gottlob believes to be a planet.}$$

As David Kaplan has explained, because Frege held that belief contexts (and similar contexts) involve *oblique* or *shifted* reference as opposed to complete “referential opacity” (to use Quine’s term), quantifying in to an oblique context should be possible, provided that the variables used for such quantification stand for *Sinne*.²⁰

This should also make it possible to capture what Quine calls “relational” or “*de re*” beliefs in addition to the “notional” or “*de dicto*” beliefs exemplified by (5) and (6) above.²¹ An example of such a relational belief would be:

view, an expression has the *Bedeutung* it has in virtue of its *Sinn*. The relation between the *Sinn* and *Bedeutung* is thus closer than that between expression and *Bedeutung*. In the terminology I employ, an expression denotes its *Bedeutung*, and a *Sinn* “picks out” or “presents” its *Bedeutung*.

¹⁹The problem of “quantifying in” has received its fullest treatment in the works of Quine. See W.V. Quine, “Reference and Modality,” in *From a Logical Point of View*, 2d. ed. (Cambridge, Mass.: Harvard University Press, 1980), 139-59, “Three Grades of Modal Involvement,” in *The Ways of Paradox and Other Essays*, rev. ed. (Cambridge, Mass.: Harvard University Press, 1976), 158-76, *Word and Object* (Cambridge, Mass.: MIT Press, 1960), §§30-2, and “Quantifiers and Propositional Attitudes,” in *Ways of Paradox*, 185-96.

²⁰Kaplan, “Quantifying In,” 351-2.

²¹For more on this distinction, see Quine, “Quantifiers and Propositional Attitudes,” 185-8.

(13) The morning star is such that Gottlob believes it to be a planet.

However, without a device such as our “ Δ ” function, capturing (13) in Frege’s Begriffsschrift would be quite impossible. The word “it” in (13) occurs in an oblique context, while its antecedent, “the morning star,” does not. Therefore, for Frege, they cannot be taken to have the same *Bedeutung*. This means that, in the Begriffsschrift, they could not be represented by the same sign. However, by using different signs and the “ Δ ” relation, this problem can be overcome. Without such a device, one could perhaps see the failure to be able to adequately capture propositions such as (13) as a serious defect in Frege’s views. Indeed, some writers on Frege have alleged just this.²²

It is only with the development of the logic of *Sinn* and *Bedeutung* that Frege could respond to certain objections against his views. However, there are other objections to his views that can only be properly formulated and assessed once the logic of *Sinn* and *Bedeutung* is in place. For example, consider Quine’s skepticism of theories of meaning such as Frege’s because of their failure to provide a clear account of the sameness of meanings or synonymy (in Frege’s case, the identity conditions of *Sinne* and *Gedanken*).²³ Unfortunately, Frege himself never provided a comprehensive account of the identity conditions of *Sinne* and *Gedanken*. However, an account of the identity conditions of *Sinne* and *Gedanken* would have to be built into the axioms of the logic of *Sinn* and *Bedeutung*. Quine would likely object that such an account of the identity conditions of *Sinne* would be *ad hoc*, because the axioms themselves are introduced on an *ad hoc* basis. However, this charge can only be evaluated once those axioms are identified and their motivation in the system fully explored.

Another way in which Frege’s views can only be fully assessed once the logic of *Sinn* and *Bedeutung* is fully developed involves the ontological commitments of the theory of *Sinn* and *Bedeutung*. At the time he wrote the *Grundgesetze*, Frege held a fairly complicated ontology of abstract objects, including functions, value-ranges of functions (including classes), truth-values, *Sinne* and *Gedanken*. However, only his commitments to functions, value-ranges and truth-values are actually captured in the Begriffsschrift he himself developed. It is important that the full range of Frege’s ontology be included within the actual commitments of the Begriffsschrift in order to fully evaluate Frege’s views. Russell’s paradox demonstrated that Frege’s broad commitments

²²See, e.g., Gregory Currie, *Frege: An Introduction to His Philosophy* (Totowa, New Jersey: Barnes and Noble Books, 1982), 91-92; Leonard Linsky, *Referring* (London: Routledge and Kegan Paul, 1967), chap. III.

²³See especially W.V. Quine, “Two Dogmas of Empiricism,” in *From a Logical Point of View*, 20-46.

to both functions and classes could not both be held together unmodified. The addition of the commitment to *Sinne* and *Gedanken* into Frege's logical system may well give rise to additional paradoxes and conundrums.

One example of historical note comes to us from Russell. Near the time he wrote *The Principles of Mathematics*, Russell held an ontology of propositions. However, he came to realize that this commitment was inconsistent with some of the other logical views he held at the time. This was due to a certain contradiction he discusses in Appendix B of the *Principles*. If propositions are objectively real entities, then they can be members of classes, i.e., there are classes of propositions. There would then be a class of all propositions believed by Kevin, and a class of all propositions about dolphins, and a class of all true propositions, and so forth. For each class of propositions m , we can generate a proposition stating that all propositions in m are true, or in Russell's terminology, a proposition "stating the logical product" of m . Some of these propositions are themselves in the class of propositions whose logical product they assert. For example, the proposition that all propositions in the class of all true propositions are true is itself a true proposition, and so, it is a member of that class. However, some propositions are not in the class whose logical product they assert. I do not myself believe that all propositions in the class of propositions believed by me are true, and so, this proposition is not in its corresponding class. However, let us consider the class of *all* propositions stating the logical product of a class they are not in. This class itself has a proposition stating its logical product. If we ask the question of whether this proposition is in its corresponding class, we arrive at a contradiction. This proposition appears to be in the class if and only if it is not.²⁴

Russell wrote to Frege about this paradox in September 1902, suggesting that it might be another paradox plaguing Frege's work.²⁵ Frege responded by pointing out that he did not hold the same sort of ontology of propositions that Russell did, and suggested that the formulation of the paradox Russell offered is solved by the theory of *Sinn* and *Bedeutung* (PMC 147-8). Russell responded in turn by pointing out that although his explicit formulation in terms of propositions might not apply, given the commitment to *Gedanken* Frege makes, the same problem should appear *mutatis mutandis*.²⁶ Although Frege then considered ways in which this might be accomplished, his ultimate conclusion

²⁴Bertrand Russell, *The Principles of Mathematics*, 2d ed. (New York: W.W. Norton, 1937), §500.

²⁵Russell to Frege, London, 29 September 1902, *Philosophical and Mathematical Correspondence*, by Gottlob Frege, 147-8.

²⁶Russell to Frege, London, 12 December 1902, *Philosophical and Mathematical Correspondence*, by Gottlob Frege, 150-1.

was that it is not clear by what inference rules the contradictory conclusion is reached, and asks Russell what they are (*PMC* 152-4).

It is true that the paradox cannot be formulated in Frege's Begriffsschrift of the *Grundgesetze*, but it by no means follows that it would not arise if that system is expanded to include the logic of *Sinn* and *Bedeutung*. Frege, therefore, was a bit unfair to Russell. It was not Russell's job to formulate the inference rules that would apply to Frege's Begriffsschrift when expanded to include commitment to *Sinne* and *Gedanken*. Russell did not share Frege's ontology. If Frege had included symbols, axioms and rules in his Begriffsschrift regarding *Sinne* and *Gedanken*, then he himself could have evaluated whether or not a corollary of this paradox plagued his work or not. As it is, without the logic of *Sinn* and *Bedeutung* in place, it is impossible to determine whether this or similar paradoxes plague Frege's logical systems. I shall discuss this and similar paradoxes in much more detail in later chapters. The crucial point here is simply once again that Frege's logical and semantic views can only be fully evaluated with the construction of a logical calculus for the theory of *Sinn* and *Bedeutung*.

FILLING THE GAP

In sum, an expansion of Frege's Begriffsschrift to include the logic of *Sinn* and *Bedeutung* is necessary for (at least) the following reasons, 1) to make the Begriffsschrift capable of expressing statements whose expression in natural language includes indirect contexts, 2) to make the Begriffsschrift capable of capturing inferences involving such contexts, 3) to provide an adequate response to certain objections raised against Frege's views, 4) to assess whether or not certain other objections against Frege's views bear weight, 5) to make the Begriffsschrift fully reflect Frege's ontological commitments, and 6) to determine whether or not Frege's metaphysical commitments are formally consistent with one another. While Frege himself never undertook such an expansion (or, if he did, the works were destroyed in the second world war²⁷), one can extract the core of such a development from his non-formal works on the theory of *Sinn* and *Bedeutung*. In the subsequent chapters, I take on the project of attempting to redress Frege's failure to develop a logical calculus for the theory of *Sinn* and *Bedeutung*, and to fashion a system that is as close as possible to the system Frege himself would have created had he done so himself. My aim will not be to develop what I myself take to be the correct logical calculus for treating indirect contexts, but rather to develop a *fully Fregean*

²⁷For more information on the destruction of portions of Frege's *Nachlaß*, see Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach, "History of the Frege *Nachlaß* and the Basis for this Edition," in *Posthumous Writings*, by Gottlob Frege, ix-xiii.

logical calculus of *Sinn* and *Bedeutung*, based upon a careful reading of Frege's own works.

I believe this is a project for which we would have Frege's full support. We have already seen that Frege viewed his *Begriffsschrift* as a mere first step towards a fuller, more comprehensive scientific language. Certainly, Frege would have seen truths about *Sinne* and *Gedanken*, about meaning, and about propositional attitudes as areas capable and worthy of scientific study.²⁸ In fact, whole sciences, such as psychology, may very well be impossible to capture in the *Begriffsschrift* unless it is expanded to be able to express belief states and the like. Thus, he would surely welcome the expansion of the *Begriffsschrift* in the ways necessary in order to include such areas of study. Yet, the expansion of the *Begriffsschrift* to include the theory of *Sinn* and *Bedeutung* cannot be considered to be entirely on a par with the expansion to include sciences such as astronomy or medicine. In Frege's mind, the theory of *Sinn* and *Bedeutung* was itself a part of logic proper.²⁹ Therefore, without its inclusion in the *Begriffsschrift*, Frege has not even succeeded in capturing the purely logical core of his *lingua characterica*.

At times, Frege even hints at engaging in such a project. This is especially true in his correspondence with Russell over the *Principles of Mathematics* Appendix B paradox, where the lack of a logical calculus for *Sinn* and *Bedeutung* is most sorely felt. Therein, Frege speaks of developing "special signs" for use "in indirect speech, though their connection with the corresponding signs in direct speech should be easy to recognize" (*PMC* 153). Indeed, at one point, after summarizing the theory of secondary *Bedeutungen* in indirect speech, Frege even seems to offer Russell an apology for not including signs dealing with oblique contexts in the *Begriffsschrift*. As he puts it, "in my *Begriffsschrift* I did not yet introduce indirect speech because I had as yet no occasion to do so" (*PMC* 149). A word of explanation is perhaps in order as to how I understand this apology. In his translation of this sentence in *Philosophical and Mathematical Correspondence*, Hans Kaal renders Frege's word "*Begriffsschrift*" as "*Conceptual Notation*," capitalized and italicized. However, this makes it appear as though Frege were speaking of his work entitled *Begriffsschrift* of 1879. However, it is quite unlikely that Frege would

²⁸We must remember that "science" here proxies for the German "*Wissenschaften*," and unlike its English counterpart, this term does not presuppose an empirical method. Mathematics, logic and philosophy would be understood as "sciences" in this sense.

²⁹We have seen how the theory was developed with largely logical motivations. However, for more on how Frege saw his semantic theories as part of logic, see Hans Sluga, "Frege and the Rise of Analytic Philosophy," *Inquiry* 18 (1975): 471-87; Dummett, *Interpretation of Frege's Philosophy*, chap. 3.

refer back to his early work of 1879 in a letter to Russell in 1902, especially given that the correspondence began by Russell expressing familiarity with the *Grundgesetze* (and not the *Begriffsschrift*).³⁰ It is much more likely that he is referring to the *Begriffsschrift*—the logical notation—developed in the *Grundgesetze*. (In the actual letter, the word is not underlined or otherwise marked as a title of a work, and is capitalized only because all German nouns are.) After all, Frege had not even developed the theory of *Sinn* and *Bedeutung* when he wrote the *Begriffsschrift*, and this would be the real reason indirect speech is not discussed therein. However, the reason indirect speech is not included in the *Begriffsschrift* of the *Grundgesetze* is another matter. The *Begriffsschrift* of the *Grundgesetze* was intended primarily to carry out that work's logicist project of demonstrating the basic truths of arithmetic from purely logical principles. Mathematics, however, does not typically call for indirect speech or the expression of propositional attitudes. This is why the *Begriffsschrift* therein is not expanded in ways dictated by the theory of *Sinn* and *Bedeutung*: it was not called for given the limited aims of *Grundgesetze*. However, in the letter to Russell, Frege seems to give his stamp of approval to including within the *Begriffsschrift* sufficient resources for capturing indirect speech and the other aspects of his semantic theory.

We seem to have Frege's full blessing in attempting to expand his *Begriffsschrift* to encompass the logic of *Sinn* and *Bedeutung*. Ironically, however, it will be found that when Frege's *Begriffsschrift* is expanded in line with his own semantic theories, indeed, in ways he himself suggests, new *internal* problems and difficulties with his overall philosophy are revealed. In particular, Frege's logical theory is subject to Cantorian antinomies analogous to Russell's paradox, but which are independent of Frege's Basic Law V, the axiom in Frege's extant *Begriffsschrift* that leads to Russell's paradox proper. The conclusion to reach from these difficulties, however, is not that Frege would not, after all, have approved of an expansion of his *Begriffsschrift*. The conclusion is rather that Frege missed certain flaws in his views only because of his failure to include his philosophical commitment to *Sinne* in his logical system. As a philosopher interested in truth, we can only assume Frege would have been appreciative of honest attempts to uncover the difficulties with his views. This is how Frege responded to Russell when Russell first began to uncover such difficulties. It is worth quoting Russell's own words:

As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work

³⁰Russell's first words to Frege are, "I have known your *Grundgesetze der Arithmetik* for a year and a half." See Russell to Frege, 16 June 1902, 130.

was on the verge of completion, much of his work has been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman, and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.³¹

³¹Russell to van Heijenoort, Penrhyndeudraeth, 23 November 1962, *From Frege to Gödel*, 127.

The Logic of the *Grundgesetze*

LOGICAL LANGUAGE AND THE CONTENT OF LOGIC

As discussed in the previous chapter, the Begriffsschrift of the *Grundgesetze* was designed to be both a *lingua characterica* and a *calculus ratiocinator*. Frege was not interested in creating merely a formal system or logical calculus, but a new, more logically perspicuous, *language*. He was aware that given its rigid formation rules, axioms and derivation rules, it would be possible to study the Begriffsschrift merely as an uninterpreted formal system for the manipulation of syntax. However, he was adamant that if it were to be studied thusly, the Begriffsschrift would lose its mathematical and logical interest (*GG* §§90-1). For him, the Begriffsschrift was interesting precisely for its ability to clearly express logical and mathematical *Gedanken* in a revealing way. In his mind, the ability of the Begriffsschrift to express such truths is not a feature of its syntax alone, but the semantics offered for each symbol included within it. Frege was quite careful to explain the semantics of the Begriffsschrift in quite some detail precisely so that its interpretation would be fixed and unambiguous.

This attitude towards logical systems is quite different from many attitudes prevalent today. There seems to be a near consensus now that it is worthwhile to study formal logical calculi as syntactic systems in isolation from their intended interpretations. This attitude is no doubt derived in part from the influence of the formalist school in mathematics, of which Frege himself was an ardent critic (see e.g., *CP* 112-21, 293-340, *PW* 165-6, 275, *GG* 89-109). The difference in attitude is perhaps also due to differing respective views on the very nature of the study of logic. Many logical positivists and verificationists, for example, argued that a proposition gets its meaning from the conditions of its empirical confirmation or disconfirmation. Given that logical truths are not susceptible to

empirical test, they were taken to be without content. Although verificationists are few and far between today, many still hold logic to be without content. Endorsement of a particular logical system is not thought to commit one to any particular body of metaphysics or other philosophical views. Logical connectives are not thought to stand for anything “real”; they are understood as purely syncategorematic marks. According to Wittgenstein, for example, logical connectives are more like punctuation marks than signs that have logical objects as their meanings.¹ Logical truths are thought to be “analytic” in a *deflationary* sense: they are intrinsically incapable of revealing anything informative about reality. These are views with which Frege would find little to agree.

For Frege, logic was thought to be a pure science with its own subject matter. Logical truths, although analytic, can be *informative*; they can tell us about objectively existing logical entities such as value-ranges, truth functions and the “objects” the True and the False. Negation, e.g., was thought by Frege to be a concept with the same ontological status as humanity or wisdom. In his eyes, the science of logic has a special relation to truth, similar to the relation of aesthetics to beauty, ethics to goodness, and even physics to motion, gravity and heat (*CP* 350). This science is not purely formal; it involves *discovering* truths about logical concepts, relations and objects (*CP* 338). The *Begriffsschrift* is designed as a language for expressing these truths. Its signs for truth functions and the like cannot simply be treated as syncategorematic symbols or be reinterpreted without doing injustice to its very purpose, any more than one can simply reread the English word “cat” as standing for dogs or standing for nothing at all. Each *Begriffsschrift* proposition is designed to express a unique *Gedanke*, and, in Frege’s eyes, is not readily amenable to multiple interpretations.²

Frege is generally credited for having founded the modern period in logic, and in particular, with developing the first modern higher-order predicate calculus. It is true that Frege’s works revolutionized logical theory and are the source of most of the major insights that are characteristic of modern higher-order logics. However, Frege’s *Begriffsschrift*, while in many ways similar to a second-order predicate calculus, enshrines aspects of his views on the nature of logic, of predication, and of meaning to such an extent that it cannot simply be relegated to the status of a standard system of predicate logic. Instead, it must be studied carefully in relation to Frege’s other views to be fully appreciated. There

¹Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*, trans. C.K. Ogden (New York: Routledge, 1922), §§5.42-5.4611.

²On Frege’s “tendency to minimize the category of the syncategorematic,” see Alonzo Church, “The Need for Abstract Entities in Semantic Analysis,” *Proceedings of the American Academy of Arts and Sciences (Daedalus)* 80 (1951): 100-12.

are important differences in the syntax and semantics of Frege's Begriffsschrift as compared to contemporary systems that must be brought to light if one is to truly understand the unique position of the Begriffsschrift as a logical language. In studying these aspects of Frege's logical work, we must bear in mind Frege's historical position. No one had attempted to create a logical language of the same sort before, and one cannot hold him accountable for failing to match his vocabulary to contemporary terminology or provide a semantic account for his system of the sort favored in wake of Tarski's formal semantics.

However, the difference between Frege's understanding of his Begriffsschrift as a language with a fixed intended interpretation and some modern understandings of logical calculi can also be overstated. Jean van Heijenoort, for example, stresses that Frege's Begriffsschrift, as a *lingua characterica* (and not merely a *calculus ratiocinator*), has a sort of *universalism* to it. Quantifiers in Frege's Begriffsschrift range over *all* objects and *all* functions in existence; because of its fixed interpretation, it is not seen as permissible to change the domain of discourse discussed, or agree to consider only a certain subset of entities for a certain time or within a certain context. This is in stark contrast with both the older approaches of such logicians as Boole and DeMorgan, who allowed such changes at will, as well as contemporary approaches to formal semantics, which typically involve discussion of different possible "models" for the semantics of a logical system. Frege allows discussion of only one domain: *the* actual universe.

While van Heijenoort is right about these aspects of Frege's attitude about his Begriffsschrift as a *lingua characterica*, he draws the following questionable conclusion:

[An] important consequence of the universality of logic is that nothing can be, or has to be, said outside of the system. And, in fact, Frege never raises any metasystematic questions (consistency, independence of axioms, completeness).³

This takes matters too far. While the domain of discourse of Frege's Begriffsschrift is universal, this does not mean he would take "metasystematic" questions to be impossible or uninteresting. Indeed, Frege was one of the first to clearly draw the now standard distinction between "object language" and "metalanguage," the former referring to the language studied, and latter referring to a *distinct* language in which that language is discussed. Frege's own terms, respectively, were "Darlegungssprache" ("explained language") and "Hilfssprache" ("helping language," *PW* 260-1). Both are *languages* (*Sprachen*)

³van Heijenoort, "Logic as Calculus and Logic as Language," 326.

in the full sense. The *Darlegungssprache* may even contain quantifiers ranging over all objects, and thus, in a sense, its domain of discourse includes everything. Yet, especially, if that language is a formal language, it is still fruitful to discuss the language and its semantics, as it were, “from the outside.” And while it is true that Frege was not always interested in the *same* metasystematic issues as contemporary logicians, this is to be expected given his place in history. Frege did raise some metasystematic issues. For example, the *Grundgesetze* contains an extensive (but flawed) attempt at a proof—in *German*—that every Begriffsschrift expression has a unique *Bedeutung* (*BL* §§28-32). Indeed, in the secondary literature, there has been increasing support for the view that some of Frege’s discussion of his Begriffsschrift represents his own peculiar way of doing a sort of formal semantics.⁴

With Frege’s unique understanding of his Begriffsschrift as a logical language in mind, let us turn to a consideration of some of the important details of his approach.

FUNCTIONALITY AND PREDICATION

In contemporary predicate logic, predicational form is taken as fundamental. A standard statement takes a form such as “Fa”, (where “F” is a predicate expression and “a” is an individual term) which would be read “a is F.” Notoriously, however, Frege abandoned the subject/predicate form of ordinary language in favor of the function/argument form prevalent in mathematical formulae. Thus, instead of predicate letters, the Begriffsschrift makes use of function signs. In the previous chapter we considered the function sign “ $\mathcal{P}()$ ”, which stands for a function that yields as value the True in case its argument is a planet, and yields the False otherwise. Instead of the formula “Fa”, which might be read as “Venus is a planet,” Frege would use a term “ $\mathcal{P}(m)$ ”, reading it as “the truth-value of Venus’s being a planet.”⁵

An important difference between these two approaches to predication is immediately noticeable. The form “Fa” of predicate logic has assertoric force; by itself it asserts that Venus is a planet. However, Frege’s “ $\mathcal{P}(m)$ ” has no similar assertoric force; it does not assert anything, but simply names a truth-value. This was quite deliberate on Frege’s part. Frege bemoaned the feature of ordinary language that assertoric force was bound up with predication (*CP* 149, 247-8, *PW* 138-43). Thus, he developed a functional notation that separated the

⁴See, e.g., Richard Heck, Jr., “Frege and Semantics,” in *The Cambridge Companion to Frege*, ed. Thomas Ricketts (Cambridge: Cambridge University Press, forthcoming).

⁵For more on how to read the propositions of the Begriffsschrift in English, see Gregory Landini, “Decomposition and Analysis in Frege’s *Grundgesetze*,” *History and Philosophy of Logic* 17 (1996): 121-39.

two. In order to assert that Venus is indeed a planet, one must add the *judgment stroke*, “ \vdash ”, to the term. The judgment stroke carries assertoric force, it asserts that the term it precedes denotes the True. Therefore,

$$\vdash \mathcal{P}(m)$$

could be read as “the truth-value of Venus’s being a planet is the True,” or more simply, “Venus is a planet.” This allowed him to transcribe predicates using expressions standing for functions that take objects as arguments and yield truth-values as value. To repeat, Frege dubbed functions, such as that denoted by “ $\mathcal{P}(\)$ ”, that take a single argument and yield a truth-value for any argument, “concepts” (*Begriffe*), and called functions of multiple arguments that yield truth-values for any set of arguments, “relations.”

Frege quite deliberately divorced assertoric force from predication, understood in the limited sense of simply considering an object as “falling under” a given concept. Far too often, readers of Frege ignore this, and as a result, have difficulties understanding how propositions in Frege’s *Begriffsschrift* are to be read or understood. Jeremy Walker, for example, would read “ $\vdash \mathcal{P}(m)$ ” ungrammatically (and perhaps even nonsensically) as “Venus is a planet is a fact,” and G.E.M. Anscombe suggests (although does not endorse) the reading “Venus is a planet is the True.”⁶ Others conclude that it is impossible to express *Begriffsschrift* propositions in English precisely because of the failures of these sorts of readings.⁷ However, the real reason these readings fail is that they erroneously attempt to maintain the assertoric nature of predicates in ordinary language. Frege insisted upon the use of the judgment stroke to make an assertion in the *Begriffsschrift*, because of the difference as he saw it between predication (again, here limited to viewing an object as falling under a certain concept) and assertion. An object can be considered as falling under a certain concept, for example, in the context of suppositions, stipulations or conditionals, where assertion is not involved. Because predication (in this sense) and assertion do not necessarily go hand in hand, and because in an ideal language, one must “endeavor to have every objective distinction reflected in symbolism” (*CP* 247), a special sign for assertion is needed, according to Frege.

Care must be taken not to confuse the judgment stroke, as Frege uses it, with the “thesis sign” used in many contemporary logical works, usually also

⁶Jeremy D.B. Walker, *A Study of Frege* (London: Oxford University Press, 1965), 16; G.E.M. Anscombe, *An Introduction to Wittgenstein’s Tractatus* (New York: Harper Torchbooks, 1959), 104.

⁷See, e.g., William C. Heck and William G. Lycan, “Frege’s Horizontal,” *Canadian Journal of Philosophy* 9 (1979): 479-92.

written “ \vdash ”. The thesis sign is used by contemporary logicians before a formula to mean that the formula is a theorem, or, more precisely, that it syntactically follows from the axioms and derivation rules (or, if a formula or formulae appear *before* the thesis sign, that it syntactically follows from the axioms, rules and the formulae that appear before it.) This is very different from Frege’s use of the judgment stroke. Firstly, Frege would use the judgment stroke in any asserted proposition, not just with theorems of the system. Moreover, strictly speaking, Frege’s judgment stroke does not precede a formula. It is instead used to *transform* what is, on its own, a term for a truth-value, *into* a formula or proposition in the full sense.

Sometimes, when Frege speaks loosely, he speaks as though there were no difference in his mind between propositions and names of truth-values (see e.g., *CP* 163, *PW* 195). However, when he is being careful, he does mark a distinction (*BL* §5). What allows him to speak loosely in this manner is that the addition of the judgment stroke, while carrying assertoric force, does not affect the *Sinn* or *Bedeutung* of the expression. The term “ $\mathcal{P}(m)$ ” and the proposition “ $\vdash \mathcal{P}(m)$ ” express the same *Gedanke* and denote the same truth-value, despite that one carries assertoric force while the other does not. As I understand Frege, while propositions with the judgment stroke *do* have truth-values as their *Bedeutungen*, they are not simply *names* of them, because propositions, as *asserted*, do more than names do.

The importance of the difference between predicational notation and functional notation can hardly be overemphasized. In normal predicate logic, there is an important difference between formulae of the language, which assert that something is the case, and terms of the language. For example, “ Fa ” is a formula, while “ a ” is a term. The latter refers to an object, while the former does not refer to anything; it asserts something. For Frege, however, both “ $\mathcal{P}(m)$ ” and “ m ” are terms, the latter standing for Venus, the former for a truth-value. The only procedure that transforms a term into something that is not a term in the *Begriffsschrift* is the addition of the judgment stroke. This approach has serious implications for the development of the *Begriffsschrift*. For example, it allows for functional embedding of a type impossible in predicate logic. For example, “ $\mathcal{P}(\mathcal{P}(m))$ ” is a well-formed term in the *Begriffsschrift*, standing for the truth-value of (the truth-value of Venus’s being a planet)’s being a planet. Since the truth-value of Venus’s being a planet is the True, and the True is not a planet, this term stands for the False.

While Frege allows the value of one function to be placed in the argument place of another function, he does not allow a function sign to stand on its own or alone in the argument place of a function of the same level. Thus, he rules out expressions such as “ $\mathcal{P}(\mathcal{P})$ ”. This is due to the essentially “unsaturated” (*ungesättigt*) or “incomplete” nature of functions discussed briefly in the

previous chapter. This important feature of functions is captured in the Begriffsschrift by requiring that a function sign always be completed by its arguments in any well-formed expression. Even when discussing a function sign of the Begriffsschrift “in the metalanguage” as it were, Frege continues to make perspicuous the unsaturated nature of functions by the use of empty parentheses to mark its argument place (e.g., “the function $\mathcal{P}(\)$ ”), or, especially in cases in which there are multiple argument places, by using the signs “ ξ ” and “ ζ ”, which are said to *indicate* but not denote possible arguments of the function (e.g., “the function $\mathcal{P}(\xi)$ ”).

Another important implication of Frege’s functional approach is that it allows him to treat truth-functional operators as *functions* in the strict sense. The three primary first-level truth function signs in the Begriffsschrift are the following:

- 1) “ $\text{— } \xi$ ” denotes the True, if what replaces the “ ξ ” denotes the True, and denotes the False otherwise. (*BL* §5)
- 2) “ $\text{— } \text{— } \xi$ ” denotes the False, if what replaces the “ ξ ” denotes the True, and denotes the True otherwise. (*BL* §6)
- 3) “ $\begin{array}{l} \text{— } \zeta \\ \text{— } \xi \end{array}$ ” denotes the False, if what replaces the “ ξ ” denotes the True and what replaces the “ ζ ” denotes something other than the True, and denotes the True otherwise. (*BL* §12)

The functions denoted by these signs take any object as argument and yield truth-values. These function signs must be flanked by terms of the Begriffsschrift, and the whole consisting of the function sign and terms will also constitute a term standing for a truth-value. Because the truth functions are functions in the strict sense—the first two are concepts, the third a relation—they are even included in the range of function quantifiers.

The first function above is called “the horizontal.” At first blush, it may seem inessential, considering that if its argument is a truth-value, that same truth-value is returned as value.⁸ However, it is important in cases in which the argument is something other than a truth-value. For example, without it, one cannot state the law of double negation, as it is not generally true that “ $\text{— } \text{— } \xi$ ” must have the same *Bedeutung* as “ ξ ”, although it is true that it must have the

⁸Indeed, some writers on Frege have been puzzled by its importance. See Heck and Lycan, “Frege’s Horizontal,” 479-92; Michael Dummett, *Frege: Philosophy of Language*, 2d ed. (Cambridge, Mass.: Harvard University Press, 1981), 315.

same *Bedeutung* as “— ξ ”.⁹ The second of the above truth functions is the negation function, and the third the conditional function. These are analogous to the contemporary truth functions written as “ \neg ” and “ \rightarrow ”, but again, are treated as functions proper, and their signs in the Begriffsschrift (unlike “ \neg ” and “ \rightarrow ”) are flanked by *terms* with the whole consisting of sign and argument terms also constituting a term. Conjunction and disjunction functions can be defined in terms of the negation and conditional functions in the normal fashion. The Begriffsschrift also contains an identity sign:

“ $\xi = \zeta$ ” denotes the True if what replaces the “ ξ ” has the same *Bedeutung* as what replaces the “ ζ ”, and denotes the False otherwise. (*BL* §7)

Because the terms flanking the identity sign can stand for truth-values as well as other objects, the identity sign is able to play the role often played by the biconditional in standard predicate logic.¹⁰

QUANTIFIERS AND GOTHIC LETTERS

Of the many ways in which Frege’s logical system represents an improvement over the logical systems that preceded it, many cite Frege’s most important advance as the introduction of quantifier notation for generality.¹¹ Yet, important as this advance is, there is a fair amount of disagreement over the details of Frege’s own understanding of quantification and what we typically now call “variables.” Questions arise regarding such things as whether Frege subscribed to (what is now called) a “substitutional” or to an “objectual” or “ontological” theory of quantification, whether or not Frege’s logic contained “free variables,” and how similar to contemporary (Tarskian) approaches we can understand the Fregean semantics for quantification and variables.¹² Some of these questions

⁹For more on the necessity of the horizontal, see Landini, “Decomposition and Analysis in Frege’s *Grundgesetze*,” 123-4.

¹⁰It is worth noting, however, that the function “ $(\text{— } \xi) = (\text{— } \zeta)$ ” is a closer analogue of the biconditional of predicate logic.

¹¹See, e.g., Leslie Stevenson, “Frege’s Two Definition of Quantification,” *Philosophical Quarterly* 23 (1973): 207; William Kneale and Martha Kneale, *The Development of Logic* (London: Oxford University Press, 1962), 511; Peter T. Geach, review of *Frege: Philosophy of Language*, by Michael Dummett, *Mind* 85 (1976): 437; Dummett, *Frege: Philosophy of Language*, chap. 2.

¹²These issues arise in such works as John N. Martin, “A Fregean Interpretation of the Quantifiers,” *Logique et Analyse* 20 (1977): 122-8, “The Semantics of Frege’s *Grundgesetze*,” *History and Philosophy of Logic* 5 (1984): 143-76; Richard G. Heck, Jr.,

are at best asked only anachronistically of Frege. Frege was working before certain contemporary debates regarding the nature of quantification had arisen, and it would not be at all surprising if Frege's views did not fall squarely within one rival camp or another. Nevertheless, in what follows, I shall attempt to explain as best as possible Frege's own understanding of the quantifiers and so-called "variables" of his system as I interpret it.

I use scare quotes when speaking of "variables" in Frege's logic because he himself did not use that term, and in fact explicitly disavowed it, claiming that "variables are not a part of the proper subject-matter of arithmetic [or logic]" (*PW* 238, see also *PMC* 81, *CP* 288, *PW* 159-64). Of course, Frege's dislike of the term stemmed principally from the sloppy way in which it was used in most contemporaneous mathematical works, and so it is not clear from this alone that the Roman and Gothic letters that appear in Frege's Begriffsschrift cannot properly be referred to as "variables" in the contemporary sense. At first blush, the Roman and Gothic letters appearing in the Begriffsschrift seem to be equivalent to the free and bound variables of modern logic, respectively. Yet, there are important disanalogies between Frege's understanding of his Roman and Gothic letters and how the post-Tarskian logician understands them that are worth exploring in some detail.

In order to grasp Frege's understanding of Roman and Gothic letters, it is necessary to look closely at the dialogical process Frege goes through in the *Grundgesetze* as he introduces them. He first discusses generality in §8. (Here, I shall follow Frege's lead in using common mathematical function signs and numerals for aiding in giving examples.) For certain functions, e.g., those denoted by " $\xi \cdot (\xi - 1)$ " and " $\xi^2 - \xi$ ", we want to assert not only that they have the same value for some particular argument (e.g., " $5 \cdot (5 - 1) = 5^2 - 5$ "), but that they have the same value for *all* arguments. Mathematicians typically write such an equation as follows, " $x \cdot (x - 1) = x^2 - x$ ". Here the inclusion of the Roman letter "x" instead of a constant numeral changes a particular proposition into a general one. Taking a cue from mathematics, Frege considers using letters in his Begriffsschrift in order to capture generality. He considers for a moment defining the *Bedeutung* of a group of signs in his Begriffsschrift containing a Roman letter "x" as follows:

By " $\Phi(x)$ " is to be understood the True, if the value of the function $\Phi(\xi)$ is the True for every argument; otherwise, it denotes the False. (*BL* §8)

"Grundgesetze der Arithmetik I, §§29-32," *Notre Dame Journal of Formal Logic* 38 (1997): 443-6, "Frege and Semantics," 25-9; Stevenson, "Frege's Two Definitions of Quantification," 207-23.

However, he quickly abandons this definition because it gives rise to scope ambiguities. If we consider the expression, “ $\neg 2 + 3x = 5x$ ”, it is unclear whether its *Bedeutung* is the True or the False. It is the True if we apply the definition of “ $\Phi(x)$ ” to the function “ $2 + 3\xi = 5\xi$ ” and then apply the definition of “ $\neg \xi$ ”. However, its *Bedeutung* is the False if we apply the definition of “ $\Phi(x)$ ” to the function “ $\neg 2 + 3\xi = 5\xi$ ”. It is essential to resolve the question of whether the scope of the negation or of the generality is wider, or else it is impossible to determine the *Bedeutungen* of expressions involving such variables.

To overcome scope ambiguity, Frege introduces a new way of expressing generality involving Gothic letters and the concavity sign. For this he introduces the following definition, obviously similar to the one just considered:

“ $\neg \ulcorner \Phi(a)$ ” is to denote the True if for every argument the value of the function $\Phi(\xi)$ is the True, and otherwise is to denote the False. (*BL* §8)

This notation allows us to resolve scope ambiguity, for the scope of the generality is to be understood as anything to the right of the concavity. Thus, “ $\neg \ulcorner 2 + 3a = 5a$ ” denotes the False, since “ $\neg 2 + 3\xi = 5\xi$ ” does not yield the True for all arguments, while “ $\ulcorner \neg 2 + 3a = 5a$ ” denotes the True, since “ $2 + 3\xi = 5\xi$ ” also does not yield the True for all arguments. One is also able to capture existential propositions by placing negation signs both before and after the concavity. For example, the truth-value of there being *something* for which the function denoted by “ $2 + 3\xi = 5\xi$ ” yields the True can be written as “ $\ulcorner \neg \ulcorner 2 + 3a = 5a$ ”.

Gothic letters, so understood, are obviously similar in nature to modern bound variables. With the exception that Frege does not allow vacuous quantification, syntactically, they are in all ways equivalent; the Gothic letters fill the syntactic argument places of the first-level function signs. However, semantically, the term “bound variable” is somewhat misplaced on Frege’s Gothic letters. Gothic letters never occur save in “bound” occurrences, nor is there anything “variable” about the *Bedeutungen* of expressions containing them. Semantically, it is perhaps best to view Gothic letters simply as parts of a larger expression for the second-level function of generality. To see this, we must delve further into how it is that Frege understands quantification. As Frege explains in §21 of the *Grundgesetze*, the universal quantifier is to be understood as a “second-level concept.” The division of levels of functions is based on Frege’s claim that the unsaturatedness of functions comes in different varieties depending upon the sort of arguments they take. First-level functions, such as the truth functions, take objects as their arguments. Second-level functions, however, are functions that take first-level functions as arguments. Frege uses

the sign “ ϕ ” for second-level functions analogously to how he uses “ ξ ” for first-level functions, with “ ϕ ” indicating the argument place of the second-level function (wherein a function name might be inserted). The truth-values denoted by the expressions “ $\ulcorner a = a \urcorner$ ” and “ $\ulcorner a^2 = 4 \urcorner$ ” are the values of the second-level function denoted by “ $\ulcorner \phi(a) \urcorner$ ” (the concept of *generality*) for the concepts of *self-identity* ($\xi = \xi$) and *square root of four* ($\xi^2 = 4$) taken as argument, respectively. Similarly, “ $\lrcorner a = a \lrcorner$ ” and “ $\lrcorner a^2 = 4 \lrcorner$ ” denote two values of the second-level function denoted by “ $\lrcorner \phi(a) \lrcorner$ ”, the concept of *existence*. In these examples, the first-level concepts play a role similar to that of logical subject in that something is being predicated of them (*BL* §21, *CP* 282-3, *PMC* 93). Just as an object is said to “fall under” a first-level concept if the concept has the True as value for the object as argument, when a second-level concept has the True as value for a first-level function as argument, the first-level function is said to “fall within” the second-level concept (*CP* 190, 283).

Given that when second-level concepts are predicated of first-level functions, the latter play the role of logical subject, one might wonder why Frege chose not to employ a simpler notation such as, “Gen ($\xi = \xi$)” to denote the truth-value of everything’s being self-identical, with “Gen” standing for generality. The problem with this notation is that it does not adequately show how it is that first and second-level functions *mutually* saturate. Since “ ξ ” appears in the expression, it appears incomplete. Moreover, the importance of using the Gothic letters is especially evident in cases involving multiple generality. Consider the claim that *every* number has *some* successor. The truth-value of this claim would be written in the Begriffsschrift as:

$$\ulcorner \lrcorner \lrcorner \lrcorner e = a + 1 \lrcorner \lrcorner \lrcorner$$

However, in the revised notation considered earlier, we could at best write this as:

$$\text{Gen} (\lrcorner \text{Gen} \lrcorner (\xi + 1 = \zeta))$$

and this would leave ambiguous which occurrence of the second-level concept denoted by “Gen” was saturating which spot in the doubly unsaturated function denoted by “ $\xi + 1 = \zeta$ ”. It is for this reason a different Gothic letter is associated with the concavity to reveal how it is that the second-level concept saturates with the first-level concept. In this way, Frege is able to differentiate the above expression from the quite different expression:

$$\lrcorner \lrcorner \lrcorner e = a + 1 \lrcorner \lrcorner \lrcorner$$

which stands for the truth-value of every number's having a predecessor. The Gothic letters thus really serve as mere *parts* of a complex sign for the second-level function sign for generality. Their task is to reveal how it is that the second-level function mutually saturates with the first-level function.

From this perspective we can perhaps begin to answer the question as to whether Frege held a substitutional or an objectual account of the semantics of quantification. Although he may have made remarks early in career that make his account seem substitutional,¹³ this is not at all true in the *Grundgesetze*. The concept of generality is explained in terms of a function's yielding the True for every object as argument, not in terms of an expression's denoting the True regardless of linguistic substitutions within it. In this regard, Frege's account is entirely objectual. However, Frege nevertheless deviates from many modern objectual understandings of quantification in that his Gothic letters are not "associated" semantically with a range of objects. They do not stand for objects at all. Instead they are simply parts of a more complex sign. In Frege's terminology, in the expression, " $\text{—}\overset{\text{a}}{\text{—}}\text{—} a = a$ ", the occurrences of "a" to the right of the concavity do not denote objects, but merely *indicate* the argument places being mutually saturated by the second-level function (*BL* §17).

I believe that many writers make far too much out of Frege's distinction between a sign's indicating something and a sign's having a *Bedeutung*, suggesting that indication is a relation between signs and objects different from, but on a par with, having a *Bedeutung*, but which needs further investigation since Frege tells us little. However, as I understand him, all Frege means by suggesting that Roman and Gothic letters "indicate" is that they reveal the sort of "supplementation that is needed" (*CP* 141) within an incomplete expression; they merely highlight the argument places within a functional expression. In this, Roman and Gothic letters are semantically no different from Frege's signs " ξ " and " ζ ". The difference in this case is only that Gothic letters are associated with a second-level concept in such a way that they reveal which argument-spot in the first-level function taken as argument is mutually saturated by the second-level concept.

In Frege's own way of understanding things, his Begriffsschrift contained no variables, only constants. As he understood it, the expression, " $\text{—}\overset{\text{a}}{\text{—}}\text{—} a = a$ " was thought to be composed of two function-constants, " $\xi = \xi$ " denoting the first-level concept of *self-identity*, and the complex second-level function sign " $\text{—}\overset{\text{a}}{\text{—}}\text{—} \phi(a)$ " denoting generality, save with the signs indicating unsaturatedness removed, and the Gothic letter filling the argument place of the first-level function sign. The Gothic letter is simply part of the second-level function *constant* for generality. The mutually saturating nature of higher-level functions

¹³For more on this see Stevenson, "Frege's Two Definitions," 207-11.

requires that each occurrence of such a function be associated with a different letter. For the sign for generality—the universal quantifier—Frege employs Gothic letters. For other higher-level functions, Frege employs Greek letters (*BL* §§9-10, *PW* 121). Frege himself did not think of these letters as “variables,” although in modern parlance, the similarity to bound variables is obvious. It is a matter of indifference what terminology we employ, provided that we maintain Frege’s own semantics. In Frege’s eyes, the *Sinn* of “ $\forall \alpha = \alpha$ ” is the *Gedanke* that self-identity falls within generality, and consists *only* of two parts, viz., the incomplete *Sinn* of “ $\xi = \xi$ ” and incomplete *Sinn* of “ $\forall \phi(\alpha)$ ”.

Frege’s Begriffsschrift, as a higher order logical system, also involves quantification over functions. The function quantifier is understood as a third-level concept, i.e., a function that takes second-level functions as argument and yields truth-values as value. These functions also make use of Gothic letters such as “ \forall ” and “ \exists ” to reveal how it is that such functions mutually saturate with their second-level arguments. The semantic characterization of expressions involving the function-quantifier reads as follows:

“ $\forall \mu_{\beta}(f(\beta))$ ” denotes the True, if the second-level function name that replaces the “ $\mu_{\beta}(\dots\beta\dots)$ ” denotes a second-level function that yields the True as value for every first-level function with one-argument place taken as argument; and denotes the False, otherwise.

Here, the sign “ $\mu_{\beta}(\dots\beta\dots)$ ” plays a role similar to that of “ ξ ” or “ ϕ ” by indicating the argument place of a third-level function (*BL* §24). It is written in such an odd manner because it will always be replaced by a sign for a second-level function, and, as we have seen in the case of the object-quantifier, second-level functions have complex signs to reveal their mutually saturating capacity. If we replace “ $\mu_{\beta}(\dots\beta\dots)$ ” by “ $\forall \phi(\alpha)$ ”, the concavity in effect replaces the “ μ ” and “ α ” replaces the occurrences of “ β ”. The resulting expression is written “ $\forall \forall f(\alpha)$ ”, and it denotes the False, since not every function yields the True for all values. The above semantic characterization, however, only applies to functions of one argument. One requires a slightly different characterization for second-level functions of multiple arguments. For two arguments, the formulation is written:

“ $\forall \mu_{\beta\gamma}(f(\beta, \gamma))$ ” denotes the True, if the second-level function name that replaces the “ $\mu_{\beta\gamma}(\dots\beta\dots\gamma\dots)$ ” denotes a second-level function that yields the True as value for every first-level function with two argument places taken as argument; and denotes the False, otherwise.

Frege himself only explicitly spoke of quantification over functions of one and two argument places. It should be clear, however, how this semantic account would be continued for quantification over functions with more than two argument places.

ROMAN LETTERS: AN ALTERNATIVE NOTATION FOR GENERALITY

Even greater controversy surrounds Roman letters in the Begriffsschrift. These appear at first blush to be free variables, but given what Frege says about them, some authors, such as Dummett, have concluded that a Roman letter is in fact “bound by a tacit initial universal quantifier.”¹⁴ However, this view is by no means universally held. One critic of Dummett’s interpretation has dubbed it “an almost universal misunderstanding of Frege’s use of Roman letters.”¹⁵ The truth is in fact somewhere in between. Frege’s Roman letters have the *syntax* of modern free variables, but the *semantics* of expressions containing them is identical to that of expressions in which they are bound by initial quantifiers. This requires some further elaboration.

Frege introduces Roman letters in §17 of the *Grundgesetze*. The impetus for their introduction is to capture inferences not possible without them, including the Begriffsschrift transcription of a proof involving the Barbara syllogism of categorical logic. Frege takes as example the inference that would be written in ordinary language as from “All square roots of one are forth roots of one,” and “All forth roots of one are eighth roots of one,” to the conclusion “All square roots of one are eighth roots of one.” Earlier, in §15, Frege had introduced an inference rule analogous to the rule of hypothetical syllogism of most natural deduction systems. This inference rule *seems* to underwrite the Barbara inference as well. Strictly speaking, the rule allows one to infer from premises of the form:

$$\begin{array}{l} \vdash B \\ \vdash A \end{array}$$

$$\begin{array}{l} \vdash C \\ \vdash B \end{array}$$

a conclusion of the form:

¹⁴Dummett, *Frege: Philosophy of Language*, 525.

¹⁵Heck, “Frege and Semantics,” 25.

$$\vdash \begin{array}{l} C \\ \quad \vdash \\ \quad \quad A \end{array}$$

where A, B and C are well-formed expressions of the Begriffsschrift. However, if the premises in our example are written using the concavity and Gothic letters, this inference rule cannot be applied to them because the premises would not have the appropriate syntax.

Frege notes, however, that if the premises were written using the first notation for generality considered and rejected in §8 (discussed above), i.e., as:

$$\vdash \begin{array}{l} x^4 = 1 \\ \quad \vdash \\ \quad \quad x^2 = 1 \end{array}$$

$$\vdash \begin{array}{l} x^8 = 1 \\ \quad \vdash \\ \quad \quad x^4 = 1 \end{array}$$

then the inference rule from §15 could indeed be used to derive the conclusion:

$$\vdash \begin{array}{l} x^8 = 1 \\ \quad \vdash \\ \quad \quad x^2 = 1 \end{array}$$

So Frege now suggests that the use of Roman letters to express generalities is acceptable within his system, with the stipulation that the “scope of the generality” is always to be understood as encompassing the whole proposition, i.e., everything to the left of the judgment stroke. This eliminates the scope ambiguity that initially lead Frege to abandon this notation for generality. As Frege writes elsewhere:

In the concept-script the judgment stroke, besides conveying assertoric force, serves to demarcate the scope of the Roman letters. In order to be able to narrow the scope over which the generality extends, I make use of Gothic letters, and with these the concavity demarcates the scope. (*PW* 195n)

Here, and elsewhere (e.g., *BL* §17, *CP* 248), Frege speaks as if the use of Roman letters and the use of Gothic letters are both ways of expressing general assertions in the Begriffsschrift, the only difference being that Gothic letters are necessary if one wants to limit the generality to encompass less than the entire proposition.

It is perhaps easy to see in this why Dummett and others conclude that Frege has no free variables, that Roman are “bound” by hidden quantifiers, because, like Gothic letters, they are used to express *general Gedanken*, and

their scope is always the entire proposition. However, Dummett is mistaken if he takes this to mean that expressions involving Roman letters are mere syntactic abbreviations for expressions involving quantifiers. The introduction of Roman marks is precisely to facilitate certain inferences. Frege's inference rules are syntactically formulated, and thus we must understand expressions involving Roman letters as being syntactically quite different from those involving Gothic letters. Roman letters do have a syntax similar to that of free variables.¹⁶

Nevertheless, Dummett is correct in thinking that the *semantics* of expressions involving Roman letters is equivalent to that of expressions containing initial universal quantifiers. Given the way in which Frege introduces Roman letters in §17, it is quite clear that he wishes to maintain the definition he gave in §8 of an expression of the form " $\Phi(x)$ ", only he now modifies the definition such that it can only be applied in the context " $\vdash \Phi(x)$ ", since the judgment stroke fixes the scope of the generality. We might then reformulate the passage from §8 quoted earlier thusly:

Where " $\Phi(x)$ " occurs in the context " $\vdash \Phi(x)$ ", by " $\Phi(x)$ ", is to be understood the True, if the value of the function $\Phi(\xi)$ is the True for every argument; otherwise, it denotes the False.

This semantic characterization of " $\Phi(x)$ " is identical to that given for " $\neg \Phi(a)$ " above.

Needless to say, the semantics of expressions involving Roman letters is therefore very different from the semantics of expressions in contemporary predicate logics involving free variables. The Begriffsschrift proposition " $\vdash x = x$ " asserts a complete general *Gedanke*, specifically, that *everything* is self-identical.¹⁷ In contemporary predicate logics, an expression such as " $x = x$ " is not at all a general assertion. In fact, such expressions only express a definite thought at all when a certain interpretation or assignment is made to the variables, in which case they express a thought about that to which the variable is assigned. It is true that such systems are typically set up such that if free variables are included in axioms and theorems, then, as *logical truths*, the axioms and theorems should remain true regardless of which interpretation is given to the variables. Thus, these systems employ rules of replacement or substitution such that in an axiom such as " $x = x$ " one can replace any other term for " x ". Frege, however, is harshly critical of the idea that there should be signs in an axiomatic system that are capable of being given multiple interpretations

¹⁶Cf. Heck, "Frege and Semantics," 25-9, "Grundgesetze §§29-32," 443-6.

¹⁷Frege is quite clear that expressions involving Roman letters express *complete, general Gedanken*. See *PMC* 21, *PW* 199, *CP* 309, *BL* §32.

or assignments (see, e.g., CP 316). Frege's quest for logical clarity forces him to disavow any signs or symbols that admit of various interpretations, meanings or assignments. As he says, "a sign without determinate *Bedeutung* is a sign without *Bedeutung*" (CP 303). There is nothing *variable* or *indeterminate* about the *Bedeutung* of expressions involving Roman letters in Frege's logic. They denote the same truth-value as do the corresponding expressions with explicit, initial, universal quantifiers.

It is very significant that propositions in Frege's Begriffsschrift containing Roman letters express complete, general *Gedanken*. If this were not the case, then Frege's axiomatic system of logic would not live up to his own standards for what an axiomatic system should be. Frege is quite adamant that an axiom can only be properly understood as a *Gedanke* whose truth is certain (CP 273). Moreover, he believes every step in an inference must be also be the recognition of a true *Gedanke*, and thus, in deriving theorems from axioms, each step in the derivation must involve grasping a true *Gedanke* (CP 314ff, PMC 16, 21-2). All of Frege's basic laws involve Roman letters, as do most of his theorems. If these propositions did not express complete *Gedanken*, his axiomatic system would be illegitimate by his own standards. If Roman letters, however, were semantically interpreted as "variables" in the sense of having an indeterminate *Bedeutung*, the *Gedanken* expressed by them would be indeterminate. Frege, however, defends the use of Roman marks in his axioms by suggesting that although they do not denote determinate objects, they nevertheless "contribute to the expression of the *Gedanke*" by "conferring generality of content" to the propositions in which they occur (PW 249, see also CP 296). Although Begriffsschrift propositions containing Gothic letters bound by initial quantifiers and Begriffsschrift propositions containing Roman letters are syntactically different, semantically, they express identical *Gedanken*.

The logical system contains Roman letters for expressing generalities with regard not only to objects, but also with regard to first-level and even second-level functions. However, in what follows, I follow Frege's practice by introducing Roman and Gothic letters only for first-level functions that take either one or two arguments. It is not that Frege did not or could not acknowledge functions with more than two arguments. However, his treatment of arithmetic did not require treatment of functions with more than two arguments, and it simplified the syntax and rules of the system to limit his treatment to simpler functions. Similarly, I introduce Roman letters only for second-level functions that take a single argument, including both those that take a single one-place first-level function as argument, and those that take a single two-place first-level function as argument. Again, while Frege certainly could have acknowledged second-level functions that take more than one argument, or take as argument first-level functions with more than two arguments, his logical

system is made much simpler by remaining limited only to these cases. It should be clear, however, how the language could be expanded to include the other kinds of functions.

VALUE-RANGES AND EXTENSIONS OF CONCEPTS

Perhaps the most significant difference between Frege's logic in the *Begriffsschrift* and the revised system of the *Grundgesetze* is the introduction in the latter of notation for what Frege calls value-ranges (*Wertverläufe*) of functions. As we have seen, for Frege, functions are essentially unsaturated, and for this reason, are not themselves objects, nor can they be subjects for predication with normal (first-level) concepts. However, Frege holds that for each first-level function, there exists an object constituting its value-range, which, in the case of concepts, Frege equates with its extension (*BL* §2, *CP* 148). Value-ranges of concepts play the role of *classes* in Frege's logic, although he is quite clear that he does not understand value-ranges or the extensions of concepts to be collections, sets, or aggregates of objects. Instead, as he puts it, "the extension of a concept is constituted in being not by the individuals but by the concept itself" (*CP* 224-5, see also *PW* 182-3, *PMC* 140-1). Although it is clear that value-ranges are not understood by Frege as aggregates or collections, it is less clear exactly what Frege *does* understand them to be. As Furth understands them, a value-range can be understood as the complete mapping of arguments onto values determined by the function. Thus the value-range of the concept *square root of four* consists of such ordered couples as 2 — *the True*; 0 — *the False*; -2 — *the True*; -4 — *the False*, etc.¹⁸ As Cocchiarella understands them, value-ranges are the *Bedeutungen* of nominalized-predicates; they are, as it were, what in fact think of when we attempt to think of concepts or other functions themselves as logical subjects.¹⁹

Despite the lack of detailed explanation of the nature of value-ranges, this much is clear. What it means for an object to be included in the extension of a concept for Frege is not for it to be a part of a collection, but rather for it to simply be one of the arguments the concept maps onto the True. Moreover,

¹⁸Montgomery Furth, Introduction to *Basic Laws of Arithmetic*, by Gottlob Frege, xxxviii.

¹⁹Nino Cocchiarella, *Logical Studies in Early Analytic Philosophy* (Columbus: Ohio State University Press, 1987), 76-80. Evidence for this view comes in that Frege claims that in an ordinary language proposition such as "the concept *horse* is instantiated," the expression "the concept *horse*" cannot actually be taken to denote a concept, since it is a complete expression and would seem to denote an object. See, e.g., *CP* 187-8. Frege strongly suggests that in such cases, what is actually denoted is the extension of the concept, see *PW* 106n.

Frege believes that value-ranges differ if and only if their corresponding functions differ in value for some arguments, which Frege takes to be an “indemonstrable” but “fundamental law of logic” (CP 142).²⁰ Beyond this, Frege tells us little. Many authors have implicated Frege’s reticence about how we are to understand the *Bedeutungen* of signs for value-ranges in the failure of his attempted proof in *Grundgesetze* §§28-31 to show that each Begriffsschrift expression is known to have a unique *Bedeutung*.²¹

Formally, in the Begriffsschrift, Frege uses what he calls the “smooth-breathing” (˘) as a sign for a second-level function that yields as value the value-range of the first-level function it takes as argument. For reasons that should be clear from the discussion of the quantifier above, Frege associates a letter—in this case, a lowercase Greek vowel—with each occurrence of the smooth-breathing in a Begriffsschrift expression to reveal how it is that the second-level function mutually saturates with its argument function. Frege gives the following semantic characterization of expressions involving the smooth-breathing:

“ $\overset{\circ}{\varepsilon}\Phi(\varepsilon)$ ” denotes the value-range of the function $\Phi(\xi)$ (BL §9)

The smooth-breathing thus denotes a second-level function that takes first-level functions as argument and yields objects, and value-ranges in particular, as value.

For use in conjunction with the smooth-breathing, Frege also introduces another primitive first-level function name, “\ \xi”, which he describes as the Begriffsschrift “substitute for the definite article” (BL §11). If the argument to this function is a value-range with a single member, the value of the function is this single member; otherwise, it yields as value the argument itself. To put this more formally:

“\ \xi” denotes the sole argument for which the function whose value-range is denoted by what replaces “\ \xi” has the True as value, if what replaces “\ \xi” denotes a value-range of a function that

²⁰This is in effect Frege’s ill-fated Basic Law V. See BL §20.

²¹For discussion of this issue, see Michael D. Resnik, “Frege’s Proof of Referentiality,” in *Frege Synthesized*, ed. L. Haaparanta and J. Hintikka (Boston: D. Reidel, 1986), 177-95; Michael Dummett, *Frege: Philosophy of Mathematics* (Cambridge, Mass.: Harvard University Press, 1991), 214-6; Edward Martin, Jr., “Referentiality in Frege’s *Grundgesetze*,” *History and Philosophy of Logic* 3 (1982): 165-91; Heck, “Grundgesetze der Arithmetik I §29-32,” 451-61.

yields the True for only one argument, and, otherwise, denotes the same as that which replaces “ ξ ”.

To give a mathematical example, the Begriffsschrift expression “ $\lambda \varepsilon(\varepsilon = 3^2)$ ” denotes the number nine, since it is the only member of the extension of the concept *square of three*, whereas the expression “ $\lambda \varepsilon(\varepsilon^2 = 4)$ ” denotes the same as the expression “ $\varepsilon(\varepsilon^2 = 4)$ ”, since the concept *square root of four* has more than one object falling under it.

THE SYNTACTIC RULES OF THE BEGRIFFSSCHRIFT

We have now discussed the semantics of all of the primitive signs of the Begriffsschrift of the *Grundgesetze*. In what follows, I shall present a full reconstruction of Frege’s logical system as a formal system, beginning with the formation rules for well-formed expressions. Frege himself never attempted a full recursive characterization of expressions in his syntax, but such an account was implicit in what he did write. I shall attempt to give one for him, making only small modifications to his syntax. The modifications include the addition of apostrophes to Greek, Roman and Gothic letters to ensure an infinite supply of such letters, and substitution of more typographically friendly notation for some of his primitive functions. Thus, for Frege’s conditional and negation strokes, I substitute the Russellian signs “ \supset ” and “ \sim ”, respectively. However, I mean these signs to retain Frege’s functional semantics. Similarly, I replace Frege’s quantifier notation with the more usual notation “ $(\forall a)(\dots a \dots)$ ”, again without changing the semantics. These changes are for typographical convenience only. Lastly, I add superscripts revealing the number of argument places for Roman and Gothic function letters, for ease in stating the syntactic rules.

In what follows, I shall use uppercase Roman letters early in the alphabet as schematic variables representing strings of Begriffsschrift signs. These are not to be confused with Frege’s second-level Roman function letters. They will sometimes stand for well-formed expressions, sometimes for components of well-formed expressions, and at times simply for individual Roman, Gothic or Greek letters depending on what is stipulated in each case. The notation $\lceil A(B) \rceil$ should be taken to mean that the string of signs represented by B occurs in the whole string of signs $\lceil A(B) \rceil$. We begin with a number of recursive definitions.

Definition: *Roman object letter*

- (i) Any lowercase Roman letter between and including *a* and *e* and between and including *m* and *z* is a *Roman object letter*, and
- (ii) if *A* is a *Roman object letter*, then $\lceil A \rceil$ is a distinct *Roman object letter*.

Part (ii) of the definition above provides us with an infinite supply of Roman object letters; because “a” is a Roman object letter (by (i)), so is “a”, and thus so are “a” and “a” (by (ii)), and so on *ad infinitum*.

Definition: *Gothic object letter*

- (i) Any lowercase Gothic letter between and including a and e and between and including m and ð is a *Gothic object letter*, and
- (ii) if A is a *Gothic object letter*, then $\ulcorner A \urcorner$ is a distinct *Gothic object letter*.

Definition: *Greek object letter*

- (i) Any lowercase Greek vowel ($\alpha, \varepsilon, \eta, \iota, \omicron, \upsilon, \omega$) is a *Greek object letter*, and
- (ii) if A is a *Greek object letter*, then $\ulcorner A \urcorner$ is a distinct *Greek object letter*.

Definition: *one-place first-level Roman function letter*

- (i) Any of the lowercase Roman letters $f^1, g^1, h^1, i^1, j^1, k^1$, and l^1 (including the superscripts) is a *one-place first-level Roman function letter*, and
- (ii) if A is a *one-place first-level Roman function letter*, then $\ulcorner A \urcorner$ is a distinct *one-place first-level Roman function letter*.

Definition: *two-place first-level Roman function letter*

- (i) Any of the lowercase Roman letters $f^2, g^2, h^2, i^2, j^2, k^2$, and l^2 (including the superscripts) is a *two-place first-level Roman function letter*, and
- (ii) if A is a *two-place first-level Roman function letter*, then $\ulcorner A \urcorner$ is a distinct *two-place first-level Roman function letter*.

Definition: *one-place Gothic function letter*

- (i) Any of the lowercase Gothic letters $f^1, g^1, h^1, i^1, j^1, k^1$, and l^1 (including the superscripts) is a *one-place Gothic function letter*, and
- (ii) if A is a *one-place Gothic function letter*, then $\ulcorner A \urcorner$ is a distinct *one-place Gothic function letter*.

Definition: *two-place Gothic function letter*

- (i) Any of the lowercase Gothic letters $f^2, g^2, h^2, i^2, j^2, k^2$, and l^2 (including the superscripts) is a *two-place Gothic function letter*, and
- (ii) if A is a *two-place Gothic function letter*, then $\ulcorner A \urcorner$ is a distinct *two-place Gothic function letter*.

Definition: *second-level Roman function letter with one-place argument*

- (i) Any uppercase Roman letter after and including M written with a subscripted β , e.g., M_β , is a *second-level Roman function letter with one-place argument*, and²²
- (ii) if A is an *second-level Roman function letter with one-place argument*, then $\ulcorner A \urcorner$ is a distinct *second-level Roman function letter with one-place argument*.

Definition: *second-level Roman function letter with two-place argument*

- (i) Any uppercase Roman letter after and including M written with subscripted β and γ , e.g., $M_{\beta\gamma}$, is a *second-level Roman function letter with two-place argument*, and
- (ii) if A is an *second-level Roman function letter with two-place argument*, then $\ulcorner A \urcorner$ is a distinct *second-level Roman function letter with two-place argument*.

With these definitions in place, we can proceed to recursively define a well-formed Begriffsschrift expression as follows:

Definition: *well-formed expression (wfe)*

- (i) Roman object letters are *wfes*;
- (ii) if A is a *wfe*, then $\ulcorner \text{— } A \urcorner$ is a *wfe*;
- (iii) if A is a *wfe*, then $\ulcorner \sim A \urcorner$ is a *wfe*;
- (iv) if A is a *wfe*, then $\ulcorner \setminus A \urcorner$ is a *wfe*;
- (v) if A and B are *wfes*, then $\ulcorner (A \supset B) \urcorner$ is a *wfe*;
- (vi) if A and B are *wfes*, then $\ulcorner (A = B) \urcorner$ is a *wfe*;
- (vii) if A is a *wfe*, and B is a *one-place first-level Roman function letter*, then $\ulcorner B(A) \urcorner$ is a *wfe*;
- (viii) if A and B are *wfes*, and C is a *two-place first-level Roman function letter*, then $\ulcorner C(A, B) \urcorner$ is a *wfe*;
- (ix) if $\ulcorner A(B) \urcorner$ is a *wfe* that contains within it B , which is itself a *wfe*, and C is a *Gothic object letter* not contained in $\ulcorner A(B) \urcorner$, then $\ulcorner (\forall C)A(C) \urcorner$ is a *wfe*, with C replacing one or more occurrences of B from $\ulcorner A(B) \urcorner$;
- (x) if $\ulcorner A(B) \urcorner$ is a *wfe* that contains within it B , which is itself a *wfe*, and E is a *Greek object letter* not contained in $\ulcorner A(B) \urcorner$, then $\ulcorner \overset{\circ}{E}A(E) \urcorner$ is a *wfe*, with E replacing one or more occurrences of B from $\ulcorner A(B) \urcorner$;
- (xi) if $\ulcorner D(A(B)) \urcorner$ is a *wfe* that contains within it $\ulcorner A(B) \urcorner$, which is itself a *wfe* containing within it B , which is a *wfe* itself, and C is a *one-place Gothic function*

²²Really, we ought to include other mutual saturation signs besides “ β ” for second-level Roman function letters for disambiguation in case one such Roman letter were to occur within the scope of another. Frege himself seems to have missed this difficulty, and I, too, overlook it for present purposes since the need never arises in the present work.

letter not contained in $\lceil D(A(B)) \rceil$, then $\lceil (\forall C)D(C(B)) \rceil$ is a *wfe*, with C replacing one or more occurrences of A from $\lceil D(A(B)) \rceil$;

(xii) if $\lceil D(A(B, E)) \rceil$ is a *wfe* that contains within it $\lceil A(B, E) \rceil$, which is itself a *wfe* containing within it B and E , both of which are *wfes* themselves, and C is a *two-place Gothic function letter* not contained in $\lceil D(A(B, E)) \rceil$, then $\lceil (\forall C)D(C(B, E)) \rceil$ is a *wfe*, with C replacing one or more occurrences of A from $\lceil D(A(B, E)) \rceil$;

(xiii) if $\lceil A(B) \rceil$ is a *wfe* that contains within it B , itself a *wfe*, and C is a *second-level Roman function letter with one-place argument*, then $\lceil C(A(\beta)) \rceil$ is a *wfe*, with β replacing one or more occurrences of B from $\lceil A(B) \rceil$, and

(xiv) if $\lceil A(B, C) \rceil$ is a *wfe* that contains within it B and C , themselves *wfes*, and D is a *second-level Roman function letter with two-place argument*, then $\lceil D(A(\beta, \gamma)) \rceil$ is a *wfe*, with β replacing one or more occurrences of B from $\lceil A(B, C) \rceil$, and γ replacing one or more occurrences of C from $\lceil A(B, C) \rceil$.

Definition: *object name*

If A is a *wfe* containing no Roman letters of any level, then A is an *object name*.

Definition: *Begriffsschrift proposition*

If A is a *wfe*, then $\lceil \vdash A \rceil$ is a *Begriffsschrift proposition*.

To understand how these recursive definitions are supposed to work, let us consider the example of the proposition, $\lceil \vdash \sim(\forall a)\sim(\forall b)(\neg(a = b) \supset (b = a)) \rceil$. By our definitions, “ a ” and “ b ” are Roman object letters. Therefore, by rule (i), they are *wfes*. Since they are *wfes*, by rule (vi), both “ $(a = b)$ ” and “ $(b = a)$ ” are *wfes*. Since “ $(a = b)$ ” is a *wfe*, by rule (ii), “ $\neg(a = b)$ ” is a *wfe*. By rule (v), since “ $\neg(a = b)$ ” and “ $(b = a)$ ” are *wfes*, likewise is “ $(\neg(a = b) \supset (b = a))$ ”. By our definition, “ a ” and “ b ” are Gothic object letters. Knowing that “ $(\neg(a = b) \supset (b = a))$ ” is *wfe*, which contains *wfe* “ b ”, and “ b ” is a Gothic object letter, by rule (ix), “ $(\forall b)(\neg(a = b) \supset (b = a))$ ” is a *wfe*. By rule (iii), likewise is “ $\sim(\forall b)(\neg(a = b) \supset (b = a))$ ”. This *wfe* contains “ a ”, itself a *wfe*. Thus, again by rule (ix), “ $(\forall a)\sim(\forall b)(\neg(a = b) \supset (b = a))$ ” is a *wfe*, and hence, by rule (iii) again, “ $\sim(\forall a)\sim(\forall b)(\neg(a = b) \supset (b = a))$ ” is a *wfe*. Thus, by our definition, $\lceil \vdash \sim(\forall a)\sim(\forall b)(\neg(a = b) \supset (b = a)) \rceil$ is a *Begriffsschrift proposition*.

In what follows, I shall make use of the following syntactic conveniences, 1) superscripts on Roman and Gothic function letters can be left off, since it should be clear from the context what they must be, 2) in a *wfe* of the form $\lceil (A \supset B) \rceil$ or $\lceil (A = B) \rceil$, matching outermost brackets can be left off in the corresponding propositions, i.e., $\lceil \vdash A \supset B \rceil$ and $\lceil \vdash A = B \rceil$, 3) successive universal quantifiers will be combined into one sign, e.g., “ $(\forall ab) \dots$ ” instead of “ $(\forall a)(\forall b) \dots$ ”, 4) to avoid a morass of brackets in complicated expressions, I

shall sometimes omit them and utilize dots surrounding connectives to signify their relative priority (the greater the number of dots surrounding a connective, the greater its scope), and also, 5) I shall sometimes employ brackets of differing types to aid the eye in finding matching brackets. In stating the inference rules, we shall also require the following definitions:

Definition: *one-place first-level function name*

If $\ulcorner A(B) \urcorner$ is a *wfe* containing B, which is itself a *wfe*, then $\ulcorner A(\xi) \urcorner$ is a *one-place first-level function name*, with ξ representing the gap created by removing one or more occurrences of B from $\ulcorner A(B) \urcorner$. (Here, the occurrences of ξ are meant simply to hold open the empty places in the function name where the occurrences of B were removed.)

Definition: *two-place first-level function name*

If $\ulcorner A(B, C) \urcorner$ is a *wfe* containing B and C, both of which are themselves *wfes*, then $\ulcorner A(\xi, \zeta) \urcorner$ is a *two-place first-level function name*, with ξ representing the gap created by removing one or more occurrences of B from $\ulcorner A(B, C) \urcorner$, and ζ representing the gap created by removing one or more occurrences of C from $\ulcorner A(B, C) \urcorner$. (Here, the occurrences of ξ and ζ are meant simply to hold open the empty places in the function name where the occurrences of B and C were removed.)

Definition: *second-level function name with one-place argument*

If $\ulcorner A(B) \urcorner$ is a *wfe* containing B, which is itself a *one-place first-level function name*, then $\ulcorner A(\phi) \urcorner$ is a *second-level function name with one-place argument*, with ϕ representing the gap created by removing one or more occurrences of B from $\ulcorner A(B) \urcorner$. (Here, the occurrences of ϕ are meant simply to hold open the empty places in the function name where the occurrences of B were removed.) (Cf. *BL* §26)

Definition: *second-level function name with two-place argument*

If $\ulcorner A(B) \urcorner$ is a *wfe* containing B, which is itself a *two-place first-level function name*, then $\ulcorner A(\phi) \urcorner$ is a *second-level function name with two-place argument*, with ϕ representing the gap created by removing one or more occurrences of B from $\ulcorner A(B) \urcorner$. (Here, the occurrences of ϕ are meant simply to hold open the empty places in the function name where the occurrences of B were removed.)

By these definitions, any expression that results from removing from a complex *wfe* one or two component parts that are themselves *wfes* is a first-level function name. Thus, even a complex expression such as “ $(\forall a)(a \supset (\xi \supset a))$ ” is to be regarded as a function name. Similarly, any expression that results from

removing a first-level function name from a wfe is a second-level function name, e.g., “ $(\forall a)\sim\phi(a)$ ”, which in this case denotes a second-level concept that can be predicated of those first-level concepts under which nothing falls.

THE AXIOMATIZATION OF FREGE’S SYSTEM

The logical system of the *Grundgesetze* as presented by Frege contains seven “basic laws” and ten inference rules (*BL* §§47-8). I shall be adding two axioms to his explicit formulation that seem presupposed by the semantics of his language, but which he leaves out, probably because they were unnecessary for his proofs of the truths of number theory. In what follows, I also divide Frege’s own system into two subsystems. The first division represents the core of Frege’s logic: its consistent treatment of higher-order logic. The second represents the (inconsistent) expansion of Frege’s logic to include axioms dealing with value-ranges. I shall refer to the first system consisting of only these axioms and rules as FC for the *function calculus*. There are six axioms of this system:

Axiom FC1. $\vdash a \supset (b \supset a)$	(Frege’s Basic Law I)
Axiom FC2. $\vdash \sim(\text{— } a = \sim b) \supset (\text{— } a = \text{— } b)$	(Frege’s Basic Law IV)
Axiom FC3. $\vdash (\forall a)f(a) \supset f(a)$	(Frege’s Basic Law IIa)
Axiom FC4. $\vdash (\forall f)M_\beta(f(\beta)) \supset M_\beta(f(\beta))$	(Frege’s Basic Law IIb)
Axiom FC5. $\vdash (\forall f)M_{\beta\gamma}(f(\beta, \gamma)) \supset M_\beta(f(\beta, \gamma))$	
Axiom FC6. $\vdash g(a = b) \supset g((\forall f)(f(b) \supset f(a)))$	(Frege’s Basic Law III)

Axiom FC1 is a simple truth-functional tautology; analogues of it are included in many axiomatic systems. Axiom FC2 states that if the value of the horizontal function for object a is not the same as the value of the negation function for object b , then the horizontal function has the same values for both a and b . The truth of this axiom can be seen from the following considerations. The consequent of this conditional, $(\text{— } a = \text{— } b)$, will only be the False if a is the True and b something other than the True, or if b is True and a something other than the True. In either case, the antecedent, $\sim(\text{— } a = \sim b)$, is also the False. In the first case, because both $\text{— } a$ and $\sim b$ are the True, and in the second case, because both $\text{— } a$ and $\sim b$ are the False.

Axiom FC3 states that for every object a , if function $f(\xi)$ has the True as value for all objects, then it has the True as value for a . Axioms FC4 and FC5 say something similar for functions. Axiom FC4 states that for every one-place first-level function $f(\xi)$, if second-level function M has the True as value for all one-place first-level functions, then M has the True as value for $f(\xi)$. Axiom FC5, which Frege himself omitted, states the same for two-place first-level functions.

Axiom FC6 captures Frege's commitment to Leibniz's law. If Leibniz's law holds, then the truth-value $(a = b)$, i.e., the truth-value of a and b 's being identical, will be the same as the truth-value $(\forall f)(f(b) \supset f(a))$, the truth-value of a 's falling under every concept under which b falls. Since $(a = b)$ and $(\forall f)(f(b) \supset f(a))$ are the same truth-value, for any function $g(\xi)$, if the value of $g(\xi)$ for the truth-value $(a = b)$ as argument is the True, then the value of $g(\xi)$ for the truth-value $(\forall f)(f(b) \supset f(a))$ is also the True. This is what this axiom states. This axiom can be used in the formal system to prove both the indiscernibility of identicals (if $g(\xi)$ is instantiated to the horizontal), and the identity of indiscernibles (if $g(\xi)$ is instantiated to the negation function). The indiscernibility of identicals makes possible the substitution of identicals in any context *salva veritate*. Frege's system, unlike many logical systems, does not require a specific inference rule for the substitution of identicals.

However, the system FC does contain a number of inference rules. They are the following:

Horizontal amalgamation rules (hor):

The following pairs of Begriffsschrift expressions are interchangeable, i.e., wherever one of these expressions occurs within a proposition, one can infer a proposition in which the other replaces it, where A and B are any wfes, C any Gothic letter, and D(C) any expression containing Gothic letter C such that $(\forall C)D(C)$ would be a wfe.

$$\begin{array}{ll}
 \lceil _ _ A \rceil & :: \lceil _ A \rceil \\
 \lceil \sim _ A \rceil & :: \lceil \sim A \rceil \\
 \lceil _ \sim A \rceil & :: \lceil \sim A \rceil \\
 \lceil _ (A \supset B) \rceil & :: \lceil (A \supset B) \rceil \\
 \lceil (_ A \supset B) \rceil & :: \lceil (A \supset B) \rceil \\
 \lceil (A \supset _ B) \rceil & :: \lceil (A \supset B) \rceil \\
 \lceil _ (A = B) \rceil & :: \lceil (A = B) \rceil \\
 \lceil _ (\forall C)D(C) \rceil & :: \lceil (\forall C)D(C) \rceil \\
 \lceil (\forall C) _ D(C) \rceil & :: \lceil (\forall C)D(C) \rceil
 \end{array}$$

Interchange (int):

Where A, B, C are wfes, from $\lceil \vdash A \supset (B \supset C) \rceil$, infer $\lceil \vdash B \supset (A \supset C) \rceil$.

Contraposition (con):

Where A and B are wfes, from $\lceil \vdash A \supset B \rceil$, infer $\lceil \vdash \sim B \supset \sim A \rceil$.

Antecedent amalgamation (amal):

Where A and B are wfes, from $\lceil \vdash A \supset (A \supset B) \rceil$, infer $\lceil \vdash A \supset B \rceil$.

Detachment (mp):

Where A and B are wfes, from $\lceil \vdash A \supset B \rceil$ and $\lceil \vdash A \rceil$, infer $\lceil \vdash B \rceil$.

Hypothetical syllogism (syll):

Where A, B and C are wfes, from $\lceil \vdash A \supset B \rceil$ and $\lceil \vdash B \supset C \rceil$, infer $\lceil \vdash A \supset C \rceil$.

Inevitability (inev):

Where A and B are wfes, from $\lceil \vdash A \supset B \rceil$ and $\lceil \vdash \sim A \supset B \rceil$, infer $\lceil \vdash B \rceil$.

Change in generality notation (gen):

a) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a Roman object letter, and C is a Gothic object letter not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.

b) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a Roman object letter, C is a Gothic object letter not contained in $\lceil A(B) \rceil$, and D is a wfe not containing B, then from $\lceil \vdash D \supset A(B) \rceil$, infer $\lceil \vdash D \supset (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.

c) Where $\lceil A(B) \rceil$ is a wfe containing B, a first-level Roman function letter, and C is a Gothic function letter with the same number of argument places as B that is not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.

d) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a first-level Roman function letter, C is a Gothic function letter with the same number of argument-places as B that is not contained in $\lceil A(B) \rceil$, and D is a wfe not containing B, then from $\lceil \vdash D \supset A(B) \rceil$, infer $\lceil \vdash D \supset (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.

Roman instantiation (ri):

a) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of Roman object letter B, and C is any wfe containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, where C replaces every occurrence of B in $\lceil A(B) \rceil$.

- b) Where $\ulcorner A(B) \urcorner$ is a wfe containing one or more occurrences of one-place first-level Roman function letter B, and C is any one-place first-level function name containing no Gothic or Greek letters contained in $\ulcorner A(B) \urcorner$, from $\ulcorner \vdash A(B) \urcorner$, infer $\ulcorner \vdash A(C) \urcorner$, with C replacing every occurrence of B from $\ulcorner A(B) \urcorner$, the empty places ξ of C being filled with the corresponding arguments to B in $\ulcorner A(B) \urcorner$.
- c) Where $\ulcorner A(B) \urcorner$ is a wfe containing one or more occurrences of two-place first-level Roman function letter B, and C is any two-place first-level function name containing no Gothic or Greek letters contained in $\ulcorner A(B) \urcorner$, from $\ulcorner \vdash A(B) \urcorner$, infer $\ulcorner \vdash A(C) \urcorner$, with C replacing every occurrence of B from $\ulcorner A(B) \urcorner$, the empty places ξ and ζ of C being filled with the corresponding arguments to B in $\ulcorner A(B) \urcorner$.
- d) Where $\ulcorner A(B) \urcorner$ is a wfe containing one or more occurrences of B, where B is a second-level Roman function letter with one-place argument, and C is any second-level function name with one-place argument containing no Gothic or Greek letters contained in $\ulcorner A(B) \urcorner$, from $\ulcorner \vdash A(B) \urcorner$, infer $\ulcorner \vdash A(C) \urcorner$, with C replacing every occurrence of B from $\ulcorner A(B) \urcorner$, the empty place ϕ of C being filled with the corresponding arguments to B in $\ulcorner A(B) \urcorner$, and whatever sign used for mutual saturation in C replacing all occurrences of β in $\ulcorner A(B) \urcorner$.
- e) Where $\ulcorner A(B) \urcorner$ is a wfe containing one or more occurrences of B, where B is a second-level Roman function letter with two-place argument, and C is any second-level function name with two-place argument containing no Gothic or Greek letters contained in $\ulcorner A(B) \urcorner$, from $\ulcorner \vdash A(B) \urcorner$, infer $\ulcorner \vdash A(C) \urcorner$, with C replacing every occurrence of B from $\ulcorner A(B) \urcorner$, the empty place ϕ of C being filled with the corresponding arguments to B in $\ulcorner A(B) \urcorner$, and whatever signs used for mutual saturation in C replacing all occurrences of β and γ in $\ulcorner A(B) \urcorner$.

Change of Gothic or Greek letter (cg):

- a) Where $\ulcorner D(A(B)) \urcorner$ is a wfe containing $\ulcorner A(B) \urcorner$, itself a wfe containing Gothic object letter B, and C is a different Gothic object letter not contained in $\ulcorner A(B) \urcorner$, then from $\ulcorner \vdash D(A(B)) \urcorner$, infer $\ulcorner \vdash D(A(C)) \urcorner$, with C replacing all occurrences of B in $\ulcorner A(B) \urcorner$.
- b) Where $\ulcorner D(A(B)) \urcorner$ is a wfe containing $\ulcorner A(B) \urcorner$, itself a wfe containing Greek object letter B, and C is a different Greek object letter not contained in $\ulcorner A(B) \urcorner$, then from $\ulcorner \vdash D(A(B)) \urcorner$, infer $\ulcorner \vdash D(A(C)) \urcorner$, with C replacing all occurrences of B in $\ulcorner A(B) \urcorner$.
- c) Where $\ulcorner D(A(B)) \urcorner$ is a wfe containing $\ulcorner A(B) \urcorner$, itself a wfe containing Gothic function letter B, and C is a different Gothic function letter with the same number of argument-places as B that is not contained in $\ulcorner A(B) \urcorner$, then from $\ulcorner \vdash D(A(B)) \urcorner$, infer $\ulcorner \vdash D(A(C)) \urcorner$, with C replacing all occurrences of B in $\ulcorner A(B) \urcorner$.

This completes the presentation of the consistent core of Frege's logic. It has been shown to be both complete and consistent in its treatment of propositional and first-order functional logic. The treatment of second-order logic is also consistent, although of course, not complete (since no higher-order system can be complete).²³ The names of the inference rules given above are my own, and in some cases the result of pure invention (e.g., "inevitability"). However, the names "Roman instantiation" and "change in generality notation" require further comment. The rule I here name "change in generality notation" is often referred to by others as the rule of "universal generalization." However, given the reading given above, expressions involving Roman letters themselves express general *Gedanken*, and it is misleading to characterize this inference rule as that of generalization. It is rather a move from one notation for expressing a general *Gedanke* to another. Similarly, most other writers refer to what I call "Roman instantiation" as "the rule of substitution." Syntactically this rule is quite similar to the substitution rules that occur in many systems involving free variables. However, since Roman letters express general propositions, the replacement of a Roman letter with some particular term must be understood semantically as a move from the general to the particular. Indeed, it is quite clear from his own usage that Frege himself understood that the replacement of a Roman letter in a proposition as representing an inference from the general to the particular (*CP* 316, *PW* 153-4, 201, *PMC* 22). Indeed, as I understand it, it is (*ri*), and not axioms FC3 and FC4, that represents Frege's real instantiation principle.

It is worth taking a closer look at the immense power of (*ri*). This rule allows us to substitute any well-formed expression for a Roman object letter, and more importantly, any function name for a Roman function letter. The way in which we defined a function name in the last section is so broad that every

²³The completeness of Frege's treatment of propositional logic was shown in 1930 by Jan Łukasiewicz. See Jan Łukasiewicz and Alfred Tarski, "Untersuchungen über den Aussagenkalkül," *Comptes rendus des séances de la Société et des Lettres de Varsovie* 23 (1930): 35; Jan Łukasiewicz, "On the History of the Logic of Propositions," in *Polish Logic 1920-1939*, ed. and trans. Storrs McCall (London: Oxford University Press, 1967), 84-7. The completeness of the first-order portion of Frege's logic was shown in Kneale and Kneale, *The Development of Logic*, chap. 8. On the consistency of various subportions of Frege's logic, see the above works and also Terence Parsons, "On the Consistency of the First-Order Portion of Frege's Logical System," *Notre Dame Journal of Formal Logic* 28 (1987): 161-8; I. Susan Russinoff, "On the Brink of a Paradox?" *Notre Dame Journal of Formal Logic* 28 (1987): 115-31; George Boolos, "Whence the Contradiction?" *Proceedings of the Aristotelian Society* 67 Supplement (1993): 213-33; Richard G. Heck, Jr., "The Consistency of Predicative Fragments of Frege's *Grundgesetze der Arithmetik*," *History and Philosophy of Logic* 17 (1996): 209-20.

open expression is included. The result is that Frege's logical system includes as a theorem every instance of the following comprehension principle (CP) for functions:

(CP) $\vdash \sim(\forall f)\sim(\forall a)(f(a) = A(a))$, where $A(a)$ is any Begriffsschrift expression containing "a" but not containing "f", and such that the whole is a wfe.

This principle posits the existence of a function for every one-place first-level function name formulable in the language, regardless of its complexity. A similar principle positing the existing of functions corresponding to two-place function names also follows. This commitment to a multiplicity of functions is indeed a powerful one within a logical system, although by itself it does not lead to contradiction.

Since Frege's system, like all standard axiomatic systems, does not contain a rule for existential instantiation, the presence of a simple or primitive Begriffsschrift sign for all the functions posited by comprehension is not guaranteed. However, Frege does allow for the unrestricted introduction of simple defined signs, which take on both the *Sinne* and *Bedeutungen* of the Begriffsschrift expressions used to define them. One writes such a definition using what we might call *the stipulation stroke*, "⌈" (*BL* §27). Indeed, Frege goes so far as to give *rules* for the introduction of definitions (*BL* §26). We might summarize these as follows:

Rules of definition:

- a) Where A is some new sign unused so far in the Begriffsschrift, and B is some wfe containing no Roman letters, then one may stipulate that the sign A is to be used to abbreviate B with the stipulation, $\lceil A = B \rceil$.
 - b) Where A is some new sign unused so far in the Begriffsschrift, and $B(\xi)$ is some one-place first-level function name containing no Roman letters, then one may stipulate that the sign A is to be used to abbreviate B with the stipulation, $\lceil A(x) = B(x) \rceil$.
 - c) Where A is some new sign unused so far in the Begriffsschrift, and $B(\xi, \zeta)$ is a two-place first-level function name containing no Roman letters, then one may stipulate that the sign A is to be used to abbreviate B with the stipulation, $\lceil A(x, y) = B(x, y) \rceil$.
 - d) Where A_τ is some new complex sign unused so far in the Begriffsschrift, and $B(\phi)$ is a one-place second-level function name containing no Roman letters, then one may stipulate that the sign A_τ is to be used to abbreviate B with the stipulation, $\lceil A_\tau(f(\tau)) = B(f) \rceil$.
- 4) and so on.

Frege then notes that once a definitional stipulation has been made using the stipulation sign, one may then regard it as a principle of the system, swapping, as it were, the stipulation stroke for the judgment stroke (*BL* §27). Partially for my own convenience in what follows, and partially to illustrate how the principles of definition are meant to work, I now define a number of signs using these rules.

- (Df. \neq) $\Vdash (x \neq y) = \sim(x = y)$
 (Df. \vee) $\Vdash (x \vee y) = (\sim x \supset y)$
 (Df. $\&$) $\Vdash (x \& y) = \sim(x \supset \sim y)$
 (Df. \equiv) $\Vdash (x \equiv y) = (\text{---} x = \text{---} y)$
 (Df. $\exists a$) $\Vdash (\exists a)f(a) = \sim(\forall a)\sim f(a)$
 (Df. $\exists \beta$) $\Vdash (\exists \beta)M_\beta(f(\beta)) = \sim(\forall \beta)\sim M_\beta(f(\beta))$
 (Df. $\exists \beta, \gamma$) $\Vdash (\exists \beta, \gamma)M_{\beta, \gamma}(f(\beta, \gamma)) = \sim(\forall \beta, \gamma)\sim M_{\beta, \gamma}(f(\beta, \gamma))$

Given the combined power of (*ri*) and unrestricted definition in Frege's Begriffsschrift, the commitment therein to a multiplicity of functions is robust.

The system FC just described is both powerful and consistent. Unfortunately, however, it does not represent the complete logical system of the *Grundgesetze*. Frege's actual logical system consists of FC plus additional axioms dealing with value-ranges. Let us call the expanded system FC^{+V} , for the *function plus value-range calculus*. The added axioms are as follows:

- Axiom $FC^{+V}7$. $\vdash (\overset{\cdot}{\varepsilon}f(\overset{\cdot}{\varepsilon}) = \overset{\cdot}{\alpha}g(\overset{\cdot}{\alpha})) = (\forall \alpha)(f(\alpha) = g(\alpha))$ (Frege's Basic Law V)
 Axiom $FC^{+V}8$. $\vdash a = \backslash \overset{\cdot}{\varepsilon}(a = \overset{\cdot}{\varepsilon})$ (Frege's Basic Law VI)
 Axiom $FC^{+V}9$. $\vdash (\forall b)\sim[a = \overset{\cdot}{\varepsilon}(b = \overset{\cdot}{\varepsilon})] \supset (\backslash a = a)$

Axiom $FC^{+V}7$, the most notorious of the axioms of the logical system of the *Grundgesetze*, asserts that the truth-value of functions $f(\xi)$ and $g(\xi)$'s having the same value-range is the same as the truth-value of $f(\xi)$ and $g(\xi)$'s having the same value for all arguments. This certainly seems to be initially plausible if value-ranges are understood as complete mappings from argument to value. The other added axioms deal with the description function named by " $\backslash \xi$ ". Given the semantic characterization given above, the value of this function, when its argument is a value-range of a concept under which a single object falls, is this single object. This is stated by axiom $FC^{+V}8$. This function, however, returns its argument as value, if the argument is not a value-range of a concept under which a single argument falls. This is stated by axiom $FC^{+V}9$. The latter axiom is also left out by Frege, again, presumably because it was not needed for his demonstration of the truths of arithmetic.

In effect, the inclusion of these axioms introduces naive class theory into the system. In fact, Frege himself proves a principle of naive class abstraction first by defining a membership sign as follows (*BL* §34):

$$(Df. \cap) \quad \Vdash (x \cap y) = \lambda \alpha \sim (\forall f)[(y = \overset{\cdot}{\varepsilon} f(\varepsilon)) \supset \sim (f(x) = \alpha)]$$

The following class abstraction principle then results as a theorem of FC^{+V} :

$$(CA) \quad \vdash f(a) = (a \cap \overset{\cdot}{\varepsilon} f(\varepsilon))^{24}$$

This asserts that the truth-value of object a 's being a member of the value-range of function $f(\xi)$ is the same as the truth-value that results as value when a is taken as argument to $f(\xi)$. Unfortunately, this class abstraction principle is too strong. If one instantiates $f(\xi)$ to the concept *not being a member of oneself*, $\sim(\xi \cap \xi)$, by (*ri*), one obtains:

$$\vdash \sim(a \cap a) = (a \cap \overset{\cdot}{\varepsilon} \sim(\varepsilon \cap \varepsilon))$$

Then, by instantiating a to the value-range of this concept, $\overset{\cdot}{\varepsilon} \sim(\varepsilon \cap \varepsilon)$, one obtains:

$$\vdash \sim(\overset{\cdot}{\varepsilon} \sim(\varepsilon \cap \varepsilon) \cap \overset{\cdot}{\varepsilon} \sim(\varepsilon \cap \varepsilon)) = (\overset{\cdot}{\varepsilon} \sim(\varepsilon \cap \varepsilon) \cap \overset{\cdot}{\varepsilon} \sim(\varepsilon \cap \varepsilon))$$

This asserts that the truth-value of the extension of the concept *not being a member of itself*'s not being a member of itself is identical to the opposite of this very truth-value. Therefore, FC^{+V} is inconsistent due to Russell's paradox.

RESPONSES TO THE PARADOX

When Frege learned of the paradox plaguing the logical system of the *Grundgesetze* while its second volume was in press, he hastily prepared an appendix in which he discussed the paradox and proposed a modification to his system. The revised system is identical to FC^{+V} except that it replaces axiom $FC^{+V}7$ (Basic Law V) with the following axiom:

$$\text{Axiom } FC^{+V}7: \vdash (\overset{\cdot}{\varepsilon} f(\varepsilon) = \overset{\cdot}{\alpha} g(\alpha)) = (\forall \alpha)[\alpha \neq \overset{\cdot}{\varepsilon} f(\varepsilon) \ \& \ \alpha \neq \overset{\cdot}{\alpha} g(\alpha) \ :\ \supset \ f(\alpha) = g(\alpha)]$$

This principle equates the value-ranges of two functions when they yield the same value for all arguments with the exceptions of the value-ranges themselves. Let us call the resulting system, $FC^{+V'}$. It was later shown that $FC^{+V'}$

²⁴A proof of this theorem can be found in *BL* §55.

is also inconsistent due to a more complicated paradox.²⁵ Given its inconsistency, and the questionable philosophical justification for its defining axiom (which Frege himself admitted from the outset, see *BL* app. 2), the relative interest of FC^{+V} as compared to FC^{+V} is slight.

However, there are other modifications to FC^{+V} that might be made. Dummett has suggested that the blame for the contradiction might be seen to lie not in its incorporation of value-ranges, but in the impredicative comprehension principle that results from *(ri)*.²⁶ Arguably in support of this contention, Heck has shown that the predicative fragments of FC^{+V} (even with its additional axioms) is consistent.²⁷ The resulting system, PFC^{+V} (for *predicative function plus value-range calculus*), would be identical to FC^{+V} , save that *(ri)* would be changed such that Roman function letters could only be replaced by expressions not including the function-quantifier. Cocchiarella has also developed a number of reconstructed Fregean systems that involve modifications to the comprehension principle, which have been shown to be consistent relative to weak Zermelo set theory.²⁸ In favor of the more usual interpretation of where things go wrong, George Boolos has pointed out that the impredicative nature of Frege's system only leads to contradiction in a system that involves the additional axioms for value-ranges, and thus, by itself, is harmless.²⁹ This might seem to suggest a return to simple FC. Indeed, later in his life, Frege seems to have concluded that the fault in his logic lay in the use of value-range notation and that his belief in such objects as extensions of concepts was an illusion created by language (*PW* 263-4, 269-70). However, one problem a Fregean might have with adopting either PFC^{+V} or simple FC is that it is likely that such

²⁵On this see B. Sobocinski, "L'analyse de l'antinomie russellienne par Lésniewski," parts 1-3, *Methodos* 1 (1949): 94-107, 220-8, 308-16; 2 (1950), 237-57; W.V. Quine, "On Frege's Way Out," *Mind* 64 (1955): 145-59; Peter T. Geach, "On Frege's Way Out," *Mind* 65 (1956): 408-9. It is worth noting that these thinkers often describe the problem with the revised system as not outright inconsistency, but only inconsistency given the additional assumption that there is more than one object. This is only, however, because they work within predicate analogues of Fregean logic and not a fully Fregean logic. The modifications do lead to outright inconsistency within a fully Fregean logic, since it is provable therein that the True is not the False.

²⁶Dummett, *Frege: Philosophy of Mathematics*, 217-22.

²⁷Heck, "The Consistency of Predicative Fragments of Frege's *Grundgesetze der Arithmetik*," 209-20.

²⁸Cocchiarella, *Logical Studies*, chaps. 2, 4, and "Cantor's Power-set Theorem Versus Frege's Double Correlation Thesis," *History and Philosophy of Logic* 13 (1992): 179-201.

²⁹Boolos, "Whence the Contradiction?" 213-33.

systems are too weak for carrying out the logicist program. Therefore, it is not surprising that Frege's rejection of value-ranges only came at a point in his life in which he had given up on logicism itself (*PW* 265-81).

In any case, it seems that the commitment within a logical system to *both* such a large number of functions, including impredicative functions, as well as value-ranges for all such functions, is untenable. However, before deciding exactly how this commitment is to be modified, it is worth noting that value-ranges, functions and truth-values are not the only abstract objects to which Frege was philosophically committed. His theory of *Sinn* and *Bedeutung* also commits him to an ontology of a "third realm" of *Sinne* (*CP* 363-72, *PW* 133-49). However, unlike his other commitments, this commitment is not incorporated even into FC^{+V} . By the semantic characterizations for the signs of FC^{+V} given above, every complete wfe of Frege's logic denotes either a value-range or a truth-value, and every incomplete expression denotes a function of some kind. Thus, these are the only sorts of objects to which the system is specifically committed. Given that the *incomplete* inclusion of his commitments to logical entities leads to such formal difficulties, one can only surmise at this point that the *full* inclusion of these commitments would lead to even more difficulties. In the subsequent chapters, we shall pursue this in much greater detail. Only once we see the full range of the problems plaguing Frege's philosophy and metaphysics of logic will we be in a position even to begin to discuss possible emendations to it. First, however, we need to examine how it is that the commitment to the "third realm" would be incorporated into the system. Before we can do this, we must first take a closer look at the nature of the supposed entities populating the third realm. This is taken up in the next chapter.

Sinne and Gedanken

THE NATURE OF SINNE GENERALLY

In the first chapter, we examined briefly the distinction Frege makes between the *Sinn* and *Bedeutung* of a linguistic expression, and the motivation for this distinction. However, in order to correctly include the philosophical commitments of this theory in a logical system, we need to discuss the nature of *Sinne* in much more detail.

Unfortunately, when it comes to spelling out exactly what we are to understand a *Sinn* to be, Frege gives us little help. He says little explicitly except that the *Sinn* of an expression provides the “mode of presentation” (*die Art des Gegebenseins*) of the *Bedeutung* (CP 158, 359), and that a difference in *Sinn* between two expressions means that they differ in “cognitive value” (*Erkenntniswert*) (CP 158, 176-7). This seems to suggest that the *Sinn* of an expression is the information it conveys, which is a natural conclusion given that Frege identifies the *Sinn* of a complete proposition with the *Gedanke* (thought) it asserts as true. Of course, *Sinne* do not merely consist of information, they also *present* something. In the case of *Gedanken*, what they present are truth-values. A *Gedanke* provides a certain set of conditions that must be satisfied in order for the True to be presented, and that *Gedanke* picks out the True just in case those conditions are satisfied (BL §32). In case of a (simple or complex) name, the *Sinn* of the name would seem to be the information the name contains about its *Bedeutung*; the *Sinn* provides a set of conditions or criteria for an object to be picked out by that *Sinn*, and the *Bedeutung* is precisely that object uniquely satisfying these conditions. As Dummett puts it, for a proper name, “to grasp its sense [*Sinn*] is to apprehend the condition which an object must satisfy for it to

be the referent [*Bedeutung*] of the name.”¹ Similarly, the *Sinn* of an expression denoting a concept can be understood as consisting of a set of conditions or criteria that an object must satisfy for it to fall under the concept, and the concept it presents will be precisely the concept that maps all and only those objects satisfying those conditions onto the True when taken as arguments.

For the moment, let us consider only those *Sinne* that present objects as their *Bedeutungen* (including *Gedanken*, whose *Bedeutungen* are truth-values), and reserve discussion of *Sinne* picking out concepts and other functions for the next section. There are different ways in which one might understand what it is for a *Sinn* to provide a “mode of presentation” or “way of determining” a *Bedeutung*. Some interpreters of Frege seem to have taken the notion of a “way” or “mode” of determining quite literally. For them, a *Sinn* actually provides a procedure or means one can carry out for determining the *Bedeutung*, or as David Shwayder puts it, “an indication of how we should set about to establish the existence of a Referent [*Bedeutung*].”² The problem with such a literal interpretation of a *Sinn* as an “Art des Gegebenseins” is that it would imply that the *Sinn* (*Gedanke*) of a whole proposition is the means of determining its truth-value, i.e., the means of verifying or falsifying it.³ This would attribute to Frege a verificationist account of meaning, which there is little evidence to support, and is most likely incompatible with the rest of his philosophy.⁴

According to the more standard reading, a *Sinn* is a set of descriptive information that determines a certain property proposed as true of only one object, and picks out its *Bedeutung* in virtue of its being that object. This reading makes Frege a forerunner of the so-called “descriptivist” theory of meaning and reference in the philosophy of language, often contrasted with the “historical” or “causal” theory of reference. This theory gets its name from Russell’s theory of descriptions, and its later application to proper names.⁵ Confirmation of this

¹Dummett, *Interpretation of Frege’s Philosophy*, 249.

²David S. Shwayder, “On the Determination of Reference by Sense,” in *Studien zu Frege*, vol. 3, *Logik und Semantik*, ed. Matthias Schirn (Stuttgart-Bad Cannstatt: Verlag-Holzboog, 1976), 91-2.

³This Shwayder admits. See Shwayder, “Determination of Reference,” 94.

⁴Consider, for example, what it would mean for logical truths. Logical truths, as *unconditionally* true, are without verifiable truth conditions. Thus, historically, the verificationists were forced to accept Wittgenstein’s conclusion in the *Tractatus* (§4.461) that logical truths are senseless. Frege, however, insists that the axioms of his Begriffsschrift express true *Gedanken* (*BL* §32, *CP* 273). *Gedanken*, therefore, cannot consist in verification conditions.

⁵The theory of descriptions was first proposed in Russell’s “On Denoting,” but fuller application of the theory by Russell to the meaning of proper names is found in Bertrand

view can be seen in the examples Frege gives when speaking of the *Sinne* expressed by ordinary proper names; he often describes such *Sinne* using descriptions of the form “the so-and-so”:

In the case of an actual proper name such as “Aristotle” opinions to the *Sinn* may differ. It might, for instance, be taken to be the following: the pupil of Plato and teacher of Alexander the Great. Anybody who does this will attach another *Sinn* to the sentence “Aristotle was born in Stagira” than will a man who takes as the *Sinn* of the name: the teacher of Alexander the Great who was born in Stagira. (CP 158n, cf. CP 358-9)

Care must be taken here to avoid misunderstanding. *Sinne* are not linguistic items; the *Sinn* of the name “Aristotle” is not (for anyone) the *words* “the pupil of Plato and teacher of Alexander the Great.” The point is only that, for Frege, a name expresses a *Sinn*, and the *Sinn* consists in some sort of descriptive information that applies only to that which is presented by the *Sinn*. The property of being the pupil of Plato and teacher of Alexander is unique to Aristotle, and thus, it may be in virtue of associating this information with the name “Aristotle” that it may be used to refer to Aristotle. As certain commentators have noted, it is not even necessary that the *Sinn* of a name be expressible by some definite description, because the descriptive information or properties in virtue of which the *Bedeutung* is determined may not be directly nameable in any natural language.⁶

In the passage quoted above, Frege was speaking of a certain to-his-mind defective feature of natural language, namely, that there are expressions to which different people attach different *Sinne*. Frege is quite clear that this is to be avoided in the Begriffsschrift and other more ideal languages (CP 158n, 359).

Russell, *The Philosophy of Logical Atomism* (Chicago: Open Court, 1985), 121; Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, 2d ed. (Cambridge: Cambridge University Press, 1927), introd. chap. 3.

⁶See Dummett, *Frege: Philosophy of Language*, 97-8; John R. Searle, *Intentionality: An Essay in the Philosophy of Mind* (Cambridge, Cambridge University Press, 1983), 233, *Speech Acts: An Essay in the Philosophy of Language* (Cambridge: Cambridge University Press, 1969), 90. Of course, the view that the *Sinne* of proper names could be simple descriptions such as “the pupil of Plato and teacher of Alexander” has been harshly criticized, especially by Saul Kripke. (See his *Naming and Necessity* (Cambridge, Mass.: Harvard University Press, 1980), lecture 1.) Searle and others have argued that extensions or modifications of the Fregean line can circumvent Kripke’s difficulties. (See especially, *Intentionality*, chap. 9.) This is a rather well-worn debate in the philosophy of language, and it would take us too far afield to discuss it here.

In the Begriffsschrift and similar languages, a unique *Sinn* is determined for every complete expression, and thus, with regard to such a language, Frege claims that one can denote the *Sinn* of any of expression A with the locution “the *Sinn* of expression ‘A’” (CP 159). (This shows that *Sinne* can themselves be the *Bedeutungen* of expressions, which means that there are some *Sinne* that pick out other *Sinne*.) Another defect of ordinary language Frege identifies is that it employs expressions with *Sinne* that present or pick out no object as *Bedeutungen* (CP 159, 162-3). That there should be such *Sinne* is not surprising given the reading of *Sinne* given here. A *Sinn* consists of a set of criteria or conditions and picks out an object in virtue of it alone satisfying those conditions. However, certainly, there are conditions or criteria that are not satisfied by any unique object. Frege gives as example the expression “the least rapidly converging series” (CP 159). This phrase expresses a *Sinn* that determines a clear condition that would have to be met by something in order for it to be the *Bedeutung*, but it is provable that there is no such thing. These *Sinne* are also important for fictional discourse. Names such as “Odysseus” and “Romulus” undoubtedly express *Sinne*, but no actual people fulfill the descriptive information determined by these *Sinne*. However, because on Frege’s view, the *Bedeutung* of an entire proposition is determined by the *Bedeutungen* of its parts, it is important in an ideal language that every expression have a *Bedeutung* so that all propositions have a determinate truth-value. Thus, in the Begriffsschrift, Frege insures that every sign expresses a *Sinn* that does pick out a *Bedeutung*. (While this does bar us from incorporating signs in the Begriffsschrift that express *Sinne* without *Bedeutungen*, it does not bar us from incorporating signs that *denote* such *Sinne*.)

There may seem to be some difficulties applying the account of *Sinne* given here to *Gedanken*. Indeed, Frege’s contention that the relationship between the *Gedanke* expressed by a complete proposition and its truth-value is the same relationship as that between the *Sinn* expressed by a name and its *Bedeutung*, is often considered one of the least plausible aspects of his philosophy. While I do not here mean to defend Frege’s contention, it does appear more plausible when one bears in mind Frege’s pains to distinguish the *Gedanke* expressed by a proposition from its assertoric force. As discussed in the previous chapter, according to Frege, these are only adequately distinguished in a language such as the Begriffsschrift, which employs a functional approach to predication and requires the use of the judgment stroke for marking assertoric force. The *Sinn* of a wfe remains unchanged regardless of whether the wfe is asserted as true. Thus, to understand the *Gedanke* expressed by a proposition such as our example “ $\vdash \mathcal{P}(m)$ ” from the previous chapters, one need only consider the *Sinn* of the content asserted as true, “ $\mathcal{P}(m)$ ”. According to the method of reading such expressions considered in the last chapter, this would be read, “the truth-value of

Venus's being a planet." This reading describes the *Sinn* of " $\mathcal{P}(m)$ " in the same way that "the pupil of Plato and the teacher of Alexander" might describe the *Sinn* of "Aristotle." If "the pupil of Plato and the teacher of Alexander" describes the *Sinn* expressed by "Aristotle," and "the truth-value of Venus's being a planet" describes the *Gedanken* expressed by a complete proposition, it is easier to see how the relationship between the former and Aristotle as *Bedeutung*, and the latter and the True as *Bedeutung*, should be the same relation (in Frege's eyes at least).

Thus far I have been describing Frege's theory of meaning as being a forerunner of the more contemporary descriptivist tradition exemplified by Russell and Searle. Perhaps the greatest difference between Frege and these thinkers, however, is that Frege reifies *Sinne*. According to Russell's post-1905 theory of descriptions, expressions such as "the present King of France," and even proper names (i.e., explicit and disguised definite descriptions) do not have any sense at all independently of the whole propositions in which they occur. For Frege, however, such expressions have *objects* corresponding to them not only as *Bedeutung* but also as *Sinn*. The *Sinn* of a name or proposition—the cluster of descriptive information associated with a name in virtue of which it denotes what it does—is considered by Frege to be an object (*Gegenstand*) in the full sense (see, e.g., *PW* 194). Perhaps to contemporary thinkers—those affected by Russell's great influence on analytic philosophy—it may seem strange to think of sets of criteria or conditions as being independently existing *objects*. Nevertheless, this view is consistent with Frege's own way of thinking, and with his similar views such as that the extensions (value-ranges) of concepts are also objects in the full sense.

Notoriously, Frege grants *Sinne* a peculiar ontological position, that of existing within a "third realm" (*CP* 363), apart from both the mental and physical. *Sinne* certainly are not objects existing within physical space. Although Frege does use expressions such as "Erkenntniswert" (cognitive value) when speaking of *Sinne*, and describes the *Sinn* of a complete proposition as a "Gedanke" (thought), he in no way thinks of *Sinne* as psychological entities, and instead argues quite strenuously that they have objective existence independent of the psychology of any person. *Sinne* and *Gedanken* are interpersonal; they can be communicated from one person to another. Multiple people cannot share the same mental state, but they can grasp the same *Gedanke* (*CP* 160, 198, 360-3). Moreover, *Gedanken* are objective in the sense that their truth or falsity is not relative to those who think them; the *Gedanke* expressed by the Pythagorean theorem is true for everyone if true for anyone. This Frege takes to be evidence for a separate realm of existence, a realm of entities that are objective but incapable of full causal interaction with the physical world. We cannot discuss the cogency of this argument here, although alternative accounts of the nature of

Sinne will be considered in later chapters. According to Frege's terminology, *Gedanken* (and by extension, other *Sinne*) are "actual" (*Wirklich*) only in the limited sense that they can have an effect on those who grasp them, but are themselves incapable of being changed or acted upon. The *Gedanken* and other *Sinne* of the third realm are timeless, and are neither created by our acts of thinking, nor destroyed by their cessation (*CP* 351-372, *PW* 128-138).⁷

The view that *Sinne* are abstract entities existing in a third realm leads to some interesting results when combined with the account of their nature given above. In particular, it follows that in order for a particular packet of descriptive information to exist as a *Sinn* it is not required that the *Sinn* be associated with any expression in any actual language, or have been grasped by any mind. The particular *Sinn* I attach to the name "Aristotle" has always existed; it existed before I ever heard the name, and for that matter, before Aristotle himself lived. It was there all the while, waiting only to be grasped. It seems natural to conclude that there do exist (and will always exist) an infinite supply of *Sinne* that never have been grasped by any mind nor ever will. Indeed, for any set of conditions or criteria, no matter how unusual or complex, there must exist a *Sinn* that presents the unique object uniquely satisfying these conditions (if there is one). Let us suppose that Aristotle was the only mammal having exactly 133,794 hairs on its head as of midnight on the first day of the year 341 BCE. If so, then due to this condition uniquely satisfied by Aristotle, in the third realm there exists a *Sinn* that picks out Aristotle in virtue of this condition. However, it is safe to assume that this *Sinn* has never been associated by any mind with the name "Aristotle" and, were it not for this paragraph, this *Sinn* would likely have never been grasped at all.

The interest of this result from the standpoint of fashioning a logical system for the theory of *Sinn* and *Bedeutung* is that it entails that for every object in existence, regardless of whether it is ever thought of by any mind or named in any language, there exists at least one *Sinn* that picks it (and only it) out. Frege, as we have seen, is committed to Leibniz's law and the identity of indiscernibles. The identity of indiscernibles requires that no two objects have every property in common. If every object has at least some property different from any other object, it is easily shown that there will be at least one set of conditions that is satisfied uniquely by that object (e.g., the logical conjunction of all of its properties). Thus, if *Sinne* are abstract objects in a third realm, and do not derive their existence from human thinking, communication or language,

⁷For a good discussion of how the third-realm theory of *Gedanken* is used by Frege to fulfill certain conditions necessary of any adequate theory of the nature of thoughts or propositions, see David Bell, "Thoughts," *Notre Dame Journal of Formal Logic* 28 (1987): 36-50.

it follows that every object is presented by some *Sinn*. Moreover, since the *Sinne* that pick out objects are themselves objects, for each such *Sinn* there is another *Sinn* picking it out. This result will be shown to be important in later chapters.

THE *SINNE* OF INCOMPLETE EXPRESSIONS AND THE COMPOSITION OF *GEDANKEN*

So far we have limited our discussion to the *Sinne* expressed by “complete” expressions in language such as complete propositions and simple and complex proper names. However, we have thus far not considered the *Sinne* of incomplete expressions, including grammatical predicates and function signs, such as “ ξ is a planet”, “ $\xi + \zeta$ ”, or such Begriffsschrift symbols as the negation and identity signs. Despite some confusion on this matter in the early secondary literature,⁸ Frege is quite clear that the distinction between *Sinn* and *Bedeutung* applies also to these expressions (see esp. *PW* 118-9, 191-2 *PMC* 63, 255). The *Bedeutungen* of such expressions, as discussed earlier, are functions, including concepts. Although it is now agreed among Frege scholars that these expressions have *Sinne* as well as *Bedeutungen*, there is less agreement as to how we are to understand the *Sinne* of function expressions.

Again, Frege tells us very little. What he does tell us is that just as complete expressions have complete entities as both their *Sinne* and *Bedeutungen*, similarly, incomplete expressions have incomplete or “unsaturated” entities for both their *Sinne* and *Bedeutungen*. We have seen already how it is that for Frege, the *Bedeutung* of an expression such as “ ξ is a planet” is a function, as well as the way in which functions are unsaturated. Frege tells us that the *Sinne* of such function signs are also incomplete, and that it is the unsaturatedness of the *Sinne*

⁸See especially the debate between Marshall, Grossmann and Dummett that occurred in the 1950s in William Marshall, “Frege’s Theory of Functions and Objects,” and “Sense and Reference: A Reply,” reprinted in *Essays on Frege*, ed. E.D. Klemke (Urbana: University of Illinois Press, 1968), 249-67, 298-320; Reinhardt Grossmann, “Frege’s Ontology,” reprinted in *Essays on Frege*, 79-98; Michael Dummett, “Frege on Functions: A Reply,” and “Note: Frege on Functions,” reprinted in *Essays on Frege*, 268-83, 295-7, *Truth and Other Enigmas* (Cambridge, Mass.: Harvard University Press, 1978), 74-86. The publication of Frege’s *Nachlaß* and correspondence, however, has left no doubt that Dummett was correct that the distinction applies also to functions. Also see Dummett, *Frege: Philosophy of Language*, chap. 7; Grossmann, *Reflections on Frege’s Philosophy* (Evanston: Northwestern University Press, 1969), 226-47; Charles E. Caton, “An Apparent Difficulty in Frege’s Ontology,” reprinted in *Essays on Frege*, 99-112; Robert Sternfeld, *Frege’s Logical Theory* (Carbondale: Southern Illinois University Press, 1966), 6, 42-4; David Bell, *Frege’s Theory of Judgment* (Oxford: Oxford University Press, 1979), 71-3.

of expressions such as “ ξ is a planet” that accounts for the unity of the *Gedanken* of complete propositions such as “the morning star is a planet” (CP 193-4, 390-393, PW 119n, 191-2, 254-5, PMC 191-2). In this case, the complete *Sinn* of “the morning star” saturates or completes the incomplete *Sinn* of “ ξ is a planet,” and the unsaturated nature of the latter provides the glue holding the complete *Gedanke* together. Frege also tells us that it is the difference in *Sinn* between expressions such as “ ξ has a heart” and “ ξ has a kidney” that explains the difference in cognitive value between propositions in which they occur, despite that, for Frege, these expressions denote the same concept since they are coextensional (PW 122).

The most common interpretation of these suggestions to be found in the secondary literature—defended by such notable Frege scholars as Church, Furth, Geach, Parsons, Tichý, Currie and Baker and Hacker—is that Frege understands the *Sinne* of function expressions themselves to be functions in the realm of *Sinn*.⁹ Frege’s ontology is usually taken to divide exhaustively between objects and functions; all complete entities are objects, and all incomplete or unsaturated entities are functions. Certainly, there is textual evidence to support this reading. In Frege’s own words, “an object is anything that is not a function” (CP 147, cf. BL §2). Since he explicitly says that the *Sinne* of incomplete expressions are incomplete or unsaturated, this would suggest that they too must be understood as functions. As we have seen, Frege also claims that an incomplete *Sinn* such as that of “ ξ is a planet” binds together with the *Sinn* of “the morning star” to form the complete *Gedanke* expressed by “the morning star is a planet.” This would suggest that we ought to understand the *Sinn* of “ ξ is a planet” as a function that takes *Sinne* picking out objects as argument and yields *Gedanken* as value. Thus, this function yields the *Gedanke* expressed by “the morning star is a planet” when the *Sinn* of “the morning star” is taken as value, but yields as value the

⁹See Howard Jackson, “Frege on Sense-Functions,” reprinted in *Essays on Frege*, ed. E.D. Klemke, 376-81; Furth, Introduction to *The Basic Laws of Arithmetic*, by Gottlob Frege, xxv; Peter T. Geach, review of *Frege: Philosophy of Language*, 440-5; Pavel Tichý, *The Foundations of Frege’s Logic* (Berlin: Walter de Gruyter, 1988), 99; Terence Parsons, “Frege’s Hierarchies of Indirect Senses and the Paradox of Analysis,” in *Midwest Studies in Philosophy*, vol. 6, *The Foundations of Analytic Philosophy*, ed. Peter A. French, Theodore Uehling, Jr. and Howard K. Wettstein (Minneapolis: University of Minnesota Press, 1981), 37-57; Gordon P. Baker and Peter M.S. Hacker, *Frege: Logical Excavations* (New York: Oxford University Press, 1984), 324-6; Currie, *Frege: An Introduction to His Philosophy*, 87; Antoni Diller, “On the Sense of Unsaturated Expressions,” *Philosophical Papers* 22 (1993), 71-9. This presupposition is also built into Church’s Logic of Sense and Denotation. (For references, see chap. 1, note 16.)

Gedanke expressed by “the Earth is a planet” when the *Sinn* of “the Earth” is taken as value. Similarly, the *Sinn* of a Begriffsschrift relation sign such as “ $\xi = \zeta$ ” might be understood as a function in the realm of *Sinn* taking two *Sinne* as arguments, and yielding a *Gedanke* as result.

In the secondary literature, these functions are usually referred to as “sense-functions,” and I shall retain this terminology. The view that the *Sinne* of incomplete expressions are sense-functions leads to a nice parallel between the realms of *Sinne* and *Bedeutungen*. Just as the *Bedeutung* of the sign “ $\xi = \zeta$ ” included in a proposition such as “the morning star = the evening star” is a function that takes as arguments the *Bedeutungen* of “the morning star” and “the evening star” and yields as value the *Bedeutung* of the whole proposition (the True), the *Sinn* of this sign is a function that takes as arguments the *Sinne* of “the morning star” and “the evening star” and yields as value the *Sinn* of the whole proposition (the *Gedanke* that the morning star is the evening star).

Although other authors have pointed out problems with the view that the *Sinne* of incomplete expressions are such sense-functions, Dummett is the only writer on Frege who has seriously challenged it as the true interpretation of what Frege understood the *Sinne* of incomplete expressions to be. He has made a number of criticisms of this view, some of which have been responded to by Geach.¹⁰ I shall limit myself to what I take to be the most important criticisms that are to found (sometimes only implicitly) in Dummett’s works.

Firstly, and perhaps ultimately most importantly, Frege’s reiterates the view that the *Sinn* of an entire proposition is *composed* of the *Sinne* of the expressions making it up, including the function expressions. (See especially *BL* §32, *CP* 378, 390-1, *PW* 225 but also *CP* 386, *PW* 151, 187, 191-2, 201, 243, 254-5, *PMC* 79-80, 98, 142, 149.) However, if we understand the relationship between the *Sinne* of parts of a proposition and the *Gedanke* of the whole as the relationship of function, argument and value, it is not clear how or why this should be. Typically, the value of a function for a certain argument cannot properly or coherently be understood as composed of the function and argument. Although twelve minus five is seven, the number seven cannot be understood as composed of the numbers twelve, five and subtraction, unless seven is similarly to be understood as composed of the infinite number of other numbers and mathematical functions for which it is yielded as value. Certainly, the same object (the True) cannot coherently simultaneously be thought to be composed out of the concept denoted by “ ξ is a planet” and the planet Venus, and also

¹⁰Dummett, *Frege: Philosophy of Language*, 291-4, *Interpretation of Frege’s Philosophy*, 249-53, 265-70, “Frege as a Realist,” in *Frege and Other Philosophers* (Oxford: Oxford University Press, 1991), 87-8, “An Unsuccessful Dig,” in *Frege and Other Philosophers*, 191; Geach, review of *Frege: Philosophy of Language*, 440-5.

thought to be composed out of the concept denoted by “ ξ is human” and Socrates, and so on. Thus it would seem to be quite a mystery how it could be that, if the *Sinn* of “ ξ is a planet” is a function that, when given the *Sinn* of “the morning star” as argument, yields the *Gedanke* expressed by “the morning star is a planet” as value, it should also be that this function and its argument could be considered components or parts of that *Gedanke*.

It is worth noting that especially early in his career, but as late as the early 1890s, Frege sometimes speaks as if it *were* his view that, even in the realm of *Bedeutung*, a function and its argument are related to the value yielded by that function for that argument as parts to whole. He states, for example that “the *Bedeutung* of a word [is] part of the *Bedeutung* of the sentence, if the word itself is a part of the sentence” (CP 165, cf. CP 281). This would imply that Socrates is part of the True, since Socrates is the *Bedeutung* of “Socrates” and the True is the *Bedeutung* of “Socrates is wise.” These passages have led some Frege interpreters to conclude that this is Frege’s final view on the matter.¹¹ However, this is not so. Perhaps realizing the incoherence of the view that objects such as the True or seven are to be understood as *composed* an infinite number of irreconcilable ways, Frege later explicitly rejects the principle that the *Bedeutung* of a whole proposition is composed out of the *Bedeutungen* of its parts or that a the value of a function is composed of the function and argument. An example he gives is that of the function denoted by “the capital of ξ .” The value of this function for Sweden taken as argument is Stockholm, yet certainly we cannot consider Sweden to be a part or component of Stockholm—quite the contrary (PW 255).¹²

In responding to Dummett’s objection to the sense-function understanding of the *Sinne* of function expressions, Geach tries to downplay the problem that seems to come from the need to see the *Sinne* of function expressions as *parts* of the *Sinne* of the propositions in which they appear by suggesting that Frege’s repeated claims to this effect should be taken no more seriously than his early suggestions that the *Bedeutungen* of parts of a proposition are parts of the

¹¹See, e.g., Tichý, *The Foundations of Frege’s Logic*, 34, Eike-Henner W. Kluge, *The Metaphysics of Gottlob Frege* (The Hague: Martinus Nijhoff, 1980), chap 3; Wolfgang Carl, “Frege’s Begriff des Logischeinfachen,” in *Studien zu Frege*, vol. 2, *Logik und Sprachphilosophie*, ed. Matthias Schirn (Stuttgart-Bad Canstatt: Verlag-Holzboog, 1976), 35, *Frege’s Theory of Sense and Reference* (Cambridge: Cambridge University Press, 1994).

¹²For more on Frege’s later disavowal of the notion that the arguments to functions form parts of their values, see Dummett, *Frege: Philosophy of Language*, 159-60, *Interpretation of Frege’s Philosophy*, 179-81.

Bedeutung of the whole.¹³ Geach implies that just as Frege rescinded the suggestion that the value of a function in the realm of *Bedeutung* is composed out of the function and argument once he had thought better of it, he would presumably also rescind the notion that *Gedanken* are composite upon further reflection. If Geach were right, this would deflect this objection to the sense-function interpretation of the *Sinne* of function expressions.

Unfortunately, however, Geach's response does not stand up to scrutiny upon examination of the textual evidence. Historically, Frege's suggestions that the *Sinne* of parts of a proposition are components of the *Gedanke* expressed by the whole become more frequent and more pronounced precisely at the time at which he realized that the *Bedeutung* of a whole proposition or expression *cannot* be understood as composed out of the *Bedeutungen* of its parts. Indeed, Frege leads into the example he gives concerning the expression "the capital of Sweden" by marking a *difference* between the realms of *Sinn* and *Bedeutung*:

We can regard a sentence as a mapping of a *Gedanke*: corresponding to the whole-part relation of a *Gedanke* and its parts we have, by and large, the same relation for the sentence and its parts. Things are different in the domain of *Bedeutung*. We cannot say that Sweden is a part of the capital of Sweden. (*PW* 255)

We simply cannot understand Frege's claims that *Gedanken* are composite *Sinne* as being the result of the same oversight that lead him to suggest at times that truth-values and other values of functions in the realm of *Bedeutung* are composed of function and argument. Indeed, there may be evidence that his espousal of the compositionality of *Sinne* was in part a reaction to his realization that *Bedeutungen* are not composite.¹⁴

According to Dummett, what is particularly worrisome about Geach's abandonment of the notion that *Gedanken* of whole propositions are composed out of the *Sinne* of their component expressions in favor of the sense-function view is that it poses perhaps an insurmountable obstacle to understanding how it is that we are able to grasp *Gedanken*, particularly the *Gedanken* of sentences we have never heard before. If one looks at the passages in which Frege suggests that *Gedanken* are composite, his argument for this thesis is sometimes based on how this enables us to grasp the *Sinne* of new propositions:

¹³Geach, review of *Frege: Philosophy of Language*, 444.

¹⁴For an interesting discussion of the evolution of Frege's views on these matters from his early work in the *Begriffsschrift* on to the doctrine of the complexity of *Gedanken*, see Gregory Currie, "Frege's Metaphysical Argument," *Philosophical Quarterly* 34 (1984): 329-42.

It is astonishing what language can do. With a few syllables it can express an incalculable number of *Gedanken*, so that even if a *Gedanke* has been grasped by an inhabitant of the Earth for the very first time, a form of words can be found in which it will be understood by someone else to whom it is entirely new. This would not be possible, if we could not distinguish parts in the *Gedanke* corresponding to the parts of a sentence, so that the structure of the sentence can serve as a picture of the structure of the *Gedanke*. (CP 390)

Suppose I have never before heard the proposition “the morning star is a planet,” nor grasped the *Gedanke* it expresses. How is it that I can nevertheless understand this *Gedanke* when I now hear this sentence spoken? Frege implies that I ought to be able to grasp the *Gedanke* by grasping the *Sinne* of the parts of the sentence, which I have already grasped in isolation, and combine them together to arrive at the new complete *Gedanke*. However, Dummett complains that this is impossible given Geach’s reading of the *Sinne* of incomplete expressions. On Geach’s reading, it must be possible for me to grasp first the *Sinn* of “the morning star” and the sense-function expressed by “ξ is a planet,” and then arrive at the particular *Gedanke* expressed by the whole by knowing that this *Gedanke* is the value of the sense-function for the *Sinn* of “the morning star” taken as argument. However, in order to know that this *Gedanke* is the value of the sense-function for this argument, I would have to already have had a prior grasp of it, which, by supposition is not true. Therefore Dummett concludes that Geach’s reading is defective because it renders unintelligible how it is that we are able to grasp *Gedanken*.¹⁵

In defense of the sense-function view, Antoni Diller argues that Dummett’s objection here rests upon adopting a particular notion of what a function is.¹⁶ Diller notes that in mathematics there are two distinct notions of a function, the set-theoretic notion and the notion of a function as a rule or procedure. On the set-theoretic notion, a function is understood as an exhaustive set of ordered couples. On this reading, the function *square of* would be understood as consisting of the ordered pairs $\langle 1, 1 \rangle$, $\langle 2, 4 \rangle$, $\langle 3, 9 \rangle$, etc. However, on the alternative understanding of functions, a function lays down a method for computing or arriving at the value given the argument or arguments. Here, the function *square of* consists not in an infinite set of ordered pairs, but rather as the process of multiplying a number by itself. Diller complains that Dummett’s objection rests on understanding sense-functions set-theoretically. If sense-functions are simply ordered couples of *Sinn* to *Sinn*, then it is true that grasping the sense-function expressed by “ξ is a planet” will provide no help unless one is

¹⁵Dummett, *Interpretation of Frege’s Philosophy*, 267-70.

¹⁶Diller, “The Sense of Unsaturated Expressions,” 71-8.

already aware of the ordered couple <the *Sinn* of “the morning star”, the *Gedanke* expressed by “the morning star is a planet”>, and thus, already aware of the *Gedanke* in question. Here, Dummett’s objection stands. However, if sense-functions are understood as procedures or rules for arriving at values, matters are different. On this reading, we can understand the sense-function expressed by “ ξ is a planet” as consisting of a method for arriving at *Gedanken* given *Sinne* as arguments, and if so, it may be possible that in grasping this sense-function, I may be able to apply it to the *Sinn* of “the morning star” and arrive at the correct *Gedanke*, even if I have never grasped it before. Diller concludes that this offers an adequate solution to Dummett’s criticism.

However, what is conspicuously absent from Diller’s discussion is any consideration at all of what Frege’s *own* conception of a function is and what it means for the sense-function view. Diller leaves Frege himself completely out of the picture. As it turns out, Frege’s understanding of functions does not fall squarely within either of these camps but if anything is closer to the set-theoretic understanding (as Dummett is quite aware). Frege does not understand a function to be a complete set of ordered couples—this, if anything, would be his understanding of the value-range of a function. However, it must not be forgot that Frege believes that functions differ only insofar as their value-ranges differ. This precludes Frege from adopting a procedure or rule-based account of functions. As we have seen, for Frege, “ ξ has a heart” and “ ξ has a kidney” must be understood as denoting the same function (in particular, the same concept). However, certainly, these two involve different *procedures* or *rules* for how one would determine a truth-value given an argument. Or, to give a mathematical example, the function denoted by “ ξ^2 ” would be understood as the same function denoted by “ $\xi \times (3\xi - ((9^2 + 2\xi) - 81))$ ”, since they would have the same value for any argument, although these two clearly involve different methods of calculation. Thus, Frege could not endorse Diller’s solution to Dummett’s problem with regard to sense-functions.

The distinction Diller makes with regard to functions considered as simple mappings of argument onto value as opposed to functions considered as procedures or rules *is* relevant to the discussion as to what the *Sinne* of function expressions are, although not in the way Diller suggests. Frege himself speaks as if the difference between function expressions such “ ξ has a heart” and “ ξ has a kidney” is a difference in *Sinn*, not *Bedeutung* (e.g., *PW* 122). Functions differ only insofar as the argument-value mappings they determine differ, but different expressions for the same function may be associated with different procedures as to how to generate that mapping, and this involves the differing *Sinne* of the function expressions. This must be borne in mind when considering what the *Sinn* of a function expression could be, but does not, as far as I see, help defend the view that these *Sinne* are themselves functions. If anything, it seems to

reinforce Dummett's criticism, since we are left in the dark as to how to generate the mapping involved with the sense-function itself.

If Dummett is correct that the *Sinne* of incomplete expressions cannot be understood as sense-functions, we are left with the task of explaining what they are instead. Dummett himself, taking seriously the seemingly exhaustive division within Fregean metaphysics between function and object, concludes that the *Sinne* of incomplete expressions are a peculiar sort of *object*. Dummett is of course quite aware that Frege himself thinks of these *Sinne* as somehow incomplete or unsaturated, and is aware of the tension that seems to arise then from concluding that these *Sinne* are objects, since the hallmark of an object as against a function is its complete and self-standing nature. Dummett concludes that the *Sinne* of incomplete expressions are incomplete in a different and very limited sense that does not preclude them from being considered objects. As he puts it, the incompleteness of such a *Sinn* "consist[s] merely in its being the sort of sense [*Sinn*] appropriate to an incomplete expression" and that it is necessary in grasping such a *Sinn* to understand it as being expressible only by "an expression containing argument-places."¹⁷ Thus, Dummett seems to make the incompleteness of these *Sinne* merely derivative on the incompleteness of the expressions whose *Sinne* they are.

There is strong textual evidence to support the view that the *Sinne* of incomplete expressions *are* incomplete or unsaturated in a *special* sense, that their incompleteness is of a fundamentally different kind from that of functions. However, this textual evidence does not support Dummett's contention that the incompleteness of these *Sinne* is derivative on the incompleteness of functional expressions. Let us examine this evidence more carefully.

As we have seen, according to Frege's mature position, when an argument saturates or completes a function, it is not in the sense of filling in a gap or empty space to form a whole. However, when the *Sinn* of "the morning star" saturates or completes the incomplete *Sinn* of "ξ is a planet," it *is* in the sense of filling a gap to form a whole. This seems to suggest that the way in which the *Sinn* of an incomplete expression is unsaturated is different from the way in which a function is unsaturated. However, if anything, it seems to imply that the *Sinne* of functions expression are more literally "incomplete" or "unsaturated" than the functions they have as their *Bedeutungen*, since it is only the former that actually form wholes with that with which they come together. Consider the following passages directly from Frege:

The words "unsaturated" and "predicative" seem *more suited* to the *Sinn* than the *Bedeutung*; still there must be something on the part of the *Bedeutung*

¹⁷Dummett, *Frege: Philosophy of Language*, 291.

which corresponds to this, and I know of no better words. (*PW* 119n, emphasis added)

If we split up a sentence into a proper name and the remainder, then this remainder has for its *Sinn* an unsaturated part of a *Gedanke*. But we call its *Bedeutung* a concept . . . We can, *metaphorically speaking*, call the concept unsaturated too; alternatively we can say that it is predicative in nature. (*PW* 193, emphasis added)

If it is true that terms such as “unsaturated” are *more* suited to the *Sinne* of function expressions than the *Bedeutungen* (and indeed apply to the latter only *metaphorically*), then this certainly counts against the sense-function view, according to which the two are unsaturated or incomplete in precisely the same way (since both are functions). However, it also counts against Dummett’s view, according to which the unsaturatedness of the *Sinne* of function expressions is derivative, since here Frege seems to imply that it is with these *Sinne* that unsaturatedness takes its purest form.

Elsewhere, when speaking of the word “and” (which, of course, is understood by Frege as a function expression), contrary to Dummett’s suggestion that the incompleteness of the *Sinne* is derivative on the incompleteness of the expressions, Frege explicitly suggests the reverse:

The “and” whose mode of employment is more precisely delimited in this way seems doubly unsaturated; to saturate it we require both a sentence preceding and another following. And what corresponds to “and” in the realm of *Sinn* must also be doubly unsaturated: inasmuch as it is saturated by *Gedanken*, it combines them together. As a mere thing, of course, the group of letters “and” is no more unsaturated than any other thing. It may be unsaturated in respect of its employment as symbol meant to express a *Sinn*, for here it can have the intended *Sinn* only when situated between two sentences: its purpose as a symbol requires completion by a preceding and a succeeding sentence. It is really in the realm of *Sinn* that unsaturatedness is found, and it is transferred from there to the symbol. (*CP* 393)

This seems to leave little doubt that Frege could not really have taken the *Sinne* of function expressions as being objects that are incomplete in a derivative sense. Instead, the *Sinne* of such expressions are the very paradigm of incompleteness for Frege.

The only other argument Dummett presents for considering the *Sinne* of incomplete expressions to be objects is that we seem to be able to denote these *Sinne* with expressions such as “the *Sinn* expressed by ‘ξ is a planet’.” Since the

expression “the *Sinn* expressed by ‘ ξ is a planet’” is itself a complete expression and can play the role of grammatical subject, on Frege’s view of the way language works, it must denote an object.¹⁸ It is indeed Frege’s view that complete expressions must always denote objects, and that names and descriptive phrases of the form “the such-and-such” must have objects as *Bedeutungen*. This is precisely why he believes, notoriously, that the expression “the concept *horse*” does not denote a concept, since it must denote an object.¹⁹ Thus far, I have been ignoring this idiosyncratic feature of the way Frege believes that language works, and allowing myself to speak loosely in my discussion of such things as “the concept of *square root of four*,” etc., as though I thereby actually referred to a concept (and I shall continue to do so). Frege, too, allows himself to speak loosely in this way, because he cannot see a way around it (see, e.g., *PW* 193). But Dummett is right that Frege’s official position would be that the phrase “the *Sinn* expressed by ‘ ξ is a planet’” denotes an object if it denotes at all. However, this does not show that the *Sinne* of incomplete expressions are objects, any more than the fact that we can form expressions such as “the *Bedeutung* of the expression ‘ ξ is a planet’” shows that the *Bedeutungen* of incomplete expressions are objects. It may simply be that we cannot actually refer to the real *Sinn* of the phrase “ ξ is a planet” by using the expression “the *Sinn* of ‘ ξ is a planet’,” just as we cannot actually refer to the real *Bedeutung* of “ ξ is a horse” by using the phrase “the concept *horse*.”

As may already be clear, my own view of how Frege understands the *Sinne* of function expressions, at least in his mature philosophy, is that they are neither functions nor objects, but a particular type of unsaturated entity in the realm of *Sinn*. Although I am aware of others who have this view, to my knowledge, it has not been defended in print by any prominent writer on Frege.²⁰ Perhaps the biggest drawback to this view is that it forces us to revise Frege’s explicit claims that everything is either a function or an object. However, it is worth noting that at the spots in which Frege explicitly makes this claim (*CP* 147, *BL* §2), he is speaking of the language of the *Begriffsschrift*. It is true that in the language of the *Begriffsschrift*, every sign stands for either a function or an object, and thus, in a sense, functions and objects exhaust the universe of Frege’s extant logical system. But it has been one of the themes of this work that Frege’s logical system is lacking in not including within it commitment to the entities of the third realm of *Sinn*. Also, the differences between incomplete *Sinne* and functions may be slight enough that Frege simply allowed himself to overlook

¹⁸Dummett, *Frege: Philosophy of Language*, 291.

¹⁹On this, see chap. 2, note 19.

²⁰Currie hints at this sort of view, but does not himself fully endorse it. See *Frege: An Introduction to His Philosophy*, 93-4.

the differences for the purposes of the works in question. Incomplete *Sinne* are similar to functions in being somehow unsaturated and, thus, if signs were introduced standing for them in the Begriffsschrift, they too would have to be written with argument-places such that when the argument-place is filled, the whole stands for an object. In any case, the view that these *Sinne* are a different sort of incomplete entity is the only plausible way of making sense of everything Frege has to say about them, such as how they are able to form wholes when saturated by other *Sinne*, and how they are unsaturated in a different, but, if anything, *purer* way, than functions.

Moreover, this view of the nature of the *Sinne* of function expressions allows us to avoid making another emendation to Frege's official view that Dummett is forced to make in taking such *Sinne* to be objects. On Dummett's view, there is a problem in understanding the unity of clauses in *oratio obliqua*. Consider the statement, "Plato believed that Socrates was wise." On Frege's official view, in *oratio obliqua*, words denote the *Sinne* they express in ordinary contexts. This means that the whole phrase "Socrates was wise" must denote a *Gedanke*. However, on Dummett's view, the component words "Socrates" and "ξ was wise" both typically express objects as their *Sinne*, and thus, in *oratio obliqua*, it appears that the whole "Socrates was wise" consists simply of the names of two objects standing next to each other, and it is unclear how these two names are supposed to come together. Dummett is forced to revise Frege's claim that the *Bedeutung* of an expression in *oratio obliqua* is always its customary *Sinn*. Instead, he suggests that the indirect *Bedeutung* of "ξ was wise" is the sense-function Geach and others believe to actually be its *Sinn*. Dummett does not deny that sense-functions exist, he only questions whether they themselves are the *Sinne* of incomplete expressions. Since he does not deny their existence, he is able to posit them as the *Bedeutungen* of expressions in *oratio obliqua*.²¹ This complication is avoided on my view, since I can take the indirect *Bedeutungen* of function expressions to be the incomplete entities they normally express as *Sinne*, and it is entirely natural how such expressions would come together with other expressions to form expressions for full *Gedanken*.

However, it is worth noting that, like Dummett, I do not deny the existence of sense-functions. Although I do not take them to be the *Sinne* of incomplete expressions, it is true that the existence of an incomplete *Sinn* does determine a unique mapping from the *Sinne* that would complete them to the complex *Sinne* that result when they are completed. Frege's view of functions seems to entail that there exists a function for every determinate mapping of objects onto other

²¹Dummett, *Frege: Philosophy of Language*, 293-4, *Interpretation of Frege's Philosophy*, 251.

objects. This does not, however, vindicate the view that sense-functions *are* the *Sinne* of incomplete expressions, only the view that sense-functions must exist.

The true test of any account of the *Sinne* of function expressions is whether or not it coheres with the nature of *Sinne* generally. In the last section, we attributed to Frege the view that *Sinne* whose *Bedeutungen* are objects should be understood as packets of descriptive information that determine a *Bedeutung* in virtue of it alone satisfying the descriptive information. This suggests an account of the nature of incomplete *Sinne*; namely, that they should be understood as packets of descriptive information that are somehow incomplete in the sense of having missing information. Here again, we must heed the warning not simply to conflate *Sinne* with descriptive *phrases*, since *Sinne* are not themselves linguistic entities. However, if we allow ourselves to use descriptive phrases to describe the contents of *Sinne*, then we might think of the *Gedanke* expressed by “Aristotle has a heart” as represented by the phrase, “the truth-value of the pupil of Plato and teacher of Alexander’s having an muscle that pumps blood through the arteries.” The incomplete *Sinn* of “ξ has a heart” thus consists of the incomplete descriptive information, “the truth-value of ()’s having an muscle that pumps blood through the arteries.” Incomplete *Sinne*, like complete *Sinne*, contain descriptive information, but unlike complete *Sinne*, they are somehow in need of supplementation. An incomplete *Sinn* picks out a function in virtue of its having as value, for any argument, whatever uniquely satisfies the descriptive information that results upon completing the incomplete descriptive information of the incomplete *Sinn* by descriptive information uniquely satisfied by the argument. This view is wholly consistent with the compositionality of *Gedanken*, since the incomplete descriptive information contained within the incomplete *Sinn* is completed by the saturating *Sinn*, and this information is a part of the whole that results. This view also explains the difference in *Sinn* between “ξ has a heart” and “ξ has a kidney,” since the latter would consist of different descriptive information from that suggested above. While the *Gedanken* that result from the saturation of these incomplete *Sinne* by the same complete *Sinn* would always present the same truth-value, this truth-value would be presented in virtue of different conditions holding.

CONTEXT PRINCIPLE, PRIORITY THESIS, AND MULTIPLE ANALYSES: CHALLENGES TO COMPOSITIONALITY

As we have seen, there is strong textual evidence to support the view that Frege takes the *Gedanke* expressed by a complete proposition to be *composed* out of the *Sinne* of the component expressions in the proposition. Let us call this the *composition principle*. I believe this to be a very important and fundamental principle within Frege’s philosophy of language. It is also a very important principle in the attempt to fashion a Fregean logical calculus for the theory of

Sinn and *Bedeutung*. For example, consider the question of whether it could be possible for the same *Gedanke*, *A*, to result from incomplete *Sinn* *B*, both when *B* is completed by complete *Sinn* *C*, and when completed by complete *Sinn* *D*, when *C* and *D* are not the same *Sinn*. I take it that the composition principle rules this out. If *A* is *composed* of exactly two components, *B* and *C*, it could not also simultaneously be thought of as being composed of *B* and *D*, unless *C* and *D* are the same. Thus, it is important to understand exactly how strong the composition principle is to be understood before engaging in the project of transcribing Frege's philosophy of *Sinne* and *Gedanken* into a formal system.

On a strong interpretation of the composition principle, *Gedanken* contain within them determinate inner structures, and are capable of a unique final analysis into parts corresponding to the parts of a sentence. However, there are certain aspects of Frege's philosophy and claims he makes that seem to contradict such a strong interpretation. Indeed, there are things he says that call the entire composition principle into question. It will be necessary to take a detailed look at what he says and how it might be thought to conflict with the composition principle.

Perhaps most infamous in this regard is the so-called context principle that appears in Frege's *Grundlagen*. There, Frege demands "never to ask for the meaning of a word in isolation, but only in the context of a proposition" (*FA* x). Later he elaborates:

But we ought always to keep before our eyes a complete proposition. Only in a proposition have the words a meaning . . . It is enough if the proposition taken as a whole has a sense; it is this that confers on its parts also their content. (*FA* §60)

Exactly what view Frege is espousing here is extraordinarily controversial. It does not help matters that he never explicitly repeated the principle outside of the *Grundlagen*. The *Grundlagen* itself is an early work, written in 1884, before Frege had first formulated the distinction between *Sinn* and *Bedeutung* in the early 1890s. Thus, although in the above passages "meaning" translates "Bedeutung," this is no guarantee that the context principle is a principle about *Bedeutung* in Frege's latter terminology, which only compounds the difficulties in interpretation. The result has been an immense amount of secondary work done on Frege's context principle, and as many interpretations of it as there are writers on the subject. Some claim that the context principle was abandoned or radically altered in Frege's later works, while others claim that it was still a

central tenet of his philosophy.²² Because of the vastness of the secondary literature on this topic, I shall not be able to discuss all of the varying interpretations of the principle in any detail, nor situate my own views among the various positions taken by others. Instead, I shall leave it to the reader to compare what I have to say to other writers on Frege.

Indeed, in what follows, I shall not attempt to supply my own interpretation of the context principle as it appears in the *Grundlagen*. The purpose of the present work is to discuss how Frege's mature semantic views would be incorporated into his mature logical system. Since the *Grundlagen* was written before Frege's mature period, its intrinsic importance to this discussion is relatively small. However, there remains the question of whether Frege retained the context principle or something similar to it in his later philosophy, and whether this has any impact on how we are to interpret the composition principle. Before we do this, however, we must delve into why it is that there

²²For works claiming that Frege abandoned or altered the context principle in his later work, see Ignacio Angelelli, *Studies on Gottlob Frege and Traditional Philosophy*, (Dordrecht: D. Reidel, 1967), 73-4; Peter Milne, "Frege's Context Principle," *Mind* 95 (1986), 491-5; Dummett, *Frege: Philosophy of Language*, 6-7, 192-6, 495-505, *Interpretation of Frege's Philosophy*, chap. 19, *Frege: Philosophy of Mathematics*, chaps. 15-8, "The Context Principle: Centre of Frege's Philosophy," in *Logik und Mathematik: Frege Kolloquium 1993*, ed. Matthias Schirn (Hawthorne: de Gruyter, 1995); Resnik, "The Context Principle in Frege's Philosophy," *Philosophy and Phenomenological Research* 27 (1967): 356-65, "Frege's Context Principle Revisited," in *Studien zu Frege*, 3:35-49, "Frege as Idealist and then Realist," *Inquiry* 22 (1979): 359-78. For a variety of other views, see Gilead Bar-Elli, "Frege's Context Principle," *Philosophia* 25 (1997): 99-129; Michael Beaney, *Frege: Making Sense* (London: Duckworth, 1996), 234-45; Leila Haaparanta, "Frege's Context Principle," in *Logic of Discovery and Logic of Discourse*, ed. Jaakko Hintikka and Fernand Vandamme (New York: Plenum Press, 1985), 83-96; Erich Reck, "Frege's Influence on Wittgenstein: Reversing Metaphysics via the Context Principle," in *Early Analytic Philosophy: Frege, Russell, Wittgenstein*, ed. William W. Tait (Chicago: Open Court, 1997), 123-87; Marco A. Ruffino, "Context Principle, Fruitfulness of Logic and the Cognitive Value of Arithmetic," *History and Philosophy of Logic* 12 (1991): 185-94; Christian Thiel, *Sense and Reference in Frege's Logic* (Dordrecht, D. Reidel, 1968), chap. 7; Crispin Wright, *Frege's Conception of Numbers as Objects* (Aberdeen: Aberdeen University Press, 1983), chap. 1; Baker and Hacker, *Frege: Logical Excavations*, 194-232, 293-9; Cocchiarella, *Logical Studies*, 66, 78-80; Currie, "Frege on Thoughts," *Mind* 89 (1980): 234-48, "The Analysis of Thoughts," *Australasian Journal of Philosophy* 63 (1985), 283-98, *Frege: An Introduction to his Philosophy*, 148-65; Sluga, *Gottlob Frege*, 94-5, 133-4, "Frege and the Rise of Analytic Philosophy," 478.

might be some tension or inconsistency between the context principle and a strong interpretation the composition principle. When translated into the language of Frege's mature semantic theory, there are at least two related but different principles we might understand the context principle to assert. On *interpretation one*, the context principle asserts something about the relationship between *Sinne* and *Gedanken*: that the *Sinn* of a word is determined by the *Gedanke* expressed by the whole proposition. On *interpretation two*, the context principle asserts something about the relationship between words and propositions: that the *Sinn* of a word can only be determined or is only recognizable within the context of a proposition.

On interpretation one, there may be a tension between the context principle and the composition principle. According to the composition principle, the *Gedanke* expressed by a proposition is composed or built up from the *Sinne* of its component parts. The way in which I recognize a *Gedanke* I have not grasped before is by putting together the *Sinne* of the component words, which I have grasped before. However, if it is true that it is impossible to determine the *Sinne* of the component words of a proposition independently of the complete *Gedanke*, then this process would be impossible, because it would be necessary for me to grasp the whole *Gedanke* before being able to grasp the component *Sinne*. Indeed, if Frege holds both the composition principle and interpretation one of the context principle, then his account of the relationship between *Sinne* and *Gedanken* is circular: the *Sinne* of the component expressions are determined by the *Gedanke* of the whole, and the *Gedanke* is determined by (composed of) the *Sinne* of the parts.

On interpretation two, however, there need not be any tension between the context principle and the composition principle. On this interpretation, it is only the case that we cannot recognize the *Sinn* expressed by a word in isolation. The same word, particularly in ordinary language, is capable of expressing multiple *Sinne*, and sometimes it is impossible to determine which *Sinn* a word expresses without placing it within the context of a proposition. Here we need not suppose that one grasps the *Sinn* of the word by first grasping the *Gedanke* expressed by the whole proposition, but rather that one only need determine in what sentential context a word appears in order to determine its *Sinn*. One is then free to regard the *Sinn* of the whole proposition, the *Gedanke*, as being composed from the *Sinne* of the component expressions, once those *Sinne* have been determined within the context in which they appear.

Again, I do not mean to take a stand on which of these two interpretations is closer to Frege's actual intention in the *Grundlagen*. However, I believe the evidence is quite clear that while Frege did hold something close to interpretation two of the context principle in his later philosophy, he could not and did not hold interpretation one. Firstly, there are no passages after he had

formulated the distinction between *Sinn* and *Bedeutung* at which Frege suggests that the *Sinne* of the component expressions of a proposition depend upon the *Gedanke* expressed by the whole, and if this were indeed one of the central and most important of his semantic principles, one would expect to find it repeated in his later works.²³ Moreover, this interpretation of the principle puts it in direct opposition to the composition principle, which is repeated at least a half dozen times in Frege's later works. Therefore, if nothing else, the principle of charity forces us to conclude that if Frege ever did accept the interpretation of the context principle that puts it at odds with the composition principle, he abandoned it in his later work.

Moreover, besides the composition principle, there are a number of other features of Frege's later philosophy that put it at odds with the acceptance of interpretation one of the context principle. Firstly, if it were true that the *Sinne* of expressions are determined by the *Gedanken* of the expressions of which they form a part, then it would be impossible to use language to express any *Sinn* without expressing a complete *Gedanke*. However, Frege explicitly claims that it is possible. For example, he suggests that what he calls "word-questions" (*Wortfragen*), by which he means questions beginning with interrogative words like "who" or "what", express incomplete *Sinne*. Thus, the question, "Who wrote the *Tractatus Logico-Philosophicus*?" expresses the incomplete *Sinn* of "ξ wrote the *Tractatus Logico-Philosophicus*." In asking such a question, we expect the respondent to say something whose *Sinn*, when used to complete the incomplete *Sinn* of the question, yields a true *Gedanke* (CP 355). Thus it must be possible to express *Sinne* that are not *Gedanken*, even incomplete *Sinne*, on their own.

Secondly, in Frege's later philosophy, it is implausible to suppose that the *Sinne* of names of objects (simple or complex proper names) would be determined by the *Gedanken* of the complete propositions in which they occur, considering that there is no crucial distinction in Frege's mind between *Sinne* that pick out objects and *Gedanken*. *Gedanken* simply are *Sinne* that pick out truth-values as their *Bedeutungen*. In the *Begriffsschrift*, the only difference between names of truth-values and complete propositions is that complete

²³Here we must bear in mind that due to the poor reception of his works during his own lifetime, Frege often found himself having to repeat the same arguments, and indeed complained about having to do so (see, e.g., CP 345). As a result, his major doctrines are repeated in many of his works. Thus, I find it difficult to accept the claims of certain philosophers that the context principle, which appears only once in one of his early works, is the "center of Frege's philosophy" (as in the title of Dummett's "The Context Principle: Centre of Frege's Philosophy,") or Frege's "most important single contribution to the philosophy of language" (Bar-Elli, "Frege's Context Principle," 99).

propositions include the judgment stroke. The judgment stroke, however, does not affect the *Sinn*. The *Sinn* of the proposition “ $\vdash \mathcal{P}(m)$ ” is the same as the name “ $\mathcal{P}(m)$ ”, viz., the *Gedanke* that the morning star is a planet. However, “ $\mathcal{P}(m)$ ” is a *term* of the Begriffsschrift and is no different in this regard from “ m ” by itself, and it can appear as argument to function signs, as in “ $\mathcal{P}(\mathcal{P}(m))$ ” or “ $\sim\mathcal{P}(m)$ ”. Here, Frege would say that the *Gedanke* expressed by “ $\mathcal{P}(m)$ ” is a part of the *Gedanken* expressed by “ $\mathcal{P}(\mathcal{P}(m))$ ” and “ $\sim\mathcal{P}(m)$ ” (CP 386-8, 390-406). However, the sense in which the *Gedanke* expressed by “ $\mathcal{P}(m)$ ” forms a part of the *Gedanke* expressed by “ $\mathcal{P}(\mathcal{P}(m))$ ” is in precisely the same sense in which the *Sinn* of “ m ” forms a part of the *Gedanke* expressed by “ $\mathcal{P}(m)$ ”. Here, the *Sinn* expressed by “ m ” and the *Gedanke* expressed by “ $\mathcal{P}(m)$ ” play precisely the same role, both provide conditions or criteria in virtue of which a certain object is determined as *Bedeutung*. The only difference between *Gedanken* and other complete *Sinne* is that the former are limited to picking out truth-values, while the latter may have other objects as *Bedeutungen*. This makes it very difficult to maintain that there is something unique or special about *Gedanken* vis-à-vis other *Sinne* such that it is only by being a component of a proposition expressing a *Gedanke* that a name of an object (or other expression) could have its own *Sinn*.

Frege does, in his later work, accept that in ordinary language, something such as interpretation two of the context principle is the case. Specifically, he says that in ordinary language, it is not always the case that the same word expresses the same *Sinn* or has the same *Bedeutung* in all contexts in which it appears, although he speaks of this as if it were a *defect* of ordinary language:

To every expression belonging to a complete totality of signs, there should certainly correspond a definite *Sinn*; but natural languages often do not satisfy this condition, and one must be content if the same word has the same *Sinn* in the same context. (CP 159, cf. PMC 115)

In a logically superior language such as the Begriffsschrift, he says instead:

Where inferences are to be drawn the case is different: for this it is essential that the same expression should occur in two propositions and should have exactly the same *Bedeutung* in both cases. It must therefore have a *Bedeutung* of its own, independent of the other parts of the proposition. (PMC 115)

Here, Frege explicitly denies that the context principle—in either of the interpretations considered—applies to the Begriffsschrift. If it is possible to create a language in which words have *Bedeutungen* independently of the other parts of the proposition, since *Bedeutung* is determined by *Sinn*, then certainly it

is not a necessary feature of the relationship between *Sinne* and *Gedanken* that an expression cannot have its own *Sinn* independently of the *Gedanke* expressed by the whole proposition. It is true of natural languages that a word does not always have the same *Sinn* and *Bedeutung*, but this is due to a feature of their construction. Thus, the context principle Frege holds in his later philosophy is not a principle about the relationship between *Sinne* and *Gedanken*, but the contingent feature of some languages that certain words have different *Sinne* and *Bedeutungen* in different propositional contexts, and thus, one needs to consider the whole proposition in order to grasp their *Sinne*.

There are a variety of ways in which words found in ordinary language vary their *Sinne* depending on context. Some ordinary language words are simply ambiguous; they have multiple, unrelated meanings, such as “bar” or “bat.” It is only by having a familiarity with all of the different *Sinne* these expressions can have individually that one is able to determine which *Sinn* is at play in any given sentence by knowing in what contexts they are likely to appear. However, there are other ways in which words in ordinary language vary their *Sinne*, some of which we have already discussed. As we have seen, when words appear in *oratio obliqua* in ordinary language, they have as *Bedeutungen* what are ordinarily their *Sinne*. Since words are related to their *Bedeutungen* only through *Sinne*, when words appear in *oratio obliqua*, not only do their *Bedeutungen* change, but likewise do their *Sinne*. In oblique contexts, the *Sinn* of word becomes a mode of presentation picking out what is ordinarily its *Sinn*. Thus, the *Sinn* of a word will vary within oblique contexts, and in understanding ordinary language, one cannot determine the *Sinn* of a word without determining what context it is in. Also, Frege holds that the *Sinn* and *Bedeutung* of a word in ordinary language vary depending upon whether or not the word appears predicatively. If a word appears predicatively, it must stand for a concept, but, as we have also seen, if a word appears in subject position, it must stand for an object. Some words, however, can occur in both contexts. Frege himself uses the word “Vienna” as example (CP 189). Normally, this word occurs in subject position and names a city, an object, as in “Vienna is the capital of Austria.” However, the word can also occur predicatively, as in “Trieste is no Vienna.” In the first example, “Vienna” must have a complete *Sinn* and complete *Bedeutung* (a city), but in the second, it has an incomplete *Sinn* and incomplete *Bedeutung* (the concept of *being a Vienna*).²⁴ Although in an ideal language one would avoid using the same sign for both a city and a concept, the practice is common enough in colloquial language.

²⁴For more on this point, see Cora Diamond, “Frege and Nonsense,” in *The Realistic Spirit: Wittgenstein, Philosophy and the Mind* (Cambridge, Mass.: MIT Press, 1991); Cocchiarella, *Logical Studies*, 78-80.

However, none of these examples calls the composition principle into doubt. Although the *Sinn* of the word “Vienna” is not the same in “Vienna is the capital of Austria” and “Trieste is not Vienna,” in each case, “Vienna” does contribute a *Sinn* to the whole *Gedanke*, and there is no difficulty in seeing this *Sinn* as a component of the *Gedanke* in question. One does not need a prior grasp of the *Gedanke* in order to grasp the *Sinn*, one only needs a sufficient familiarity with language in order to determine whether or not a word appears predicatively. To use another example, although the words “London is pretty” do not express the same *Sinn* when those words appear in the direct context, “London is pretty and New York is not,” as they do in the *oblique* context, “Pierre believes that London is pretty,” in each case there is no barrier to understanding the *Sinn* it does express as a part of the *Gedanke* expressed by the whole. In the first example, the *Gedanke* expressed by “London is pretty” is a part of the composite *Gedanke* expressed by the whole. In the second example, the *indirect Sinn* of “London is pretty” (which is not itself a *Gedanke*, but a *Sinn* that picks out a *Gedanke* as *Bedeutung*) comes together with the direct *Sinn* of “Pierre believes that . . .” to form a complete *Gedanke*. Thus, a close reading of Frege’s later philosophical works does seem to support the contention that he held a weak form of that context principle, but not a form that poses any problems for the composition principle.

However, there are other theses to be found in Frege’s later philosophy that might also be seen to pose a problem for the composition principle, some of which have been linked by some writers to a stronger form of the context principle. Consider for example, what is sometimes called the *priority thesis*, the claim that the content of a judgment is logically prior to the concepts involved in such a judgment. This claim is found even in Frege’s later writings, such as his “Notes for Ludwig Darmstädter,” which also contains the *Sinn/Bedeutung* distinction and even the composition principle. Therein, he writes:

What is distinctive about my conception of logic is that I begin by giving pride of place to the content of the word “true”, and then immediately go on to introduce a *Gedanke* as that to which the question “Is it true?” is in principle applicable. So I do not begin with concepts and put them together to form a *Gedanke* or judgment; I come by the parts of a *Gedanke* by analysing the *Gedanke*. This marks off my Begriffsschrift from the similar inventions of Leibniz and his successors, despite what the name suggests; perhaps it was not a very happy choice on my part. (*PW* 253, cf. *PMC* 101)

This passage almost seems a vindication of interpretation one of the context principle, because it seems to imply that *Gedanken* are logically prior to the *Sinne* contained as parts, and perhaps even that recognition of any *Sinn* requires

analyzing a previously grasped *Gedanke*. In fact, if taken to the extreme, the priority thesis may even be seen as implying that *Gedanken* do not have a determinate and unique inner structure or composition as such, and it is only through our acts of analysis that they can come to be viewed as such. Proponents of this reading also appeal to Frege's numerous claims that the same *Gedanke* can be expressed by different propositions with different grammatical structures (e.g., *CP* 188) and that *Gedanken* are capable of being analyzed in multiple ways (*CP* 188-9, *PW* 187, 192, 202). If it is true that *Gedanken* do not have a determinate inner structure, then it is possible for them to be "carved up" in radically different ways. This poses a definite problem for a strong reading of the composition principle.²⁵

However, I believe this is the wrong interpretation of the priority thesis. Let us take a closer look at the passage from Frege's "Notes for Ludwig Darmstädter" quoted above. There, Frege was attempting to distinguish his approach to logic from the approach taken in classical logic. The logical approach of Leibniz and Boole, continuous with the Aristotelian tradition, analyzes arguments in terms of judgments, and judgments into one fixed form: *subject concept – copula – predicate concept*. Here, the concepts are taken as basic and complete, and it is the copula that provides the unity of the judgment. In Frege's approach, however, concepts are incomplete, and this displaces the need for a copula in his approach. It is really the incompleteness of concepts in particular Frege is stressing in this passage. On the reading of the *Sinne* of function expressions (including, of course, concept expressions) given above, the *Sinn* of a concept expression would be nothing other than a set of description information describing a truth-value—i.e., a *Gedanke*—from which a component is removed or certain information remains to be supplied. The *Sinn* of a concept can thus be seen as a *Gedanke* from which a component has been removed. It is in this sense that *Gedanken* are prior to the *Sinne* of concepts; the *Sinne* of concepts are *Gedanken* with parts removed. This differs from classical

²⁵For works that interpret Frege in this way, see Harold T. Hodes, "The Composition of Fregean Thoughts," *Philosophical Studies* 41 (1982): 161-78; Pieranna Garavaso, "Frege and the Analysis of Thoughts," *History and Philosophy of Logic* 12 (1991): 195-210; Bernard Linsky, "A Note on the 'Carving Up Content' Principle in Frege's Theory of Sense," *Notre Dame Journal of Formal Logic* 33 (1992): 126-35; Ruffino, "Context Principle, Fruitfulness of Logic," 185-94; Currie, "The Analysis of Thoughts," 283-98; Geach, "Names and Identity," in *Mind and Language*, ed. S. Guttenplan (Oxford: Oxford University Press, 1975), 139-58; Sluga, "Semantic Content and Cognitive Sense," in *Frege Synthesized* (Dordrecht: D. Reidel, 1986), 47-64. For contrasting interpretations, see Bell, "Thoughts," 45-6; Dummett, *Interpretation of Frege's Philosophy*, chap. 15; Landini, "Decomposition and Analysis in Frege's *Grundgesetze*," 132-9.

logic, which treats concepts as involving complete or self-standing descriptive criteria, which are then copulated together in a judgment. However, this does not mean that the incomplete descriptive information provided by the *Sinne* of concepts are not unequivocally parts of the structure of a complete *Gedanke*, or that *Gedanken* are capable of radically divergent analyses.

Frege does say that *Gedanken* can be decomposed in different ways, but one must be careful to look at the sorts of examples he gives in order to correctly understand why this is so. Usually, when Frege speaks of *Gedanken* as being analyzed in different ways, he has in mind propositions that include more than one object name along with a relation expression:

If several proper names occur in a sentence, the corresponding *Gedanke* can be analyzed into a complete part and an unsaturated part in different ways. The *Sinn* of each of these proper names can be set up as the complete part over against the rest of the *Gedanke* as the unsaturated part. (*PW* 192, cf. *CP* 281)

The example Frege gives here is the sentence, “Jupiter is larger than Mars.” One can initially divide the *Gedanke* expressed by this proposition in at least two ways. On one division, the *Gedanke* is seen as composed of the complete *Sinn* of “Jupiter” and the incomplete *Sinn* of “ ξ is larger than Mars.” On another, it is seen as composed of the incomplete *Sinn* of “Jupiter is larger than ξ ” and the complete *Sinn* of “Mars.” These partial analyses are indeed different. The complete *Sinn* involved in one analysis is not the same as that involved in the other analysis, and they pick out different planets. The two incomplete *Sinne* pick out different concepts, in one case the concept of being larger than Mars, in the other, the concept of being something than which Jupiter is larger (*PW* 192). However, it is clear that these different analyses are only possible given the consequence of the comprehension principle that any complete expression with one term missing denotes a concept or function. Really, the *Sinne* of both “ ξ is larger than Mars” and “Jupiter is larger than ξ ” are themselves composite and capable of further analysis. They each consist of the doubly incomplete *Sinn* of “ ξ is larger than ζ ,” with one of its two unsaturated parts filled by a complete *Sinn*. Thus, in both cases, if the analysis of *Sinne* into parts is carried further, one arrives at the same ultimate constituents, the complete *Sinne* of “Jupiter” and “Mars” and the doubly unsaturated *Sinn* of “ ξ is larger than ζ .” That *Gedanken* can be decomposed in different ways does not in itself show that they do not have a unique and determinate composition. In Dummett’s terminology, while there may be multiple possible (partial) “decompositions” of a *Gedanke*, there is only one final “analysis” into basic and elementary constituents.²⁶

²⁶Dummett, *Interpretation of Frege’s Philosophy*, 271.

Frege, however, does not limit discussion of propositions having multiple analyses to cases containing multiple proper names. At one point he says something that at first blush seems much more radical:

Language has means of presenting now one, now another part of the *Gedanke* as the subject; one of the most familiar is the distinction of active and passive forms. It is thus not impossible that one way of analysing a given *Gedanke* should make it appear as a singular judgment; another, as a particular judgment; and a third, as a universal judgment. It need not then surprise us that the same sentence may be conceived as saying something about an object; only we must observe that *what* is being said is different. (CP 188-9)

However, this too can be explained by showing how the same *Gedanke*, with the same *ultimate* constituents, can be decomposed in different ways. In the case of more complex propositions, especially those involving second-level concepts such as quantifiers, the possibility of different decompositions of the same *Gedanke* becomes even greater. Another example Frege gives of a proposition that can be decomposed in different ways is, “Christ converted some people to his teachings.” (PW 187). It is easier to see how the different decompositions of such a proposition are possible when it is transcribed into logical notation, because the structure of Begriffsschrift propositions is meant to represent much more closely the structure of *Gedanken*. If we include in the language of the Begriffsschrift, the new function constants “ $C(\xi, \zeta)$ ” for the relation of ξ converting ζ to ξ 's teachings and “ $\mathcal{H}(\xi)$ ” for the concept of being a person, and a new object constant “ c ” for Christ, then this proposition would be written as:

$$\vdash (\exists a)(\mathcal{H}(a) \ \& \ C(c, a))$$

Here, too, the *Gedanke* expressed by the whole can be decomposed in various ways. On one reading, it decomposes into the incomplete *Sinn* of the concept sign, “ $(\exists a)(\mathcal{H}(a) \ \& \ C(\xi, a))$ ”, standing for the concept of *having converted some people*, and the complete *Sinn* of “ c ”, standing for Christ. On this analysis, the proposition appears as a singular judgment, a judgment about Christ falling under a certain concept. On another analysis, the *Gedanke* divides into the incomplete *Sinn* of the second-level concept sign “ $(\exists a)(\mathcal{H}(a) \ \& \ \phi(a))$ ”, picking out the second-level concept that yields the True as value when the first-level function it takes as value applies to some people, and the incomplete *Sinn* of the first-level concept sign, “ $C(c, \xi)$ ”, denoting the concept of *having been converted by Christ*. (Here, the two incomplete *Sinne*, since one corresponds to a

second-level function, are capable of mutual saturation.) On this analysis, the proposition appears particular.²⁷

In ordinary language this *Gedanke* is expressible in multiple ways, and indeed, certain ways of expressing it may suggest certain decompositions as more natural. If the *Gedanke* considered above is expressed simply as “Christ converted some people to his teachings,” the first of the decompositions above may seem more natural. However, if it is expressed instead as “Some people were converted by Christ to his teachings,” the second decomposition may seem more natural. However, this difference in what is suggested in natural language does not touch the structure of the *Gedanke* itself. Although it is capable of being decomposed in either of these ways, ultimately it is built of the same basic constituents. This does nothing to call the composition principle into question.

There is, however, one troubling example that is not easily explained even by the distinction between decomposition and analysis. At one point, Frege claims that the Begriffsschrift propositions, “ $\vdash x^2 - 4x = x(x - 4)$ ” and “ $\vdash \varepsilon(\varepsilon^2 - 4\varepsilon) = \alpha(\alpha(\alpha - 4))$ ”, express “the same *Sinn*, but in a different way” (CP 143). However, these propositions seem to have quite different forms, even in the Begriffsschrift, and there is no way of showing how they can be seen as two decompositions of the same structured whole. My own view is that this was a simple slip on Frege’s part. What he should have said is not that these propositions express the same *Sinn* but in a different way, but rather, that they have the same *Bedeutung*, but arrive at it differently. This claim appears in “Function und Begriff,” the first work Frege published after forming the *Sinn/Bedeutung* distinction, and it may be that he had not fully mastered the distinction or the terminology.²⁸ In any case, the claim that these two propositions express the same *Sinn*, but differently, conflicts with his other views. It is Frege’s Basic Law V that puts the truth-value denoted by an identity of value-ranges, such as “ $\varepsilon(\varepsilon^2 - 4\varepsilon) = \alpha(\alpha(\alpha - 4))$ ”, as equal to the truth-value of a universal agreement in values, such as “ $x^2 - 4x = x(x - 4)$ ” (which, as we saw in the last chapter, is semantically equivalent to “ $(\forall a)(a^2 - 4a = a(a - 4))$ ”). Presumably, if the two sides of Basic Law V have the same *Sinn* in this case, they do in all other instances as well. However, this would mean that it would be impossible for anyone to doubt the truth of any instance of Basic Law V! If the two halves express the same *Gedanke*, then they are interchangeable even in oblique contexts. This has the result that if the proposition “Kevin believes that $(\forall a)(a^2 - 4a = a(a - 4))$ ” is true, then the proposition “Kevin believes that

²⁷Cf. Landini, “Decomposition and Analysis in Frege’s *Grundgesetze*,” 132-5.

²⁸The claim also occurs in the article prior to the discussion of the *Sinn/Bedeutung* distinction, and Frege may not have been using the word “*Sinn*” with its technical meaning at this point.

$\dot{\varepsilon}(\varepsilon^2 - 4\varepsilon) = \dot{\alpha}(\alpha(\alpha - 4))$ ” must also be true, and Basic Law V becomes indubitable. However, even before Russell had shown that Basic Law V leads to contradictory results, Frege admitted that it was possible to entertain doubts about its truth (*BL* 3-4). Thus, Frege must not have seriously maintained that the two halves of Basic Law V express the same *Gedanke*. Indeed, in later works, where Frege seems more versed in the vocabulary of his theory, he claims not that the two halves of an instance of Basic Law V have the same *Sinn* but that they have the same *Bedeutung* (see, e.g., *BL* §10).²⁹

Unfortunately, there is a misapprehension on the part of some Frege scholars that he believed that a logical truth taking the form of an identity statement, such as Basic Law V, as *analytic*, must have, on the two sides of the identity sign, expressions that have the same *Sinn*. This is argued, for example, by Sluga, who takes Frege’s mature account of analyticity to be a modification of Kant’s understanding, according to which a judgment is analytic if the predicate concept is “contained” in the subject concept.³⁰ My interpretation differs radically. As noted in the previous chapter, Frege’s account of analyticity is not of a *deflationary* sort; truths can be analytic and cognitively informative. For Frege, analytical truths are simply logical truths, but this does not mean that they are true simply in virtue of form, because logic itself has content (*CP* 338). He does not require that logical truths have any particular form, and his abandonment of the subject/predicate style of logical analysis would, in any case, make a Kantian understanding of analyticity difficult to apply. This is especially the case of logical truths that do not take the form of identity statements. Moreover, because arithmetical truths were understood by Frege as analytic truths, Sluga’s interpretation would force Frege to claim that the two halves of an arithmetical equation such as “ $5 + 7 = 12$ ” express the same *Sinn*, a view Frege explicitly denies (see e.g., *PW* 224-5, *PMC* 128, 163-4) and is inconsistent with the rest of his philosophy. In the next section, however, I take up the issue of the identity conditions of *Sinne*, and it should be clear from the discussion below that Frege could not hold that all logically true identity statements must have expressions expressing the same *Sinn* on the two halves of the equation.

In conclusion, I believe that, despite any initial appearance to the contrary, the context principle, the priority thesis, and the possibility of there existing multiple decompositions of the same *Gedanke*, do not pose any great difficulty for the composition principle or the understanding of *Gedanken* as having determinate inner structures.

²⁹Dummett argues similarly. See his *Interpretation of Frege’s Philosophy*, 531-7.

³⁰Sluga, *Gottlob Frege*, 150-4, “Semantic Content and Cognitive Sense,” 59-63.

THE IDENTITY CONDITIONS OF SINNE AND GEDANKEN

The conclusions with regard to the nature of *Sinne* and *Gedanken* reached so far put certain constraints upon how we can understand their identity conditions. We know, for example, that since *Sinne* pick out unique *Bedeutungen*, if the *Bedeutung* of *Sinn* A differs from the *Bedeutung* of *Sinn* B, A and B are not identical. We know, furthermore, that if *Gedanke* A is composed of incomplete *Sinn* B and complete *Sinn* C, then *Gedanke* A cannot also be composed of incomplete *Sinn* B and complete *Sinn* D, unless C is identical with D. However, it is worth delving further into the identity conditions of *Sinne* and *Gedanken* more generally. What is it in virtue of which two expressions express the same *Sinn*, and what is it that determines the conditions under which *Sinne* are identical?

Although he does leave us with certain hints, nowhere in his published writings does Frege attempt to lay out systematically an account of the identity conditions of *Sinne*. However, in certain unpublished works and his correspondence, he does make some relevant remarks. The most thorough remark Frege makes on the subject appears in "A Brief Survey of My Logical Doctrines," in which he writes:

Now two sentences *A* and *B* can stand in such a relation that anyone who recognizes the content of *A* as true must thereby also recognize the content of *B* as true and, conversely, that anyone who accepts the content of *B* must straight-away accept that of *A*. (*Equipollence*). It is here being assumed that there is no difficulty in grasping the content of *A* and *B* . . . I assume there is nothing in the content of either of the two equipollent sentences *A* and *B* that would have to be immediately accepted as true by anyone who had grasped it properly . . .

So one has to separate off from the content of a sentence the part that alone can be accepted as true or rejected as false. I call this part the *Gedanke* expressed by the sentence. (*PW* 197-8)

Here Frege strongly suggests that if two propositions express the same *Gedanke*, then anyone who accepts one as true must straight-away accept the other as true (provided that he or she correctly grasps the *Gedanke* expressed in the two cases). At first blush, this might appear to be a strange suggestion. For Frege, *Gedanken* are not psychological entities; they are neither created by our psychological attitudes nor destroyed by their cessation. Thus, it would seem odd that the identity conditions of *Gedanken* would depend upon psychological factors.

However, once Frege is correctly understood on this point, this suggestion is not surprising nor counterintuitive. Frege does not mean that being believed

by all and only the same people is *definitive* or *constitutive* of the identity of *Gedanken*, but nevertheless it is a necessary condition for it. As objectively existing entities, what it means for *Gedanken* and other *Sinne* to be identical is no different from that for other entities. For Frege, identity is logically basic, and cannot be defined or explicated by other notions (CP 200). Nevertheless, as we saw in Chapter 1, Frege was a proponent of Leibniz's law and, more than anything else, he saw it as capturing the essence of identity. This means that for *Gedanken*, no less than anything else, what it means for A and B to be identical is for everything true of A to also be true of B and vice-versa. Frege's account of *oratio obliqua*, that in statements of propositional attitudes, expressions denote their customary *Sinne*, is in effect the view that beliefs and other propositional attitudes are relations between believers and *Gedanken*. As discussed briefly in the previous section, this means that if two propositions express the same *Sinn*, it is impossible for anyone who understands both to believe one and not the other. This is why, for Frege, it is a necessary condition for propositions A and B to express identical *Gedanken* that anyone who accepts A must straight-away also accept B. This is not to reveal something new and important about what constitutes the identity of *Gedanken* in particular. It is simply a necessary consequence of Leibniz's law and Frege's theory of *oratio obliqua*.

Generalizing from this result, we can conclude the following: propositions A and B express the same *Gedanke* if and only if A and B can be substituted for each other in all (singly) oblique contexts without change of truth-value. More generally, we can say that expressions A and B express the same *Sinn* if and only if A and B can be substituted for each other in all (singly) oblique contexts without changing the truth-value of the whole. This, I believe, is the core of Frege's understanding of the identity conditions of *Sinne*.

In the passage quoted above, Frege focuses heavily on a *Gedanke* as that component of the content of a proposition that alone determines its truth-value. This is of course consistent with the account of *Gedanken* given earlier as descriptive information picking out either the True or the False. It stands to reason then that two propositions express the same *Gedanke* if they contain the same information or determine a truth-value in virtue of the same conditions holding. Of course, it obviously need not be the case of two propositions with the same truth-value that they determine that truth-value the same way or that they express the same *Gedanke*. Although having contingently the same truth-value is no guarantee that two propositions have the same *Sinn*, it might be wondered whether the case is different with two propositions having *necessarily* the same truth-value.

Although, as we shall see, Frege does toy with this idea, it seems clear that he could not consistently hold even that propositions with necessarily the same truth-value always express the same *Gedanke*. This is perhaps most evident

when considering Frege's views on the nature of arithmetic and logic. It is true that Frege thought that both the truths of logic and those of arithmetic were analytic. However, as we have already seen, Frege's notion of analyticity was not of a deflationary sort. Frege believed that a proposition could be both analytic and informative. Moreover, he thought that in an axiomatic system, one could, using a small number of logically true axioms, arrive at new and interesting truths simply by applying logical operations (such as double-negation, commutation, contraposition, etc.) If Frege held that propositions with necessarily the same truth-value express the same *Gedanke* he would have to hold that all logically true propositions, including, for him, all of the truths of arithmetic, express the same *Gedanke*. Given the interchangeability of propositions expressing the same *Gedanke* in oblique contexts such as, "Kevin believes that . . ." and "It is informative that . . .", this view would have the absurd result that if someone believes one truth of arithmetic, he or she believes them all, and that all arithmetical and logical truths are equally informative. However, as we have seen, part of the impetus of the *Sinn/Bedeutung* distinction was to explain the difference in cognitive content between a proposition such as "12 = 12" and one such as "5 + 7 = 12". Thus, necessary coincidence in truth-values is not sufficient for two propositions to express the same *Sinn* for Frege.

Frege's views on the informativity of arithmetic seem to require two things, 1) not all logical truths express the same *Gedanke*, and 2) the application of a purely logical rule of transformation to a proposition, although resulting in a proposition with necessarily the same truth-value, may result in a proposition expressing a different *Gedanke*. It is precisely in order to retain the first of these principles that in the passage from "A Brief Survey of My Logical Doctrines" quoted earlier, Frege includes the proviso about it being assumed that neither proposition has a content that will automatically be accepted as true by anyone. Frege does not want to commit himself to the view that the *Gedanken* expressed by two propositions will be same just because a person will assent to both unhesitatingly. A rational person will always unhesitatingly assent to a simple logical truth, but this does not mean that all such truths are the same.

However, at one point in his correspondence with Husserl, Frege does seem to endorse the view that in the case of *contingent* propositions, having necessarily the same truth-value does entail that two propositions express the same *Gedanke*. He writes:

It seems to me that an objective criterion is necessary for recognizing a *Gedanke* again as the same, for without it logical analysis is impossible. Now it seems to me that the only possible means of deciding whether proposition A expresses the same *Gedanke* as proposition B is the following, and here I assume that neither of the two propositions contains a logically self-evident

component part in its *Sinn*. If *both* the assumption that the content of A is false and that of B true *and* the assumption that the content of A is true and that of B false lead to a logical contradiction, and if this can be established without knowing whether the content of A or B is true or false, and without requiring other than purely logical laws for this purpose, then . . . what is capable of being judged true or false in the contents of A and B is identical, and this alone is of concern to logic, and this is what I call the *Gedanke* expressed by both A and B. (*PMC* 70-1)

Here, Frege does claim that in the case of contingent propositions, if they have necessarily the same truth-value, they express the same *Gedanke*. Because Frege here excludes the case in which one of the two propositions “contains a logically self-evident component,” this does not violate the first principle listed above. However, it does seem to violate or at least qualify the second, for it implies that in the case of contingent propositions, purely logical manipulations will always preserve the *Sinn* of a proposition. If A is contingent, then it expresses the same *Sinn* as $\lceil \sim A \rceil$, $\lceil A \vee A \rceil$ and $\lceil A \& A \rceil$, and if A and B are contingent, then $\lceil A \vee B \rceil$ expresses the same *Sinn* as $\lceil B \vee A \rceil$, etc. Indeed, there are places in Frege’s writings in which he seems to imply just this (see, e.g., *CP* 393, 404).

However, despite these passages, I do not believe we can take this to be Frege’s considered view. Firstly, it conflicts with the composition principle. The *Gedanke* expressed by $\lceil A \vee A \rceil$ must be composed of the incomplete *Sinn* of “ $\xi \vee \xi$ ” and the complete *Sinn* of A, which would mean that the *Sinn* of A is a partial component of the *Sinn* of $\lceil A \vee A \rceil$. But if these two expressions express the same *Sinn*, then the *Sinn* of A is a partial component of itself. This is simply impossible. Secondly, if this view were carried to its logical conclusion, it would hold that A expresses the same *Gedanke* as the complicated expression $\lceil \sim(A \& \sim B) \supset \sim(B \supset \sim A) \rceil$, which necessarily has the same truth-value. However, clearly, it is not the case that these two expressions would be interchangeable in belief contexts: someone who believes A need not also believe $\lceil \sim(A \& \sim B) \supset \sim(B \supset \sim A) \rceil$. It requires a special mental act or calculation to realize that these two must have the same truth-value, just as it requires a special mental act to realize that “ $5 + 7 = 12$ ” must have the same truth-value as “ $12 = 12$ ”. Clearly, what holds of mathematical functions must hold also of truth functions. If “ $7 + 5 = 12$ ” and “ $12 = 12$ ” differ in *Sinn* because it requires an act of the intellect to realize that they must have the same *Bedeutung*, as Frege clearly tells us they must, then it likewise holds that $\lceil A \vee A \rceil$ must differ in *Sinn* from A, because here too an act of the intellect, albeit small, is required to determine that they must have the same *Bedeutung*. This must be taken to be Frege’s considered view on the matter. The textual evidence to the contrary is small, and is based mainly on the letter to Husserl. It is always problematic to

attribute a certain view to a philosopher based upon a claim made in correspondence not intended for publication. It is indeed striking that the claim is not explicitly made in any of his published writings. Perhaps after the letter to Husserl, Frege realized his mistake.³¹

Here again we have evidence against Sluga's view that Frege requires the two halves of logically or analytically true identity statements to express the same *Sinn*. The following is a theorem of the system FC described in the last chapter:

$$\vdash (\neg a) = [\neg(a \ \& \ \sim b) \supset \sim(b \supset \sim a)]$$

However, for reasons that should be clear from the preceding discussion, the two halves of an instance of this identity statement certainly need not express the same *Sinn*.³² In fact, if I am right, they *cannot*, as they would be composed of different constituents.

Thus, in order to be charitable to Frege and present a coherent account of his semantic views, we must not attribute to him even the view that contingent propositions necessarily alike in truth-value express the same *Gedanke*. We are left with only the only primary criterion for *Sinn*-identity being interchangeability in oblique contexts. This, along with the constraints put upon the identity conditions of *Sinne* by the composition principle, forms the basis of the axioms dealing with the nature of *Sinne* and their identity in the Fregean calculus for the theory of *Sinn* and *Bedeutung* presented in Chapter 5. Although at this point, we surely have not answered every question there is to ask concerning the nature of Fregean *Sinne*, I believe we have presented a sufficiently rich and coherent interpretation of Frege's philosophical semantics to begin to consider how to go about incorporating his theory in a logical system and to evaluate other systems that attempt to do so.

³¹For more on these issues, see Eva Picardi, "A Note on Dummett and Frege on Sense-Identity," *European Journal of Philosophy* 1 (1993): 69-80; Jean van Heijenoort, "Frege on Sense Identity," *Journal of Philosophical Logic* 6 (1977): 103-8; Dummett, *Frege: Philosophy of Language*, 228, *Interpretation of Frege's Philosophy*, chap. 17; Beaney, *Frege: Making Sense*, 225-34. For contrasting views, see Currie, "The Analysis of Thoughts," 292-8; Garavaso, "Frege and the Analysis of Thoughts," 203-4.

³²See also the argument given by Landini ("Decomposition and Analysis in Frege's *Grundgesetze*," 137-8) against Sluga's view, in which he shows that the supposition that two halves of a logical truth that takes the form of an identity statement must have the same *Sinn*, would, given certain plausible assumptions regarding the decomposition of complex *Sinne*, lead to the conclusion that the two halves of some *false* identity statements have the same *Sinn*.

Church's Logic of Sense and Denotation

OVERVIEW

We now turn to an examination of the previous attempts to capture the distinction between *Sinn* and *Bedeutung* within logical calculi. These consist primarily of the systems developed by Alonzo Church under the description, “the Logic of Sense and Denotation,” and the various reformulations and revisions to these systems devised by David Kaplan, C.A. Anderson, Pavel Tichý and Charles Parsons.¹ The discussion of these systems in this context serves two primary purposes. Firstly, it provides a general strategy for how to incorporate the commitments of the theory of *Sinn* and *Bedeutung* in a logical system, and this overall strategy will be carried forward to the system presented in the next chapter. Secondly, scrutinizing how well these systems reflect

¹For references to Church's own works, see chap. 1, note 16. For the others mentioned, see David Kaplan, “Foundations of Intensional Logic,” (Ph.D. diss., University of California, Los Angeles, 1964), “How to Russell a Frege-Church,” *The Journal of Philosophy* 72 (1975): 716-29; C. Anthony Anderson, “Some Models for the Logic of Sense and Denotation with an Application to Alternative (0),” (Ph.D. diss., University of California, Los Angeles, 1977), “Some New Axioms for the Logic of Sense and Denotation, Alternative (0),” *Noûs* 14 (1980): 217-34, “General Intensional Logic,” in *Handbook of Philosophical Logic*, vol. 2, *Extensions of Classical Logic*, ed. Dov Gabbay and Franz Guenther (Boston: D. Reidel, 1984), 355-85, “Semantical Antinomies in the Logic of Sense and Denotation,” *Notre Dame Journal of Formal Logic* 28 (1987): 99-114; Charles Parsons, “Intensional Logic in Extensional Language,” *Journal of Symbolic Logic* 47 (1982): 289-328; Tichý, *Foundations of Frege's Logic*, chap. 10.

Frege's own theories in logic and semantics provides insight into what would be required of a logical calculus if it were to capture Frege's views accurately. The conclusions reached in this regard will loom large in the following chapters.

However, it should be noted that in originally formulating his Logic of Sense and Denotation, it was not Church's intention to devise a fully Fregean logic of *Sinn* and *Bedeutung*. Rather, he was interested in attempting to develop a *feasible* intensional logic, something that might be a candidate for the truth, and because he knew that Frege's own logical system (even before its expansion) contains flaws, he quite explicitly deviated from Frege's own approach at several points.² Thus, it should come as no surprise that one conclusion of this chapter will be that Church's systems do not adequately reflect the views of the historical Frege. They were never meant to do so. We point out the deviations between Church's approach and Frege's views not as criticism of Church, but rather because it is crucial to do so given that one of the purposes of the present work *is* to describe a logical calculus true to Frege's own views. Nevertheless, Church remains one of the most (and perhaps one of the only) prominent proponents of *broadly* Fregean semantics, and the general distinction between the *Sinne* and *Bedeutungen* of linguistic expressions. There is much to be learned from his attempts to capture these views within systems of formal logic.

It is worth first noting that in discussing Church's systems, I shall not always follow Church's own terminology. This is because Church often uses terms in ways that are inconsistent with ways they are used in the present work. For example, Church (following Carnap) uses the word "concept" to mean something capable of being that in virtue of which a name has the denotation it does, which is much closer to Frege's word "Sinn" than it is to Frege's "Begriff." He also uses the word "proposition" synonymously with Frege's "Gedanke." Thus, to avoid confusion, I replace Church's terminology with that adopted in the previous chapters.

THE METHOD OF TRANSPARENT INTENSIONAL LOGIC

It will be recalled from Chapter 1 that Frege's proposed solution to the apparent inconsistency between Leibniz's law and the failure of the substitutivity of customarily coreferential expressions in such contexts as the expression of propositional attitudes is his doctrine of *indirect Bedeutung*. According to this doctrine, when a word in ordinary language appears in a context such as that following "believes that . . .", the word denotes its customary *Sinn* rather than its customary *Bedeutung*.

²See, e.g., Church, "A Formulation," 4.

If one were to try to devise a logical language into which one might transcribe statements of propositional attitudes and the like, then, taking this solution into account, there are two possible alternatives one might take.³ On the first alternative, which is dubbed the method of *indirect discourse*,⁴ the same ordinary language expression would be transcribed into the logical language using the same symbol or expression in all occurrences. Like ordinary language, then, the logical language would have oblique contexts and the *Bedeutung* of a symbol occurring in it would depend on the context wherein it appears. One would then have to fashion the inference rules of the logical system to be sensitive to these contextual ambiguities. For example, in such a system, Leibniz's law would have to be set up in such a way that its application would be prohibited or restricted when an oblique context is involved.

The second approach—the method of *direct discourse*—advocates a different approach to transcription; since, according to the theory at hand, ordinary language expressions have different *Bedeutungen* in different contexts, it would transcribe the same ordinary language expression using different notations in the logical language depending on the context in which the ordinary language expression appears. An example of this approach was considered in the first chapter. Although in a Fregean system, one might transcribe the complete proposition “the morning star is a planet” as “ $\vdash P(m)$ ”, one would use different signs when transcribing “the morning star is a planet” when it appears as a component of the proposition, “Gottlob believes that the morning star is a planet,” because here the component phrase “the morning star is a planet” does not *express* the *Gedanke* that the morning star is a planet and denote the True, it denotes the *Gedanke* itself. As noted in the first chapter, it then becomes necessary to introduce some method of showing the relationship between the different signs used (such as the relation sign “ Δ ”).

Although there may be a certain naturalness to the method of indirect discourse given its greater similarity to natural language, the advantages of the method of direct discourse are many. The language then itself is devoid of contextual ambiguities; the same expression always has the same *Bedeutung*. This avoids the necessity of having to relativize the rules of inference to different contexts. Moreover, this seems to be the method that Frege himself would have preferred. As we have seen, he describes the contextual ambiguities of natural language as defects to be corrected in a language concerned with precise inferential rules (*CP* 159, *PMC* 115). Moreover, in one of the few cases

³For more on these alternatives, see Furth, Introduction to *Basic Laws of Arithmetic*, by Gottlob Frege, xxiv-xxv; Kaplan, “Foundations of Intensional Logic,” 23-9.

⁴The names of the two methods are due to Kaplan. See “Foundations of Intensional Logic,” 23.

in which he himself speaks of expanding his Begriffsschrift to include indirect speech, he explicitly advocates use of “special signs” to “avoid ambiguity” (*PMC* 153). In any case, it is the method of direct discourse that Church adopts in his Logic of Sense and Denotation.

The result, oddly enough, is a system that is “intensional” under one definition, and “extensional” under another. It is worth taking a moment to discuss these terms, as they can cause considerable confusion given the various ways they can be defined or used in this context. Normally, “intensional logic” and “extensional logic” are thought to be mutually exclusive domains of study, yet in this context one finds odd results, with one striking example being Kaplan’s work entitled, “Foundations of Intensional Logic,” which develops a number of logical languages (based on those of Church), *all* of which are “extensional” by Kaplan’s *own* definition.⁵ How can this be?

On one way of understanding the relationship between intension and extension, some sorts of entities (real or purported) are thought to be “extensional” and others are thought to be “intensional.” In the realm of intensions, we find finely grained entities such as Russellian propositions, universals or properties, mental representations and various forms and guises of reified meanings or senses. In the realm of extensions, we find more coarsely grained entities such as sets, classes and ordinary physical objects. With this understanding of the divide between intension and extension, it becomes natural to think of intensional logic as logic that involves inferences involving these intensions, their identity conditions and so on. According to Anderson, for example, a logical system is *extensional* if it . . .

. . . requires for the statement and justification of its general principles only such concepts as truth and falsity, identity and difference of truth values (of sentences or propositions), sets or classes, and coextensiveness or divergence (of predicates or properties).⁶

Correspondingly, a logical system is *intensional* if it requires, in *addition* to those listed above, “such notions as synonymy, identity and difference of intension, proposition, property or concepts.”⁷ Let us call systems that meet these definitions *extensional*₁ or *intensional*₁. Under these definitions, although they adopt the method of direct discourse, Church’s systems are nevertheless *intensional*₁, because many of their axioms deal with, for example, the identity conditions of *Sinne* and *Gedanken* (i.e., synonymy) and their nature generally.

⁵See Kaplan, “Foundations of Intensional Logic,” 16-20.

⁶Anderson, “General Intensional Logic,” 355.

⁷Anderson, “General Intensional Logic,” 355.

This is not unexpected, as Church himself describes the Logic of Sense and Denotation as a “tentative beginning” towards developing of a broadly Fregean “intensional logic.”⁸

However, there is a related but distinct understanding of the phrases “extensional” and “intensional.” This understanding can be best appreciated by considering a few results that follow in a logical system if it is *extensional*₁. If *extensional*₁, a logical system will treat properties or concepts identically so long as they are coextensive (apply to all and only the same things), since nothing other than their coextensiveness is involved in the basic principles of the system. Similarly, if a logical system is *extensional*₁, it will treat propositions identically if they are alike in truth-value. This being the case, it is nearly always true of a logical system that is *extensional*₁ that both coextensive predicates and propositions with the same truth-value can be swapped for one another *salve veritate*. At the very least, Leibniz's law will hold of such a system; expressions denoting the same object will be replaceable *salve veritate*. These features, however, are sometimes themselves taken to be the defining features of an “extensional” logical system. Let us then say that a logical system is *extensional*₂ if the system has these features. A system will then be *intensional*₂ if it is not true of it that coextensive predicates, coreferential names or propositions alike in truth-value can always replace another without modifying the truth-value of the whole.

With these definitions in place, it is easy to see that, for example, C.I. Lewis's systems of modal logic are *intensional*₂, because they are non-truth-functional. One cannot infer from the propositions “ $p \equiv q$ ” and “ $\Box p$ ” the conclusion “ $\Box q$.”⁹ If the method of indirect discourse were employed for a logical calculus of *Sinn* and *Bedeutung*, it too would be *intensional*₂, because in belief contexts and the like, expressions for propositions having the same truth-value could not be substituted for one another. However, because the method of direct discourse is employed in Church's Logic of Sense and Denotation (as well as my own system developed in the next chapter), it remains *extensional*₂. In an *intensional*₁ system employing the method of direct discourse, it may still be true that if “ p ” and “ q ” are propositions with the same truth-value, they can replace each other in every context, because other symbols—symbols not having truth-values as their *Bedeutungen*—are employed in their place in the problematic contexts.

It will be recalled that Frege's understanding of Leibniz's law is that expressions with the same *Bedeutung* are replaceable *salva veritate*. Because, on

⁸Church, “A Formulation,” 3.

⁹See principally C.I. Lewis and C.H. Langford, *Symbolic Logic* (New York: Dover, 1932).

his understanding, the *Bedeutungen* of propositions are their truth-values, and the *Bedeutungen* of predicates are concepts whose identity conditions depend only on their extensions, the hallmarks of *extensionality*₂ are built in to the inference rules of his Begriffsschrift. Indeed, these principles represent the core of what Carnap has called “Frege’s principles of interchangeability.”¹⁰ The theory of indirect *Bedeutung* is designed precisely to reconcile these *extensional*₂ principles with *oratio obliqua*. Therefore, it should not be surprising that logical systems inspired by Frege could be both *intensional*₁ and *extensional*₂.

Parsons describes this peculiar mix of intensionality and extensionality as “intensional logic in extensional language,” and Tichý employs the phrase “transparent intensional logic.” I find this last description particularly apt. It reveals that such systems are extensional in the sense of being *referentially transparent*: expressions having the same *Bedeutung* are interchangeable in all contexts. At the same time, it reveals that these systems also serve the purpose usually attributed to the domain of “intensional logic,” by being able to express propositional attitudes, and by dealing with such things as synonymy (the identity conditions of *Gedanken* and *Sinne*). The method of transparent intensional logic is one of the most important contributions of Church’s Logic of Sense and Denotation to the attempt to fashion a Fregean logical calculus for the theory of *Sinn* and *Bedeutung*.

Nevertheless, one must be careful when utilizing the expressions “intension” and “extension” with regard to Frege’s theories. Especially when thinking of the notion of “intension” involved in the characterization of *intensionality*₁, it is tempting to equate the intension/extension distinction with the *Sinn/Bedeutung* distinction. However, there are some reasons to be wary of doing so. Frege himself cautions against equating the two distinctions (*PW* 121-4). Consider, for example, a predicate. On the usual understanding of the divide between intension and extension, a property or (Platonic) universal is its intension, and the class of things having that property is its extension. However, the closest thing to a property in Frege’s metaphysics is a concept, and concepts are the *Bedeutungen*, not the *Sinne*, of linguistic predicates. In effect, Frege has a much more complicated division, and using the terminology of “intension” and “extension” with regard to Frege’s semantics can be misleading. Also, the division of some entities as “intensional” and others “extensional,” when correlated with the *Sinn/Bedeutung* distinction suggests that *Sinne* or *Bedeutungen* are two distinct categories of entities in Frege’s theory. However, Frege does think that *Sinne* can themselves be *Bedeutungen* (such as in oblique

¹⁰Rudolf Carnap, *Meaning and Necessity*, 2d. ed. (Chicago: University of Chicago Press, 1956), 121-4.

contexts), and thus *Sinne* and *Bedeutungen* do not represent wholly distinct classes of entities. This is why in general I avoid using these terms in favor of Frege's own, and prefer to speak not of "Fregean intensional logic" but of "the logic of *Sinn* and *Bedeutung*."

ALTERNATIVES (0), (1) AND (2) AND SYNONYMOUS ISOMORPHISM

Church's "Logic of Sense and Denotation" is not a single, unified system of logic. From the outset, Church outlined three alternative systems differing in their treatment of the identity conditions of *Sinne*. Moreover, each of the systems initially outlined underwent a number of revisions or reformulations at the hands of Church or others in the wake of semantical paradoxes and other difficulties. Let us speak firstly, however, of the three different alternatives Church initially suggests, which he calls "Alternative (0)," "Alternative (1)," and "Alternative (2)."

Alternative (2) is perhaps the simplest. According to Alternative (2), propositions A and B are thought to express the same *Sinn* if it is *provable within the logical system* that "A = B". Of course, in a functional calculus, such as that of Frege or Church, in which propositions are thought to denote their truth-values, this only states that A and B have the same truth-value. Thus, on Alternative (2), two propositions are thought to have the same *Sinn* if it is a *logical truth* that they have the same truth-value (logical equivalence). It was noted, however, in the last chapter that logical equivalence is too weak to serve as the identity criterion for *Gedanken* when propositional attitudes are involved. If this criterion of identity were adopted in a language involving propositional attitudes, it would follow from it that if someone believed one logical truth, she or he must believe all others (since all would have the same *Sinn*). This Church himself admits. Alternative (2) was suggested primarily with the intent of developing direct discourse systems of modal logic. Within modal contexts, interchangeability of logical equivalents does not pose a problem. However, the result is that the systems Church develops to conform to Alternative (2), and the related systems of Kaplan and Parsons, which also employ Alternative (2), are much more relevant to the development of modal logic in the tradition of C.I. Lewis than to development of the logic of the Fregean theory of *Sinn* and *Bedeutung*.

Alternatives (1) and (0) are more promising. At the core of the identity conditions of *Sinne* endorsed within both of these alternatives is a revision of Carnap's notion of *intensional isomorphism* that Church calls *synonymous*

isomorphism.¹¹ It is helpful here to understand how this concept would be applied to natural language before considering its application to formal languages. In natural language, two propositions A and B are synonymously isomorphic if and only if there is some finite number of synonym replacements that could be made to the expressions in A such that B results. Thus, if the English expressions “brother” and “male sibling” are taken to be synonymous (expressive of the same *Sinn* in the same contexts), then the propositions:

- (1) Some bachelors have brothers.
- (2) Some bachelors have male siblings.

would be synonymously isomorphic, because (2) results from (1) by one synonym replacement. Similarly, if “bachelor” and “unmarried man” are thought to be synonymous in English, then (1) and (2) would also be synonymously isomorphic to the proposition:

- (3) Some unmarried men have male siblings.

Here, two synonym replacements are required to arrive at (3) from (1), but they nevertheless are isomorphic.

It has been suggested by Dummett among others that synonymous or intensional isomorphism maps very well on to Frege’s own understanding of the conditions under which propositions express identical *Sinne*.¹² In particular, it seems to conform well with the compositionality of *Gedanken* discussed in the previous chapter. Frege explicitly states that different natural language sentences can express the same *Gedanke* (see e.g., *PW* 140-1, *CP* 357). However, since *Gedanken* have determinate inner structures corresponding to the structures of the sentences that express them, different sentences expressing the same *Gedanke* must at least have some commonality in structure. Synonymously isomorphic propositions seem to have precisely the appropriate sort of structural similarity required. Of course, as discussed in the previous chapter, Frege does allow that at least some deviance in the structure of synonymous propositions is possible, such as between actives and passives. However, there may be a way of incorporating this within synonymous isomorphism. For instance, one might stipulate that “ ξ gives ζ to γ ” is to be regarded as a synonym of “ ζ is given to γ ”

¹¹Carnap, *Meaning and Necessity*, 56-64; Alonzo Church, “Intensional Isomorphism and Identity of Belief,” in *Propositions and Attitudes*, ed. Nathan Salmon and Scott Soames (Oxford: Oxford University Press, 1988), 159-68.

¹²Dummett, *Frege: Philosophy of Language*, 228-9, 379.

by ξ ," and that the move from one to another is to be regarded as a synonym replacement on par with those considered earlier.

One apparent problem with taking synonymous isomorphism to be the primary criterion for sameness of *Sinn* is that, as a criterion, it seems incomplete. It provides an account of the identity conditions of *Sinne* of whole propositions, but it relies on there having been established a set of prior synonymies among the primitive expressions. Synonymous isomorphism cannot explain why "bachelor" and "unmarried man" are synonyms; such "synonym pairs" seemingly have to be simply stipulated. This is precisely the sort of criticism of this theory of synonymy one finds in the work of Quine.¹³ Consider for example, how such an account of synonymy would apply to formalized languages such as those here under consideration. Church's account of synonymous isomorphism begins with a set of pairs of constants that are taken as primitive synonymies, and two complex expressions are isomorphic if one can be derived from the other by a series of replacements of one member of such a synonym pair by another, one letter for another for bound variables, or, under Alternative (1), through lambda conversion. However, the primitive synonym pairs must simply be stipulated. Thus, the notion of synonymous isomorphism built into Alternatives (0) and (1) does not provide us with a full account of synonymy. However, within a Fregean system this is perhaps as it should be. The notion that two simple expressions express the same *Sinn* may simply be logically primitive for Frege, and therefore, to attempt to give an account of the conditions under which two simple expressions have the same *Sinn* in terms of features of the propositions in which they appear would be to reverse the logical order.

The difference between Alternative (1) and Alternative (0) lies in the precise details of their treatment of synonymous isomorphism within a formal language involving lambda abstracts. Under Alternative (1), lambda conversion is thought to preserve the *Sinn* of a formula, while under Alternative (0), it is not. This cannot be made clear until we have discussed the details of the lambda calculus and the lambda abstraction operator. In our discussion, let us use **A**, **B** and **D** schematically for expressions of the language. The lambda abstraction operator, " λ ", is a variable binding operator, and a variable such as " x " is said to be free or bound according to whether it appears in a formula of the form $\lceil (\lambda x A) \rceil$. The abstraction operator appears solely in formulae of this form, and a formula of the form $\lceil (\lambda x A) \rceil$ is treated as sign for a function that takes any argument of the same type as the variable " x ". The value of this function, if it takes as argument the *Bedeutung* of **B**, is the *Bedeutung* of **D**, where **D** results from **A** by replacing all occurrences of " x " in **A** with **B**. In effect, the abstraction

¹³Especially in Quine, "Two Dogmas," 24-7, 32-6.

operator provides us with a complex function sign for every open formula in the language. Of course, a language containing the comprehension principle is in effect committed to the existence of functions corresponding to all open formulae; in such a language, the use of the lambda abstraction to form expressions for these functions seems straightforwardly innocuous. If we again allow ourselves mathematical signs to aid in giving examples, and temporarily ignore the need for type indices, the expression “ $(\lambda x x = 7)$ ” would stand for *the concept of being equal to seven*, and thus “ $(\lambda x x = 7)(2 + 5)$ ” would denote the True, while “ $(\lambda x x = 7)(2 + 4)$ ” would denote the False. The process of *lambda conversion* refers to the move from a proposition involving a lambda abstract to the corresponding proposition in which the argument to the lambda abstract simply replaces the bound variable, e.g., from “ $(\lambda x x = 7)(2 + 5)$ ” to “ $(2 + 5) = 7$ ”, or the reverse. In Alternative (1), lambda coverts are treated as synonymously isomorphic. In Alternative (0), they are not.

Church does not raise the question of which of Alternative (1) or Alternative (0) better reflects Frege’s own understanding of synonymy, although he hints that Alternative (1) might be closer. He says, for example, that in Alternative (1), those *Sinne* with truth-values as their *Bedeutungen* are identified with “Fregean *Gedanken*,” whereas he normally simply refers to such entities by the more generic term “proposition.”¹⁴ However, in some ways the question is moot. Frege’s logical system did not have the lambda operator, and thus Frege would never have considered the question of whether lambda conversion preserves the *Sinn* of a proposition. Nevertheless, it may shed some light on the matter to recall Frege’s remarks about different ways of expressing the same *Gedanke*. The question in Church’s notation is whether “ $(\lambda x 9 > x)7$ ”, “ $(\lambda x x > 7)9$ ” and “ $9 > 7$ ” should all be taken to express the same *Gedanke*. The natural renderings of these three propositions into English are, respectively, “seven has the property of being something of which nine is greater,” “nine has the property of being greater than seven,” and “nine is greater than seven.” Frege, however, is quite explicit that propositions differing in this way represent different ways of expressing the same *Gedanke* (*PW* 188, 281n), although it is true that the different renderings, in the terminology employed in the previous chapter, seem to make certain “decompositions” of that *Gedanke* more natural. While these sentences have different grammatical subjects and predicates, it must be recalled that Frege does not think that this grammatical distinction dictates the logical structure of the *Gedanke* expressed by a proposition.

The possibility of “alternative analyses” or subject/predicate renderings of the same *Gedanke*, as discussed in the previous chapter, is a consequence of functional comprehension. Frege, too, is committed to functions existing for

¹⁴Church, “A Revised Formulation,” 141.

each of Church's lambda abstracts. For Church's " $(\lambda x x > 7)$ ", Frege has simply " $\xi > 7$ ", and for Church's " $(\lambda x 9 > x)$ ", Frege has " $9 > \zeta$ ". The difference, however, is that Frege's notation clearly shows the unsaturatedness of these functions, whereas Church's does not. The notation " $(\lambda x x > 7)9$ " would seem wholly unnatural for Frege; it would be akin to writing " $(\xi > 7)9$ ", which would only obscure the fact that the *Sinn* of "9" is thought to *complete* the incomplete *Sinn* of " $\xi > 7$ " to form a whole, and, in so doing, it *eliminates* the unsaturatedness marked by " ξ ". The resulting *Gedanke* is better expressed simply as " $9 > 7$ ". Note that the same whole is formed when the incomplete *Sinn* of " $9 > \zeta$ " is completed by the *Sinn* of "7". Indeed, in a letter to Russell (*PMC* 161-2), Frege considers a notation in many ways similar to Church's lambda notation. Instead of writing " $(\lambda x x > 7)$ ", Frege considers something such as " $\varepsilon(\varepsilon > 7)$ ", with a rough-breathing over the initial " ε " instead of a smooth-breathing to mark that the result is a function, not a value-range. However, he finds it unworkable, because it obscures the unsaturatedness of functions. Frege would transcribe *both* Church's " $(\lambda x x > 7)9$ " and his " $(\lambda x 9 > x)7$ " simply as " $9 > 7$ ". This too leads us to the conclusion that Frege would see no difference in the *Sinne* of lambda converts, and thus that we ought to favor Alternative (1) over Alternative (0) as the more Fregean alternative. Clearly, however, we cannot unproblematically attribute to Frege a view that he never had the opportunity to accept *or* reject. Indeed, Frege probably would himself likely have concluded that the question arises only because of the misguided decision to introduce lambda abstracts in the first place.

THE FORMULATION OF THE SYSTEMS

Here we begin to consider the actual formulation of Church's various systems. For reasons that should be clear from the preceding discussion, we need not consider the details of Alternative (2). Alternatives (0) and (1) seem more interesting to our present purposes. In fact, I shall focus mostly on Alternative (1) for two reasons. Firstly, as I have just argued, there is some reason to think it is the more Fregean alternative. Secondly, it is simpler. However, we shall not ignore Alternative (0) altogether, especially as, when one factors in the work of others (Anderson especially), more work has been done on Alternative (0) than on Alternative (1). Unfortunately, most of Church's work on the Logic of Sense and Denotation has focused on Alternative (2), and in his initial works on the subject, he presents no more than brief sketches of Alternatives (0) and (1). Similarly, both Kaplan and Parsons base their systems on Alternative (2). The result is that Alternative (1) did not receive full attention until Church's last article on the subject, published as recently as 1993, and that Alternative (0) has been treated in much more detail in the works of Anderson than in Church's

own. However, all of these systems are sufficiently similar that we can begin by considering their common core.

Church builds his Logic of Sense and Denotation upon a functional calculus of lambda conversion employing the simple theory of types. However, the simple types are further divided according to the *Sinn* hierarchy. The types of non-functions include two hierarchies, ι_ω and o_ω . Type ι_0 consists of individuals that are not *Sinne*, and type ι_{i+1} consists of *Sinne* that determine or present individuals of type ι_i . Type o_0 consists of the two truth-values, the True and the False, and type o_{i+1} consists of *Sinne* that determine or present individuals of type o_i . Thus, *Gedanken*, as *Sinne* that determine truth-values as *Bedeutungen*, fall into type o_1 . For any types α and β , there exists a type $(\alpha\beta)$ consisting of functions that take arguments of type β and yield values of type α . Thus, concepts would have the type $(o_0\iota_0)$, taking individuals as argument and yielding truth-values as value. Functions of multiple arguments are treated in the method suggested by Schönfinkel as functions with one argument that yield functions as value.¹⁵ For example, the conditional function has the type $((o_0o_0)o_0)$; it is understood as function that takes one truth-value as argument and yields as value a function with one remaining argument place and having a truth-value as result. The *Sinne* of functions are treated as sense-functions in accordance with the theory of Geach and others discussed in the previous chapter. Thus, the *Sinn* of a concept expression would have type $(o_1\iota_1)$, taking as value *Sinne* picking out individuals and yielding *Gedanken* as value. Type indices are required on all variables and constants of the system, with the conventions that subscripts of indices can be left off when the subscript is 0, and parentheses can sometimes be dropped from compound types with the convention of association to the left. Thus, the type-symbol mentioned above for the conditional function $((o_0o_0)o_0)$, can be abbreviated simply “ ooo ”.

The language consists of an infinite supply of variables ($a_\alpha, a'_\alpha, a''_\alpha, \dots, b_\alpha, b'_\alpha$, etc.) of each type α . The four primary constants of the system are:

$$C_{ooo}, \Pi_{o(o\alpha)}, \iota_{\alpha(o\alpha)}, \Delta_{o\alpha\alpha_1}$$

The first, “ C_{ooo} ”, is the material conditional function. Strictly speaking, the system is written in Polish notation, but defined signs are introduced into the system such that “ $A_o \supset B_o$ ” is defined as “ $C_{ooo}A_oB_o$ ”. The second, “ $\Pi_{o(o\alpha)}$ ”, represents the universal quantifiers; one exists for every type α , which yields as value the True just in case the function it takes as argument has the True as value for all arguments of type α . The third, “ $\iota_{\alpha(o\alpha)}$ ”, represents the description

¹⁵Moses Schönfinkel, “On the Building Blocks of Mathematical Logic,” in *From Frege to Gödel*, 355-66.

operators. (Again, one exists for all types α .) Unlike Frege's description operator, "\", which takes value-ranges as argument, Church's description operator takes functions as argument, and yields as value the sole member of type α for which the function yields the True, and some chosen member of type α if there is no such unique member. Lastly, there is the symbol for the relation between a *Sinn* and the *Bedeutung* it picks out or presents, " $\Delta_{\alpha\alpha_1}$ ". This function yields as value the True just in case its first argument, of type α_1 , is a *Sinn* that picks out or determines its second argument (of type α) as *Bedeutung*, and yields the False otherwise.¹⁶

In addition to the constants listed above, the language also contains a hierarchy of constants for sense-functions corresponding to the *Sinne* of these constants, viz.,

$$C_{o_n o_n o_n}, \Pi_{o_n(o_n \alpha_n)}, \iota_{\alpha_n(o_n \alpha_n)}, \Delta_{o_n \alpha_n \alpha_{n+1}}$$

Let us consider as example $C_{o_1 o_1 o_1}$. This is a function from *Gedanken* as arguments to a *Gedanke* as value. Unlike the simple conditional, C_{ooo} , which takes truth-values as argument and yields a truth-value, the expressions that appear in the argument positions of " $C_{o_1 o_1 o_1}$ " will actually *denote Gedanken*, rather than express them, and the whole that is formed with " $C_{o_1 o_1 o_1}$ " and its arguments will also denote a *Gedanke* (again, rather than expressing one.) Thus, if A_{o_1} denotes the *Gedanke* that the morning star is a planet, and B_{o_1} denotes the *Gedanke* that the evening star is planet, then " $C_{o_1 o_1 o_1} A_{o_1} B_{o_1}$ " denotes the *Gedanke* that if the morning star is a planet, then the evening star is a planet. Note, however, that this expression is quite different from one that actually expresses or could be used to assert this *Gedanke* (which would employ " C_{ooo} "). " $C_{o_1 o_1 o_1} A_{o_1} B_{o_1}$ " is the name of an entity in the realm of *Sinn*. While the *Bedeutung* of this expression is a *Gedanke*, the *Sinn expressed by* this expression, however is not a *Gedanke*, but a *Sinn* picking out a *Gedanke*. If we wanted to form a name for *this Sinn*, we would use " $C_{o_2 o_2 o_2} A_{o_2} B_{o_2}$ ".

Lastly, the language contains the abstraction operator λ , which was discussed in some detail above. Here it is worth noting that formulae of the form $(\lambda x_\beta M_\alpha)$ are treated as signs for functions of type $(\alpha\beta)$. (Alternative (0) is actually forced to employ a hierarchy of such signs λ_n , with λ_0 representing the normal abstraction operator, and λ_{i+1} representing the *Sinn* of the sign λ_i .)

¹⁶Actually, in Church's own notation, this function has type $\Delta_{\alpha_1 \alpha}$, thus with the *Bedeutung* half of the relation as first argument and the *Sinn* half of the relation as second argument. However, I have swapped their order to accord with the notation adopted in the other parts of the present work. (See chap. 1, note 17.)

With the abstraction operator in place, Church then proceeds to define more usual notation for quantification; $\lceil (\forall x_\alpha) A_\alpha \rceil$ is defined as $\lceil \Pi_{o(\alpha\alpha)} (\lambda x_\alpha A_\alpha) \rceil$. Negation is defined as implying the false proposition $\lceil (\forall a_\alpha) a_\alpha \rceil$ (the proposition that all truth-values are the True). The existential quantifier, disjunction and conjunction signs are defined as one would expect. Identity is simply defined in the language as indiscernibility. Church does not, like Frege, employ a judgment stroke, although the functional nature of his systems seems to call for it. (Church sometimes employs the sign “-” instead as a thesis sign, although I shall not do so in setting out his system, in order to avoid confusion.) Inference rules of the system include the standard set (alphabetic change of bound variable, universal generalization, *modus ponens*) plus the rules of lambda conversion. (Lambda conversion in Alternative (0) is limited to cases in which the lambda operator is λ_0 .)

The axioms of the system vary between the three alternatives and the reformulations of each. Since it is not relevant to our present interests, we need not be concerned with those specific to Alternative (2). Let us consider first the axioms for Church’s original sketch of Alternative (1). These were meant only as a sketch, and in setting out the system Church was not convinced that his axiom list was comprehensive. Let us call this system LSD(1)¹, for the first formulation of the Logic of Sense and Denotation, Alternative (1). The axioms proposed fall into three categories. First there are axioms for a standard extensional higher-order function-calculus employing the simple theory of types. Since these axioms are in no way specific to the Logic of Sense and Denotation, we shall not discuss them here.¹⁷ Secondly, there are those axioms that deal with the very notion of a *Sinn* and those constants for *Sinne* introduced in the language. (These axioms were originally thought to apply regardless of which of the three alternatives is adopted.) The axioms fall into the following axiom schemata (with a version of each schema existing for the different types α and β and type-indices n):

$$\begin{array}{ll}
\text{Axiom LSD(1) 11:} & \Delta_{o(o_n o_n o_n)(o_{n+1} o_{n+1} o_{n+1})} C_{o_{n+1} o_{n+1} o_{n+1}} C_{o_n o_n o_n} \\
\text{Axiom LSD(1) 12:} & \Delta_{o(o_n(o_n \alpha_n))(o_{n+1}(o_{n+1} \alpha_{n+1}))} \Pi_{o_{n+1}(o_{n+1} \alpha_{n+1})} \Pi_{o_n(o_n \alpha_n)} \\
\text{Axiom LSD(1) 13:} & \Delta_{o(\alpha_n(o_n \alpha_n))(\alpha_{n+1}(o_{n+1} \alpha_{n+1}))} \iota_{\alpha_{n+1}(o_{n+1} \alpha_{n+1})} \iota_{\alpha_n(o_n \alpha_n)} \\
\text{Axiom LSD(1) 14:} & \Delta_{o(o_n \alpha_n \alpha_{n+1})(o_{n+1} \alpha_{n+1} \alpha_{n+2})} \Delta_{o_{n+1} \alpha_{n+1} \alpha_{n+2}} \Delta_{o_n \alpha_n \alpha_{n+1}} \\
\text{Axiom LSD(1) 15:} & (\forall f_{\alpha\beta})(\forall f_{\alpha_1\beta_1})(\forall x_\beta)(\forall x_{\beta_1})(\Delta_{o(\alpha\beta)(\alpha_1\beta_1)} f_{\alpha_1\beta_1} f_{\alpha\beta} \supset \\
& \qquad \qquad \qquad \Delta_{o\beta\beta_1} x_{\beta_1} x_\beta \supset \Delta_{o\alpha\alpha_1} f_{\alpha_1\beta_1} x_{\beta_1})(f_{\alpha\beta} x_\beta)) \\
\text{Axiom LSD(1) 16:} & (\forall f_{\alpha\beta})(\forall f_{\alpha_1\beta_1})[(\forall x_\beta)(\forall x_{\beta_1})(\Delta_{o\beta\beta_1} x_{\beta_1} x_\beta \supset \\
& \qquad \qquad \qquad \Delta_{o\alpha\alpha_1} f_{\alpha_1\beta_1} x_{\beta_1})(f_{\alpha\beta} x_\beta)) \supset \Delta_{o(\alpha\beta)(\alpha_1\beta_1)} f_{\alpha_1\beta_1} f_{\alpha\beta}]
\end{array}$$

¹⁷The interested reader should consult Church, “A Formulation,” 17-8, “A Formulation of the Simple Theory of Types,” *Journal of Symbolic Logic* 5 (1940): 56-68.

Axiom LSD(1) 17: $(\forall x_\alpha y_\alpha)(\forall x_{\alpha_1})(\Delta_{\alpha\alpha_1} x_{\alpha_1} x_\alpha \supset \Delta_{\alpha\alpha_1} x_{\alpha_1} y_\alpha \supset x_\alpha = y_\alpha)$

Axioms 11-14 are quite simple. They express the relation between such constants as “ $C_{\alpha\alpha}$ ” and the constants such as “ $C_{\alpha_1\alpha_1}$ ” introduced to stand for the *Sinne* of the former. Thus, one instance of axiom 11 is “ $\Delta_{\alpha(\alpha\alpha)(\alpha_1\alpha_1\alpha_1)} C_{\alpha_1\alpha_1\alpha_1} C_{\alpha\alpha}$ ”, which asserts that $C_{\alpha_1\alpha_1\alpha_1}$ is a *Sinn* picking out $C_{\alpha\alpha}$ as *Bedeutung*. Axioms 15 and 16 enshrine Church's commitment to the sense-function view of *Sinne* presenting functions as *Bedeutung*. Axiom 15 states that if f^* is a *Sinn* picking out a function f as *Bedeutung*, then f^* itself, as a function in the realm of *Sinn*, has as value for any *Sinn* x^* , whose corresponding *Bedeutung* is x , a *Sinn* whose *Bedeutung* is the value of f for x as argument. Axiom 16 states the converse: if f^* is a function in the realm of *Sinn* whose value for every *Sinn* x^* itself determining x as *Bedeutung* is itself a *Sinn* determining as *Bedeutung* the value of some function f for x as argument, then f^* is a *Sinn* determining f as *Bedeutung*. Lastly, axiom 17 states that *Sinne* have unique *Bedeutungen*. This is of course essential if the *Bedeutung* of an expression is determined by its *Sinn* alone; the *Sinn* itself must be sufficient to determine a unique *Bedeutung*. Of course, it is not similarly true that *Bedeutungen* are picked out by unique *Sinne*. The morning star/evening star example and countless others show this.

In addition to these axioms, Church suggests a number of axioms that seem appropriate to Alternative (1)'s equation of synonymy with synonymous isomorphism. Firstly there are a number of axioms stating that *Gedanken* with differing compositions are non-identical. The first such is:

Axiom LSD(1) 39: $(\forall p_{\alpha_{n+1}} q_{\alpha_{n+1}})(\forall f_{\alpha_{n+1}\alpha_{n+1}})(C_{\alpha_{n+1}\alpha_{n+1}\alpha_{n+1}} p_{\alpha_{n+1}} q_{\alpha_{n+1}} \neq \Pi_{\alpha_{n+1}(\alpha_{n+1}\alpha_{n+1})} f_{\alpha_{n+1}\alpha_{n+1}})$

This asserts that conditional *Gedanken* are never identical to universal *Gedanken*. In a formal language, this means that propositions whose “main operator” is the conditional never express the same *Gedanken* as propositions whose “main operator” is the universal quantifier. LSD(1)¹ also contains similar axioms stating that conditional *Gedanken* are not identical to descriptive *Sinne* (the *Sinne* of expressions whose “main operator” is the description function, ι), that conditional *Gedanken* are not identical to denotative *Gedanken* (the *Gedanken* expressed by propositions whose “main operator” is the denotation function Δ), that universal *Gedanken* are not identical to descriptive *Sinne*, and so on. We need not list all such axioms individually here.

In addition to these, there are a number of axioms to the effect that identical *Gedanken* have identical compositions. The first two read:

$$\begin{aligned} \text{Axiom LSD(1) 45: } & (\forall p_{o_{n+1}} q_{o_{n+1}} r_{o_{n+1}} s_{o_{n+1}}) (C_{o_{n+1}o_{n+1}o_{n+1}} p_{o_{n+1}} q_{o_{n+1}} = \\ & C_{o_{n+1}o_{n+1}o_{n+1}} r_{o_{n+1}} s_{o_{n+1}} \cdot \supset \cdot p_{o_{n+1}} = r_{o_{n+1}}) \\ \text{Axiom LSD(1) 46: } & (\forall p_{o_{n+1}} q_{o_{n+1}} r_{o_{n+1}} s_{o_{n+1}}) (C_{o_{n+1}o_{n+1}o_{n+1}} p_{o_{n+1}} q_{o_{n+1}} = \\ & C_{o_{n+1}o_{n+1}o_{n+1}} r_{o_{n+1}} s_{o_{n+1}} \cdot \supset \cdot q_{o_{n+1}} = s_{o_{n+1}}) \end{aligned}$$

These two axioms assert that identical conditional *Gedanken* have identical *Gedanken* as their antecedents and consequents. The next axiom reads:

$$\text{Axiom LSD(1) 47: } (\forall f_{o_{n+1}\alpha_{n+1}} g_{o_{n+1}\alpha_{n+1}}) (\Pi_{o_{n+1}(o_{n+1}\alpha_{n+1})} f_{o_{n+1}\alpha_{n+1}} = \Pi_{o_{n+1}(o_{n+1}\alpha_{n+1})} g_{o_{n+1}\alpha_{n+1}} \cdot \supset \cdot f_{o_{n+1}\alpha_{n+1}} = g_{o_{n+1}\alpha_{n+1}})$$

This asserts that identical universal *Gedanken* must be composed of the same *Sinne* picking out the argument concepts. To use a natural language example, if the (false) propositions that “everything is a bachelor” and “everything is an unmarried man” express the same *Gedanke*, then “() is a bachelor” and “() is an unmarried man” must express the same *Sinn*. The system contains similar axioms for descriptive *Sinne* and denotative *Gedanken*. Like universal *Gedanken*, identical descriptive *Sinne* must be composed of the same *Sinne* picking out the same argument functions. Similarly, identical denotative *Gedanken* must be composed of identical *Sinne* picking out the objects involved in the *Sinn* and *Bedeutung* halves of the Δ -relation.

Lastly, the system contains a number of axioms to the effect that *Sinne* that are composed of components of differing types are never identical. We here only list one such example, and here α and β are required to be different types:

$$\text{Axiom LSD(1) 53: } (\forall x_{\alpha_{n+1}}) (\forall x_{\alpha_{n+2}}) (\forall y_{\beta_{n+1}}) (\forall y_{\beta_{n+2}}) (\Delta_{o_{n+1}\alpha_{n+1}\alpha_{n+2}} x_{\alpha_{n+2}} x_{\alpha_{n+1}} \neq \Delta_{o_{n+1}\beta_{n+1}\beta_{n+2}} y_{\beta_{n+2}} y_{\beta_{n+1}})$$

This axiom asserts that if there are two denotative *Gedanken*, and the first is the *Gedanke* that *Sinn* x^* picks out x as *Bedeutung*, and the second is the *Gedanke* that *Sinn* y^* picks out y as *Bedeutung* where x^* is not the same type as y^* and x is not the same type as y , then the *Gedanken* are not identical. A *Gedanke* must have unique composition, and thus cannot be formed out of *Sinne* of different types.

What is missing from the incomplete axiomatization Church initially offers for Alternative (1) are axioms capturing the identity conditions of *Sinne* and *Gedanken* generally, as opposed to those specifically dealing with the sense-function constants he introduces, $C_{o_{n+1}o_{n+1}o_{n+1}}$, $\Pi_{o_{n+1}(o_{n+1}\alpha_{n+1})}$, $\iota_{\alpha_{n+1}(o_{n+1}\alpha_{n+1})}$, and $\Delta_{o_{n+1}\alpha_{n+1}\alpha_{n+2}}$. In fact, nowhere does Church (or anyone else) attempt to complete the system by adding such axioms. However, in his outline of Alternative (0) from 1974, Church begins to formulate such axioms, and in his

own reformulations of Alternative (0), Anderson offers what he takes to be a complete axiomatization of Alternative (0).¹⁸ Perhaps the most important of such axioms is the following:

$$\text{Axiom LSD(0) 64: } (\forall f_{\alpha\beta})(\forall f_{\alpha_1\beta_1})(\forall x_{\beta}y_{\beta})(\forall x_{\beta_1}y_{\beta_1})[\Delta^m_{o(\alpha\beta)(\alpha_1\beta_1)}f_{\alpha_1\beta_1}f_{\alpha\beta} :: \supset :: \Delta^m_{o\beta\beta_1}x_{\beta_1}x_{\beta} :: \supset :: \Delta^m_{o\beta\beta_1}y_{\beta_1}y_{\beta} :: \supset :: f_{\alpha_1\beta_1}x_{\beta_1} = f_{\alpha_1\beta_1}y_{\beta_1} \cdot \supset \cdot x_{\beta_1} = y_{\beta_1}]$$

(The superscripts m on the Δ operators will be explained in the next section.) This axiom asserts precisely the principle discussed in the last chapter in conjunction with the composition principle. Because *Sinne* are composite wholes with a determinate structure, if f^* is a sense-function, and x^* and y^* are *Sinne*, then the composite *Sinn* $f^*(x^*)$ is the same as the composite *Sinn* $f^*(y^*)$ only if x^* and y^* are the same. In fact, from this principle, along with axioms 10-17, many of the axioms from Alternative (1) discussed above can be derived as theorems.

PROBLEMS IN CHURCH'S LOGIC OF SENSE AND DENOTATION

Even in his first attempts at formulating the Logic of Sense and Denotation, Church was aware that including commitment to entities such as *Sinne* and *Gedanken* within a logical system held its perils. In a footnote to his first paper on the subject, he wrote:

The writer is indebted to Leon Henkin for raising the question of the cardinal number of the . . . [*Sinne*] . . . of a given type, in connection with the answer to which an antinomy may easily appear, at least unless appropriate caution has been exercised in regard to the assumptions which are made (in the form of axioms and rules). Because of this and other possibilities of self-contradiction, no logistic treatment of sense and denotation can be accepted as more than provisional until its consistency has been thoroughly studied.¹⁹

Here, Church alludes to the possibility of semantical antinomies arising within the Logic of Sense and Denotation if the cardinal number of *Sinne* to which the system is committed is too great. Ironically, this very problem was soon discovered to plague his first formulation of Alternative (1).

¹⁸Church, "Outline of a Revised Formulation," part 2, 149-53; Anderson, "Some New Axioms," 228-30, "Some Models," 55-9.

¹⁹Church, "A Formulation," 6n.

As early as 1958, John Myhill showed system LSD(1)¹ to be formally inconsistent.²⁰ Like so many antinomies, the problem stems from Cantor's theorem: the principle that the (cardinal) number of sets or classes of entities of a certain type is always greater than the number of entities of that type. Myhill began by showing that extensional class theory could in effect be proxied in systems utilizing the lambda calculus. Since Church's system employs the simple theory of types, the resulting class theory is not fully naive and is not subject to Russell's paradox. However, problems arise for the type α_1 of *Gedanken*, because the system is committed to a cardinal number of *Gedanken* as great as the cardinal number of classes of *Gedanken*. There are of course as many classes as *Gedanken* as there are *Gedanken*; however, it is also possible to generate a *Gedanke* for every class, such as the *Gedanke* that all *Gedanken* are in that class. This means that there would exist *Gedanken* and classes of *Gedanken* in equal number, in violation of Cantor's theorem. Myhill uses this to show that Church's system falls prey to an antinomy very similar to Russell's paradox from Appendix B of *The Principles of Mathematics* discussed briefly in Chapter 1. Some *Gedanken* take the form of picking out the True if and only if all *Gedanken* fall into some class. If we consider the class of all such *Gedanken* that are not in their corresponding classes, we arrive at an antinomy by considering the *Gedanke* that all *Gedanken* fall into *this* class. As Russell himself was already aware, this antinomy is not solved by adopting a simple type-theory.²¹ We need not repeat here the formal details of Myhill's rediscovery of this antinomy,²² but nevertheless it is clear that LSD(1)¹ must be modified if the Logic of Sense and Denotation is to fulfill Church's goal of representing a feasible logic of intensions. Because Myhill seems to have "rediscovered" this antinomy independently of Russell, Anderson dubs this contradiction "the Russell-Myhill antinomy."

However, this was not the only problem Church's original formulation faced. Additionally, a modified version of the Epimenides or "liar" paradox has been found to be formulable in the system.²³ Consider the *Gedanke* expressed by the proposition "Church's favorite *Gedanke* is not true." Now it seems possible that Church's favorite *Gedanke* is this very *Gedanke*. We then ask whether

²⁰Myhill, "Problems Arising in the Formalization of Intensional Logic," *Logique et Analyse* 1 (1958): 82.

²¹Russell, *Principles of Mathematics*, §500.

²²In addition to Myhill's paper, this antinomy as it relates to the Logic of Sense and Denotation is treated in greater formal detail in Anderson's work. See Anderson, "Some Models," chap. 3, "Some New Axioms," 221-3, "Semantical Antinomies," 107-8.

²³This observation was reportedly first made by David Kaplan. See Anderson, "Semantical Antinomies," 113n.

Church's favorite *Gedanke* is true (presents the True as *Bedeutung*), and arrive at the familiar contradiction. The Logic of Sense and Denotation provides us with the resources for transcribing this paradox.

First we can define the predicate "Tr₀₀₁" as standing for the function $(\lambda x_{01} \Delta_{0001} x_{01} (\forall p_0)(p_0 \supset p_0))$. This function is of type 00₁; it takes *Gedanke* to truth-values, and its value will be the True just in case its argument is a *Gedanke* that has the True as *Bedeutung* (since $(\forall p_0)(p_0 \supset p_0)$ is the True). In Church's system, for any constant, it is possible to define the *Sinn* that constant expresses simply by raising the subscripts on all the type-symbols by one. Therefore, Church's system then gives us resources to define a sense-function Tr₀₁₀₂, the *Sinn* of "Tr₀₀₁", which takes as value a *Sinn* whose *Bedeutung* is itself a *Gedanke* and yields a *Gedanke* as value. (To be precise, since "Tr₀₀₁" is a defined sign, so is "Tr₀₁₀₂"; it is defined as $(\lambda x_{02} \Delta_{010102} x_{02} (\Pi_{01(0101)} (\lambda p_{01} C_{010101} p_{01} p_{01})))$.) If we use c_{01} to stand for Church's favorite *Gedanke*, and c_{02} as the *Sinn* of the expression "Church's favorite *Gedanke*" (a *Sinn* whose *Bedeutung* is the *Gedanke* c_{01}), we then assume:

$$c_{01} = \sim_{0101} \text{Tr}_{0102} c_{02}$$

This states that Church's favorite *Gedanke* is the *Gedanke* expressed by "Church's favorite *Gedanke* is not true" (or more precisely, that expressed by "Church's favorite *Gedanke* does not pick out the True as *Bedeutung*.") It then follows from the axioms of LSD(1)¹ that:²⁴

$$\text{Tr}_{001} c_{01} \ \&_{000} \ \sim_{00} \text{Tr}_{001} c_{01}$$

This states that Church's favorite *Gedanke* both is and is not true.

Of course, it is neither a theorem nor an axiom of LSD(1)¹ that Church's favorite *Gedanke* is that expressed by "Church's favorite *Gedanke* is not True." Thus, this not does not lead to an outright inconsistency or contradiction within the system; it is, in the distinction made by Quine, only a (contingent) "paradox" and not an "antinomy."²⁵ Nevertheless, because this assumption does lead to a contradiction if assumed, its *negation* becomes a theorem of the system. However, the assumption in question does not seem logically impossible, and it

²⁴For more on the technical details of this paradox, see Anderson, "Semantical Antinomies," 100-1, "Some Models," 72-5. The derivation of the contradiction is parallel to that given in the next chapter for my own system.

²⁵W.V. Quine, "The Ways of Paradox," in *The Ways of Paradox and Other Essays*, rev. ed. (Cambridge, Mass.: Harvard University Press, 1976), 1-18.

is obviously unbecoming for a logical system to contradict any possible matter of fact. Thus, this too seems to pose the need for reforming LSD(1)¹.

There are a number of possible tactics one could take within the context of devising a logical calculus for resolving such semantical antinomies and paradoxes. We shall discuss many such tactics in greater detail in Chapter 7. However, it is worth discussing two tactics briefly here. One important strategy stems from the work of Russell and Whitehead.²⁶ Their response is the adoption of a *ramified* theory of types (as opposed to the *simple* theory of types that formed the basis of Church's initial systems.) Ramified systems involve a notion not only of *type* but of *order*. To use our Fregean terminology, ramified systems would hold there to be an essential difference in *order* between *Gedanken* of a certain order and any *Gedanke* that is *about* all or some *Gedanken* of that order. For example, if we think of the problematic *Gedanke* of the Russell-Myhill antinomy, it is a *Gedanke* that is itself *about* all *Gedanken*, including, it would appear, *itself*. Ramified type-theory rules this out; *Gedanken* cannot be about a range of *Gedanken* in which they themselves are included. Classes of propositions are thus limited to propositions of a certain order, and thus the question does not arise as to whether the *Gedanke* that all *Gedanken* of order *n* fall into the problematic class (of *Gedanken* of order *n*) itself falls into this class, because this *Gedanke* is not of order *n*. In the case of the modified Epimenides paradox, one would not be able to speak of "Church's favorite *Gedanke*" simpliciter, but only "Church's favorite *Gedanke* of order *n*." The *Gedanke* expressed by "Church's favorite *Gedanke* of order *n* is not true" would not itself be of order *n*, and so it could not itself be the *Gedanke* it mentions. Thus, no contradiction would result.

Another resolution of such semantical paradoxes comes from the work of Tarski.²⁷ Tarski's solution derives from the common distinction between metalanguage and object language, the latter representing the language under study, the former the language in which it is studied. Tarski maintains that when it comes to formal languages, semantic concepts such as "truth" and "meaning" must always be relativized to a language, and moreover, that a consistent language cannot contain its own semantic concepts. One cannot, *within* language

²⁶Whitehead and Russell, *Principia Mathematica*, introd. chap. 2; Bertrand Russell, "Mathematical Logic as Based on the Theory of Types," in *From Frege to Gödel*, ed. Jean van Heijenoort (Cambridge, Mass.: Harvard University Press, 1967), 150-82.

²⁷See, e.g., Alfred Tarski, "The Semantic Conception of Truth and the Foundations of Semantics," in *The Philosophy of Language*, 3d. ed., ed. A.P. Martinich (New York: Oxford University Press, 1996), 61-84, "The Concept of Truth in Formalized Languages," in *Logic, Semantics and Metamathematics: Papers from 1923 to 1938* (Oxford: Clarendon Press, 1956), 152-278.

L, discuss the truth or meaning of propositions of L. It is likely that from this perspective, Tarski would object to language-independent semantic entities such as Frege's *Sinne* and *Gedanken*. If Tarski were to allow talk of such semantic entities at all, he would limit them to categories such as "*Gedanken* expressible in L." The result is a solution to the semantical antinomies that is in some ways similar to that of Whitehead and Russell.²⁸ The problematic statement of the modification of the Epimenides becomes "Church's favorite *Gedanke* expressible in L is not true-in-L." The *Gedanke* expressed by this proposition, however, is not expressible in L, since it involves the semantic concepts of L, and again, contradiction is avoided. Similarly, in the case of the Russell-Myhill antinomy, presumably the *Gedanke* that all *Gedanken* expressible in L are in the problematic class would not itself be a *Gedanke* expressible in L, and the paradox is avoided.

In his first attempt to revise his Logic of Sense and Denotation, rather than opting for a system employing a fully ramified theory of types, Church attempts to fashion a modification to his systems based on Tarski's notion of a hierarchy of languages.²⁹ It might seem at first that Tarski's solution is antithetical to the very notion of the Logic of Sense and Denotation, which is by its nature a system that treats semantical notions within the object language. In particular, it employs the semantic operator, Δ . Church's solution is to require that each occurrence of Δ include a superscript m , where m is either l , $l + 1$, $l + 2$, etc., with this hierarchy representing something akin to Tarski's hierarchy of languages and metalanguages. If we suppose that x_α is some object presented by *Sinn* x_{α_1} then one might assert, in language l , that x_{α_1} picks out x_α by writing " $\Delta^l_{\text{oo}\alpha_1}x_{\alpha_1}x_\alpha$ ". According to the direct discourse method of using different expressions for the *Sinne* and *Bedeutungen* of expressions, and assuming x_{α_2} is a *Sinn* picking out x_{α_1} , then one might use the formula " $\Delta^l_{\text{o}_1\alpha_1\alpha_2}x_{\alpha_2}x_{\alpha_1}$ " to denote the *Gedanke* expressed by the proposition, " $\Delta^l_{\text{oo}\alpha_1}x_{\alpha_1}x_\alpha$ ". Church's idea is that, since this *Gedanke* involves semantic notions, it should be seen as impossible, in language l , to say that the *Gedanke* denoted by " $\Delta^l_{\text{o}_1\alpha_1\alpha_2}x_{\alpha_2}x_{\alpha_1}$ " picks out the truth-value denoted by " $\Delta^l_{\text{oo}\alpha_1}x_{\alpha_1}x_\alpha$ ". Rather, it is only in $l + 1$ (and higher languages) that this can be said. Thus it does not hold that " $\Delta^l_{\text{oo}_1}\Delta^l_{\text{o}_1\alpha_1\alpha_2}x_{\alpha_2}x_{\alpha_1}\Delta^l_{\text{oo}\alpha_1}x_{\alpha_1}x_\alpha$ " but only " $\Delta^{l+i}_{\text{oo}_1}\Delta^l_{\text{o}_1\alpha_1\alpha_2}x_{\alpha_2}x_{\alpha_1}\Delta^l_{\text{oo}\alpha_1}x_{\alpha_1}x_\alpha$ ", where $i \geq 1$.

To arrive at this result, Church modifies his Axiom LSD(1) 14 to the following:

²⁸For more on the similarity between these approaches, see Church, "Comparison of Russell's Resolution of the Semantical Antinomies with that of Tarski," *Journal of Symbolic Logic* 41 (1976): 747-60.

²⁹Church, "Outline of a Revised Formulation," part 2, 150-1.

Axiom LSD(1) 14': $\Delta^{m+1}_{o(o_n\alpha_n\alpha_{n+1})(o_{n+1}\alpha_{n+1}\alpha_{n+2})}\Delta^m_{o_{n+1}\alpha_{n+1}\alpha_{n+2}}\Delta^m_{o_n\alpha_n\alpha_{n+1}}$

The purpose of axiom LSD(1) 14 was to assert that the constants introduced to stand for the *Sinne* of the semantic-operators pick out or present as *Bedeutung* the functions for which those semantic-operators stand. The import of the superscripts, however, is that the constants introduced to stand for the *Sinne* of the semantic-operators for language m can only be asserted to pick out or present their *Bedeutungen* within a language higher than m in the metalinguistic hierarchy. (Axiom LSD(1) 14' in fact only asserts that it holds in the metalanguage of m , viz., $m + 1$, but an additional axiom holds that whatever holds in $m + 1$ holds also in $m + 2$ and so on.)

It is clear how it is that Church intends these changes to his system to block the semantical paradoxes discussed above. First, in the case of the modified Epimenides paradox, because the predicate “ Tr_{oo_1} ” is defined using the Δ operator, it too must be given a superscript. If we assume it holds that “ $\text{Tr}^m_{oo_1c_{o_1}}$ ”, it does not follow that “ $\sim_{oo}\text{Tr}^m_{oo_1c_{o_1}}$ ” but only “ $\sim_{oo}\text{Tr}^{m+1}_{oo_1c_{o_1}}$ ”. Since “ $\text{Tr}^m_{o_1o_2}$ ” is defined using a constant for the *Sinn* of the Δ -operator used in the definition of “ $\text{Tr}^m_{oo_1}$ ”, it follows only that “ $\Delta^{m+1}_{o(o_{o_1})(o_1o_2)}\text{Tr}^m_{o_1o_2}\text{Tr}^m_{oo_1}$ ” and not “ $\Delta^m_{o(o_{o_1})(o_1o_2)}\text{Tr}^m_{o_1o_2}\text{Tr}^m_{oo_1}$ ”, and hence only “ $\Delta^{m+1}_{ooo_1}\text{Tr}^m_{o_1o_2c_{o_2}}\text{Tr}^m_{oo_1c_{o_1}}$ ” and not “ $\Delta^m_{ooo_1}\text{Tr}^m_{o_1o_2c_{o_2}}\text{Tr}^m_{oo_1c_{o_1}}$ ”. Similarly, if we assume “ $\sim_{oo}\text{Tr}^m_{oo_1c_{o_1}}$ ” it follows only “ $\text{Tr}^{m+1}_{oo_1c_{o_1}}$ ” and not “ $\text{Tr}^m_{oo_1c_{o_1}}$ ”. Since the problematic *Gedanke* in the Russell-Myhill antinomy would also likely be defined using a constant for the *Sinn* of the Δ -operator involved in the definition of the problematic class, a similar resolution follows in that case.

However, despite this proposed fix, Anderson soon showed that the semantic paradoxes still plagued the revised system.³⁰ While adding the superscripts to the Δ -operators and modifying Axiom LSD(1) 14 block the contradictory results when the semantical antinomies are formulated using the constants $\Delta^m_{o_{n+1}\alpha_{n+1}\alpha_{n+2}}$ standing the *Sinne* of the Δ -operators, Anderson proved that the semantical antinomies could be formulated without using these constants. The problem is the following theorem, an unforeseen consequence of his first seventeen axioms:³¹

Theorem LSD(0) 1: $(x_\beta)(\exists x_{\beta_1})\Delta^m_{o\beta\beta_1}x_{\beta_1}x_\beta$

This theorem states that for every entity of type β there is at least one *Sinn* of type β_1 that picks out the former as *Bedeutung* in language m , or, in other words,

³⁰Anderson, “Some Models,” 22-9, “Some New Axioms,” 221-3.

³¹For the derivation of this theorem, see Anderson, “Some New Axioms,” 221.

that everything has a *Sinn* in language m . One instance of this theorem is the following principle:

$$(\exists f_{o_1\alpha_1\alpha_2})\Delta^m_{o(o\alpha\alpha_1)(o_1\alpha_1\alpha_2)}f_{o_1\alpha_1\alpha_2}\Delta^m_{o\alpha\alpha_1}$$

This states that there is a sense-function f , which picks out the semantical functions of language m ($\Delta^m_{o\alpha\alpha_1}$) in language m itself. Thus, while in the revised system, the *constant* $\Delta^m_{o_1\alpha_1\alpha_2}$ introduced as the *Sinn* of " $\Delta^m_{o\alpha\alpha_1}$ " can only be proved to pick out the function $\Delta^m_{o\alpha\alpha_1}$ in language $m + 1$, the system is nevertheless committed to at least one sense-function that picks out $\Delta^m_{o\alpha\alpha_1}$ in m itself. If this function is used to formulate the semantical paradoxes or antinomies, Church's solution does not block the resulting contradictions.

In his own reformulations of Alternative (0), Anderson's recommended response is to modify the axioms that lead to theorem LSD(0) 1. Central in the proof of this theorem is the following axiom (carried over, slightly modified, from LSD(1)¹):

$$\text{Axiom LSD(0) 16: } (\forall f_{\alpha\beta})(\forall f_{\alpha_1\beta_1})[(\forall x_{\beta})(\forall x_{\beta_1})(\Delta^m_{o\beta\beta_1}x_{\beta_1}x_{\beta} \supset \Delta^m_{o\alpha\alpha_1}(f_{\alpha_1\beta_1}x_{\beta_1})(f_{\alpha\beta}x_{\beta})) \supset \Delta^m_{o(\alpha\beta)(\alpha_1\beta_1)}(\lambda_1x_{\beta_1}f_{\alpha_1\beta_1}x_{\beta_1})f_{\alpha\beta}]$$

This axiom is central to the presumption taken in the Logic of Sense and Denotation that sense-functions are the *Sinne* of function expressions. Anderson, however, advocates dropping this axiom in favor of a revised axiom schema that does not result in theorem 1, hoping that it would be possible to salvage those theorems of this axiom that are desired in some other way.³² In his own final reformulation of Alternative (1), however, Church instead advocates adopting a fully ramified theory of types for the system to avoid semantical paradoxes.³³ Church is quite reticent about why he now adopts a Russellian approach to resolving the semantical paradoxes rather than attempting to salvage his earlier more Tarskian approach, but presumably he is unwilling to drop axiom 16, believing it to be too central to the basic assumptions of the system to abandon.

THE UNFREGEAN ELEMENTS OF CHURCH'S LOGIC OF SENSE AND DENOTATION

It is perhaps fair to say that none of the various formulations of the Logic of Sense and Denotation that have been developed thus far have succeeded in their goal of representing a workable account of intensional₁ logic. However, what is more important in this context is what they can teach us about the development

³²Anderson, "Some New Axioms," 224-5.

³³Church, "A Revised Formulation," 152-5

of a fully Fregean logical calculus of *Sinn* and *Bedeutung*. There are a number of features of the Logic of Sense and Denotation Church develops that I believe to be contrary to how Frege himself would have approached the subject, and it is important to highlight these aspects before turning to my own attempt to develop a more truly Fregean calculus.

Firstly, it was argued in Chapter 2 that Frege's understanding of the semantics of expressions involving Roman letters is that they express general *Gedanken*; i.e., their *Sinne* are the same as the corresponding expressions containing Gothic letters and initial universal quantifiers. Church adopts an interpretation of the semantics of his free variables according to which each is assigned not only a range of objects as their *Bedeutungen*, but also a "sense range," a range of *Sinne*.³⁴ On Church's understanding, it is only relative to a certain assignment of variables that one can speak of *the Sinn* of an expression containing free variables. It should be clear from my previous discussion of this issue that Frege could not have accepted this approach. This, however, may not be seen as a substantial critique of the Logic of Sense and Denotation as a formal system, since it would always be possible to reinterpret its semantics in the more Fregean way.

Secondly, Church adopts the sense-function interpretation of the *Sinne* of incomplete expressions. I argued in the previous chapter against this interpretation. Of course, as I noted there, it is admitted both by me and other opponents of the sense-function interpretation that these sense-functions exist; we merely object to regarding such functions as the *Sinne* of functional expressions. Thus, it might be possible to accept some of the commitments Church makes in his systems to such functions and the claims made about them, even under an alternative interpretation. However, the details of Church's treatment of the sense-function view give rise to serious problems. Perhaps the most objectionable aspect of Church's endorsement of this interpretation in his system is the inclusion of formulae such as " $\Delta_{o(o_1)(o_1t_1)}g_{o_1t_1}f_{o_1}$ ". This stands for the truth-value of sense-function g being a *Sinn* that picks out concept f as *Bedeutung*. Church then has to introduce Axioms LSD(1) 15 and 16 to equate the truth-value of this formula with that of " $(\forall x_1)(\forall x_{t_1})(\Delta_{o_1t_1}x_1x_{t_1} \supset \Delta_{o_1o_1}(g_{o_1t_1}x_{t_1})(f_{o_1}x_1))$ ", which stands for the truth-value of g being so as to yield for any *Sinn* x^* with x as *Bedeutung* taken as argument a *Gedanke* that picks out as its *Bedeutung* the truth-value yielded by f for x as argument. This latter formulation is not entirely objectionable even on my own view, considering that even I admit that sense-functions exist, and any sense-function g determined to exist by an incomplete *Sinn* that picks out function f will satisfy this complicated formula. However, the formulation " $\Delta_{o(o_1)(o_1t_1)}g_{o_1t_1}f_{o_1}$ ", which Church stipulates

³⁴Church, "The Need for Abstract Entities," 102-3.

to be equivalent, suggests that sense-functions are themselves *Sinne* that pick out functions as *Bedeutungen* just as *Sinne* presenting objects pick out their *Bedeutungen*. This is what is not so on the interpretation offered in the previous chapter.

Moreover, even if the sense-function interpretation of the nature of the *Sinne* of function expressions is adopted, contrary to Church's axiom 16, it is not true that satisfying the formula " $(\forall x_i)(\forall x_{i_1})(\Delta_{o_{i_1}x_{i_1}x_i} \supset \Delta_{o_{o_{o_1}}(g_{o_{i_1}x_{i_1}})(f_{or}x_i))$ " is a *sufficient* condition for g to be a sense-function of f . Here f has type o_i ; it is a function from individual to truth-values. Let us suppose that f is the concept of being human. If so, then it generates a mapping of arguments onto to values such that for Socrates as argument, the True is yielded as value, for Boston as argument, the False is yielded as value, and so on. We could then imagine f to be the *Bedeutung* of the predicate expression " ξ is human" in natural language. On the sense-function view, the *Sinn* of an expression such as " ξ is human" is to be understood as a sense-function that would take *Sinne* as arguments and yield *Gedanken* as value. For the *Sinn* of "Socrates" it would yield the *Gedanke* expressed by "Socrates is human," for the *Sinn* of "Boston" it would yield the *Gedanke* expressed by "Boston is human," etc. If g in the above formula were understood as this sense-function, it is easy to see that it would satisfy the formula " $(\forall x_i)(\forall x_{i_1})(\Delta_{o_{i_1}x_{i_1}x_i} \supset \Delta_{o_{o_{o_1}}(g_{o_{i_1}x_{i_1}})(f_{or}x_i))$ ". However, let us imagine g to not be this sense-function, but a function very much like it, having the same value for every argument, with one exception. Rather than yielding the *Gedanke* expressed by "Russell is human" for the *Sinn* of "Russell" taken as argument, it yields the *Gedanke* expressed by "Stockholm is in Sweden." Oddly enough, if g takes this form, then g still satisfies " $(\forall x_i)(\forall x_{i_1})(\Delta_{o_{i_1}x_{i_1}x_i} \supset \Delta_{o_{o_{o_1}}(g_{o_{i_1}x_{i_1}})(f_{or}x_i))$ ". For even where x_i is instantiated to Russell and x_{i_1} to the *Sinn* of "Russell", the g and f still satisfy " $\Delta_{o_{o_{o_1}}(g_{o_{i_1}x_{i_1}})(f_{or}x_i)$ ", because here the *Bedeutung* of " $f_{or}x_i$ " is the True (since Russell is human), and " $g_{o_{i_1}x_{i_1}}$ " stands for the *Gedanke* expressed by "Stockholm is in Sweden," which *does* pick out the True as *Bedeutung*. However, it seems clear that g should not be considered a sense-function of f in this case. This seems to provide an independent argument against Church's axiom 16 in addition to the problems found with this axiom by Anderson.

From a Fregean standpoint, expressions such as " $\Delta_{o(o_i)(o_{i_1})}g_{o_{i_1}f_{o_i}}$ " are objectionable for another reason. Removing the type indices, this expression simply reads " $\Delta g f$ ". This expression places signs for functions in subject-position without any indication of the unsaturatedness of these functions. This Frege would not allow. Of course, there is a sense in which Frege does allow function expressions to occur in subject position. This is when they occur as argument to a higher-level function. For example, in Frege's expression " $(\forall a)\mathcal{P}(a)$ ", in a sense, " $\mathcal{P}(\xi)$ " occurs in subject position as argument to the

second-level function sign “ $(\forall a)\phi(a)$ ”. This may not seem to differ in any significant respects from Church’s simpler formulation, “ $\Pi_{o(o_i)}f_{o_i}$ ”. However, Frege prefers notation that reveals clearly the way in which the higher-level function *mutually* saturates with its argument function, as described in Chapter 2. This is clearly missing from Church’s notation. Indeed, Church’s notation would be inadequate if his system included functions with multiple arguments, because if f had two argument places, in a formula such as “ $\Pi_{o(o_i)}f_{o_i(t, i)}$ ”, it would be unclear which of f ’s argument places (or both) is to be understood as mutually saturated by the quantifier. Church avoids this by allowing functions to have functions as value. Frege rejects such functions, presumably because for him, completed expressions cannot have functions as their *Bedeutung*, and thus if a function expression is completed by an argument expression, its *Bedeutung* must be an object.³⁵ Church’s notation simply does not respect Frege’s desired isomorphism between completeness or incompleteness of an expression and completeness or incompleteness of its *Bedeutung*. We saw earlier that Frege would likely object to Church’s lambda notation because it hides the unsaturatedness of the functions such expressions denote. In Church’s system, a sign for a function is considered a well-formed formula completely on its own, a conclusion Frege would wholeheartedly resist.

Let us return to a consideration of formulae that appear in Church’s system such as “ $\Delta_{o(o_i)(o_1t_1)}g_{o_1t_1}f_{o_i}$ ”. In Frege’s own terminology, the sign “ Δ ” here must be understood as higher-level function, since it takes functions as arguments. However, in an expression such as “ $\Delta_{o_1t_1}x_{t_1}y_{t_1}$ ”, the sign “ Δ ” stands for a first-level function, since it takes individuals (and *Sinne* of individuals) as arguments. Of course, it was not Church’s intention that “ $\Delta_{o_n\alpha_n\alpha_{n+1}}$ ” should denote a single function, but rather a different function for every type α and natural number n . Where α is a functional type, then “ $\Delta_{o_n\alpha_n\alpha_{n+1}}$ ” is meant to stand for a higher-level function; where n is equal to zero, then it is meant to stand for a relation, where n is greater than zero, it is meant to stand for a sense-function. Frege would probably find it objectionable that the same sign be used to denote so many different functions. At the very least he would insist that a different sign be used for functions of different level, and insist that the signs used for higher-level functions reveal their capacity for mutual saturation in some way.

Church, himself aware that his allowance of function signs in subject position is contrary to Frege’s own practice, attempts to downplay the division this creates between his own approach and Frege’s by noting that in many of those cases in which he places a function name in subject position, Frege would place instead a name of the value-range of that function, and thus, in a sense,

³⁵On this, see Dummett, *Frege: Philosophy of Language*, 43-4.

Frege allows for nominalized functions.³⁶ It is true that in many of the places in which Church places the name for a function in subject position, Frege would have a corresponding form that would utilize the name for the value-range of that function. Compare, for example, Church's use of his description function, " ι ", with the corresponding function in Frege's system, " λ ". The former would be followed simply by the name of a function occurring as argument; the latter would be followed by the name of a value-range. Moreover, it was noted in Chapter 2 that for Frege, value-ranges derive their existence from their functions; the value-range and the function in some sense share their being. Nevertheless, for Frege, there are important differences between the two; e.g., one is an object, the other is a function, one is unsaturated, the other complete. Church's notation blurs the distinction between the two in ways Frege himself would almost certainly find objectionable.

Another significant way in which Church's Logic of Sense and Denotation deviates from Frege's own understanding of logic and semantics involves the complicated theory of types it adopts. It is not simply that Frege would object to a theory of types in any form. In fact, Frege's distinction between objects and functions, and the hierarchy of levels of functions, which are built into his logical system, create a sort of theory of types even in the *Begriffsschrift* of the *Grundgesetze*. Objects, functions that take objects as argument, functions that take as argument functions that take objects as argument, etc., are distinct in Frege's system. This is why, for example, the formulation of Russell's paradox involving properties that do not apply to themselves is not formulable even in the *Begriffsschrift* of the *Grundgesetze* (see *PMC* 132-3). However, this division of entities has solid philosophical and metaphysical motivations for Frege. This division is made according to the saturatedness or classification of unsaturatedness of the entities in question. The division is not made simply as an *ad hoc* maneuver to avoid contradictions; in fact, the distinction was in place even before Frege had concerned himself with logical or semantical paradoxes at all.

However, not all of the type-distinctions involved in Church's Logic of Sense and Denotation seem to have solid metaphysical or philosophical motivations. Firstly, at the core of Church's type system is the division between types o and ι , the type of truth-values and the type of individuals, respectively. In Frege's system, the truth-values are simply treated as objects in the full sense, and these types are not divided. Church's rationale for the division he employs is not at all clear. It is possible that he makes the distinction in order to preserve a difference in syntax between truth-functional connectives and signs for monadic and relational predicates. But given that Church is working within a function

³⁶Church, "A Formulation," 4.

calculus wherein both logical connectives and monadic and relational predicates are transcribed as functions, it is a mystery why it would be important to preserve such a distinction. It is true that doing so obviates the need to define the truth functions in such a way that the result is known even when the arguments to them are not truth-values, but Frege's own work shows that the truth functions can be so defined without insuperable difficulties.

Church also draws a distinction in type between individuals (t) and *Sinne* that pick out individuals (t_1), and similarly, between *Sinne* that pick out individuals and *Sinne* that pick out such *Sinne* (t_2), and so on, and similarly between truth-values (o), *Sinne* that pick them out (*Gedanken*, o_1), *Sinne* that pick out such *Sinne* (o_2), etc. Here, too, the division in type between these entities is unclear, at least within Fregean philosophy. As noted in the previous chapter, Frege himself holds *Sinne* that pick out objects as *Bedeutungen* to themselves be objects. He insists that all functions should be defined for all objects, which seems contrary to holding non-*Sinn* objects, *Sinne* picking out such objects, and *Sinne* picking out such *Sinne* to each fall within different syntactic types. One of the hallmarks of the objects within Frege's third realm of *Sinn* is that the objects that exist therein are non-actual (*Wirklich*), or to be more precise, actual in a very limited way. While our grasp of *Gedanken* and other *Sinne* can have an impact within the material and mental realms, the entities in realm of *Sinn* themselves are timeless and unchangeable, impervious to being affected by actions in the material or mental worlds (*CP* 370, *PW* 138). Yet, Frege explicitly says that actuality is only "one predicate out of many" and that it has "no special relevance to logic" (*BL* 18). This would seem to suggest that there is no reason that *Gedanken* and other *Sinne* should be treated as falling within an entirely different ontological category from other objects; they simply have the peculiar properties characteristic of existing in the third realm. The rationale for holding *Sinne* as falling into different types according to whether or not their *Bedeutungen* are themselves *Sinne* is perhaps even more mysterious. According to the interpretation offered in the last chapter, a *Sinn* contains a packet of descriptive information and presents a *Bedeutung* in virtue of it alone satisfying the criteria contained in this information. This is unchanged whether the *Bedeutung* presented by a *Sinn* is or is not itself a *Sinn*, and thus there seems little reason to classify *Sinne* into different metaphysical categories based on what their *Bedeutungen* happen to be.

Consider expressions such as "Frege's favorite *Sinn*," "Church's favorite *Sinn*," and "Kaplan's favorite *Sinn*." On Frege's understanding of language, "()'s favorite *Sinn*" would be understood as denoting a function taking individuals as argument and yielding as value that individual's favorite *Sinn*. But it is certainly possible that Frege's favorite *Sinn* might be a *Sinn* picking out a non-*Sinn* object, while Church's favorite *Sinn* might be *Sinn* picking out another

Sinn. If these *Sinne* were of different logical types, it would be impossible for them to be values of the same function for different arguments, since the values of a given function must all be of the same type. Functions such as the one purportedly denoted by “()’s favorite *Sinn*” may be thought to be problematic for a variety of reasons, but the important thing in this context is that the same sorts of things can seemingly be true of *Sinne* of both proposed types. When speaking of the properties of the *Sinne* themselves, the same sorts of properties and predicates seem applicable, whereas it is usually the case of two entities of different types that it is a category mistake to predicate of one entity the same things as one predicates of the other.

Ramified type-theory, which Church adopts in his final reformulation, seems even more at odds with the Fregean mindset. While later in his life, Frege certainly would have been aware of ramified type-theory as it appears in the work of Whitehead and Russell, there is no indication that he ever regarded it as the proper solution to the paradoxes. Indeed, the philosophical justifications typically given for ramification in terms of Poincaré’s vicious circle principle or Russell’s multiple relations theory of judgment are not readily available with Frege’s philosophy. Consider the principle that a proposition must never have as part of its truth conditions anything having to do with that proposition itself, else there be a “vicious circle” in the definition of those truth-conditions. In Fregean philosophy, the truth-conditions of a proposition are contained within the *Gedanke* it expresses. Is there any difficulty in assuming that the truth-conditions of a *Gedanke* might depend in some way on that *Gedanke* itself? If *Gedanken* and other *Sinne* only came into being with our definitions and linguistic practices, it is easy to see why there might be a difficulty. However, on Frege’s understanding, *Gedanken* are abstract entities that have their existence (presumably a necessary existence) in a third realm. *Gedanken* are not “defined” into being, and as such, I see no principled reason why they cannot involve themselves in their own truth-conditions. In any case, much more would have to be said to motivate a ramified hierarchy within Fregean metaphysics. Without solid philosophical motivations, the adoption of a ramified type-theory seems to be nothing more than an *ad hoc* dodge designed to avoid paradoxes and contradictions, and not a natural part of the attempt to create the ideal *calculus ratiocinator*.

However, ramified type-theory is no less Fregean than Church’s other proposed solution to the antinomies, that of placing superscripts on the semantic relation sign “ Δ ” to represent a Tarskian hierarchy of languages. Although it might be thought that Frege’s notion of Begriffsschrift does not involve a distinction between object language and metalanguage, I do not think this is the case. In fact, as we saw in Chapter 2, Frege himself anticipates Tarski’s distinction in his notions of *Hilfssprache* (“helping language”) and

Darlegungssprache (“explained language”). Although Frege might have been able to appreciate the Tarskian hierarchy, however, he would not have found it appropriate to relativize the Δ -relation—the relation between a *Sinn* and the *Bedeutung* it presents—to a language. It will be recalled that Frege does not consider *Sinne* to be linguistic entities; indeed, he explicitly says that the same *Sinne* are expressible in different languages and that *Sinne* would exist even if language did not. Thus it makes little sense to say that *Sinn* A picks out *Bedeutung* B only with regard to language L. The relation between *Sinne* and their *Bedeutungen* must be understood as language independent. To deny this is to abandon the heart of Frege’s understanding of *Sinne* as objectively existing entities.

Thus, neither of Church’s proposed solutions to the semantical antinomies is in accord with Frege’s own philosophical views. Again, this is not criticism of Church. It may very well be that Frege’s philosophical views simply do not afford any adequate solution to the semantical antinomies. This will be discussed in much more detail in the forthcoming chapters of the present work. The important point in the present context is that there are many features of Church’s various formulations of the Logic of Sense and Denotation that preclude us from regarding it as a truly *Fregean* logical calculus for the theory of *Sinn* and *Bedeutung*. Again, this should not be altogether surprising given that this was not Church’s intent. However, since the present work is aimed in part at providing such a calculus, the differences must be kept in mind. Nevertheless, there are aspects of Church’s systems that are helpful for the present project, not least of which is the very strategy involved in what we have called “transparent intensional logic.” In devising a logical calculus for Frege’s own understanding of *Sinne* and *Bedeutungen*, we can learn from Church’s systems in studying both the ways they accord and the ways they do not accord with Fregean logic and philosophy of language. In the next chapter, we turn to the application of what we have learned.

A Logical Calculus for the Theory of *Sinn* and *Bedeutung*

BASIC FEATURES

In this chapter, the attempt is made to formulate a logical calculus for the theory of *Sinn* and *Bedeutung* that is as close as possible to the sort of system Frege himself would have developed had he had “the occasion to do so,” given what has been concluded in the preceding chapters. Our primary aim is not to create a feasible intensional logical calculus, but only one that reflects Frege’s views. In fact, a number of formal difficulties will be found with the calculus formulated, and these problems reveal flaws in Frege’s semantic views that have scarcely been discussed by philosophers.

Because our aim is to develop a fully Fregean logical calculus of *Sinn* and *Bedeutung*, we build our system upon Frege’s own Begriffsschrift, as outlined in Chapter 2 (employing the typographical changes therein). We shall consider expansions of both systems FC and FC^{+V}.

While some of the decisions with regard to the development of the calculus will be made so as to improve on ways in which Church’s Logic of Sense and Denotation fails to reflect Frege’s own views, many of the strategies and approaches taken here are borrowed from Church. Firstly, and perhaps most importantly, we adopt the method of direct discourse. Rather than transcribing ordinary language expressions with the same signs in the Begriffsschrift regardless of the context of their occurrence, sometimes they are transcribed using signs that stand for their customary *Bedeutungen*, and sometimes they are transcribed using signs that stand for their customary *Sinne* (their indirect *Bedeutungen*). We then introduce new function signs, principally “ Δ ”, to express relationships between *Sinne* and the *Bedeutungen* they present.

Let us return to the example considered in the first chapter. Here, again, in our informal discussion, I use lowercase Cyrillic letters for new object constants and uppercase script letters for new function constants. (I do this instead of using Roman letters, because, on the reading given in Chapter 2, propositions involving Roman letters must be understood as expressing generalities.) If we imagine “ $\mathcal{P}()$ ” to stand for the concept of being a planet, and “ \mathfrak{m} ” to stand for the morning star (i.e., Venus itself), then “ $\mathcal{P}(\mathfrak{m})$ ” can be understood as standing for the True (the *Bedeutung* of the proposition, “the morning star is a planet.”) The term “ $\mathcal{P}(\mathfrak{m})$ ” expresses the *Gedanke* that the morning star is a planet, but denotes the True. It can be used to transcribe “the morning star is a planet” in direct speech, where this expression has its customary *Bedeutung*. However, in order to transcribe a sentence in which “the morning star is a planet” appears in *oratio obliqua*, in a system of direct discourse, we would use instead a different sign understood as denoting not a truth-value, but the *Gedanke* “ $\mathcal{P}(\mathfrak{m})$ ” expresses. In Chapter 1, we used the sign “ \mathfrak{p} ” to stand for this *Gedanke*. Therefore, we transcribe the complete English proposition:

(1E) The morning star is a planet.

as:

(1B) $\vdash \mathcal{P}(\mathfrak{m})$

However, we transcribe the English proposition:

(2E) Gottlob believes that the morning star is a planet.

as (with “ $\mathcal{B}(\xi, \zeta)$ ” standing for the belief relation, and “ \mathfrak{g} ” for Gottlob):

(2B) $\vdash \mathcal{B}(\mathfrak{g}, \mathfrak{p})$

And then, in order to capture the relationship between “ $\mathcal{P}(\mathfrak{m})$ ” (standing for the truth-value of the morning star’s being a planet) and “ \mathfrak{p} ” (standing for the *Gedanke* that the morning star is a planet), we write:

(3B) $\vdash \Delta(\mathfrak{p}, \mathcal{P}(\mathfrak{m}))$

This asserts that the *Gedanke* is a *Sinn* presenting the truth-value as *Bedeutung*.

In Chapter 3, it was concluded that Frege’s primary criterion for the identity conditions of *Sinne* is that phrases expressing them should be interchangeable even in all singly oblique contexts. One of the advantages of the method of

direct discourse is that this follows directly from Leibniz's law. Above we used "p" to stand for the *Gedanke* expressed by "the morning star is a planet" in direct speech (and denoted by that phrase in indirect speech). Let us use the sign "q" to stand for the *Gedanke* expressed by the German sentence, "der Morgenstern ist ein Planet." If we assume that the German sentence and the English sentence express the same *Sinn*, we can express that p and q are the same *Gedanke* in the Begriffsschrift using the identity sign:

$$(4B) \vdash p = q$$

The following conclusion then results from (2B) and (4B) and Leibniz's law (axiom FC6 from Chapter 2):

$$(5B) \vdash \mathcal{B}(g, q)$$

I.e., Gottlob believes the *Gedanke* expressed by "der Morgenstern ist ein Planet," or more simply, if allow ourselves to mix languages, that:

$$(5E/G) \text{ Gottlob believes that der Morgenstern ist ein Planet.}$$

This is simply (2E) with expressions for the same *Gedanke* substituted in *oratio obliqua*.

For reasons that should be clear from the previous chapter, we do not incorporate a complicated theory of types into the present system, although of course we do retain Frege's distinction of *levels* of functions. We do, however, need to add some new styles of variables to stand for the entities of the realm of *Sinn*.

NAMES AND VARIABLES FOR SINNE

In Chapter 3, it was concluded that the *Sinne* of function expressions should be understood neither as functions nor as objects, but as a particular kind of incomplete entity in the realm of *Sinn*. This suggests that we ought to add a new kind of symbol to the language of the Begriffsschrift to stand for such incomplete *Sinne*. In our informal discussion here, I use Hebrew characters to stand for constants of this type. Let us, for example, use "᠔" to stand for the *Sinn* of the English predicate "ξ is a planet" or the Begriffsschrift function sign " $\mathcal{P}(\xi)$ ". Since this *Sinn* is incomplete, it is appropriate to use parentheses and a sign indicating its unsaturatedness, just as for function signs we use parentheses and the sign "ξ". For incomplete *Sinne*, let us use the sign "δ" instead of "ξ". (Later, we shall also use "χ" for this purpose for doubly unsaturated incomplete *Sinn* signs.) Thus, we write it as "᠔(δ)". If a sign for a complete *Sinn* replaces

the “ δ ”, what results is a sign for a complete *Gedanke*. Therefore, if we imagine “ s ” to stand for the *Sinn* of the English phrase “the morning star” or our sign “ m ”, then we can imagine “ $\mathfrak{Q}(s)$ ” as standing for the *Gedanke* expressed by “the morning star is a planet.” Indeed, we might write something such as:

$$(7B) \vdash \mathfrak{Q}(s) = \mathfrak{p}$$

And in transcribing sentences such as (2E) into the Begriffsschrift, we might prefer to use the complex sign “ $\mathfrak{Q}(s)$ ” instead of “ \mathfrak{p} ”, as it might be thought to better reveal the logical structure of the *Gedanke* believed.

In the system described in this chapter, I employ uppercase Roman and Gothic letters between *F* and *L* inclusive (with possible added apostrophes) as variables ranging over incomplete *Sinne*. The inclusion of this new category of Roman and Gothic letters, however, complicates the syntax of the logical language. This is made manifest by considering that such signs will, like the constant “ $\mathfrak{Q}(\delta)$ ”, be written with one or more signs indicating their unsaturatedness, e.g., “ $F(\delta)$ ”. In any well-formed expression of which they form a part, the “ δ ” must be replaced with the name of a saturating *Sinn*. However, it will quickly be noted that in stating the formation rules of the syntax of such expressions, we must find a way to restrict what sort of expressions can replace the “ δ ” in Roman incomplete *Sinn* letters such as “ $F(\delta)$ ” and incomplete *Sinn* constants such as “ $\mathfrak{Q}(\delta)$ ”. While “ $F(s)$ ” and “ $\mathfrak{Q}(s)$ ” are presumably well-formed, expressions such as “ $F(2)$ ” or “ $\mathfrak{Q}(2)$ ” seem to be simply nonsensical. Incomplete *Sinne*, by their very nature, seem capable of being saturated only by *Sinne*. Because “ s ” stands for a *Sinn*, it is natural to regard “ $\mathfrak{Q}(s)$ ” as standing for the composite *Sinn* that results when the incomplete *Sinn* denoted by “ $\mathfrak{Q}(\delta)$ ” is completed by the *Sinn* denoted by “ s ”. However, the expression “ $\mathfrak{Q}(2)$ ” is problematic. It would seem to stand for what results when the incomplete *Sinn* denoted by “ $\mathfrak{Q}(\delta)$ ” is saturated by the number two. However, the number two is not a *Sinn*; it seems intrinsically incapable of saturating “ $\mathfrak{Q}(\delta)$ ”. Frege himself is quite clear that only *Sinne* can be components of *Gedanken*; numbers, people, mountains and other non-*Sinn* objects cannot (*PW* 187, *PMC* 163). But here there is a problem. Both “2” and “ s ” are object constants; it would seem that they fall in the same syntactic category.

In order to solve this problem, it will be necessary to complicate the syntax of the Begriffsschrift further to allow a syntactic distinction between two types of object names, those that denote *Sinne*, and those that may denote other objects. For the most part, this is an undesirable move to have to make. Complete *Sinne* (including *Gedanken*) are objects on Frege’s view, and were it not for this problem, there would seem little reason to introduce a distinct style of variable for them. However, in the calculus proposed herein, Roman and

Gothic letters that include an asterisk (e.g., a^* , b^*) are to be understood as ranging only over complete *Sinne*. They may be instantiated only to names of *Sinne*. However, since complete *Sinne* are themselves objects, they also fall in the range of quantification involving Roman or Gothic letters without such asterisks, although, in this case, along with other objects that are not *Sinne*. However, we shall form the syntactic rules of the system such that symbols for incomplete *Sinne* can be completed only by Roman and Gothic letters including asterisks, or by constants stipulated to stand for *Sinne*. This should be clearer once the formal syntactic definitions of the expanded system are spelled out in the upcoming sections.

There is, however, one additional odd result of the introduction of signs for incomplete *Sinne* worth discussing. According to the definition of a first-level function name given in Chapter 2, any object name from which a component object name is removed is a function name. Oddly, then, in addition to being a sign for an incomplete *Sinn* “ $\mathfrak{D}(\)$ ” would *also* be considered a function name. More precisely, “ $\mathfrak{D}(\delta)$ ” is an incomplete *Sinn* name and “ $\mathfrak{D}(\xi)$ ” is a function name. On one way of looking at it, “ $\mathfrak{D}(\)$ ” is “ $\mathfrak{D}(\mathfrak{s})$ ” (the name of a *Gedanke*) from which “ \mathfrak{s} ” (the name of a *Sinn*) has been removed. Of course, the very point of introducing signs such as “ $\mathfrak{D}(\delta)$ ” as opposed to simply using signs for sense-functions is that, on the present reading, incomplete *Sinne* are not functions. It seems odd that signs for them fall under the definition of a function name.

But here caution is called for. The sign “2” is of course the name of an object. However, consider axiom $FC^{+SB}4$:

$$\vdash (\forall f)M_{\beta}(f(\beta)) \supset M_{\beta}(f(\beta))$$

By the rule of Roman instantiation, from this, one can infer:

$$\vdash (\forall f)f(2) \supset f(2)$$

Here it appears that we have replaced the second-level Roman function letter $M_{\beta}(\dots\beta\dots)$ with the constant “2”. This makes it appear as if “2” is both the name of an object and the name of a second-level function. This is misleading. It is true that there is a second-level function, which is more perspicuously written as “ $\phi(2)$ ”, that yields as value, for any first-level function taken as argument, the value of the first-level function for the number two taken as argument. But this does not show that the number two is both an object and a second-level function, only that the existence of the object also guarantees the existence of such a second-level function. The case of “ $\mathfrak{D}(\delta)$ ” and “ $\mathfrak{D}(\xi)$ ” is similar. “ $\mathfrak{D}(\delta)$ ” is first and foremost the name of an incomplete *Sinn*, but there is also a sense-function

that yields as value, for any *Sinn* taken as argument, the complex *Sinn* that results when the argument *Sinn* completes the incomplete *Sinn*. Here we can define a *sense-function* as a function for which there is an incomplete *Sinn* such that the function has as value, for any *Sinn* as argument, the complex *Sinn* that results from the completion of the incomplete *Sinn* by the argument *Sinn*. A sense-function corresponding to the incomplete *Sinn* $\varrho(\delta)$ can be written simply as " $\varrho(\xi)$ ". Because all incomplete *Sinn* names are also, considered in a different light, function names, it turns out that function variables can in a sense be instantiated to incomplete *Sinn* names by the rule of Roman instantiation.

This is not as problematic as it may seem. To see this, suppose for a moment that instead of including function "comprehension" in his logical calculus through the Roman instantiation or "substitution" rule, Frege had included an explicit comprehension principle in a form such as:

(CP) $\vdash (\exists f)(\forall a)(f(a) = A(a))$ where $A(a)$ is any Begriffsschrift expression containing "a" but not containing "f", and such that the whole is a wfe.

We might also imagine a corollary of this principle, limited only to *Sinne*, that would read:

(CP*) $\vdash (\exists f)(\forall a^*)(f(a^*) = A(a^*))$ where $A(a^*)$ is any Begriffsschrift expression containing " a^* " but not containing "f", and such that the whole is a wfe.

Consider then the following instance of (CP*):

(8B) $\vdash (\exists f)(\forall a^*)(f(a^*) = \varrho(a^*))$

This states that there is a sense-function whose value for any *Sinn* taken as argument is the complex *Sinn* that results when the argument *Sinn* completes the incomplete *Sinn* " $\varrho(\delta)$ ". Let us call the sense-function posited by (8B) " $\mathcal{P}^*(\xi)$ ".

We then have:

(9B) $\vdash (\forall a^*)(\mathcal{P}^*(a^*) = \varrho(a^*))$

This says that for every *Sinn* as argument, the sense-function $\mathcal{P}^*(\xi)$ has as value the *Gedanke* that results when $\varrho(\delta)$ is completed by that argument. Due to Leibniz's law, this means that in any Begriffsschrift proposition, wherever " $\varrho(\delta)$ " or " $\mathcal{P}^*(\xi)$ " appears, the other can be replaced *salva veritate*. If comprehension were included in the system by means of an explicit comprehension principle, it would not be true that one could directly instantiate

function variables to incomplete *Sinn* names. However, in a case such as this, while one not could directly instantiate a function variable to “ $\mathfrak{D}(\delta)$ ”, because “ $\mathcal{P}^*(\xi)$ ” is a function name, one could instantiate the function variable to “ $\mathcal{P}^*(\xi)$ ”, and then, by Leibniz’s law, replace this occurrence with “ $\mathfrak{D}(\delta)$ ”. Frege’s system, wherein comprehension is effected not through an explicit comprehension principle, but by allowing “Roman instantiation” to any incomplete expression, cuts out the intermediate step. But given what has been said in this paragraph, it is not so odd that one can, in effect, instantiate function variables to incomplete *Sinn* names. This is not because incomplete *Sinne* are functions, but rather because they determine sense-functions to exist that are such that names for the incomplete *Sinn* and names for the sense-functions are interchangeable. However, instantiation to incomplete *Sinn* names will have to be limited in such a way that it is impossible to instantiate a well-formed expression involving a Roman function letter such as “ $f(2)$ ” to an ill-formed expression such as “ $\mathfrak{D}(2)$ ”.

Unfortunately, because capital letters are already used for second-level Roman function letters, we cannot use Roman uppercase letters after *M simpliciter* to stand for second-level incomplete *Sinne* (incomplete *Sinne* whose *Bedeutungen* are second-level functions). Thus, in what follows, I employ capital Roman letters after *M* with asterisks for expressing generalities with regard to such incomplete *Sinne*. Just as we limited ourselves in Chapter 2 to Roman and Gothic letters only for one-place and two-place first-level functions, and to Roman letters only for one-place second-level functions that take either one-place or two-place first-level functions as argument, we limit ourselves here only to one-place and two-place first-level incomplete *Sinne*, and only to second-level incomplete *Sinne* that mutually saturate with one-place and two-place first-level incomplete *Sinne*. Again, it should be easy to discern how the language could be expanded to include other varieties of incomplete *Sinne*.

THE NEW CONSTANTS OF THE EXPANDED LANGUAGE

At the core of the expansion of the Begriffsschrift to include the commitments of the theory of *Sinn* and *Bedeutung* is the inclusion of a number of semantic constants into the language. The most important is the function sign “ Δ ”, which has already been discussed in some detail. However, let us formally state the semantics of this sign here.

“ $\Delta(\xi, \zeta)$ ” denotes the True if what replaces the “ ξ ” denotes a *Sinn* that picks out as *Bedeutung* the object denoted by what replaces the “ ζ ”; and denotes the False, otherwise.

Because this function yields a truth-value for any pair of arguments, it is, in Frege's terminology, a "relation." In particular, it is the relation holding between a *Sinn* and the *Bedeutung* it picks out or presents.

Next, we need to define the semantics of expressions involving quantifiers ranging over complete or incomplete *Sinne*. The quantifier for complete *Sinne* is a second-level function sign; the quantifier for incomplete *Sinne* is a third-level function sign. The semantic characterization for the former reads as follows:

" $(\forall \alpha^*)\phi(\alpha^*)$ " denotes the True, if what replaces the " ϕ " is the name is a function that yields the True for all complete *Sinne* taken as argument; and yields the False, otherwise.

Compare this to the explanation of the standard quantifier given in Chapter 2. Also, we have the following characterization:

" $(\forall \mathfrak{F})\mu_\beta(\mathfrak{F}(\beta))$ " denotes the True, if the second-level function name that replaces the " $\mu_\beta(\dots\beta\dots)$ " denotes a second-level function that yields the True as value for all first-level sense-functions with one-argument place (determined to exist by any incomplete *Sinn* with one space of incompleteness); and denotes the False, otherwise.

We also introduce a similar quantifier ranging over two-place incomplete *Sinne*. Also, given the interpretation of expressions involving Roman letters given in Chapter 2, an expression of the form " $\Phi(\alpha^*)$ " that appears in the context " $\vdash \Phi(\alpha^*)$ " denotes the same truth-value as the corresponding expression of the form " $(\forall \alpha^*)\Phi(\alpha^*)$ ", and an expression of the form " $M_\beta(F(\beta))$ " that appears in the context " $\vdash M_\beta(F(\beta))$ " denotes the same truth-value as the corresponding expression of the form " $(\forall \mathfrak{F})M_\beta(\mathfrak{F}(\beta))$ ".

Additionally, we shall add a number of constants for particular *Sinne*. Of course, in a sense, all *Sinne* are logical objects; their existence is logically necessary. Thus, there is no *Sinn* for which it would be inappropriate to introduce a constant in a logical language in the way it would be inappropriate to introduce a constant for a contingently existing entity. It would be impossible, of course, to introduce a constant for every possible *Sinn*. However, it seems prudent at least to introduce constants standing for the *Sinne* of the logical constants such as " \supset ", " \sim ", " $=$ ", the quantifier, and so on. These constants would be used to transcribe the logical operators of ordinary language when they appear in *oratio obliqua*. Because all such logical constants stand for functions

(of some level), the constants we introduce here stand for their incomplete *Sinne*. In order to fulfill Frege's suggestion that signs used for indirect speech be such that "their connection with the corresponding signs in direct speech should be easy to recognize" (PMC 153), we use the sign "... δ " for the *Sinn* of "—", " \rightarrow " for the *Sinn* of " \supset ", " \neg " for the *Sinn* of " \sim ", " \approx " for the *Sinn* of "=", and " \uparrow " for the *Sinn* of " \setminus ". We then have the following semantic characterizations:

- "... δ " denotes the *Gedanke* that results when the incomplete *Sinn* of "—" is completed by the *Sinn* denoted by what replaces the " δ ".
- " $\neg\delta$ " denotes the *Gedanke* that results when the incomplete *Sinn* of " \sim " is completed by the *Sinn* denoted by what replaces the " δ ".
- " $\delta \rightarrow \chi$ " denotes that *Gedanke* that results when the doubly incomplete *Sinn* of " \supset " is completed in its first (antecedent) incomplete spot by the *Sinn* denoted by what replaces the " δ ", and in its second (consequent) incomplete spot by the *Sinn* denoted by what replaces the " χ ".
- " $\delta \approx \chi$ " denotes that *Gedanke* that results when the doubly incomplete *Sinn* of "=" is completed in its first incomplete spot by the *Sinn* denoted by what replaces the " δ ", and in its second incomplete spot by the *Sinn* denoted by what replaces the " χ ".
- " $\uparrow\delta$ " denotes the complex *Sinn* that results when the incomplete *Sinn* of " \setminus " is completed by the *Sinn* denoted by what replaces the " δ ".

Perhaps additional explanation of these symbols is necessary. Because these are signs for incomplete *Sinne*, expressions involving them are only well-formed if the expressions completing them name *Sinne*. Because "2" is not the same of a *Sinn*, " ~ 2 " is ill-formed, despite that, in Frege's function calculus, " ~ 2 " is a well-formed expression naming the True (since two is not the True). However, because our sign " $\mathfrak{D}(\mathfrak{s})$ " is the name of a *Sinn*, (and in particular, a *Gedanke*), " $\neg\mathfrak{D}(\mathfrak{s})$ " is a well-formed expression and denotes the *Gedanke* that results when the incomplete *Sinn* of the negation sign is completed by the *Gedanke* denoted by " $\mathfrak{D}(\mathfrak{s})$ ". The resulting *Gedanke* denoted by " $\neg\mathfrak{D}(\mathfrak{s})$ " could be understood as the *Gedanke* that the morning star is not a planet. The expression " $\neg\mathfrak{D}(\mathfrak{s})$ " denotes the *Gedanke* that " $\sim\mathcal{P}(\mathfrak{m})$ " would express. Here " \neg " denotes the incomplete *Sinn* of " \sim ", " \mathfrak{D} " denotes the incomplete *Sinn* of " \mathcal{P} ", and " \mathfrak{s} " denotes the (complete)

Sinn of “m”. While an expression such as “ $\mathcal{P}(m) \supset \mathcal{P}(m)$ ” must be understood as expressing a *Gedanke* and denoting a truth-value—in this case, a tautological *Gedanke* and the True, respectively—an expression such as “ $\mathcal{Q}(s) \rightarrow \mathcal{Q}(s)$ ” does not denote a truth-value. It denotes a *Gedanke* that picks out the True as *Bedeutung*; in fact, it denotes the *Gedanke* expressed by “ $\mathcal{P}(m) \supset \mathcal{P}(m)$ ”. Therefore, while the following:

$$\vdash \mathcal{P}(m) \supset \mathcal{P}(m)$$

would be considered a true Begriffsschrift proposition, the following:

$$\vdash \mathcal{Q}(s) \rightarrow \mathcal{Q}(s)$$

would not be, because it must be understood as asserting that a *Gedanke* is the True (which of course is different than asserting that a *Gedanke* picks out the True), much like “ $\vdash 2$ ” would be taken as asserting that the number two is the True. However, this expression can nevertheless appear within a true Begriffsschrift proposition, such as:

$$\vdash \Delta(\mathcal{Q}(s) \rightarrow \mathcal{Q}(s), \mathcal{P}(m) \supset \mathcal{P}(m))$$

This asserts that the *Gedanke* named by “ $\mathcal{Q}(s) \rightarrow \mathcal{Q}(s)$ ” picks out as *Bedeutung* the truth-value named by “ $\mathcal{P}(m) \supset \mathcal{P}(m)$ ”.

So far we have introduced constants standing for the *Sinne* of the first-level function constants of the system outlined in Chapter 2. Let us now proceed to introduce the *Sinne* of our higher-level function constants, the quantifier and the smooth-breathing. The *Sinne* of higher-level function signs must also be understood as incomplete or unsaturated, but their incompleteness is of a different sort from the incomplete *Sinne* of first-level function signs. Just as a second-level function is only saturated by something incomplete, a first-level function, a *Sinn* picking one out will also only be saturated by something incomplete, in this case, an incomplete *Sinn* picking out a first-level function. Thus, a sign for an incomplete *Sinn* picking out a second-level function will also have an incomplete spot, and this spot would be filled with the name of an incomplete *Sinn* of the first-level. Let us use the sign “ $(\Pi a^*)\theta(a^*)$ ” as a name for the *Sinn* of the sign “ $(\forall a)\phi(a)$ ”. The sign “ θ ” is then used to indicate the unsaturatedness of this incomplete *Sinn*, and it is to be replaced by the name of a first-level incomplete *Sinn*. We then characterize its intended semantics as follows:

“(Πa^*) $\theta(a^*)$ ” denotes the *Gedanke* that results when the incomplete *Sinn* of “($\forall a$) $\phi(a)$ ” mutually saturates with the incomplete *Sinn* denoted by what replaces the “ θ ”.

Thus, “(Πa^*) $\vartheta(a^*)$ ” is the name not of a truth-value but of a *Gedanke*, just like “ $\vartheta(\$) \rightarrow \vartheta(\$)$ ”. “(Πa^*) $\vartheta(a^*)$ ” names the *Gedanke* that would be expressed by “($\forall a$) $\mathcal{P}(a)$ ”, i.e., the *Gedanke* that everything is a planet. Although this *Gedanke* is false (or, to be more precise, it picks out the False), it nevertheless exists. Care must be taken to differentiate an expression such as “(Πa^*) $\vartheta(a^*)$ ”, which names a *Gedanke*, from one such as “($\forall a^*$) $\vartheta(a^*)$ ”. While the former is a perfectly natural expression that denotes a false *Gedanke*, the latter is a rather unnatural expression that expresses a very strange (and false) *Gedanke* but actually denotes the False. Given our semantic characterizations, “($\forall a^*$) $\vartheta(a^*)$ ” denotes the truth-value of the sense-function “ $\vartheta(\xi)$ ” being such as to yield the True as value for all *Sinne* as arguments. Since sense-functions never have the True as value for *Sinne* as arguments (their values are always *Sinne* for *Sinne* as arguments), this truth-value is the False.

Similarly, we introduce a constant for the *Sinn* of the third-level function-quantifier “($\forall f$) $\mu_\beta(f(\beta))$ ”, which we write “($\Pi \mathfrak{F}$) $\mu_\beta^*(\mathfrak{F}(\beta))$ ”. Its semantic characterization is given as follows:

“($\Pi \mathfrak{F}$) $\mu_\beta^*(\mathfrak{F}(\beta))$ ” denotes the *Gedanke* that results when the incomplete *Sinn* of “($\forall f$) $\mu_\beta(f(\beta))$ ” mutually saturates with the incomplete *Sinn* denoted by what replaces the “ $\mu_\beta^*(\dots\beta\dots)$ ”.

Thus, while the term “($\forall f$) $f(m)$ ” expresses the *Gedanke* that every function maps the morning star onto the True, the expression “($\Pi \mathfrak{F}$) $\mathfrak{F}(\$)$ ” denotes this *Gedanke*. We also add a similar sign for the *Sinn* of the quantifier ranging over two-place first-level functions. Lastly, we introduce a sign “ $\overset{\circ}{\varepsilon} \theta(\varepsilon)$ ” for the *Sinn* of the smooth-breathing class-forming operator “ $\overset{\circ}{\varepsilon} \phi(\varepsilon)$ ”. Thus, we have:

“ $\overset{\circ}{\varepsilon} \theta(\varepsilon)$ ” denotes the complex *Sinn* that results when the incomplete *Sinn* of “ $\overset{\circ}{\varepsilon} \phi(\varepsilon)$ ” mutually saturates with the incomplete *Sinn* denoted by what replaces the “ θ ”.

Here, while “ $\overset{\circ}{\varepsilon} \vartheta(\varepsilon)$ ” would not denote a *Gedanke*, it similarly does not denote a value-range. Rather, it denotes a *Sinn* that picks out a value-range, namely the value-range denoted by “ $\overset{\circ}{\varepsilon} \mathcal{P}(\varepsilon)$ ” (the class of all planets).

We have now introduced constants for the *Sinne* of all the primitive signs of the Begriffsschrift as explicated in Chapter 2. However, we could go on to add

constants standing for the *Sinne* of our new constants, or perhaps, à la Church, a hierarchy of such signs. Thus, we might consider adding a hierarchy of signs “ \rightarrow^n ” such that “ \rightarrow^{m+1} ” stands for the *Sinn* of “ \rightarrow^m ”. However, I shall not attempt to do so. As it turns out, the axioms of the system will be such that they are nevertheless committed to *Sinne* picking out these functions, even without constants standing for them. Therefore, our system is not made weaker by leaving out such constants. However, I do propose adding a constant for the *Sinn* of our new semantic constant “ $\Delta(\xi, \zeta)$ ”, which I write as “ $\blacktriangle(\delta, \chi)$ ”. Its semantics are as follows:

“ $\blacktriangle(\delta, \chi)$ ” denotes the *Gedanke* that results when the doubly incomplete *Sinn* of “ $\Delta(\xi, \zeta)$ ” is completed in its first (*Sinn*) spot by the *Sinn* denoted by what replaces the “ δ ”, and in its second (*Bedeutung*) spot by the *Sinn* denoted by what replaces the “ χ ”.

This rounds out our discussion of the new constants added to Frege’s Begriffsschrift in this chapter.

SYNTACTIC RULES OF THE EXPANDED LANGUAGE

In stating the semantic rules of our expanded language, the same methods are employed as in Chapter 2. Thus, uppercase letters prior to F in the alphabet are to be understood as metalinguistic schematic variables standing for strings of one or more Begriffsschrift signs, with the convention that $\lceil A(B) \rceil$ stands for a string of Begriffsschrift sign in which string B is included as part. As in Chapter 2, we begin with a number of definitions. All definitions given in Chapter 2, with the exception of the definition of a *well-formed expression (wfe)* remain unchanged. We shall offer a revised definition below. We begin, however, by adding some new definitions.

Definition: *Roman complete Sinn letter*

If A is a *Roman object letter*, then $\lceil A^* \rceil$ is a *Roman complete Sinn letter*.

Definition: *Gothic complete Sinn letter*

If A is a *Gothic object letter*, then $\lceil A^* \rceil$ is a *Gothic complete Sinn letter*.

Definition: *one-place first-level Roman incomplete Sinn letter*

(i) Any of the uppercase Roman letters $F^1, G^1, H^1, I^1, J^1, K^1$, and L^1 (including the superscripts) is a *one-place first-level Roman incomplete Sinn letter*, and
(ii) if A is a *one-place first-level Roman incomplete Sinn letter*, then $\lceil A^1 \rceil$ is a distinct *one-place first-level Roman incomplete Sinn letter*.

Definition: *two-place first-level Roman incomplete Sinn letter*

- (i) Any of the uppercase Roman letters $F^2, G^2, H^2, I^2, J^2, K^2,$ and L^2 (including the superscripts) is a *two-place first-level Roman incomplete Sinn letter*, and
- (ii) if A is a *two-place first-level Roman incomplete Sinn letter*, then $\ulcorner A \urcorner$ is a distinct *two-place first-level Roman incomplete Sinn letter*.

Definition: *one-place first-level Gothic incomplete Sinn letter*

- (i) Any of the uppercase Gothic letters $\mathfrak{F}^1, \mathfrak{G}^1, \mathfrak{H}^1, \mathfrak{I}^1, \mathfrak{J}^1, \mathfrak{K}^1,$ and \mathfrak{L}^1 (including the superscripts) is a *one-place first-level Gothic incomplete Sinn letter*, and
- (ii) if A is a *one-place first-level Gothic incomplete Sinn letter*, then $\ulcorner A \urcorner$ is a distinct *one-place first-level Gothic incomplete Sinn letter*.

Definition: *two-place first-level Gothic incomplete Sinn letter*

- (i) Any of the uppercase Gothic letters $\mathfrak{F}^2, \mathfrak{G}^2, \mathfrak{H}^2, \mathfrak{I}^2, \mathfrak{J}^2, \mathfrak{K}^2,$ and \mathfrak{L}^2 (including the superscripts) is a *two-place first-level Gothic incomplete Sinn letter*, and
- (ii) if A is a *two-place first-level Gothic incomplete Sinn letter*, then $\ulcorner A \urcorner$ is a distinct *two-place first-level Gothic incomplete Sinn letter*.

Definition: *Second-level Roman incomplete Sinn letter with one-place argument*

If A is a *second-level Roman function letter with one-place argument*, then $\ulcorner A^* \urcorner$ is a *second-level Roman incomplete Sinn letter with one-place argument*.

Definition: *Second-level Roman incomplete Sinn letter with two-place argument*

If A is a *second-level Roman function letter with two-place argument*, then $\ulcorner A^* \urcorner$ is a *second-level Roman incomplete Sinn letter with two-place argument*.

Definition: *complete Sinn expression*

- (i) Any *Roman complete Sinn letter* is a *complete Sinn expression*;
- (ii) if A is a *complete Sinn expression*, then $\ulcorner \dots A \urcorner$ is a *complete Sinn expression*;
- (iii) if A is a *complete Sinn expression*, then $\ulcorner \neg A \urcorner$ is a *complete Sinn expression*;
- (iv) if A is a *complete Sinn expression*, then $\ulcorner \lrcorner A \urcorner$ is a *complete Sinn expression*;
- (v) if A and B are *complete Sinn expressions*, then $\ulcorner (A \rightarrow B) \urcorner$ is a *complete Sinn expression*;
- (vi) if A and B are *complete Sinn expressions*, then $\ulcorner (A \approx B) \urcorner$ is a *complete Sinn expression*;
- (vii) if A and B are *complete Sinn expressions*, then $\ulcorner \blacktriangle(A, B) \urcorner$ is a *complete Sinn expression*;
- (viii) if A is a *complete Sinn expression*, and B is a *one-place first-level Roman incomplete Sinn letter*, then $\ulcorner B(A) \urcorner$ is a *complete Sinn expression*;

- (ix) if A and B are *complete Sinn expressions*, and C is a *two-place first-level Roman incomplete Sinn letter*, then $\ulcorner C(A, B) \urcorner$ is a *complete Sinn expression*;
- (x) if $\ulcorner A(B) \urcorner$ is a *complete Sinn expression* that contains within it B, which is itself a *complete Sinn expression*, and C is a *Gothic complete Sinn letter* not contained in $\ulcorner A(B) \urcorner$, then $\ulcorner (\Pi C)A(C) \urcorner$ is a *complete Sinn expression*, with C replacing one or more occurrences of B from $\ulcorner A(B) \urcorner$;
- (xi) if $\ulcorner A(B) \urcorner$ is a *complete Sinn expression* that contains within it B, which is itself a *complete Sinn expression*, and E is a *Greek object letter* not contained in $\ulcorner A(B) \urcorner$, then $\ulcorner \overset{\circ}{E}A(E) \urcorner$ is a *complete Sinn expression*, with E replacing one or more occurrences of B from $\ulcorner A(B) \urcorner$;
- (xii) if $\ulcorner D(A(B)) \urcorner$ is a *complete Sinn expression* that contains within it $\ulcorner A(B) \urcorner$, which is itself a *complete Sinn expression* containing within it B, which is a *complete Sinn expression* itself, and C is a *one-place Gothic incomplete Sinn letter* that is not contained in $\ulcorner D(A(B)) \urcorner$, then $\ulcorner (\Pi C)D(C(B)) \urcorner$ is a *complete Sinn expression*, with C replacing one or more occurrences of A from $\ulcorner D(A(B)) \urcorner$;
- (xiii) if $\ulcorner D(A(B, E)) \urcorner$ is a *complete Sinn expression* that contains within it $\ulcorner A(B, E) \urcorner$, which is itself a *complete Sinn expression* containing within it B and E, both of which are *complete Sinn expressions* themselves, and C is a *two-place Gothic incomplete Sinn letter* that is not contained in $\ulcorner D(A(B, E)) \urcorner$, then $\ulcorner (\Pi C)D(C(B, E)) \urcorner$ is a *complete Sinn expression*, with C replacing one or more occurrences of A from $\ulcorner D(A(B, E)) \urcorner$;
- (xiv) if $\ulcorner A(B) \urcorner$ is a *complete Sinn expression* that contains within it B, itself a *complete Sinn expression*, and C is a *second-level incomplete Sinn letter with one-place argument*, then $\ulcorner C(A(\beta)) \urcorner$ is a *complete Sinn expression*, with β replacing one or more occurrences of B from $\ulcorner A(B) \urcorner$, and
- (x) if $\ulcorner A(B, C) \urcorner$ is a *complete Sinn expression* that contains within it B and C, both of which are themselves *complete Sinn expressions*, and D is a *second-level incomplete Sinn letter with two-place argument*, then $\ulcorner D(A(\beta, \gamma)) \urcorner$ is a *complete Sinn expression*, with β replacing one or more occurrences of B from $\ulcorner A(B, C) \urcorner$, and γ replacing one or more occurrences of C from $\ulcorner A(B, C) \urcorner$.¹

Definition: *one-place first-level incomplete Sinn expression*

If $\ulcorner A(B) \urcorner$ is a *complete Sinn expression* containing B, which is itself a *complete Sinn expression*, then $\ulcorner A(\delta) \urcorner$ is a *one-place first-level incomplete Sinn expression*, with δ representing the gap created by removing one or more occurrences of B from $\ulcorner A(B) \urcorner$. (Here, the occurrences of δ are meant simply to

¹Notice the isomorphism between this definition and the definition given of a *well-formed expression* in Chapter 2. The isomorphism between well-formed expressions and expressions standing for their *Sinne* is not unexpected given the structural similarity Frege believes to exist between expressions and their *Sinne*.

hold open the empty places in the *incomplete Sinn expression* where the occurrences of B have been removed.)

Definition: *two-place first-level incomplete Sinn expression*

If $\lceil A(B, C) \rceil$ is a *complete Sinn expression* containing B and C, each of which is itself a *complete Sinn expression*, then $\lceil A(\delta, \chi) \rceil$ is a *two-place first-level incomplete Sinn expression*, with δ representing the gap created by removing one or more occurrences of B from $\lceil A(B, C) \rceil$, and χ representing the gap created by removing one or more occurrences of C from $\lceil A(B, C) \rceil$. (Here, the occurrences of δ and χ are meant simply to hold open the empty places in the *incomplete Sinn expression* where the occurrences of B and C have been removed.)

Definition: *second-level incomplete Sinn expression with one-place argument*

If $\lceil A(B) \rceil$ is a *complete Sinn expression* containing B, which is itself a *one-place first-level incomplete Sinn expression*, then $\lceil A(\theta) \rceil$ is a *second-level incomplete Sinn expression with one-place argument*, with θ representing the gap created by removing one or more occurrences of B from $\lceil A(B) \rceil$. (Here, the occurrences of θ are meant simply to hold open the empty places in the *incomplete Sinn expression* where the occurrences of B were removed.)

Definition: *second-level incomplete Sinn expression with two-place argument*

If $\lceil A(B) \rceil$ is a *complete Sinn expression* containing B, which is itself a *two-place first-level incomplete Sinn expression*, then $\lceil A(\theta) \rceil$ is a *second-level incomplete Sinn expression with two-place argument*, with θ representing the gap created by removing one or more occurrences of B from $\lceil A(B) \rceil$. (Here, the occurrences of θ are meant simply to hold open the empty places in the *incomplete Sinn expression* where the occurrences of B were removed.)

With these definitions in place, we can now proceed to redefine a *well-formed expression* for the expanded language. This is done as follows:

Definition: *well-formed expression (wfe)*

- (i) Any Roman object letter is a *wfe*;
- (ii) any *complete Sinn expression* is a *wfe*;
- (iii) if A is a *wfe*, then $\lceil _ A \rceil$ is a *wfe*;
- (iv) if A is a *wfe*, then $\lceil \sim A \rceil$ is a *wfe*;
- (v) if A is a *wfe*, then $\lceil \setminus A \rceil$ is a *wfe*;
- (vi) if A and B are *wfes*, then $\lceil (A \supset B) \rceil$ is a *wfe*;
- (vii) if A and B are *wfes*, then $\lceil (A = B) \rceil$ is a *wfe*;
- (viii) if A and B are *wfes*, then $\lceil \Delta(A, B) \rceil$ is a *wfe*;

- (ix) if A is a *wfe*, and B is a *one-place first-level Roman function letter*, then $\ulcorner B(A) \urcorner$ is a *wfe*;
- (ix) if A and B are *wfes*, and C is a *two-place first-level Roman function letter*, then $\ulcorner C(A, B) \urcorner$ is a *wfe*;
- (x) if $\ulcorner A(B) \urcorner$ is a *wfe* that contains within it B, which is itself a *wfe*, and C is a *Gothic complete Sinn letter* not contained in $\ulcorner A(B) \urcorner$, then $\ulcorner (\forall C)A(C) \urcorner$ is a *wfe*, with C replacing one or more occurrences of B from $\ulcorner A(B) \urcorner$;
- (xi) if $\ulcorner A(B) \urcorner$ is a *wfe* that contains within it B, which is itself a *wfe* but *not* a *complete Sinn expression*, and C is a *Gothic object letter* not contained in $\ulcorner A(B) \urcorner$, then $\ulcorner (\forall C)A(C) \urcorner$ is a *wfe*, with C replacing one or more occurrences of B from $\ulcorner A(B) \urcorner$;
- (xii) if $\ulcorner A(B) \urcorner$ is a *wfe* that contains within it B, which is itself a *wfe* but *not* a *complete Sinn expression*, and E is a *Greek object letter* not contained in $\ulcorner A(B) \urcorner$, then $\ulcorner EA(E) \urcorner$ is a *wfe*, with E replacing one or more occurrences of B from $\ulcorner A(B) \urcorner$;
- (xiii) if $\ulcorner D(A(B)) \urcorner$ is a *wfe* that contains within it $\ulcorner A(B) \urcorner$, which is itself a *complete Sinn expression* containing within it B, which is a *complete Sinn expression* itself, and C is a *one-place Gothic incomplete Sinn letter* that is not contained in $\ulcorner D(A(B)) \urcorner$, then $\ulcorner (\forall C)D(C(B)) \urcorner$ is a *wfe*, with C replacing one or more occurrences of A from $\ulcorner D(A(B)) \urcorner$;
- (xiii) if $\ulcorner D(A(B, E)) \urcorner$ is a *wfe* that contains within it $\ulcorner A(B, E) \urcorner$, which is itself a *complete Sinn expression* containing within it B and E, both of which are *complete Sinn expressions* themselves, and C is a *two-place Gothic incomplete Sinn letter* that is not contained in $\ulcorner D(A(B, E)) \urcorner$, then $\ulcorner (\forall C)D(C(B, E)) \urcorner$ is a *wfe*, with C replacing one or more occurrences of A from $\ulcorner D(A(B, E)) \urcorner$;
- (xiv) if $\ulcorner D(A(B)) \urcorner$ is a *wfe* that contains within it $\ulcorner A(B) \urcorner$, which is itself a *wfe* but *not* a *complete Sinn expression* containing within it B, which is itself a *wfe*, and C is a *one-place Gothic function letter* not contained in $\ulcorner D(A(B)) \urcorner$, then $\ulcorner (\forall C)D(C(B)) \urcorner$ is a *wfe*, with C replacing one or more occurrences of A from $\ulcorner D(A(B)) \urcorner$;
- (xiv) if $\ulcorner D(A(B, E)) \urcorner$ is a *wfe* that contains within it $\ulcorner A(B, E) \urcorner$, which is itself a *wfe* but *not* a *complete Sinn expression* containing within it B and E, both of which are themselves *wfes*, and C is a *two-place Gothic function letter* not contained in $\ulcorner D(A(B, E)) \urcorner$, then $\ulcorner (\forall C)D(C(B, E)) \urcorner$ is a *wfe*, with C replacing one or more occurrences of A from $\ulcorner D(A(B, E)) \urcorner$;
- (xv) if $\ulcorner A(B) \urcorner$ is a *wfe* that contains within it B, itself a *wfe*, and C is a *second-level Roman function letter with one-place argument*, then $\ulcorner C(A(\beta)) \urcorner$ is a *wfe*, with β replacing one or more occurrences of B from $\ulcorner A(B) \urcorner$, and
- (xvi) if $\ulcorner A(B, C) \urcorner$ is a *wfe* that contains within it B and C, themselves *wfes*, and D is a *second-level Roman function letter with two-place argument*, then

$\lceil D(A(\beta, \gamma)) \rceil$ is a *wfe*, with β replacing one or more occurrences of B from $\lceil A(B, C) \rceil$, and γ replacing one or more occurrences of C from $\lceil A(B, C) \rceil$.

It is worth noting some of the results of these definitions. While both “ $a \supset a$ ” and “ $a^* \rightarrow a^*$ ” are *wfes* on the definition above, “ $a \rightarrow a$ ” is not. Only *Sinne* expressions can flank an incomplete *Sinn* sign such as “ \rightarrow ”, and “ a ” is not a *Sinn* expression. While all *Sinne* expressions are *wfes*, not all *wfes* are *Sinn* expressions. Given the rules of *Roman instantiation (ri)* to be listed later, while a Roman object letter such as “ a ” can be instantiated to any *wfe*, and thus to any complete *Sinn* expression, a Roman complete *Sinn* letter such as “ a^* ” can only be instantiated to *Sinn* expressions, and thus not to “ a ” or a longer expression such as “ $a \supset a$ ”. Similarly, given the way function names were defined in Chapter 2, while both “ \sim ” and “ $-$ ” are defined as function names, only the latter is defined as an incomplete *Sinn* expression, and thus while a Roman function letter such as “ $f(\xi)$ ” could be instantiated to either, a Roman incomplete *Sinn* letter such as “ $F(\delta)$ ” could not be instantiated to “ \sim ”.

The same syntactic conveniences with regard to combining quantifiers, leaving off superscripts, replacing brackets with dots, etc., mentioned in Chapter 2, continue to be used here. Superscripts are also left off of Roman and Gothic incomplete *Sinn* letters, as it will be apparent from the number of argument places written after them what the superscript must be.

AXIOMS AND INFERENCE RULES OF THE EXPANDED CALCULI

Now that we have laid out the syntax of the expanded language, we can proceed to list its axioms and inference rules. Here we consider expansions of both the system FC and the system FC^+V from Chapter 2. We shall call the system consisting of the axioms of system FC along with the new axioms listed below and the new inference rules, system FC^{+SB} (for *function plus Sinn and Bedeutung calculus*), and we shall call the similar expansion containing all of the axioms of FC^+V and the new axioms and inference rules, FC^{+SB+V} (for *function plus Sinn and Bedeutung plus value-range calculus*). Thus, axioms FC1 through FC6 form the core of all of the systems here under consideration. We also maintain all of the inference rules of system FC, although a number of them will have to be expanded or revised to accommodate the new syntactic divisions and quantifiers. The revised rules are listed below; those not listed below remain unchanged from Chapter 2.

Horizontal amalgamation rules (hor):

In addition to those pairs listed in Chapter 2, the following pair of expressions are also interchangeable. Here, A and B may be any *wfes* of the language.

$$\lceil \neg \Delta(A, B) \rceil :: \lceil \Delta(A, B) \rceil$$

Change in generality notation (gen):

In addition to cases (a)-(d) from Chapter 2, this rule contains also the following cases.

e) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a Roman complete *Sinn* letter, and C is a Gothic complete *Sinn* letter not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.

f) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a Roman complete *Sinn* letter, C is a Gothic complete *Sinn* letter not contained in $\lceil A(B) \rceil$, and D is a wfe not containing B, then from $\lceil \vdash D \supset A(B) \rceil$, infer $\lceil \vdash D \supset (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.

g) Where $\lceil A(B) \rceil$ is a wfe containing B, a first-level Roman incomplete *Sinn* letter, and C is a Gothic incomplete *Sinn* letter with the same number of argument places as B that is not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.

h) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a first-level Roman incomplete *Sinn* letter, C is a Gothic incomplete *Sinn* letter with the same number of argument places as B that is not contained in $\lceil A(B) \rceil$, and D is a wfe not containing B, then from $\lceil \vdash D \supset A(B) \rceil$, infer $\lceil \vdash D \supset (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.

Roman instantiation (ri):

Parts (a)-(e) of this rule are changed as follows, and parts (f)-(j) are newly added.

a) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of Roman object letter B, and C is any wfe containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, where C replaces every occurrence of B in $\lceil A(B) \rceil$, *provided that* $\lceil \vdash A(C) \rceil$ *is a wfe*.

b) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of one-place first-level Roman function letter B, and C is any one-place first-level function name containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty places ξ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$, *provided that* $\lceil \vdash A(C) \rceil$ *is a wfe*.

c) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of two-place first-level Roman function letter B, and C is any two-place first-level function name containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer

$\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty places ξ and ζ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$, provided that $\lceil \vdash A(C) \rceil$ is a wfe.

d) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of B, where B is a second-level Roman function letter with one-place argument, and C is any second-level function name with one-place argument containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty place ϕ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$, and whatever sign used for mutual saturation in C replacing all occurrences of β in $\lceil A(B) \rceil$, provided that $\lceil \vdash A(C) \rceil$ is a wfe.

e) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of B, where B is a second-level Roman function letter with two-place argument, and C is any second-level function name with two-place argument containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty place ϕ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$, and whatever signs used for mutual saturation in C replacing all occurrences of β and γ in $\lceil A(B) \rceil$, provided that $\lceil \vdash A(C) \rceil$ is a wfe.

f) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of Roman complete Sinn letter B, and C is any complete Sinn expression containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, where C replaces every occurrence of B in $\lceil A(B) \rceil$.

g) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of one-place first-level incomplete Sinn letter B, and C is any one-place first-level incomplete Sinn expression containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty places δ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$.

h) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of two-place first-level incomplete Sinn letter B, and C is any two-place first-level incomplete Sinn expression containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty places δ and χ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$.

i) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of B, where B is a second-level incomplete Sinn letter with one-place argument, and C is any second-level incomplete Sinn expression with one-place argument containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty place θ of C being

filled with the corresponding arguments to B in $\ulcorner A(B) \urcorner$, and whatever sign used for mutual saturation in C replacing all occurrences of β in $\ulcorner A(B) \urcorner$.

j) Where $\ulcorner A(B) \urcorner$ is a wfe containing one or more occurrences of B, where B is a second-level incomplete *Sinn* letter with two-place argument, and C is any second-level incomplete *Sinn* expression with two-place argument containing no Gothic or Greek letters contained in $\ulcorner A(B) \urcorner$, from $\ulcorner \vdash A(B) \urcorner$, infer $\ulcorner \vdash A(C) \urcorner$, with C replacing every occurrence of B from $\ulcorner A(B) \urcorner$, the empty place θ of C being filled with the corresponding arguments to B in $\ulcorner A(B) \urcorner$, and whatever signs used for mutual saturation in C replacing all occurrences of β and γ in $\ulcorner A(B) \urcorner$.

Change of Gothic or Greek letter (cg):

In addition to parts (a)-(c) from Chapter 2, this rule also contains the following instances:

d) Where $\ulcorner D(A(B)) \urcorner$ is a wfe containing $\ulcorner A(B) \urcorner$, itself a wfe containing Gothic complete *Sinn* letter B, and C is a different Gothic complete *Sinn* letter not contained in $\ulcorner A(B) \urcorner$, then from $\ulcorner \vdash D(A(B)) \urcorner$, infer $\ulcorner \vdash D(A(C)) \urcorner$, with C replacing all occurrences of B in $\ulcorner A(B) \urcorner$.

e) Where $\ulcorner D(A(B)) \urcorner$ is a wfe containing $\ulcorner A(B) \urcorner$, itself a wfe containing Gothic incomplete *Sinn* letter B, and C is a different Gothic incomplete *Sinn* letter with the same number of argument places as B that is not contained in $\ulcorner A(B) \urcorner$, then from $\ulcorner \vdash D(A(B)) \urcorner$, infer $\ulcorner \vdash D(A(C)) \urcorner$, with C replacing all occurrences of B in $\ulcorner A(B) \urcorner$.

The additions to (*hor*), (*gen*) and (*cg*) should be easily understood. The changes to parts (a)-(c) of (*ri*) perhaps require further comment. The addition of the clauses “*provided that* $\ulcorner A(C) \urcorner$ *is a wfe*” are meant to rule out an inference from, for example, axiom FC2:

$$\vdash (\forall a)f(a) \supset f(b)$$

to an ill-formed expression such as:

$$\vdash (\forall a)F(a) \supset F(b)$$

Although “ $F()$ ” can, for reasons given earlier, be considered both a function name and an incomplete *Sinn* expression, the expressions “ $F(b)$ ” and “ $(\forall a)F(a)$ ” are not well-formed, because only a sign for a *Sinn* can complete an incomplete *Sinn* expression. The new parts (d) and (e) allow the instantiation from Roman *Sinn* letters (both complete and incomplete) to any *Sinne* expressions, simple or

complex, although do not allow instantiation to other expressions that might not stand for *Sinne*.

Before turning to the new axioms, let us add some defined signs.

$$\begin{aligned} \text{(Df. } \exists d) \Vdash (\exists a^*)f(a^*) &= \sim(\forall a^*)\sim f(a^*) \\ \text{(Df. } \exists e) \Vdash (\exists \mathfrak{F})M_\beta(\mathfrak{F}(\beta)) &= \sim(\forall \mathfrak{F})\sim M_\beta(\mathfrak{F}(\beta)) \\ \text{(Df. } \exists f) \Vdash (\exists \mathfrak{F})M_{\beta\gamma}(\mathfrak{F}(\beta, \gamma)) &= \sim(\forall \mathfrak{F})\sim M_{\beta\gamma}(\mathfrak{F}(\beta, \gamma)) \\ \text{(Df. } \mathcal{S}) \Vdash \mathcal{S}(x) &= (\exists a^*)(x = a^*) \end{aligned}$$

The definitions of the existential versions of the new quantifiers should be straightforward. The last sign, “ $\mathcal{S}(\xi)$ ”, is the sign for a function that yields as value for any object as argument the True just in case that object is a complete *Sinn*. “ $\mathcal{S}(\xi)$ ” can be understood as standing for the concept of being a complete *Sinn*.

The core axioms of the expanded systems are listed below. These are the axioms that deal with the denotation function, Δ , the new quantifiers and the core features of *Sinne* and *Gedanken*. I shall first list the axioms, then discuss them individually.

$$\begin{aligned} \text{Axiom FC}^{+SB}10. \Vdash (\forall a^*)f(a^*) \supset f(a^*) \\ \text{Axiom FC}^{+SB}11. \Vdash (\forall \mathfrak{F})M_\beta(\mathfrak{F}(\beta)) \supset M_\beta(F(\beta)) \\ \text{Axiom FC}^{+SB}12. \Vdash (\forall \mathfrak{F})M_{\beta\gamma}(\mathfrak{F}(\beta, \gamma)) \supset M_{\beta\gamma}(F(\beta, \gamma)) \\ \text{Axiom FC}^{+SB}13. \Vdash \Delta(a, b) \ \& \ \Delta(a, c) \ \supset \ (b = c) \\ \text{Axiom FC}^{+SB}14. \Vdash \Delta(a, b) \supset \mathcal{S}(a) \\ \text{Axiom FC}^{+SB}15. \Vdash \sim \mathcal{S}(\varepsilon f(\varepsilon)) \\ \text{Axiom FC}^{+SB}16. \Vdash \sim \mathcal{S}(\neg a) \\ \text{Axiom FC}^{+SB}17. \Vdash F(a^*) = F(b^*) \ \supset \ a^* = b^* \\ \text{Axiom FC}^{+SB}18. \Vdash F(a^*, b^*) = F(c^*, d^*) \ \supset \ (a^* = c^*) \ \& \ (b^* = d^*) \\ \text{Axiom FC}^{+SB}19. \Vdash [M_\beta^*(F(\beta)) = M_\beta^*(G(\beta))] \ \& \\ \quad (\forall a^*\mathfrak{F}\mathfrak{G})[M_\beta^*(\mathfrak{F}(\beta)) \neq \mathfrak{G}(\mathfrak{F}(a^*))] \ \supset \ (\forall a^*)(F(a^*) = G(a^*)) \\ \text{Axiom FC}^{+SB}20. \Vdash [M_{\beta\gamma}^*(F(\beta, \gamma)) = M_{\beta\gamma}^*(G(\beta, \gamma))] \ \& \\ \quad (\forall a^*b^*\mathfrak{F}\mathfrak{G})[M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)) \neq \mathfrak{G}(\mathfrak{F}(a^*, b^*))] \\ \quad \supset \ (\forall a^*b^*)(F(a^*, b^*) = G(a^*, b^*)) \\ \text{Axiom FC}^{+SB}21. \Vdash [F(a^*) = G(a^*) \ \& \ F(b^*) = G(b^*) \ \& \ a^* \neq b^*] \supset \\ \quad (\forall a^*)(F(a^*) = G(a^*)) \\ \text{Axiom FC}^{+SB}22. \Vdash [F(a^*, b^*) = G(a^*, b^*) \ \& \ F(c^*, d^*) = G(c^*, d^*) \ \& \\ \quad (a^* \neq b^*) \ \& \ (c^* \neq d^*) \ \& \ (a^* \neq c^*) \ \vee \ (b^* \neq d^*)] \\ \quad \supset \ (\forall a^*b^*)(F(a^*, b^*) = G(a^*, b^*)) \\ \text{Axiom FC}^{+SB}23. \Vdash F(a^*) = G(a^*) \ \& \ (\forall \mathfrak{H})(\mathfrak{H}(a^*) \neq F(b^*)) \ \& \\ \quad (\forall \mathfrak{H})(\mathfrak{H}(a^*) \neq G(c^*)) \ \supset \ (\forall a^*)(F(a^*) = G(a^*)) \\ \text{Axiom FC}^{+SB}24. \Vdash F(a^*, b^*) = G(a^*, b^*) \ \& \ (a^* \neq b^*) \ \& \end{aligned}$$

Axioms $FC^{+SB}19$ and $FC^{+SB}20$ state something similar for second-level incomplete *Sinn*. Axiom $FC^{+SB}19$ asserts that if the complex $Sinn M_{\beta}^*(F(\beta))$, composed of the second-level incomplete $Sinn M_{\beta}^*(\theta(\beta))$ and the first-level incomplete $Sinn F(\delta)$, is the same as complex $Sinn M_{\beta}^*(G(\beta))$, then $F(\delta)$ and $G(\delta)$ must be the same incomplete *Sinn*. The clause “ $(\forall \alpha^* \mathfrak{F} \mathfrak{G}) [M_{\beta}^*(\mathfrak{F}(\beta)) \neq \mathfrak{G}(\mathfrak{F}(\alpha^*))]$ ” in the antecedent serves to rule out this inference in cases in which the second-level incomplete $Sinn M_{\beta}^*(\theta(\beta))$ takes a simple form such as $\theta(b^*)$. If the axiom read simply:

$$\vdash M_{\beta}^*(F(\beta)) = M_{\beta}^*(G(\beta)) \therefore (\forall \alpha^*)(F(\alpha^*) = G(\alpha^*))$$

then, by the rule of Roman instantiation, one would be able to infer:

$$\vdash F(b^*) = G(b^*) \therefore (\forall \alpha^*)(F(\alpha^*) = G(\alpha^*))$$

This asserts that if two first-level incomplete *Sinne* yield the same complex *Sinn* when saturated by any one complete *Sinn*, then they yield the same complex *Sinn* for any complete *Sinn* (or in effect, that they are the same incomplete *Sinn*). This may at first seem unproblematic. After all, if $F(b^*)$ is *composed* of $F(\delta)$ and b^* , and $G(b^*)$ is composed of $G(\delta)$ and b^* , given that both complex *Sinne* $F(b^*)$ and $G(b^*)$ have b^* as one component, how could they be the same unless their other component is also the same? The problem, however, is that $F(\delta)$ and $G(\delta)$ might themselves have complex inner structures that include the *Sinn* b^* itself. The Roman instantiation rule allows us to instantiate Roman letters such as “ $F(\delta)$ ” and “ $G(\delta)$ ” directly to complex expressions of the same type. Suppose in the above we instantiate “ $F(\delta)$ ” directly to the complex incomplete *Sinn* expression “ $(\delta \rightarrow b^*)$ ” and “ $G(\delta)$ ” to “ $(b^* \rightarrow \delta)$ ”. We would then have the following:

$$\vdash (b^* \rightarrow b^*) = (b^* \rightarrow b^*) \therefore (\forall \alpha^*)[(\alpha^* \rightarrow b^*) = (b^* \rightarrow \alpha^*)]$$

The antecedent of this conditional is obviously a logical truth. We would then have the following theorem:

$$\vdash (\forall \alpha^*)[(\alpha^* \rightarrow b^*) = (b^* \rightarrow \alpha^*)]$$

Applying the rule of (*gen*), we arrive at:

$$\vdash (\forall b^*)(\forall \alpha^*)[(b^* \rightarrow \alpha^*) = (b^* \rightarrow \alpha^*)]$$

This states, in effect, that no matter what *Sinne* the expressions A and B express, that the *Gedanke* expressed by $\lceil A \supset B \rceil$ is the same as that expressed by $\lceil B \supset A \rceil$. This is demonstrably false. The incomplete *Sinn* denoted by “ $(\delta \rightarrow b^*)$ ” is not the same as the incomplete *Sinn* denoted by “ $(b^* \rightarrow \delta)$ ”, despite that, they do form the same *Gedanke* when saturated by b^* . Therefore, the more complicated form of axiom FC^{+SB}19 is adopted. Axiom FC^{+SB}20 states roughly the same as axiom FC^{+SB}19 save for second-level incomplete *Sinne* that mutually saturate with two-place rather than one-place first-level incomplete *Sinne*.

Notice, however, for the example given in the previous paragraph, while the incomplete *Sinne* denoted by “ $(\delta \rightarrow b^*)$ ” and “ $(b^* \rightarrow \delta)$ ” result in the same complex *Sinn* when saturated by one particular complete *Sinn* (b^*), they result in different complex *Sinne* when saturated by any other complete *Sinn*. Axiom FC^{+SB}21 states that if complex *Sinn* $F(a^*)$ is the same as complex *Sinn* $G(a^*)$, and complex *Sinn* $F(b^*)$ is the same as complex *Sinn* $G(b^*)$, where a^* and b^* are not the same complete *Sinn*, then the incomplete *Sinn* $F(\delta)$ must be the same as the incomplete *Sinn* $G(\delta)$. One cannot conclude that incomplete *Sinne* are the same simply because they result in the same complex *Sinn* for one particular *Sinn*, but if they result in the same complex *Sinn* when completed by multiple complete *Sinne*, there is no such problem. Axiom FC^{+SB}21 thus only asserts the sameness of incomplete *Sinne* when they yield the same complex *Sinn* for multiple complete *Sinne*.² Axiom FC^{+SB}22 states roughly the same, save for two-place incomplete *Sinne*.

Note also in the example used to show that the less complicated form of FC^{+SB}19 cannot be adopted, that the problematic conclusion can only be reached if we instantiate $F(\delta)$ and $G(\delta)$ to complex incomplete *Sinne* that contain b^* . However, if incomplete *Sinne* $F(\delta)$ and $G(\delta)$ yield the same complex *Sinn* for some complete *Sinn* that is contained in neither $F(\delta)$ nor $G(\delta)$, then it is unproblematic to conclude that $F(\delta)$ and $G(\delta)$ must be the same incomplete *Sinn*. This is stated by axiom FC^{+SB}23. Axiom FC^{+SB}24 asserts something very similar for two-place incomplete *Sinne*.

Axioms FC^{+SB}25 and FC^{+SB}26 state something very similar to FC^{+SB}21, save at the next higher level. They state sufficient conditions for the sameness of second-level incomplete *Sinne* just as axiom FC^{+SB}21 states them for first-level incomplete *Sinne*. It would also be nice to be able to state also axioms similar to FC^{+SB}23, except for second-level incomplete *Sinne*. Unfortunately, there is no way to do so without quantifiers for second-level functions and for second-level incomplete *Sinne*. The expansion of Frege’s Begriffsschrift here considered does not include such quantifiers. While I do not myself see any reason why Frege

²This point is briefly mentioned by Anderson in his discussion of Church’s Alternative (0). See Anderson, “General Intensional Logic,” 379.

could not have accepted such quantifiers—understood as *fourth-level* functions—he himself did not include functions of this level in his system, and it is a matter of controversy whether he would have welcomed expanding his system to include such high level functions. Moreover, including fourth-level functions in the system would greatly complicate its syntax, semantics and inference rules. For these reasons, I do not include them here. Unfortunately, as we shall see, this precludes us from capturing a number of truths of higher-order incomplete *Sinne* that it would have otherwise been desirable to capture.

Axioms FC^{+SB}27 through FC^{+SB}30 codify the notion that a *Sinn* is never identical to a complex *Sinn* of which it forms only a part. Axiom FC^{+SB}27 asserts that if complete *Sinn* a^* saturates incomplete *Sinn* $F(\delta)$, the result is identical to a^* only if $F(\delta)$ is the trivial, content-less incomplete *Sinn* one would write simply as “ δ ”, which corresponds to the sense-function that maps every *Sinn* onto itself. Axiom FC^{+SB}28 states that a *Sinn* is never identical to a relational complex *Sinn* of which it forms a part. Axiom FC^{+SB}29 states that the merging of the first-level incomplete *Sinn* $F(\delta)$ and some second-level incomplete *Sinn* is never identical to the merging of $F(\delta)$ and some complete *Sinn* a^* unless the second-level incomplete *Sinn* takes the form $\theta(a^*)$. Axiom FC^{+SB}30 states the same for two-place incomplete *Sinne*.

Axiom FC^{+SB}31 also involves the compositionality of complex *Sinne*. It states that if the complex *Sinn* $F(a^*)$ is identical to the complex *Sinn* $G(b^*)$, where a^* and b^* are different complete *Sinne*, then the complete *Sinn* b^* must somehow be “included” in the incomplete *Sinn* F . For example, if $F(\delta)$ is the incomplete *Sinn* $\delta \rightarrow b^*$, and $G(\delta)$ is the incomplete *Sinn* $a^* \rightarrow \delta$, then $F(a^*)$ is the same as $G(b^*)$ even if a^* and b^* are different, but only because b^* is included as partial constituent of $F(\delta)$. Here too, there would be seem to be a higher-level analogue of this principle, but it is impossible to formulate without higher-level quantifiers.

Axioms FC^{+SB}32 through FC^{+SB}34 together assert that if a complex *Sinn* picks out a *Bedeutung*, then each of its component *Sinne* picks out a *Bedeutung*. This corresponds to the linguistic fact that if a proposition has a truth-value, then the expressions occurring in it must have *Bedeutungen*. As noted in Chapter 3, Frege believes that natural languages contain a number of expressions that express *Sinne* but have no *Bedeutungen*, e.g., “Odysseus” or “Romulus.” Frege further tells us that while the complete propositions in which such empty names occur can be understood as expressing *Gedanken*, these *Gedanken* do not pick out truth-values (CP 162). The introduction of empty names is to be avoided in a logically superior language; thus we do not add into the logical language signs such as these. However, nothing prevents us from introducing signs that actually denote *Sinne* without *Bedeutungen*; we are only barred for introducing signs that express such *Sinne*. However, if such a *Sinn* occurs as part of a complex

Sinn, the complex *Sinn* does not denote anything. Thus, axiom FC^{+SB}32 asserts that if a complete *Sinn* occurs as part of a complex *Sinn*, the complex *Sinn* picks out some *Bedeutung* only if the complete *Sinn* picks out some *Bedeutung*.

Axiom FC^{+SB}33 asserts something similar about one-place first-level incomplete *Sinne*. More specifically, it says that the complex *Sinn* $M_{\beta}^*(F(\beta))$ picks out some *Bedeutung* only if there is some function $f(\xi)$ for which what results when $F(\delta)$ saturates with a complete *Sinn* is a complex *Sinn* picking out the value of function $f(\xi)$ for the *Bedeutung* of the *Sinn* saturating $F(\delta)$. Thus, in this system, the expression:

$$(\forall \alpha \alpha^*)[\Delta(\alpha^*, \alpha) \supset \Delta(F(\alpha^*), f(\alpha))]$$

is used more or less to stand for the truth-value of incomplete *Sinn* $F(\xi)$ picking out function $f(\xi)$ as *Bedeutung*. However, we noted in the previous chapter why it is that satisfying Church's similar formula:

$$(\forall x_1)(\forall x_{1_1})(\Delta_{01_1} x_1 x_{1_1} \supset \Delta_{000_1}(g_{0_1_1} x_{1_1})(f_{0_1} x_1))$$

is not a sufficient condition for function g to be a sense-function of f . However, the problems with Church's formula are avoided when the formulation is done in terms of incomplete *Sinne* rather than sense-functions. As noted in the previous chapter, there may be a sense-function g of type $0_1 1_1$ that yields for any *Sinn* of a name as argument the *Gedanke* expressed by the proposition that results when that name completes the incomplete expression "ξ is human," except in the case of the *Sinn* of the name "Russell", for which it yields the *Gedanke* expressed by "Stockholm is in Sweden." Such a function g could satisfy Church's formula, where f is the concept of being human, despite its not being a sense-function of f . However, there can be no incomplete *Sinn* $F(\delta)$ that yields the *Gedanke* expressed by "Socrates is human" when completed by the *Sinn* of "Socrates" but the *Sinn* of "Stockholm is in Sweden" when saturated by the *Sinn* of "Russell." When incomplete *Sinne* saturate with complete *Sinne*, they form wholes, and thus the wholes formed by any one incomplete *Sinn* must all have a common constituent. Hence, our formulation does not share the problems of Church's formulation.

Axiom FC^{+SB}34 states the same as FC^{+SB}33 save for two-place first-level incomplete *Sinne*. It would be nice to be able to state a similar axiom for second-level incomplete *Sinne* that might occur as part of a complex *Sinn*. Again, however, it is not possible without quantifiers ranging over them.

Axioms FC^{+SB}35 through FC^{+SB}37 may be somewhat controversial. Axiom FC^{+SB}35 asserts that there is at least one *Sinn* picking out any object. It was argued to be a result of Frege's theory of *Sinne* in Chapter 3 that every object

has at least one *Sinn* picking it out. On the interpretation defended there, *Sinne* are understood as packets of descriptive information, and pick out the *Bedeutungen* they do in virtue of them alone satisfying the descriptive information. Given the identity of indiscernibles, for any object there will always be at least one complex set of descriptive information true of that object alone. Consequently, at least one *Sinn* picks it out. Presumably, something similar is true of functions. Therefore, axiom FC^{+SB}36 states that every one-place first-level function has at least one one-place first-level incomplete *Sinn* picking it out, and FC^{+SB}37 asserts that every two-place first-level function has at least one two-place first-level incomplete *Sinn* picking it out. If we had a quantifier for second-level incomplete *Sinne*, we might include an axiom very much like these axioms postulating a second-level incomplete *Sinn* for each second-level function. For reasons given above, however, this is not done.

In addition to these core axioms, our expanded system also includes a number of axioms governing the constants added for the *Sinne* of our logical symbols. Here, again, I shall first list the axioms and then discuss them.

- Axiom FC^{+SB}38. $\vdash \Delta(a^*, a) \supset \Delta(\dots a^*, _ a)$
 Axiom FC^{+SB}39. $\vdash \Delta(a^*, a) \supset \Delta(\neg a^*, \sim a)$
 Axiom FC^{+SB}40. $\vdash \Delta(a^*, a) \supset \Delta(\lrcorner a^*, \setminus a)$
 Axiom FC^{+SB}41. $\vdash \Delta(a^*, a) \ \& \ \Delta(b^*, b) \ \supset \ \Delta(a^* \rightarrow b^*, a \supset b)$
 Axiom FC^{+SB}42. $\vdash \Delta(a^*, a) \ \& \ \Delta(b^*, b) \ \supset \ \Delta(a^* \approx b^*, a = b)$
 Axiom FC^{+SB}43. $\vdash \Delta(a^*, a) \ \& \ \Delta(b^*, b) \ \supset \ \Delta(\blacktriangle(a^*, b^*), \Delta(a, b))$
 Axiom FC^{+SB}44. $\vdash (\forall a a^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \supset \Delta((\Pi a^*)F(a^*), (\forall a)f(a))$
 Axiom FC^{+SB}45. $\vdash (\forall a a^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \supset \Delta(\varepsilon F(\varepsilon), \varepsilon f(\varepsilon))$
 Axiom FC^{+SB}46. $\vdash (\forall f \mathfrak{F})\{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{F}(a^*), f(a))] \supset \Delta(M_{\beta}^*(\mathfrak{F}(\beta)), M_{\beta}(f(\beta)))\} \supset \Delta((\Pi \mathfrak{F})M_{\beta}^*(\mathfrak{F}(\beta)), (\forall f)M_{\beta}(f(\beta)))$
 Axiom FC^{+SB}47. $\vdash (\forall f \mathfrak{F})\{(\forall a a^* b b^*)[\Delta(a^*, a) \ \& \ \Delta(b, b^*) \ \supset \ \Delta(\mathfrak{F}(a^*, b^*), f(a, b))]\} \supset \Delta(M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)), M_{\beta\gamma}(f(\beta, \gamma))) \supset \Delta((\Pi \mathfrak{F})M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)), (\forall f)M_{\beta\gamma}(f(\beta, \gamma)))$
 Axiom FC^{+SB}48. $\vdash (\Pi a^*)F(a^*) \neq G(F(a^*))$
 Axiom FC^{+SB}49. $\vdash \varepsilon F(\varepsilon) \neq G(F(a^*))$
 Axiom FC^{+SB}50. $\vdash (\Pi \mathfrak{F})M_{\beta}^*(\mathfrak{F}(\beta)) \neq G(M_{\beta}^*(F(\beta)))$
 Axiom FC^{+SB}51. $\vdash (\Pi \mathfrak{F})M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)) \neq G(M_{\beta\gamma}^*(F(\beta, \gamma)))$
 Axiom FC^{+SB}52. $\vdash (\Pi \mathfrak{F})M_{\beta}^*(\mathfrak{F}(\beta)) = (\Pi \mathfrak{F})N_{\beta}^*(\mathfrak{F}(\beta)) \ \supset \ .$
 $(\forall \mathfrak{F})[M_{\beta}^*(\mathfrak{F}(\beta)) = N_{\beta}^*(\mathfrak{F}(\beta))]$
 Axiom FC^{+SB}53. $\vdash (\Pi \mathfrak{F})M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)) = (\Pi \mathfrak{F})N_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)) \ \supset \ .$
 $(\forall \mathfrak{F})[M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)) = N_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma))]$

Although some axioms in this series may be difficult to read and understand, they are all quite simple in meaning. Axioms FC^{+SB}38 through FC^{+SB}47 simply state for each of our *Sinn* constants what function the *Sinn* it stands for picks out.

For example, axiom $FC^{+SB}39$ can be understood as stating that $\neg\delta$ picks out the negation function for its *Bedeutung*, or, more precisely, that for any complete *Sinn* a^* , picking out object a , with which it saturates, the complex *Sinn* formed by the completion of the incomplete *Sinn* by a^* will be a *Sinn* picking out as *Bedeutung* the value of the negation function for a as argument. Axioms $FC^{+SB}38$ through $FC^{+SB}43$ correlate our first-level incomplete *Sinn* constants with the functions they present. Axioms $FC^{+SB}44$ and $FC^{+SB}45$ do something similar for our second-level incomplete *Sinn* constants. For example, axiom $FC^{+SB}44$ states that if $F(\delta)$ is an incomplete *Sinn* picking out function $f(\xi)$, then the *Gedanke* $(\Pi a^*)F(a^*)$, formed by the mutual saturation of the first-level incomplete *Sinn* $F(\delta)$ and the second-level incomplete *Sinn* $(\Pi a^*)\theta(a^*)$, picks out for its *Bedeutung* the truth-value $(\forall a)f(a)$, i.e., the value of the second-level function $(\forall a)\phi(a)$ for $f(\xi)$ as argument. Axioms $FC^{+SB}46$ and $FC^{+SB}47$ say something very similar about our third-level incomplete *Sinn* constants. Although difficult to read, the import of $FC^{+SB}46$ is that if the second-level incomplete *Sinn* M_β^* picks out second-level function M_β , then the *Gedanke* $(\Pi \mathfrak{F})M_\beta^*(\mathfrak{F}(\beta))$ picks out as its *Bedeutung* the truth-value $(\forall f)M_\beta(f(\beta))$, i.e., the value of $(\forall f)\mu_\beta(f(\beta))$ for M_β taken as argument.

With regard to axioms $FC^{+SB}38$ through $FC^{+SB}43$, it should be noted that none of these axioms can be taken to make the metaphysical commitments of the system stronger. These axioms introduce a number of constants for incomplete *Sinne*. However, even without these axioms, the system is already committed to incomplete *Sinne* having their properties due to axioms $FC^{+SB}36$ and $FC^{+SB}37$. This is because the functions denoted by our first-level logical constants are themselves within the range of quantification over first-level functions. Consider the following instance of axiom $FC^{+SB}36$:

Theorem $FC^{+SB}36.1 \vdash (\exists \mathfrak{F})(\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{F}(a^*), \sim a)]$

This theorem postulates the existence of an incomplete *Sinn* that picks out the negation function. Therefore, even without the sign “ \sim ” and axiom $FC^{+SB}39$, the system is already committed to a *Sinn* picking out the negation function. Indeed, this theorem is just as deductively powerful as axiom $FC^{+SB}39$. However, within an axiomatic system that does not contain a rule of “existential instantiation,” having constants standing for the *Sinne* of the logical constants is quite a convenience. In a system already containing a strong ontological axiom such as $FC^{+SB}36$, their presence is innocuous, as are the axioms $FC^{+SB}38$ through $FC^{+SB}43$. Axioms $FC^{+SB}44$ through $FC^{+SB}47$ would similarly not add to the deductive power or commitments of the system if we had been able to formulate higher-level versions of axioms $FC^{+SB}35$ through $FC^{+SB}37$. This is important when considering the sources of the difficulties of the expanded systems,

because the source of any undesired results that result in the system cannot be traced back to axioms $FC^{+SB}38$ through $FC^{+SB}43$. Even if those axioms are used in the demonstration of an undesired result, it would be possible to find a similar demonstration for the undesired result that makes use instead of axiom $FC^{+SB}36$ or axiom $FC^{+SB}37$.

Axioms $FC^{+SB}48$ through $FC^{+SB}51$ assert that our higher-level incomplete *Sinn* constants, as primitive, are not reducible to lower-level *Sinne*. For example, axiom $FC^{+SB}48$ in effect asserts that the incomplete *Sinn* $(\Pi a^*)\theta(a^*)$ is not definable with any lower-level *Sinn* constants; there are no *Sinne* a^* , $G(\delta)$ and $F(\delta)$ such that $(\Pi a^*)F(a^*)$ is the same as $G(F(a^*))$. Part of the significance of this axiom is that, when combined with axiom $FC^{+SB}19$, the following theorem results:

Theorem $FC^{+SB}48.1 \vdash (\Pi a^*)F(a^*) = (\Pi a^*)G(a^*) \therefore (\forall a^*)(F(a^*) = G(a^*))$

The universal *Gedanke* $(\Pi a^*)F(a^*)$ is the same the universal *Gedanke* $(\Pi a^*)G(a^*)$ only if $F(\delta)$ and $G(\delta)$ are the same incomplete *Sinn*. A similar theorem follows for *Gedanken* of the form $\overset{\circ}{\varepsilon} \theta(\varepsilon)$ from $FC^{+SB}19$ and $FC^{+SB}49$. Axioms $FC^{+SB}50$ and $FC^{+SB}51$ assert roughly the same for our higher-level incomplete *Sinne* $(\Pi \mathfrak{F})\mu_{\beta}^*(\mathfrak{F}(\beta))$ and $(\Pi \mathfrak{F})\mu_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma))$, as $FC^{+SB}48$ states for $(\Pi a^*)\theta(a^*)$. Axioms $FC^{+SB}52$ and $FC^{+SB}53$ state roughly the same for $(\Pi \mathfrak{F})\mu_{\beta}^*(\mathfrak{F}(\beta))$ and $(\Pi \mathfrak{F})\mu_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma))$ as theorem $FC^{+SB}48.1$ states for $(\Pi a^*)\theta(a^*)$. It would be preferable if these axioms were to follow as theorems from $FC^{+SB}50$ and $FC^{+SB}51$ and higher-level analogues of $FC^{+SB}19$. Unfortunately, there is no way of formulating such axioms without higher-level quantifiers.

Together, axioms $FC1$ through $FC6$ and axioms $FC^{+SB}10$ through $FC^{+SB}53$ can be taken as a complete axiomatization of the system FC^{+SB} . These, in addition to axioms $FC^{+V}7$, $FC^{+V}8$ and $FC^{+V}9$, can be understood as a complete axiomatization of FC^{+SB+V} . Given that axioms $FC^{+SB}15$, $FC^{+SB}45$ and $FC^{+SB}49$ involve value-ranges, one might wish to limit them to system FC^{+SB+V} . However, these axioms alone are not sufficient to introduce naive-class theory into the system, and their presence in FC^{+SB} is relatively innocuous.

INFERENCES INVOLVING PROPOSITIONAL ATTITUDES AND QUANTIFYING IN

In Chapter 1, we discussed some of the principal motivations for expanding Frege's logic in the direction we have. It is only by doing so that we are able to capture inferences involving *oratio obliqua*, including statements of propositional attitudes. Before turning to a consideration of the difficulties of

our systems FC^{+SB} and FC^{+SB+V} , let us first consider how they are successful in filling the gap in Frege's Begriffsschrift discussed in Chapter 1.

We have seen how it is that our expanded language is able to transcribe statements of propositional attitudes. Let us consider how it tackles inferences involving them. Consider again this inference discussed in Chapter 1:

(1E) Gottlob believes that the morning star is a planet.

(2E) The morning star is a planet.

Therefore:

(3E) Gottlob believes something true.

Earlier, we transcribed (1E) and (2E) as:

(1B) $\vdash \mathcal{B}(g, \mathfrak{s})$

(2B) $\vdash \mathcal{P}(m)$

Frege would understand (3E) as saying that Gottlob believes a *Gedanke* that picks out the True. In order to transcribe (3E), we need to define a function sign that stands for a concept that yields the True just in case its argument is a True *Gedanke*. Since “ $(\forall a)(a \supset a)$ ” is a name for the True, we can form such a definition as follows:

(Df. \mathcal{T}) $\Vdash \mathcal{T}(x) = \Delta(x, (\forall a)(a \supset a))$

We then transcribe (3E) as:

(3B) $\vdash (\exists a^*)(\mathcal{B}(g, a^*) \ \& \ \mathcal{T}(a^*))$

There is something Gottlob believes that is true.

Because the system employs a method of direct discourse, the English expressions “the morning star is a planet” occurring in both (1E) and (2E) are transcribed differently into the Begriffsschrift in the two cases. This makes it difficult to jump right to the conclusion. However, it does not make such a move impossible, contrary to the criticism offered by Blackburn and Code discussed in Chapter 1. Firstly, the *Sinn* denoted by “ \mathfrak{s} ” picks out the morning star. Hence:

(4B) $\vdash \Delta(\mathfrak{s}, m)$

Similarly, the incomplete *Sinn* denoted by “ $\mathfrak{Q}(\delta)$ ” picks out the concept of *being a planet*.

$$(5B) \vdash \Delta(a^*, a) \supset \Delta(\mathfrak{Q}(a^*), \mathcal{P}(a))$$

The conclusion is then derived as follows (taking 1B, 2B, 4B and 5B as premises):

6.	$\vdash \Delta(\mathfrak{s}, \mathfrak{m}) \supset \Delta(\mathfrak{Q}(\mathfrak{s}), \mathcal{P}(\mathfrak{m}))$	5B, <i>ri, ri</i>
7.	$\vdash \Delta(\mathfrak{Q}(\mathfrak{s}), \mathcal{P}(\mathfrak{m}))$	4B, 6, <i>mp</i>
8.	$\vdash a : \supset b \supset a \equiv b$	(theorem of FC)
9.	$\vdash (\forall a)(a \supset a)$	(theorem of FC)
10.	$\vdash \mathcal{P}(\mathfrak{m}) : \supset (\forall a)(a \supset a) \supset \mathcal{P}(\mathfrak{m}) \equiv (\forall a)(a \supset a)$	8, <i>ri, ri</i>
11.	$\vdash (\forall a)(a \supset a) \supset \mathcal{P}(\mathfrak{m}) \equiv (\forall a)(a \supset a)$	2B, 10, <i>mp</i>
12.	$\vdash \mathcal{P}(\mathfrak{m}) \equiv (\forall a)(a \supset a)$	9, 11, <i>mp</i>
13.	$\vdash \neg \mathcal{P}(\mathfrak{m}) = \neg (\forall a)(a \supset a)$	12, <i>df. \equiv</i>
14.	$\vdash \mathcal{P}(\mathfrak{m}) = (\forall a)(a \supset a)$	13, <i>hor, hor</i> ³
15.	$\vdash \Delta(\mathfrak{Q}(\mathfrak{s}), (\forall a)(a \supset a))$	7, 14, <i>sub. of identicals</i>
16.	$\vdash \mathcal{T}(\mathfrak{Q}(\mathfrak{s}))$	15, <i>df. \mathcal{T}</i>
17.	$\vdash a : \supset b \supset a \& b$	(theorem of FC)
18.	$\vdash \mathcal{B}(\mathfrak{g}, \mathfrak{Q}(\mathfrak{s})) : \supset \mathcal{T}(\mathfrak{Q}(\mathfrak{s})) \supset \mathcal{B}(\mathfrak{g}, \mathfrak{Q}(\mathfrak{s})) \& \mathcal{T}(\mathfrak{Q}(\mathfrak{s}))$	17, <i>ri, ri</i>
19.	$\vdash \mathcal{T}(\mathfrak{Q}(\mathfrak{s})) \supset \mathcal{B}(\mathfrak{g}, \mathfrak{Q}(\mathfrak{s})) \& \mathcal{T}(\mathfrak{Q}(\mathfrak{s}))$	1B, 18, <i>mp</i>
20.	$\vdash \mathcal{B}(\mathfrak{g}, \mathfrak{Q}(\mathfrak{s})) \& \mathcal{T}(\mathfrak{Q}(\mathfrak{s}))$	16, 19, <i>mp</i>
21.	$\vdash (\forall a^*)f(a^*) \supset f(a^*)$	axiom FC ^{+SB} 10
22.	$\vdash \neg f(a^*) \supset \neg (\forall a^*)f(a^*)$	19, <i>con</i>
23.	$\vdash \neg [\mathcal{B}(\mathfrak{g}, \mathfrak{Q}(\mathfrak{s})) \& \mathcal{T}(\mathfrak{Q}(\mathfrak{s}))] \supset \neg (\forall a^*) \neg [\mathcal{B}(\mathfrak{g}, a^*) \& \mathcal{T}(a^*)]$	20, <i>ri, ri</i>
24.	$\vdash (a \& b) = \neg \neg (a \& b)$	(theorem of FC)
25.	$\vdash [\mathcal{B}(\mathfrak{g}, \mathfrak{Q}(\mathfrak{s})) \& \mathcal{T}(\mathfrak{Q}(\mathfrak{s}))] = \neg \neg [\mathcal{B}(\mathfrak{g}, \mathfrak{Q}(\mathfrak{s})) \& \mathcal{T}(\mathfrak{Q}(\mathfrak{s}))]$	22, <i>ri, ri</i>
26.	$\vdash \neg \neg [\mathcal{B}(\mathfrak{g}, \mathfrak{Q}(\mathfrak{s})) \& \mathcal{T}(\mathfrak{Q}(\mathfrak{s}))]$	20, 23, <i>sub. of identicals</i>
27.	$\vdash \neg (\forall a^*) \neg [\mathcal{B}(\mathfrak{g}, a^*) \& \mathcal{T}(a^*)]$	23, 26, <i>mp</i>
28.	$\vdash (\exists a^*) [\mathcal{B}(\mathfrak{g}, a^*) \& \mathcal{T}(a^*)]$	27, <i>df. \exists d</i>

Blackburn and Code had suggested that this sort of inference is impossible on Frege’s theory of indirect *Bedeutung*. While this criticism does show that his extant system can be considered to be lacking, if Frege had expanded his logical system to include the theory of *Sinn* and *Bedeutung*, he would have been able to reply adequately to this criticism.

³Because “ $\mathcal{P}(\xi)$ ” stands for a concept (a function whose value is always a truth-value), the rule of *horizontal amalgamation* would also apply to it if it were included within the formal language.

In the above demonstration, I allowed myself to use the rule of substitution of identicals, hereafter abbreviated as “*i*”. This is not actually an inference rule of the system, but is a nearly direct consequence of axiom FC6. I also allowed myself to simply refer to “theorems of FC.” These are simple truths of the function calculus, usually truth-functional tautologies, and many of those I invoke in this demonstration and in the other demonstrations that follow are proven by Frege himself in the *Grundgesetze* (see esp. *BL* §§49-52). It would make my demonstrations too long to include within them the demonstration of the functional analogues of the theorems of first-order logic. Since FC has been proven to be complete in its treatment of first-order logic, I allow myself to refer to such truths in my demonstrations, hereafter with the abbreviation “*fc*”.

We have seen one example of how inferences involving propositional attitudes are treated in the logical calculi of *Sinne* and *Bedeutungen*. Some of the more notorious examples of inferences involving propositional attitudes, however, deal with the questions surrounding the notion of “quantifying in” to the statement of a propositional attitude. An example of quantifying in occurs in the inference from:

(1E) Gottlob believes that the morning star is a planet.

to the conclusion:

(6E) There is something Gottlob believes to be a planet.

Here, the inference seems unproblematic. However, consider the case of the spurious planet Vulcan, once believed to exist in rotation between Mercury and the Sun. The inference from:

(7E) Gottlob believes that Vulcan is a planet.

to the same conclusion is problematic. Vulcan does not exist, and if all we know is that (7E) is true, we cannot infer (6E), because this commits us existentially to the object believed by Gottlob to be a planet. In this case, there is no such object.

Sorting out the difficulties with the problem of quantifying in is a problem faced by any who hope for a logical treatment of propositional attitudes. It was suggested in Chapter 1 that the development of a logical calculus for the theory of *Sinn* and *Bedeutung* would provide an answer to how Frege would treat the problem of “quantifying in.” First, let us consider the unproblematic inference from (1E) to (6E). Above we gave (1B) as the Begriffsschrift transcription of (1E). From (1B), we can infer (by way of axiom FC^{+SB}10):

$$(6B^*) \vdash (\exists a^*) \mathcal{B}(g, \varrho(a^*))$$

However, it would be very misleading to give this as a transcription of (6B). Here, the quantifier ranges over *Sinne*. What this says is not that there is some thing believed by Gottlob to be a planet; rather, it says only that there is some *Sinn* that, when saturated with the incomplete *Sinn* of “ ξ is a planet,” yields a *Gedanke* believed by Gottlob. It does not commit us to an object picked out by that *Sinn*. A more proper transcription of (6E) into the expanded Begriffsschrift would be:

$$(6B) \vdash (\exists a a^*) [\mathcal{B}(g, \varrho(a^*)) \ \& \ \Delta(a^*, a)]$$

This commits us not only to a *Sinn* that, when saturated with the incomplete *Sinn* of “ ξ is a planet,” yields a *Gedanke* believed by Gottlob, but also the existence of an object picked out by that *Sinn*. However, (6B) does not follow from (6E) alone; it requires also (4B) as a premise. Without the premise that the *Sinn* expressed by “the morning star” has a *Bedeutung*, the inference does not go through.

Let us compare this with what happens in case of (7E). While Frege disallows us from introducing a sign standing for the planet Vulcan, since all Begriffsschrift signs must denote something, the word “Vulcan” nevertheless expresses a *Sinn*, and we can introduce a sign for this *Sinn*. Let us use the sign “ v ” for this *Sinn*. We can then transcribe (7E) as follows:

$$(7B) \vdash \mathcal{B}(g, \varrho(v))$$

While it is true that we can infer (6B*) from (7B), again, this does not commit us to an *object* believed by Gottlob to be a planet, but only a *Sinn* that, when saturated with the incomplete *Sinn* of “ ξ is a planet,” yields a *Gedanke* believed by Gottlob. One *cannot* infer (6B) itself from this, because we do not have an analogue to (4B) used in the previous example. The *Sinn* v does not pick any object as *Bedeutung*. Thus, instead of something such as (4B), we have instead:

$$(8B) \vdash \sim(\exists a)(v, a)$$

Obviously, one cannot infer (6B) from (7B) and (8B).

Thus it appears that our system does have at least a *minimally* adequate treatment of the problem of “quantifying in.” The problem of “quantifying in” has received its fullest attention in the work of Quine. Quine himself, however, infers from the problem of inferring (6E) from (7E) that expressions are “referentially opaque” in statements of propositional attitudes, by which he

means that they do not refer at all in any standard way.⁴ The present account of Frege's solution is much less radical. For Frege, it is not that words occurring in propositional attitudes (in *oratio obliqua*) do not refer at all; rather, they refer to something other than they would normally. Because they *refer*, quantification over them is possible. It is only that care must be taken to remember what the quantifiers would range over. One can existentially generalize from "v" in (7B), but here we must remember that this stands for a *Sinn*, not for a planet, nor even for a purported planet. For Frege, *oratio obliqua* is not referentially *opaque*, but referentially *oblique*.

Here we can see some of the successes of this logical calculus of *Sinn* and *Bedeutung*, based upon Frege's own theories. It shows that Frege would have been able to respond adequately to certain criticisms, as well as provide satisfactory answers to longstanding philosophical questions. These victories, however, will be relatively short-lived.

DIFFICULTIES WITH THE EXPANDED CALCULI

Above, two systems, FC^{+SB} and FC^{+SB+V} , are outlined. Let us consider the latter first. I believe that FC^{+SB+V} , because it represents an expansion of the entire logical system of the *Grundgesetze* (as opposed to an expansion of a subportion), is as close as possible to the sort of system Frege himself would have devised if he had had an opportunity to create a logical calculus for his theory of *Sinn* and *Bedeutung* during the 1890s. The axioms, inference rules and signs we have added are based on his own understanding of the nature of *Sinne* and *Gedanken*. Of course, because the system FC^{+V} whereupon it is based is itself inconsistent due to Russell's paradox, FC^{+SB+V} is also inconsistent. This is not an unexpected result. The aim of our exposition herein was not to devise a feasible intensional logic, but one reflecting the views of the historical Frege.

However, it is interesting to note that besides Russell's paradox, system FC^{+SB+V} is subject to other antinomies that derive from its naive class theory in conjunction with some of the new axioms of the system dealing with *Sinne*. The addition of the many entities of the third realm of *Sinn* to the commitments of a logical calculus provide the resources for constructing new Cantorian diagonal constructions between different types of entities. The simplest such construction leading to a semantical antinomy involves a diagonal construction between *Sinne* and classes of *Sinne*. (We shall be considering a similar but more complicated antinomy in the next chapter.) Since *Sinne* that pick out classes are themselves objects, they may or may not be in the classes they pick out. Consider the class of all those *Sinne* that pick out classes they are not in.

⁴Quine, *Word and Object*, §30, "Quantifiers and Propositional Attitudes," 188 and *passim*.

Consider also a *Sinn* picking out this class. Is this *Sinn* in the class it picks out? It is if and only if it is not. The problematic class in this antinomy would be defined as follows:

$$(Df. \spadesuit) \Vdash \spadesuit = \varepsilon^2(\exists\alpha)[\Delta(\varepsilon, \alpha) \ \& \ \sim(\varepsilon \cap \alpha)]$$

Consider then the following instance of FC^{+SB}35, stating that there is at least one *Sinn* picking out this class:

$$\vdash (\exists\alpha^*)\Delta(\alpha^*, \spadesuit)$$

Let us use the sign “ \spadesuit^* ” for the *Sinn* (or one of the *Sinne*) postulated by this axiom. We then have the following principle:

$$(Pp. \spadesuit^*) \vdash \Delta(\spadesuit^*, \spadesuit)$$

This principle is not strictly a theorem of the system. However, it can be thought of as an “existential instantiation” of the instance of FC^{+SB}35 above, and it is easily shown that in systems such as those under consideration, whenever something is provable when a use of this rule is involved, a longer proof for a similar conclusion can be found that does not make use of the rule.⁵ The following contradiction is then provable in the system:

$$\vdash (\spadesuit^* \cap \spadesuit) \ \& \ \sim(\spadesuit^* \cap \spadesuit)$$

Proof:

- | | |
|--|--------------------------------|
| 1. $\vdash f(a) = (a \cap \varepsilon^2 f(\varepsilon))$ | (CA) (see p. 56) |
| 2. $\vdash f(\spadesuit^*) = (\spadesuit^* \cap \varepsilon f(\varepsilon))$ | 1, <i>ri</i> |
| 3. $\vdash (\exists\alpha)[\Delta(\spadesuit^*, \alpha) \ \& \ \sim(\spadesuit^* \cap \alpha)] = (\spadesuit^* \cap \varepsilon^2(\exists\alpha)[\Delta(\varepsilon, \alpha) \ \& \ \sim(\varepsilon \cap \alpha)])$ | 2, <i>ri</i> |
| 4. $\vdash (\exists\alpha)[\Delta(\spadesuit^*, \alpha) \ \& \ \sim(\spadesuit^* \cap \alpha)] = (\spadesuit^* \cap \spadesuit)$ | 3, <i>df. \spadesuit</i> |
| 5. $\vdash \Delta(a, b) \ \& \ \Delta(a, c) \ \therefore (b = c)$ | axiom FC ^{+SB} 13 |
| 6. $\vdash \Delta(\spadesuit^*, \spadesuit) \ \& \ \Delta(\spadesuit^*, c) \ \therefore (\spadesuit = c)$ | 5, <i>ri, ri</i> |
| 7. $\vdash a \ \therefore \therefore a \ \& \ b \ \therefore c \ \therefore b \supset c$ | <i>fc</i> |
| 8. $\vdash \Delta(\spadesuit^*, \spadesuit) \ \therefore \therefore \Delta(\spadesuit^*, \spadesuit) \ \& \ b \ \therefore c \ \therefore b \supset c$ | 7, <i>ri</i> |
| 9. $\vdash \Delta(\spadesuit^*, \spadesuit) \ \& \ b \ \therefore c \ \therefore b \supset c$ | <i>Pp. \spadesuit^*, 7, mp</i> |
| 10. $\vdash \Delta(\spadesuit^*, \spadesuit) \ \& \ \Delta(\spadesuit^*, c) \ \therefore (\spadesuit = c) \ \therefore \Delta(\spadesuit^*, c) \ \supset (\spadesuit = c)$ | 9, <i>ri, ri</i> |
| 11. $\vdash \Delta(\spadesuit^*, c) \ \supset (\spadesuit = c)$ | 6, 10, <i>mp</i> |

⁵See, e.g., Irving M. Copi, *Symbolic Logic*, 4th ed. (New York: Macmillan, 1973), 260-3.

12. $\vdash a \supset b \therefore (a \& d) \supset (d \& b)$ *fc*
13. $\vdash \Delta(\spadesuit^*, c) \supset (\spadesuit = c) \therefore [\Delta(\spadesuit^*, c) \& \sim(\spadesuit^* \cap c)] \supset [\sim(\spadesuit^* \cap c) \& (\spadesuit = c)]$
1, 12, *ri, ri, ri*
14. $\vdash [\Delta(\spadesuit^*, c) \& \sim(\spadesuit^* \cap c)] \supset [\sim(\spadesuit^* \cap c) \& (\spadesuit = c)]$ 11, 13, *mp*
15. $\vdash [f(c) \& (a = c)] \supset f(a)$ *fc*
16. $\vdash [\sim(\spadesuit^* \cap c) \& (\spadesuit = c)] \supset \sim(\spadesuit^* \cap \spadesuit)$ 15, *ri, ri*
17. $\vdash [\Delta(\spadesuit^*, c) \& \sim(\spadesuit^* \cap c)] \supset \sim(\spadesuit^* \cap \spadesuit)$ 14, 16, *syll*
18. $\vdash (\forall a)\{\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)\} \supset \sim(\spadesuit^* \cap \spadesuit)$ 17, *gen*
19. $\vdash (\forall a)(f(a) \supset b) \therefore (\exists a)f(a) \supset b$ *fc*
20. $\vdash (\forall a)\{\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)\} \supset \sim(\spadesuit^* \cap \spadesuit) \therefore$
 $(\exists a)[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)] \supset \sim(\spadesuit^* \cap \spadesuit)$ 19, *ri, ri*
21. $\vdash (\exists a)[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)] \supset \sim(\spadesuit^* \cap \spadesuit)$ 18, 20, *mp*
22. $\vdash (\spadesuit^* \cap \spadesuit) \supset \sim(\spadesuit^* \cap \spadesuit)$ 4, 21, *i*
23. $\vdash a \supset \sim a \therefore \sim a$ *fc*
24. $\vdash (\spadesuit^* \cap \spadesuit) \supset \sim(\spadesuit^* \cap \spadesuit) \therefore \sim(\spadesuit^* \cap \spadesuit)$ 23, *ri*
25. $\vdash \sim(\spadesuit^* \cap \spadesuit)$ 22, 24, *mp*
26. $\vdash \sim(\exists a)[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)]$ 4, 25, *i*
27. $\vdash \sim(\forall a)\sim[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)]$ 26, *df. \exists a*
28. $\vdash \sim\sim a = \text{---} a$ *fc*
29. $\vdash \sim(\forall a)\sim[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)] = \text{---} (\forall a)\sim[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)]$
28, *ri*
30. $\vdash \text{---} (\forall a)\sim[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)]$ 27, 29, *i*
31. $\vdash (\forall a)\sim[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)]$ 30, *hor*
32. $\vdash (\forall a)f(a) \supset f(a)$ axiom FC3
33. $\vdash (\forall a)\sim[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)] \supset \sim[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)]$ 32, *ri*
34. $\vdash \sim[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)]$ 31, 33, *mp*
35. $\vdash \sim(b \& \sim c) \supset (b \supset c)$ *fc*
36. $\vdash \sim[\Delta(\spadesuit^*, a) \& \sim(\spadesuit^* \cap a)] \supset [\Delta(\spadesuit^*, a) \supset (\spadesuit^* \cap a)]$ 35, *ri, ri*
37. $\vdash \Delta(\spadesuit^*, a) \supset (\spadesuit^* \cap a)$ 34, 36, *mp*
38. $\vdash \Delta(\spadesuit^*, \spadesuit) \supset (\spadesuit^* \cap \spadesuit)$ 37, *ri*
39. $\vdash \spadesuit^* \cap \spadesuit$ Pp. \spadesuit^* , 38, *mp*
40. $\vdash a \supset b \therefore a \& b$ *fc*
41. $\vdash \spadesuit^* \cap \spadesuit \supset b \therefore (\spadesuit^* \cap \spadesuit) \& b$ 40, *ri*
42. $\vdash b \therefore (\spadesuit^* \cap \spadesuit) \& b$ 39, 41, *mp*
43. $\vdash \sim(\spadesuit^* \cap \spadesuit) \therefore (\spadesuit^* \cap \spadesuit) \& \sim(\spadesuit^* \cap \spadesuit)$ 42, *ri*
44. $\vdash (\spadesuit^* \cap \spadesuit) \& \sim(\spadesuit^* \cap \spadesuit)$ 25, 43, *mp*

The system $\text{FC}^{\text{SB+V}}$ thus has more to contend with than the simple Russell's paradox. In the next chapter, we shall also see that the system $\text{FC}^{\text{SB+V}}$ succumbs to an analogue of Russell's paradox from Appendix B of the *Principles of Mathematics*.

The disastrous consequences of adding the metaphysical commitments of Frege’s third realm of *Sinn* to a logical calculus, however, do not appear only in a logical system already committed to value-ranges or classes. As it turns out, semantic paradoxes and antinomies also arise in the more modest system FC^{+SB} . For example, it can be shown that FC^{+SB} falls prey to the modified Epimenides paradox considered in the previous chapter. Again, the paradox comes from assuming that Church’s favorite *Gedanke* is the *Gedanke* that Church’s favorite *Gedanke* is not true. Our expanded logical language allows us to formulate this hypothesis. We begin by considering the function $\mathcal{T}(\xi)$ defined in the previous section, which yields the True as value just in case its argument is a true *Gedanke*. We then consider an incomplete *Sinn* picking out this function. Of course, axiom $FC^{+SB}36$ guarantees that there is at least one such incomplete *Sinn*. However, as it turns out, we need not use $FC^{+SB}36$ to arrive at such a *Sinn*. “ $\mathcal{T}(\xi)$ ” is a defined sign, and because we have constants standing for the *Sinne* of all the signs used in defining it, we can define a sign for such an incomplete *Sinn*, “ $\mathfrak{T}(\delta)$ ”, as follows:

$$(Df. \mathfrak{T}) \Vdash \mathfrak{T}(x^*) = \blacktriangle(x^*, (\Pi a^*)(a^* \rightarrow a^*))$$

We now introduce a sign “ Υ ” standing for Church’s favorite *Gedanke*, and a sign “ Υ^* ” standing for the *Sinn* of the expression “Church’s favorite *Gedanke*.” We then have as one of our premises:

$$(P1) \vdash \Delta(\Upsilon^*, \Upsilon)$$

We now assume for the sake of the paradox that Church’s favorite *Gedanke* is the *Gedanke* expressed by “ $\sim\mathcal{T}(\Upsilon)$ ”, which we denote with the expression “ $\neg\mathfrak{T}(\Upsilon^*)$ ”:

$$(P2) \vdash \Upsilon = \neg\mathfrak{T}(\Upsilon^*)$$

From these premises we can derive the following contradictory result:

$$\vdash \mathcal{T}(\Upsilon) \ \& \ \sim\mathcal{T}(\Upsilon)$$

Proof:

- | | |
|--|--------------------------|
| 1. $\vdash \Delta(a^*, a) \ \& \ \Delta(b^*, b) \ \therefore \Delta(a^* \rightarrow b^*, a \supset b)$ | axiom $FC^{+SB}41$ |
| 2. $\vdash \Delta(a^*, a) \ \& \ \Delta(a^*, a) \ \therefore \Delta(a^* \rightarrow a^*, a \supset a)$ | 1, <i>ri</i> , <i>ri</i> |
| 3. $\vdash p \ \& \ p \ \therefore q \ \therefore p \supset q$ | <i>fc</i> |

33.	$\vdash \mathcal{T}(Y) \supset \sim \mathcal{T}(Y)$	30, 32, <i>i</i>
34.	$\vdash \mathcal{T}(Y) \supset \sim \mathcal{T}(Y)$	33, <i>hor</i>
35.	$\vdash p \supset \sim p \therefore \sim p$	<i>fc</i>
36.	$\vdash \mathcal{T}(Y) \supset \sim \mathcal{T}(Y) \therefore \sim \mathcal{T}(Y)$	35, <i>ri</i>
37.	$\vdash \sim \mathcal{T}(Y)$	34, 36, <i>mp</i>
38.	$\vdash a \therefore a = (\forall a)(a \supset a)$	<i>fc</i>
39.	$\vdash \sim \mathcal{T}(Y) \therefore \sim \mathcal{T}(Y) = (\forall a)(a \supset a)$	38, <i>ri</i>
40.	$\vdash \sim \mathcal{T}(Y) = (\forall a)(a \supset a)$	37, 39, <i>mp</i>
41.	$\vdash \Delta(\neg \mathcal{T}(Y^*), (\forall a)(a \supset a))$	22, 40, <i>i</i>
42.	$\vdash \Delta(Y, (\forall a)(a \supset a))$	P2, 41, <i>i</i>
43.	$\vdash \mathcal{T}(Y)$	42, df. \mathcal{T}
44.	$\vdash p \therefore q \therefore p \& q$	<i>fc</i>
45.	$\vdash \mathcal{T}(Y) \therefore \sim \mathcal{T}(Y) \therefore \mathcal{T}(Y) \& \sim \mathcal{T}(Y)$	44, <i>ri, ri</i>
46.	$\vdash \sim \mathcal{T}(Y) \therefore \mathcal{T}(Y) \& \sim \mathcal{T}(Y)$	43, 45, <i>mp</i>
47.	$\vdash \mathcal{T}(Y) \& \sim \mathcal{T}(Y)$	37, 46, <i>mp</i>

This result does not *by itself* show our system FC^{+SB} to be inconsistent (although see below). The demonstration given here only shows that a contradiction results if we assume premises (P1) and (P2). In order to stave off contradiction, however, we cannot deny (P1). Although not itself a theorem of the system, axiom $FC^{+SB}35$ guarantees us at least one *Sinn* picking out Y , and whatever *Sinn* we take this to be can be used in formulating a similar paradox. Thus, we are left with denying (P2). The negation of (P2) thus becomes a *theorem* of the system. However, (P2) seems to state something that could at least be contingently true of Church's preferences with regard to *Gedanken*. It is obviously undesirable for a logical system to lead to results that are not logical truths. Thus, there seem to be problems even with system FC^{+SB} .

The problems do not end there. As it turns out, an argument can be found for the outright inconsistency of system FC^{+SB} . The argument comes from considering a certain variant of the *Principles of Mathematics* Appendix B or Russell-Myhill antinomy. Specifically, it derives from a Cantorian diagonal construction between *Gedanken* and concepts. Let us first state the antinomy informally, and then discuss the formal details. Consider universal *Gedanken*, such as those expressed by the English sentences, "everything is human" or "everything is a planet" or "everything is self-identical." For every concept expression, such a *Gedanke* exists, even if it is not true. Some of these *Gedanken* are such that they themselves fall under the concept which they generalize, some do not. For example, the *Gedanke* expressed by "everything is self-identical" is itself self-identical. However, the *Gedanke* expressed by "everything is a planet" is not itself a planet. Let us now define the concept F , which is such that an object falls under it just in case, as with our previous

example, it is a universal *Gedanke* that itself does *not* fall under its corresponding concept. However, consider now the *Gedanke* expressed by “Everything is F .” Does this *Gedanke* fall under concept F ? A universal *Gedanke* falls under F just in case it does not fall under its corresponding concept, and so this universal *Gedanke* falls under F just in case it does not.

Let us now turn to the formalization of this antinomy in system FC^{+SB} . Consider incomplete *Sinne* such as our $\mathfrak{D}(\delta)$, that pick out one-place functions such as our $\mathcal{P}(\xi)$. When these incomplete *Sinne* mutually saturate with the second-level incomplete *Sinn* of “ $(\forall a)\phi(a)$ ”, viz., $(\Pi a^*)\theta(a^*)$, a *Gedanke* is formed. When $\mathfrak{D}(\delta)$ mutually saturates with $(\Pi a^*)\theta(a^*)$, the *Gedanke* formed, $(\Pi a^*)\mathfrak{D}(a^*)$, picks out the truth-value of everything’s being a planet. If $(\Pi a^*)\theta(a^*)$ merges with the incomplete *Sinn* $\delta \approx \delta$, the resulting *Gedanke*, $(\Pi a^*)(a^* \approx a^*)$, picks out the truth-value of everything’s being self-identical. These *Gedanken* are objects. First-level functions, however, must be defined over all objects. Some of these universal *Gedanken* will be such that, when they are taken as argument to the first-level function picked out by the incomplete *Sinn* that mutually saturates with $(\Pi a^*)\theta(a^*)$, the True is given as value. For example, the *Gedanke* $(\Pi a^*)(a^* \approx a^*)$, when taken as argument to the function picked out by the incomplete *Sinn* $\delta \approx \delta$, viz., $\xi = \xi$, yields the True as value, since “ $(\Pi a^*)(a^* \approx a^*) = (\Pi a^*)(a^* \approx a^*)$ ” denotes the True. However, some such *Gedanken* will not be so as to yield the True when taken as argument to the function picked out by the incomplete *Sinn* mutually saturated with $(\Pi a^*)\theta(a^*)$. Here, $(\Pi a^*)\mathfrak{D}(a^*)$, is such a *Gedanke*. The incomplete *Sinn* $\mathfrak{D}(\delta)$ picks out the function $\mathcal{P}(\xi)$. When the *Gedanke* $(\Pi a^*)\mathfrak{D}(a^*)$ is taken as argument to $\mathcal{P}(\xi)$, the value is the False, since $(\Pi a^*)\mathfrak{D}(a^*)$ is not a planet, i.e., “ $\mathcal{P}((\Pi a^*)\mathfrak{D}(a^*))$ ” denotes the False.

Now, consider the concept $F(\xi)$, whose value is the True in case its argument is a *Gedanke* of this form that, like $(\Pi a^*)\mathfrak{D}(a^*)$, does not yield the True when it saturates the function picked out by the incomplete *Sinn* mutually saturated with $(\Pi a^*)\theta(a^*)$, and whose value is the False otherwise. This concept can be characterized as follows:

$$\text{(Df. } F) \quad \Vdash F(x) = (\exists f\mathfrak{F})\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{F}(a^*), f(a))] \ \& \\ (x = (\Pi a^*)\mathfrak{F}(a^*)) \ \& \sim f(x)\}$$

Consider now an incomplete *Sinn* picking out this function. Axiom $FC^{+SB}36$ guarantees us such an incomplete *Sinn*. One instance of $FC^{+SB}36$ reads:

$$\vdash (\exists \mathfrak{F})(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{F}(a^*), F(a))]$$

Let us introduce a sign for the incomplete *Sinn* (or one of the incomplete *Sinne*) postulated by this proposition, “ $\mathfrak{A}(\delta)$ ”. Thus we have the following:

$$(Pp. \mathfrak{A}) \vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), F(a))]$$

As a first-level incomplete *Sinn*, $\mathfrak{A}(\delta)$ itself can mutually saturate with $(\Pi a^*)\theta(a^*)$. We now ask the question of what is yielded as value when the resulting *Gedanke*, $(\Pi a^*)\mathfrak{A}(a^*)$, is taken as argument to our function $F(\xi)$. The result will be the True just in case $\mathfrak{A}(\delta)$ picks out a function that yields something other than the True when $(\Pi a^*)\mathfrak{A}(a^*)$ is taken as argument. But $\mathfrak{A}(\delta)$ picks out $F(\xi)$, and so, $F(\xi)$ will yield the True as value for $(\Pi a^*)\mathfrak{A}(a^*)$ as argument just in case it does not.

The demonstration of this antinomy is given below. Again, while Pp. \mathfrak{A} is not strictly speaking a theorem of the system FC^{+SB} , due to the above instance of $FC^{+SB}36$ (which is a theorem), for any proof carried out by means of this principle, there will be a longer proof for the same (or similar) conclusion that does not make use of it. With it, the antinomy is formulated as follows:

$$\vdash \neg F((\Pi a^*)\mathfrak{A}(a^*)) \ \& \ \sim \neg F((\Pi a^*)\mathfrak{A}(a^*))$$

To prove this contradiction, we first need to prove the following theorem of FC^{+SB} . This theorem states that incomplete *Sinne* pick out unique functions.

$$\text{Theorem } FC^{+SB}13.1 \vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \ \therefore (\forall a)(f(a) = g(a))$$

Proof:

1. $\vdash \Delta(a, b) \ \& \ \Delta(a, c) \ \therefore (b = c)$ axiom $FC^{+SB}13$
2. $\vdash \Delta(F(a^*), b) \ \& \ \Delta(F(a^*), c) \ \therefore (b = c)$ 1, *ri*
3. $\vdash \Delta(F(a^*), f(a)) \ \& \ \Delta(F(a^*), g(a)) \ \therefore (f(a) = g(a))$ 2, *ri, ri*
4. $\vdash (\forall aa^*)[\Delta(F(a^*), f(a)) \ \& \ \Delta(F(a^*), g(a)) \ \therefore (f(a) = g(a))]$ 3, *gen*
5. $\vdash (\forall aa^*)[h(a, a^*) \supset i(a, a^*)] \ \& \ (\forall aa^*)[h(a, a^*) \supset j(a, a^*)] \ \therefore$
 $(\forall aa^*)[h(a, a^*) \supset i(a, a^*) \ \& \ j(a, a^*)]$ *fc*
6. $\vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))]$
 $\therefore (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a)) \ \& \ \Delta(F(a^*), g(a))]$ 5, *ri, ri, ri*
7. $\vdash p \supset (\forall aa^*)[h(a, a^*) \supset i(a, a^*)] \ \therefore$
 $(\forall aa^*)[i(a, a^*) \supset j(a, a^*)] \ \therefore p \supset (\forall aa^*)[h(a, a^*) \supset j(a, a^*)]$ *fc*
8. $\vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))]$
 $(\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a)) \ \& \ \Delta(F(a^*), g(a))] \ \therefore$
 $(\forall aa^*)[\Delta(F(a^*), f(a)) \ \& \ \Delta(F(a^*), g(a)) \ \therefore j(a)] \ \therefore$

- $$(\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \\ \therefore (\forall aa^*)[\Delta(a^*, a) \supset j(a)] \quad 7, \text{ri}, \text{ri}, \text{ri}$$
9. $\vdash (\forall aa^*)[\Delta(F(a^*), f(a)) \ \& \ \Delta(F(a^*), g(a)) \ \supset \ j(a)] \ \therefore:$
 $(\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \ \therefore.$
 $(\forall aa^*)[\Delta(a, a^*) \supset j(a)] \quad 6, 8, \text{mp}$
10. $\vdash (\forall aa^*)[\Delta(F(a^*), f(a)) \ \& \ \Delta(F(a^*), g(a)) \ \supset \ (f(a) = g(a))] \ \therefore:$
 $(\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \ \therefore.$
 $(\forall aa^*)[\Delta(a^*, a) \supset (f(a) = g(a))] \quad 9, \text{ri}$
11. $\vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \\ \therefore (\forall aa^*)[\Delta(a^*, a) \supset (f(a) = g(a))] \quad 4, 10, \text{mp}$
12. $\vdash (\forall aa^*)[h(a, a^*) \supset g(a)] \supset (\forall a)[(\exists a^*)h(a, a^*) \ \therefore \ g(a)] \quad \text{fc}$
13. $\vdash (\forall aa^*)[\Delta(a^*, a) \supset (f(a) = g(a))] \supset (\forall a)[(\exists a^*)\Delta(a^*, a) \supset (f(a) = g(a))] \\ 12, \text{ri}, \text{ri}$
14. $\vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \ \therefore.$
 $(\forall a)[(\exists a^*)\Delta(a^*, a) \supset (f(a) = g(a))] \quad 11, 13, \text{syll}$
15. $\vdash (\forall a)(h(a) \supset i(a)) \ \therefore (\forall a)h(a) \supset (\forall a)i(a) \quad \text{fc}$
16. $\vdash (\forall a)[(\exists a^*)\Delta(a^*, a) \supset (f(a) = g(a))] \ \therefore.$
 $(\forall a)(\exists a^*)\Delta(a^*, a) \supset (\forall a)(f(a) = g(a)) \quad 15, \text{ri}, \text{ri}$
17. $\vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \ \therefore.$
 $(\forall a)(\exists a^*)\Delta(a^*, a) \supset (\forall a)(f(a) = g(a)) \quad 14, 16, \text{syll}$
18. $\vdash q \ \therefore \therefore \ p \ \therefore \ q \supset r \ \therefore \ p \supset r \quad \text{fc}$
19. $\vdash (\forall a)(\exists a^*)\Delta(a^*, a) \ \therefore \therefore \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \\ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \ \therefore (\forall a)(\exists a^*)\Delta(a^*, a) \supset \\ (\forall a)(f(a) = g(a)) \ \therefore \therefore \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \\ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \ \therefore (\forall a)(f(a) = g(a)) \quad 18, \text{ri}, \text{ri}, \text{ri}$
20. $\vdash (\exists a^*)\Delta(a^*, b) \quad \text{axiom FC}^{\text{SB}35}$
21. $\vdash (\forall a)(\exists a^*)\Delta(a^*, a) \quad 20, \text{gen}$
22. $\vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \ \therefore.$
 $(\forall a)(\exists a^*)\Delta(a^*, a) \supset (\forall a)(f(a) = g(a)) \ \therefore \therefore \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \\ \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \ \therefore (\forall a)(f(a) = g(a)) \quad 19, 21, \text{mp}$
23. $\vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \ \therefore.$
 $(\forall a)(f(a) = g(a)) \quad 17, 22, \text{mp}$

With this theorem in place, we derive the contradiction as follows:

$$\vdash \overline{F}((\Pi a^*)\mathfrak{A}(a^*)) \ \& \ \sim \overline{F}((\Pi a^*)\mathfrak{A}(a^*))$$

Proof:

1. $\vdash \overline{F}((\Pi a^*)\mathfrak{A}(a^*)) = (\exists f\mathfrak{F})\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{F}(a^*), f(a))] \ \& \\ [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathfrak{F}(a^*)] \ \& \ \sim f((\Pi a^*)\mathfrak{A}(a^*))\} \quad \text{df. } \overline{F}, \text{ri}$

2. $\vdash (\Pi a^*)F(a^*) = (\Pi a^*)G(a^*) \supset (\forall a^*)(F(a^*) = G(a^*))$ theorem FC^{SB}48.1
3. $\vdash (\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)G(a^*) \supset (\forall a^*)(\mathfrak{A}(a^*) = G(a^*))$ 2, *ri*
4. $\vdash p \supset q \supset (s \ \& \ p \ \& \ t) \supset (s \ \& \ t \ \& \ q)$ *fc*
5. $\vdash (\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)G(a^*) \supset (\forall a^*)(\mathfrak{A}(a^*) = G(a^*)) \supset$
 $\{s \ \& \ [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)G(a^*)] \ \& \ t\} \supset [s \ \& \ t \ \& \ (\forall a^*)(\mathfrak{A}(a^*) = G(a^*))]$
4, ri, ri
6. $\vdash \{s \ \& \ [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)G(a^*)] \ \& \ t\} \supset [s \ \& \ t \ \& \ (\forall a^*)(\mathfrak{A}(a^*) = G(a^*))]$
3, 5, mp
7. $\vdash \{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(G(a^*), f(a))] \ \& \ t \ \& \ (\forall a^*)(F(a^*) = G(a^*))\} \supset$
 $\{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \ \& \ t\}$ *fc*
8. $\vdash \{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(G(a^*), f(a))] \ \& \ t \ \& \ (\forall a^*)(\mathfrak{A}(a^*) = G(a^*))\} \supset$
 $\{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), f(a))] \ \& \ t\}$ 7, *ri*
9. $\vdash \{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(G(a^*), f(a))] \ \& \ [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)G(a^*)] \ \& \ t\} \supset$
 $[(\forall a a^*)[\Delta(a^*, a) \supset \Delta(G(a^*), f(a))] \ \& \ t \ \& \ (\forall a^*)(\mathfrak{A}(a^*) = G(a^*))]$ 6, *ri*
10. $\vdash \{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(G(a^*), f(a))] \ \& \ [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)G(a^*)] \ \& \ t\} \supset$
 $\{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), f(a))] \ \& \ t\}$ 8, 9, *syll*
11. $\vdash (\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), f(a))] \ \& \ (\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), \mathcal{F}(a))] \supset$
 $(\forall a)(f(a) = \mathcal{F}(a))$ Theorem FC^{SB}13.1, *ri, ri*
12. $\vdash q \supset \supset p \ \& \ q \supset r \supset \supset p \supset r$ *fc*
13. $\vdash (\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), \mathcal{F}(a))] \supset \supset (\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), f(a))]$
 $\ \& \ (\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), \mathcal{F}(a))] \supset \supset (\forall a)(f(a) = \mathcal{F}(a)) \supset \supset$
 $(\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), f(a))] \supset \supset (\forall a)(f(a) = \mathcal{F}(a))$ 12, *ri, ri, ri*
14. $\vdash (\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), f(a))] \ \& \ (\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), \mathcal{F}(a))] \supset$
 $(\forall a)(f(a) = \mathcal{F}(a)) \supset \supset (\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), f(a))] \supset \supset (\forall a)(f(a) = \mathcal{F}(a))$
Pp. 13, mp
15. $\vdash (\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), f(a))] \supset \supset (\forall a)(f(a) = \mathcal{F}(a))$ 11, 14, *mp*
16. $\vdash x \supset y \supset \supset (x \ \& \ t) \supset \supset (y \ \& \ t)$ *fc*
17. $\vdash (\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), f(a))] \supset \supset (\forall a)(f(a) = \mathcal{F}(a)) \supset \supset$
 $\{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), f(a))] \ \& \ t\} \supset \supset [(\forall a)(f(a) = \mathcal{F}(a)) \ \& \ t]$ 16, *ri, ri*
18. $\vdash \{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), f(a))] \ \& \ t\} \supset \supset [(\forall a)(f(a) = \mathcal{F}(a)) \ \& \ t]$
15, 17, mp
19. $\vdash \{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(G(a^*), f(a))] \ \& \ [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)G(a^*)] \ \& \ t\} \supset$
 $[(\forall a)(f(a) = \mathcal{F}(a)) \ \& \ t]$ 10, 18, *syll*
20. $\vdash [(\forall a)(f(a) = \mathcal{F}(a)) \ \& \ \sim f(x)] \supset \supset \sim \mathcal{F}(x)$ *fc*
21. $\vdash \{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(G(a^*), f(a))] \ \& \ [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)G(a^*)]$
 $\ \& \ \sim f(x)\} \supset \supset [(\forall a)(f(a) = \mathcal{F}(a)) \ \& \ \sim f(x)]$ 19, *ri*
22. $\vdash \{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(G(a^*), f(a))] \ \& \ [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)G(a^*)]$
 $\ \& \ \sim f(x)\} \supset \supset \sim \mathcal{F}(x)$ 20, 21, *syll*
23. $\vdash \{(\forall a a^*)[\Delta(a^*, a) \supset \Delta(G(a^*), f(a))] \ \& \ [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)G(a^*)] \ \& \$
 $\ \sim f((\Pi a^*)\mathfrak{A}(a^*))\} \supset \supset \sim \mathcal{F}((\Pi a^*)\mathfrak{A}(a^*))$ 22, *ri*

24. $\vdash \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*) \supset \sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(G(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)G(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\}$ 23, *con*
25. $\vdash \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*) \supset (\forall f\mathcal{F})\sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathcal{F}(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathcal{F}(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\}$ 24, *gen, gen*
26. $\vdash \sim(\forall f\mathcal{F})\sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathcal{F}(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathcal{F}(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\} \supset \sim \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*)$ 25, *con*
27. $\vdash (\exists f\mathcal{F})\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathcal{F}(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathcal{F}(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\} \supset \sim \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*)$ 26, *df. $\exists b, \exists e$*
28. $\vdash \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*) \supset \sim \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*)$ 1, 27, *i*
29. $\vdash p \supset \sim p \therefore \sim p$ *fc*
30. $\vdash \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*) \supset \sim \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*) \therefore \sim \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*)$ 29, *ri*
31. $\vdash \sim \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*)$ 28, 30, *mp*
32. $\vdash \sim(\exists f\mathcal{F})\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathcal{F}(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathcal{F}(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\}$ 1, 31, *i*
33. $\vdash (\forall f)(\forall \mathcal{F})\sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathcal{F}(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathcal{F}(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\}$ 32, *df. $\exists b, \exists e$*
34. $\vdash (\forall f)M_\beta(f(\beta)) \supset M_\beta(f(\beta))$ axiom FC4
35. $\vdash (\forall f)(\forall \mathcal{F})\sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathcal{F}(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathcal{F}(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\} \supset (\forall \mathcal{F})\sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathcal{F}(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathcal{F}(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\}$ 34, *ri*
36. $\vdash (\forall \mathcal{F})\sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathcal{F}(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathcal{F}(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\}$ 33, 35, *mp*
37. $\vdash (\forall \mathcal{F})M_\beta(\mathcal{F}(\beta)) \supset M_\beta(\mathcal{F}(\beta))$ axiom FC^{+SB}11
38. $\vdash (\forall \mathcal{F})\sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathcal{F}(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathcal{F}(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\} \supset \sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)F(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\}$ 37, *ri*
39. $\vdash \sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)F(a^*)] \& \sim f((\Pi a^*)\mathfrak{A}(a^*))\}$ 36, 38, *mp*
40. $\vdash \sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), F(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathfrak{A}(a^*)] \& \sim \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*)\}$ 39, *ri, ri*
41. $\vdash \sim(p \& q \& \sim r) \therefore p \therefore q \supset r$ *fc*
42. $\vdash \sim\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), F(a))] \& [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathfrak{A}(a^*)] \& \sim \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*)\} \therefore (\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), F(a))] \therefore [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathfrak{A}(a^*)] \supset \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*)$ 41, *ri*
43. $\vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{A}(a^*), F(a))] \therefore [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathfrak{A}(a^*)] \supset \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*)$ 40, 42, *mp*
44. $\vdash [(\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathfrak{A}(a^*)] \supset \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*)$ Pp. \mathfrak{A} , 43, *mp*
45. $\vdash a = a$ *fc*
46. $\vdash (\Pi a^*)\mathfrak{A}(a^*) = (\Pi a^*)\mathfrak{A}(a^*)$ 45, *ri*
47. $\vdash \mathcal{F}(\Pi a^*)\mathfrak{A}(a^*)$ 44, 46, *mp*

48. $\vdash p \supset: \sim p \supset. p \ \& \ \sim p$ *fc*
 49. $\vdash \mathcal{F}(\Pi a^*) \mathfrak{A}(a^*) \supset: \sim \mathcal{F}(\Pi a^*) \mathfrak{A}(a^*) \supset. \mathcal{F}(\Pi a^*) \mathfrak{A}(a^*) \ \& \ \sim \mathcal{F}(\Pi a^*) \mathfrak{A}(a^*)$ *48, ri*
 50. $\vdash \sim \mathcal{F}(\Pi a^*) \mathfrak{A}(a^*) \supset. \mathcal{F}(\Pi a^*) \mathfrak{A}(a^*) \ \& \ \sim \mathcal{F}(\Pi a^*) \mathfrak{A}(a^*)$ *47, 49, mp*
 51. $\vdash \mathcal{F}(\Pi a^*) \mathfrak{A}(a^*) \ \& \ \sim \mathcal{F}(\Pi a^*) \mathfrak{A}(a^*)$ *31, 50, mp*

This completes the argument that even system FC^{+SB} is inconsistent.

This is a quite striking result. Since the system FC^{+SB} is built upon, FC , is consistent, the addition of the new axioms of this chapter must be considered the source of the difficulty. Frege's theory of *Sinn* and *Bedeutung*, while not uniformly plausible, seems to be at least *internally* consistent. However, the inconsistency of FC^{+SB} shows that the addition of axioms in line with Frege's own understanding of the nature of *Sinne* and *Gedanken* leads to formal difficulties when added to an otherwise consistent system. In this instance, that consistent system is the core of Frege's own Begriffsschrift of the *Grundgesetze*. The system FC^{+SB} , however, does not contain the portion of the system of the *Grundgesetze* that has been identified by most writers on Frege as the source of the previously identified difficulties. In particular, FC^{+SB} does not contain Frege's Basic Law V (axiom FC^{+V7}), usually blamed for Russell's paradox in Frege's extant system. The difficulties with FC^{+SB} are quite independent of those previously discussed difficulties. We can only conclude that the expansion of Frege's Begriffsschrift in line with his own semantic theories—indeed, in ways he himself suggests—reveals internal difficulties and flaws within his overall philosophical position that have hitherto gone unnoticed and unaddressed. Frege himself seems to have missed these flaws only because he himself never attempted to include the commitments of his theory of *Sinn* and *Bedeutung* within a logical calculus.

Now that we have redressed the failure on Frege's part to complete his Begriffsschrift, we are in a better position to evaluate his philosophy and to look for possible solutions to the difficulties plaguing it. In the previous section, we saw ways in which the expansion of Frege's logical system suggested here would actually help him respond to objections and provide solutions to philosophical conundrums. However, because these responses and solutions derive from an understanding of the nature of *Sinne* and *Gedanken* we have now found to be riddled with purely formal difficulties, they cannot be understood to be fully adequate. Nevertheless, the question remains as to whether a *broadly* Fregean philosophy of language and mind can fare better, incorporating the positive aspects of his views while circumventing the formal difficulties. In hopes of answering this question, we turn to an examination of the source of the formal difficulties within Frege's theory of *Sinn* and *Bedeutung* identified in this chapter. We first attempt to do this by considering Frege's views in light of

those of like-minded thinkers, the similar difficulties such thinkers have faced, and the solutions they have offered. This may be the best place to look for fixes to the formal problems identified here.

Comparison with Russell and Other Thinkers

“SOLATIUM MISERIS, SOCIOS HABUISSE MALORUM.”

The title of this section is Latin for the proverb, “it is a comfort for the wretched to have companions in misery.” Frege includes this sentiment in the appendix of the *Grundgesetze* in which he first responded to Russell’s paradox (BL 127). Frege was relieved that he was not the only thinker to have to face the paradox. We can only assume that, had he been aware of the problems in his theory of *Sinn* and *Bedeutung* discussed in the last chapter, he would have been similarly comforted that parallel problems plague the theories of others. In this chapter, we turn to an examination of some theories of logical entities and language that are in some ways close to Frege’s own; close enough, at least, that they face some of the same challenges. Through an examination of similar theories, and how the thinkers who held them struggled with the sorts of problems now seen to plague Frege’s philosophy of language, we can gain insight both into the ultimate source of the difficulties, as well as into solutions to them that might be adopted from within a broadly Fregean standpoint.

The works of Bertrand Russell are especially helpful in this regard. As seen in the first chapter, Frege’s primary philosophical project was the articulation and defense of logicism. No prominent thinker shared both the goals and strategies of Frege’s logicism more than Russell. To be sure, there are important differences between Frege’s and Russell’s versions of logicism, as well as their logical and philosophical views (many of which we explore in this chapter). However, the core shared by the two thinkers is remarkable. Of particular interest in this context is that both Frege and Russell were realists with regard to logical entities; neither held the widespread contemporary view that logic must be without ontological import. As we have seen, Frege was committed to the

objective existence of a wide assortment of logical entities including truth-values, concepts, functions, value-ranges (including classes), as well as the many denizens of the third realm of *Sinn*. These commitments, however, pose serious problems for Frege. The commitments Russell made to logical entities varied during different periods of his career; but especially early on, he too was a realist about many sorts of logical entities. For this reason, he too faced difficulties.

After the discovery of the paradox bearing his name, however, no one was more aware than Russell of the propensity of the inclusion of such commitments within a logical system to lead to paradox. In the years between his discovery of the paradox while working on *The Principles of Mathematics* and the publication of the first volume of *Principia Mathematica*, Russell seems to have been engaged in little else but an examination of the sources of the paradoxes plaguing the foundations of logic and consideration of possible resolutions. We find him engaged in examination of contradictions very similar to the sorts considered in the present work and desperately searching for philosophically adequate solutions. He did have a number of successes in finding solutions to various paradoxes. However, he fell into a pattern that as soon as a solution to one paradox was found, another would rear its head. Searching for solace in his own misery, Russell wrote to Frege complaining that “from Cantor’s proposition that any class contains more subclasses than objects we can elicit constantly new contradictions.”¹

It is worth taking a closer look at exactly what sorts of contradictions Russell had in mind as well as how they arose within Russell’s own philosophy, and the steps he took to attempt to solve them. It is also worth taking a look at Frege’s response to some of Russell’s concerns. Our primary purpose here is to consider the problems plaguing Russell’s logic from the standpoint of someone interested in the Fregean logic of *Sinn* and *Bedeutung*. What can be learned from Russell’s efforts that might be relevant to this project? Can company help alleviate our misery?

RUSSELL’S ONTOLOGY OF PROPOSITIONS

At least until some time in 1907, Russell held an ontology of propositions. While the details of his understanding of propositions (especially after “On Denoting” in 1905) went through a number of changes from 1900 to 1907, the core remained the same. “Propositions,” in Russell’s sense, are not at all similar to “propositions” (*Sätze*) in Frege’s vocabulary, which are just declarative sentences. Russell’s propositions are closer to states of affairs or possible facts.

¹Russell to Frege, London, 29 September 1902, *Philosophical and Mathematical Correspondence*, by Gottlob Frege, 147.

They are thought to exist independently of mind and language. The proposition corresponding to the sentence “Socrates is wise” is to be understood as a whole consisting of Socrates the person and wisdom, understood as a Platonic universal. By and large, the constituents of a Russellian proposition are the entities named in a sentence that expresses it. This is complicated only by the pre-1905 theory of denoting concepts or the post-1905 theory of descriptions, although we can overlook these complications for present purposes.

This theory has implications for Russell’s logic. Variables in Russell’s early logic were unrestricted; they were taken to range over whatever is. Propositions, as entities, thus fall in the range of the quantifiers. It would not be uncommon to find Russell writing formulae such as “ $(p)(p \supset p)$ ”, i.e., p implies p , whatever p may be. Indeed, given Russell’s understanding of propositions as entities, in his early logic, logical connectives such as “ \supset ” should be understood as primitive *relation* symbols. In an expression such as “ $p \supset q$ ”, “ p ” and “ q ” are understood as *terms* standing for propositions (or other entities), and thus “ $p \supset q$ ” can be understood as having the same logical form as a relational expression such as “ aRb ”. Because “ \supset ” is to be understood as a relation sign flanked by terms, in the complex expression “ $p \supset (q \supset r)$ ”, the consequent clause, “ $(q \supset r)$ ”, must similarly be understood as a *term* standing for a proposition. Thus, for every well-formed formula of his system, Russell is committed to a proposition existing corresponding to it. In certain reconstructions of his early logical systems, Russell scholars have advocated using nominalizing brackets, “{” and “}”, around formulae for terms for such propositions,² although Russell himself leaves off such brackets. In Russell’s early logical systems, an expression such as “ $p = \{q \supset r\}$ ” could be understood as stating that p is the proposition that q implies r . Because variables in Russell’s early logic are unrestricted, it is permissible to instantiate them to terms formed using such brackets.

Owing to this, there are features of Russell’s logic shared by Frege’s logic but not by most contemporary logical systems. In standard contemporary logic, there is a difference between terms and formulae, and thus a corresponding difference between the syntax of relation signs (which are flanked by terms) and the syntax of logical connectives (which are flanked by formulae). Thus, formulae such as “ $(1 + 2 = 3) = (4 = 4)$ ” and “ $2 \supset 1$ ” are ill-formed. As a relation sign, “ $=$ ” can be flanked only by individual variables or other terms, not by complete propositions, and “ \supset ”, as a connective, can be flanked only by formulae. In both Russellian and Fregean logic, these distinctions are somewhat more complicated. As we have seen, in Russell’s early logical systems,

²See, e.g., Gregory Landini, “Logic in Russell’s *Principles of Mathematics*,” *Notre Dame Journal of Formal Logic* 37 (1996): 561, *Russell’s Hidden Substitutional Theory* (New York: Oxford University Press, 1998), 43.

connectives are understood as having the same syntax as relations. Moreover, formulae can be made into terms with the addition of nominalizing brackets. In Fregean Begriffsschrift, both relations and connectives are transcribed using function signs, and both are defined over all objects as argument. Thus, they too have the same syntax. While Frege does acknowledge a difference between terms and propositions, the expressions that appear after Frege's judgment stroke are understood as *terms* standing for objects, typically truth-values. Roman letters in Frege's logic can be instantiated to any well-formed expressions (in the sense defined in Chapter 2).

While Frege and Russell would understand the *semantics* of formulae such as " $2 \supset 1$ " or " $(1 + 2 = 3) = (4 = 4)$ " (in Russell's case, the latter would be better written as " $\{1 + 2 = 3\} = \{4 = 4\}$ ") very differently, both could understand these as well-formed. For Frege, both of these are names of the True. In the first case, this is because the conditional function always has the True as value if its first argument is not a truth-value. In the second case, it stands for the True because the expressions on both sides of the main identity sign, " $1 + 2 = 3$ " and " $4 = 4$ " are themselves names for the True, and the True is self-identical. For Russell, the first asserts that two implies one. Given the way Russell uses the " \supset " sign in his very early work, such as *The Principles of Mathematics*, Russell would regard this as false, because on the view held at that time, only propositions can stand in true implications.³ However, at least by the time of his paper "The Theory of Implication,"⁴ written in 1905, and perhaps before, he reinterpreted the horseshoe such that " $x \supset y$ " is to be regarded as true if x is not a proposition. The second asserts that the proposition that one plus two is three is the same proposition as the proposition that four is self-identical, which is unquestionably false.

Frege's *Gedanken* are often taken to be the closest analogues in his philosophy to Russell's propositions. Indeed, in the appendix of *The Principles of Mathematics* dedicated to Frege, Russell wrote that "the *Gedanke* is what I have called an unasserted proposition."⁵ This may be, but it is important to realize the important differences between Fregean *Gedanken* and Russellian propositions. For Russell, a proposition can be thought to be the "meaning" of a sentence. Unlike Frege, Russell does not have a dualism between two aspects of meaning; the proposition in some ways plays the role of Frege's *Sinn* and in other ways plays the role of Frege's *Bedeutung*. The components of a Russellian proposition such as that expressed by "Socrates is wise" are the actual entities

³Russell, *The Principles of Mathematics*, §16.

⁴Bertrand Russell, "The Theory of Implication," *American Journal of Mathematics* 28 (1906): 159-202.

⁵Russell, *Principles of Mathematics*, §477.

named by the words, including, in this case, Socrates.⁶ On Russell's understanding, these entities come together to form a whole. For Frege, while the actual entities named by the words in a sentence, their *Bedeutungen*, determine the *Bedeutung* of the whole sentence, as we have seen, this is not in the sense of coming together to form a whole. For Frege, it is only in the realm of *Sinn* that what corresponds to parts of sentences come together to form a whole, viz., a *Gedanke*. Socrates himself is not a part of the whole *Gedanke* expressed by "Socrates is wise"; rather, it is the *Sinn* of the name "Socrates" that is part of the whole. Frege's system does not in any way commit him to a whole in which Socrates himself is contained either in the realm of *Sinn* or in the realm of *Bedeutung*.

In not adopting a *Sinn/Bedeutung* distinction, Russell's ontology may be thought to be simpler. Whether it appears in direct or indirect speech, for Russell, the meaning of a sentence is a proposition. For Russell, a belief involves a relation between a believer and a proposition, just as for Frege a belief involves a relation between a believer and a *Gedanke*. In expressing a propositional attitude, Russell could use a term for a proposition flanking a sign for the belief relation. This term might be no different from the name of a proposition that occurs flanking a logical connective. Since Russell regards the meaning of an expression to be the same in every occurrence, he requires a different solution to the puzzles about belief and quantifying in. These puzzles are treated in different ways at different points in his career. But because the details of the account he offered from 1905 on, viz., the theory of descriptions, are well known, we need not dwell on them here.⁷

RUSSELL AND THE *PRINCIPLES OF MATHEMATICS*

APPENDIX B PARADOX

Although in some ways simpler than Frege's ontology, Russell's commitment to propositions still leads to difficulties, as he himself began to discover as early as 1902. In the first chapter we discussed briefly the paradox of propositions Russell outlined in Appendix B of his 1903 *Principles of Mathematics*. This paradox was also discussed in Chapter 4 as the "Russell-Myhill antinomy." It is worth examining this paradox in greater detail, especially as it brings to the fore both the differences between Frege and Russell on the nature of logical entities,

⁶Given the theory of ordinary proper names Russell eventually adopts, a name such as "Socrates" might be taken as a disguised definite description, in which case Socrates the person would not actually be a constituent of the proposition in question. For present purposes, however, this can be overlooked.

⁷See "On Denoting," 113-5.

as well as ways in which, despite their differences, their logical systems face at least some similar challenges.

Russell's views on the nature of propositions committed him implicitly to certain views regarding their identity conditions and logical relations. Russell himself never explicitly axiomatized his views regarding the identity conditions of propositions within a logical calculus; instead, he seems to have operated with an informal understanding of the logical principles governing propositional identities.⁸ However, certain Russell scholars have attempted to make explicit Russell's commitments by devising a number of principles governing the intensional logic of propositions as held by Russell in *The Principles of Mathematics* and other works of the period. This was first attempted by Church.⁹ In a pair of related papers, Anderson criticized some of the details of Church's original formulation, and offered another in its place.¹⁰ The differences between their versions are not important for our present purposes. Their formulations consist of a number of principles that could be added as additional axioms to a standard higher-order system.

Let us consider some examples. In stating these principles, I deviate somewhat from Church and Anderson's own syntax. Firstly, I employ nominalizing brackets as discussed above to make more evident the occurrences of propositions as terms. Church built his formulation of Russellian intensional logic on his reconstruction of the system of *Principia Mathematica* simplified from a ramified to a simple type-theory.¹¹ In his reconstruction, the identity sign "=" can be flanked only by individual variables or constants and not by complicated formulae. Thus, Church introduces a new primitive sign, "≡", used to express identities of propositions, but with the syntax of a connective rather than a relation. However, there is no reason to follow Church in this. As noted above, in his early logical systems, Russell himself did not draw a distinction in syntax between connectives and relation signs, and he himself allowed the

⁸However, Gregory Landini has shown that many of the principles concerning propositional identities can be proven as theorems within the logic of Russell's substitutional theory held between 1905 and 1907. See his *Russell's Hidden Substitutional Theory*, 112-26.

⁹Alonzo Church, "Russell's Theory of Identity of Propositions," *Philosophia Naturalis* 21 (1984): 513-22.

¹⁰C. Anthony Anderson, "Some Difficulties Concerning Russellian Intensional Logic," *Noûs* 20 (1986): 35-43, "Russellian Intensional Logic," in *Themes from Kaplan*, ed. Joseph Almog, John Perry and Howard Wettstein (New York: Oxford University Press, 1989), 67-103.

¹¹This system is expounded in Alonzo Church, "Russellian Simple Type Theory," *Proceedings and Addresses of the American Philosophical Association* 47 (1974): 21-33.

identity sign “=” to be flanked by complicated formulae. Especially with the use of nominalizing brackets to make explicit the occurrences of propositional expressions as terms, the sign “≡” is unnecessary; “=” can be used instead. Consider, then, the following examples of such principles regarding propositional identity (with RIL for *Russellian intensional logic*):

Principle RIL1. $p = q \supset p \equiv q$

Principle RIL2. $\{p \supset q\} = \{r \supset s\} \supset p = r$

Principle RIL3. $\{p \supset q\} = \{r \supset s\} \supset q = s$

Principle RIL4. $\{(x)Fx\} = \{(x)Gx\} \supset (x)(\{Fx\} = \{Gx\})$

Principle RIL1 states that identical propositions are materially equivalent. Obviously, if p and q are the same proposition, one can be true if and only if the other is. Principle RIL2 and RIL3 state that identical conditional propositions have the same entities for their antecedents and consequents. Principle RIL4 asserts that if the proposition that everything is F is the same as the proposition that everything is G, then for any individual x , the proposition that x is F is the same as the proposition that x is G. These four principles are either included in the list of principles presented by both Church and Anderson, or are derivable from them. Church’s formulation contains fifteen axioms governing the identity of propositions altogether; Anderson’s system contains fourteen. Together these principles codify the Russellian notion that identical propositions consist of identical constituents as parts in the same positions.

With these principles in place, we can show how the *Principles of Mathematics* Appendix B paradox is formulated in Russellian intensional logic.¹² Again, the paradox involves a Cantorian diagonal construction between propositions and classes of propositions. First we define w , the class of all propositions stating the logical product of some class they are not in. This can be written as follows:

(Df. w) $w =_{\text{df}} \hat{p}: (\exists m)[p = \{(q)(q \in m \supset q)\} \ \& \ p \notin m]$ ¹³

Recall that “ $\hat{p}: \dots$ ” is to be read as “the class of all p such that \dots ” We then define the proposition r , stating the logical product of w , as follows:

(Df. r) $r =_{\text{df}} \{(q)(q \in w \supset q)\}$

¹²For a fuller discussion and derivation of this antinomy in Russellian intensional logic, see Church, “Russell’s Theory of Identity of Propositions,” 516-22.

¹³Cf. Russell to Frege, 29 September 1902, 147. I have, however, replaced Russell’s early notation borrowed from Peano with more contemporary notation.

We can then prove the following contraction:

$$r \in w \ \& \ r \notin w$$

Because we have not worked out a formal system of deduction, I shall only sketch the demonstration informally. Assume that $r \in w$. Then, by class abstraction, $(\exists m)[r = \{(q)(q \in m \supset q)\} \ \& \ r \notin m]$. Consider now the m such that $r = \{(q)(q \in m \supset q)\} \ \& \ r \notin m$. From this we can infer both $r = \{(q)(q \in m \supset q)\}$ and $r \notin m$. By the definition of r , $\{(q)(q \in w \supset q)\} = \{(q)(q \in m \supset q)\}$. By RIL4, then, $(q)[\{q \in w \supset q\} = \{q \in m \supset q\}]$. It then follows that $\{r \in w \supset r\} = \{r \in m \supset r\}$. By RIL2, it follows that $\{r \in w\} = \{r \in m\}$. By RIL1, then $(r \in w) \equiv (r \in m)$. Since $r \notin m$, we can infer $r \notin w$. This contradicts our assumption. Thus, we know that $r \notin w$. But now, by class abstraction, we know that $\sim(\exists m)[r = \{(q)(q \in m \supset q)\} \ \& \ r \notin m]$. From this, and the rules of propositional logic and the quantifiers, we can infer that $(m)[r = \{(q)(q \in m \supset q)\} \supset r \in m]$. Thus, $r = \{(q)(q \in w \supset q)\} \supset r \in w$. From the definition of r we have $r = \{(q)(q \in w \supset q)\}$. By *modus ponens*, we can conclude $r \in w$. Thus, $r \in w \ \& \ r \notin w$, QED.

Russell realized early on that this paradox plagued his philosophy. It proved to be particularly frustrating for him, largely because he also soon realized that, unlike Russell's paradox, it is not solved by adopting a simple theory of types.¹⁴ Moreover, there is a version of the paradox that does not make use of class terms.¹⁵ Instead of defining a class w of all those propositions stating the logical product of a class they are not in, we consider instead a propositional function W that is true of something just in case it is a proposition stating that all propositions for which some propositional function M is true are true, but itself does not have the M in question. We then consider r , the proposition stating that all propositions having W are true, and arrive at a similar result. Thus, we introduce W by means of an instance of Russell's comprehension principle for propositional functions:

$$(CP \ W) \ (x)(Wx \equiv (\exists M)[x = \{(q)(Mq \supset q)\} \ \& \ \sim Mx])$$

The proposition r is then defined in terms of W .

$$(Df. \ r) \ r =_{df} \ \{(q)(Wq \supset q)\}$$

¹⁴See Russell, *Principles of Mathematics*, §500.

¹⁵Russell hints at this in a later letter to Frege. See Russell to Frege, 24 May 1903, 159-60.

By a parallel demonstration, it follows that:

$$Wr \ \& \ \sim Wr$$

Because W is defined as propositional function applying to propositions, and r is a proposition, it is clear that the simple theory of types will not prevent this paradox. Nor, given the latter formulation, can the paradox be blamed on class theory.

Russell's main preoccupation in the years following the publication of the *Principles* was the attempt to find a philosophically adequate solution of Russell's paradox, this paradox of propositions, and similar antinomies (such as the liar paradox and Burali-Forti paradox). In 1905, with the help of the theory of descriptions, Russell finally discovered a way of reconciling the theory of types that seemed necessary to solve Russell's paradox with the view defended in the *Principles* that variables ought to be unrestricted. This was the logic of substitution. Strictly speaking, Russell's substitutional theory resulted in a type-free language with only one style of variable that at the same time was able to build a simple theory of types into the grammar by doing away with explicit class and propositional function terms but instead *proxying* them in terms of substitutional matrices.¹⁶ The resulting logic of substitution provided a genuine solution to Russell's paradox. However, the propositional paradoxes such as the *Principles of Mathematics* Appendix B paradox (formulated in terms of propositional functions) remained to be solved. Indeed, multiple versions of this sort of Cantorian propositional paradox are formulable in the initial versions of the logic for the substitutional theory, as presented in such papers as "On Some Difficulties on the Theory of Transfinite Numbers and Order Types," and "On the Substitutional Theory of Classes and Relations."¹⁷ One such version has been named the " p_0/a_0 paradox" by Landini.¹⁸

To solve this problem, in the 1906 paper published in French as "Les paradoxes de la logique,"¹⁹ Russell advocated abandoning general propositions. On this approach, while quantifiers could be regarded as acceptable components

¹⁶For the details of how this is accomplished, see Landini, *Russell's Hidden Substitutional Theory*, chaps. 4-7.

¹⁷Bertrand Russell, "On Some Difficulties in the Theory of Transfinite Numbers and Order Types," in *Essays in Analysis*, ed. D. Lackey (London: Open Court, 1973), 135-64, "On the Substitutional Theory of Classes and Relations," in *Essays in Analysis*, 165-89.

¹⁸Landini, *Russell's Hidden Substitutional Theory*, chap. 8, "New Evidence Concerning Russell's Substitutional Theory of Classes," *Russell* 9 (1989): 26-42.

¹⁹This paper was published in English as "On 'Insolubilia' and Their Solution by Symbolic Logic," in *Essays in Analysis*, 190-214.

of asserted statements, there would be no ontological commitment to general propositions understood as language-independent entities. In the logical language, this would mean that while quantifiers would be acceptable when placed at the beginning of a formula, one would no longer be able to form terms by placing nominalizing brackets around formulae involving quantifiers. For example, while one could assert that all propositions are in class w using the expression " $(q)(q \in w \supset q)$ ", there would be no term " $\{(q)(q \in w \supset q)\}$ " to which the variables could be instantiated. Since abandoning the theory of denoting concepts of the *Principles* in "On Denoting," Russell had been unsure of how to understand the constituents of general propositions. "Les paradoxes" offered a solution: there are no general propositions, only general assertions. The logical system that resulted from "Les paradoxes" succeeds in avoiding the version of the Appendix B paradox discussed above. In this version, the problematic proposition, r , is defined as a general proposition. With the abandonment of general propositions, r becomes indefinable; the paradox cannot be formulated in this form. Thus, Russell was able to solve this version of the *Principles of Mathematics* Appendix B paradox.

However, this success was short-lived. The abandonment of general propositions greatly weakened the system of "Les paradoxes." In order to recover any hope of reaching mathematics, Russell added a "mitigating axiom," positing, for any formula of the system (general or non-general), the existence of a (non-general) proposition true if and only if the formula holds.²⁰ However, soon thereafter, Russell discovered that this axiom, while still dealing adequately with the traditional version of the Appendix B paradox, results in the " p_0/a_0 paradox," a Cantorian antinomy very similar to the propositional functions version of the Appendix B paradox but formulated using a non-general proposition.²¹ The details of latter antinomy need not concern us here.

The important thing is that Russell was unable to solve the paradoxes of propositions while maintaining the robust ontology of propositions as logical entities initially adopted in the *Principles*. In 1908, Russell adopted the *ramified* theory of types that eventually appeared in the *Principia*.²² The ramified theory of types, its difference from the simple theory of types, and the ways in which it might be employed to block an antinomy such as the *Principles of Mathematics* Appendix B paradox, were discussed briefly in Chapter 4. The philosophical

²⁰Russell, "On 'Insolubilia'," 212.

²¹On this see Russell to Hawtrey, 22 January 1907, reprinted in *Russell's Hidden Substitutional Theory*, by Gregory Landini, ii.

²²Although Russell hinted at the underlying idea behind orders as early as §500 of the *Principles*, a worked out theory of orders appeared first in the 1908 "Mathematical Logic as Based on the Theory of Types."

justification of orders, however, only came from Russell's adoption of a new correspondence theory of truth related to his new multiple relations theory of judgment.²³ On this theory, the truth conditions of a statement are recursively defined; there is not one sense of "truth" but a whole hierarchy of senses of "truth." Orders are then explained in terms of which sense of "truth" in the hierarchy applies to the formula in question. However, this theory of truth requires substantial modification to the ontology of propositions. Russell now adopts an ontology of facts; he is no longer committed to an ontology of false propositions. Statements that seem to assert such false propositions are instead understood as "incomplete symbols," much like definite descriptions that fail to denote.²⁴ However, this step taken, Russell has taken a step even further away from a Fregean ontology of abstract objects.

FREGE AND THE *PRINCIPLES OF MATHEMATICS* APPENDIX B PARADOX

Let us turn back to Frege. As noted in Chapter 1, Russell wrote to Frege concerning the *Principles of Mathematics* Appendix B paradox in September 1902. Although clearly devastated by Russell's paradox, Frege was not similarly impressed by the Appendix B paradox. The way in which Russell discussed the paradox and formulated the contradiction made the differences between Russell's ontology of propositions and Frege's theory of *Gedanken* glaringly evident. Unsurprisingly, Frege came to the conclusion that the contradiction was due to Russell's poor grasp of the importance of the *Sinn/Bedeutung* distinction. It is true that the particular formulation of the antinomy utilized by Russell could not be used in Fregean logic. However, as we shall see, the antinomy can be reformulated within the logical calculus for the theory of *Sinn* and *Bedeutung* developed in the last chapter. Frege seems to have missed that his philosophy is subject to a paradox of *Gedanken* analogous to Russell's paradox of propositions only because he failed to develop his logical system sufficiently in order to express truths about *Gedanken*.

Let us first examine Frege's explicit response. We saw above that, in Russell's formulation, the antinomy is derived from the following two definitions:

²³The new theory of truth first appeared in "On the Nature of Truth," *Proceedings of the Aristotelian Society* 7 (1907): 28-49. It was spelled out in much more detail in Whitehead and Russell, *Principia Mathematica*, introd. chap. 2.

²⁴For more on this development, see Landini, *Russell's Hidden Substitutional Theory*, chap. 10, "A New Interpretation of Russell's Multiple Relations Theory of Judgment," *History and Philosophy of Logic* 12 (1991): 37-69.

(Df. w) $w =_{\text{df}} \hat{p}: (\exists m)[p = \{(q)(q \in m \supset q)\} \ \& \ p \notin m]$

(Df. r) $r =_{\text{df}} \{(q)(q \in w \supset q)\}$

If we simply ignored the nominalizing brackets, it would be possible to replace the syntax of these definitions with more Fregean syntax (using “ \cap ” instead of “ \in ” and the smooth-breathing instead of Russell’s class notation, etc.), while retaining them as well-formed definitions. However, Frege would not interpret such definitions as saying what Russell intends them to say. Consider the definition of r . It is defined as “ $\{(q)(q \in w \supset q)\}$ ”. Ignoring the brackets, Frege would regard this not as the name of a proposition or *Gedanke*, but as the name of a truth-value. The proof of the contradiction in Russellian logic requires that from the assumption that $\{(q)(q \in w \supset q)\} = \{(q)(q \in m \supset q)\}$, i.e., that the proposition $\{(q)(q \in w \supset q)\}$ is identical to the proposition $\{(q)(q \in m \supset q)\}$, we can conclude that w and m are the same class. This is unproblematic in Russellian logic, since identical propositions must have identical components. However, Frege would regard something such as “ $\{(q)(q \in w \supset q)\} = \{(q)(q \in m \supset q)\}$ ” as expressing an identity of truth-values, not of propositions. Obviously, one cannot conclude from this that w and m are the same class.

Of course, one might argue that Frege ought to understand the nominalizing brackets to create an oblique context. Russell’s “ r ” stands for a proposition; thus, Frege ought to understand it as a *Gedanke*. Thus perhaps Frege should read “ $\{\alpha\}$ ” as standing for the *Sinn* of “ α ”. However, if this step is taken, then care must be taken with regard to the expressions that appear in the oblique context. In oblique contexts, an expression has its customary *Sinn* as *Bedeutung*. Thus, in the context “ $\{(q)(q \in m \supset q)\}$ ”, “ m ” stands not for a class, but a *Sinn* picking out a class. However, if this is the case, then an expression such as the “ $(\exists m)[p = \{(q)(q \in m \supset q)\} \ \& \ p \notin m]$ ” that occurs in Russell’s definition of w is somewhat problematic. In the first instance after the existential quantifier, “ m ” stands for a *Sinn*; in the second instance, it stands for a class. If these two occurrences of “ m ” do not stand for the same thing, it is extremely problematic that they be bound by the same quantifier.

Frege’s distinction between *Sinn* and *Bedeutung* is made precisely in order to explain what goes wrong in an ordinary language inference such as:

(1) Gottlob believes that the morning star is a planet.

(2) The morning star = the evening star.

Therefore, (3) Gottlob believes that the evening star is a planet.

According to Frege, this inference is not valid because in (1), since “the morning star” occurs in *oratio obliqua*, it denotes a *Sinn*, whereas in (2), it denotes the morning star itself. Since the two occurrences of “the morning star” do not denote the same thing, no inference is possible. In Frege’s mind, Russell’s formulation of the Appendix B paradox is flawed in precisely the same way. The antinomy requires using “*w*” and “*m*” ambiguously at times to stand for classes, and as times for *Sinne*. Russell, who thinks that the constituents of a proposition are the very entities named in the sentence expressing the proposition, thinks that *w* can both be a class and a constituent of a proposition. However, from Frege’s standpoint, if it is at all possible that *r* be a member of *w*, then *w* must be a class, but if *w* is a constituent of the *Gedanke* $\{(q)(q \in w \supset q)\}$, then *w* must be a *Sinn*, because only a *Sinn* can be a constituent of a *Gedanke*. Frege would doubtlessly insist that no sign used in a properly formed Begriffsschrift could stand for both a class and for a *Sinn*. Frege seems to have concluded that this paradox plagues Russell only because he fails to adequately distinguish between *Sinn* and *Bedeutung* in his logical system (*PMC* 149-158).

In their correspondence, Frege relayed his doubts concerning the formulation of the paradox to Russell. In conclusion, Frege writes:

I do not quite know how you arrive at the equation ‘ $r \in w \equiv r \sim \varepsilon w$ ’ [‘ $r \in w \equiv r \notin w$ ’] nor what it is supposed to mean: whether a coincidence of *Gedanken* or of truth-values. By what methods of inference do you get your equation? (*PMC* 154)

However, there is no answer to the question Russell could have given that would have been acceptable to Frege. For Russell, the “inference methods” would involve the nature of *propositions* and their identity conditions. If an analogue of the antinomy does apply to Frege, it would involve rather the nature of *Gedanken* and their identity conditions. Russell, however, was in no position to tell Frege what “methods of inference” apply to *Sinne* and *Gedanken*. Not surprisingly, then, the discussion between the two at this point moved away from the contradiction in particular and into the realm of “philosophical logic,” and their respective positions on language, meaning, the nature of propositions and *Gedanken*. For example, they debate whether classes themselves can be constituents of propositions or *Gedanken*. Since they were unable to see eye to eye on these questions, they were obviously unable to agree upon the sorts of “methods of inference” that would be involved in an antinomy such as the *Principles* Appendix B paradox. Their discussion simply moved on to other things.

Unfortunately, Frege himself did not develop his logical system sufficiently in order even to answer the question of whether a paradox such as this applies.

The Begriffsschrift of the *Grundgesetze* contained only axioms and rules governing truth-values, truth functions and value-ranges. Indeed, without the addition of further constants, the language is not rich enough even to *express* truths regarding anything else. Thus, it is true that the *Principles of Mathematics* Appendix B paradox is not formulable in Frege's extant logical system. However, this is only because that system is not rich enough to express the truths regarding the nature of *Gedanken* necessary to formulate the paradox. The question then becomes whether or not the antinomy is formulable in Frege's Begriffsschrift if it is expanded in accord with the theory of *Sinn* and *Bedeutung*. We saw in Chapter 4 that Myhill discovered independently that this antinomy plagued Church's Logic of Sense and Denotation. Is the same true of our system FC^{+SB+V} from the previous chapter? At it turns out, the answer is yes. The system FC^{+SB+V} in effect provides the missing "methods of inference" Frege wanted from Russell.

Before showing this, it is worth noting that Russell actually formulated the antinomy in a slightly more complicated form than was actually necessary. Russell considered, for each class of propositions, the proposition that every proposition in the class is true. The paradox is actually slightly easier to formulate, and the contradiction slightly easier to prove, if we simply consider, for each class of propositions, the proposition that everything is in that class. That is, we can redefine w and r as:

$$\text{(Df. } w) \quad w =_{\text{df}} \hat{p}: (\exists m)[p = \{(x)(x \in m)\} \ \& \ p \notin m]$$

$$\text{(Df. } r) \quad r =_{\text{df}} \{(x)(x \in w)\}$$

Here, w is defined as the class of all propositions that state that everything is in some class m that they themselves are not in, and r as the proposition that states that everything is in w . Again, it follows that r is in w just in case it is not. The differences between this formulation of the antinomy and the one given above are relatively unimportant. A little reflection reveals that the core of the contradiction is that one can generate a unique proposition for every class of propositions, and that it is relatively indifferent what form we imagine those propositions to have, so long as we are consistent in our characterizations of r and w . When Myhill formulated the contradiction in Church's Logic of Sense and Denotation, this is actually closer to the form he used. In showing that our system FC^{+SB+V} is also subject to a version of the *Principles* Appendix B paradox, it simplifies matters, especially in the context of Fregean logic, to formulate it in this form.

Our system FC^{+SB+V} employs the method of direct discourse; it contains no oblique contexts. In a system of indirect discourse, the same sign might be used

at times to stand for a *Sinn* and at times for the *Bedeutung* picked out by that *Sinn*. In a system of direct discourse, we instead employ different signs for the *Sinn* and for the *Bedeutung*. Thus, we do not use nominalizing brackets to transform an expression from a name of a truth-value to the name of the *Gedanke* expressed by that expression. Instead, we employ a process we might call *Sinn*-transformation. We begin with an expression of the system FC. We then replace each function constant with the *Sinn* constant introduced to stand for the *Sinn* of the function constant in question. For example, we replace each occurrence of “ \supset ” with “ \rightarrow ”, each occurrence of “ \sim ” with “ \neg ”, and each occurrence of “ $(\forall a)\phi(a)$ ” with “ $(\Pi a^*)\theta(a^*)$ ” and so on. If the expression in question is closed, this is all that is required. If it is open, then we must replace each Roman object or function letter with a Roman complete or incomplete *Sinn* letter and stipulate somewhere that what the latter stands for picks out what the former stands for. If the expression contains a defined sign such as “ \cap ”, we can define a new incomplete *Sinn* constant using the *Sinn*-transform of the definiens. Recall that Frege defines his membership sign, “ \cap ”, thusly:

$$\text{(Df. } \cap) \Vdash (x \cap y) = \lambda \alpha \sim (\forall f)[(y = \overset{\cdot}{\varepsilon} f(\varepsilon)) \supset \sim (f(x) = \alpha)]$$

Therefore, we can define a sign for the *Sinn* of this sign thusly:

$$\text{(Df. } \in) \Vdash (x^* \in y^*) = \lambda \alpha \neg (\Pi \mathfrak{F})[(y^* \approx \overset{\circ}{\varepsilon} \mathfrak{F}(\varepsilon)) \rightarrow \neg (\mathfrak{F}(x^*) \approx \alpha)]$$

It then follows from axioms FC^{+SB}39 through FC^{+SB}46 and definitions of “ \in ” and “ \cap ” that:

$$\text{Theorem FC}^{+SB}46.1. \vdash \Delta(a^*, a) \ \& \ \Delta(b^*, b) \ \supset \ \Delta(a^* \in b^*, a \cap b)$$

(The proof is elementary and left to the reader.) With such a definition in place, we can use the defined incomplete *Sinn* constant in a *Sinn*-transform. Thus, instead of writing “ $\{(\forall x)(x \cap m)\}$ ” to stand for the *Gedanke* expressed by “ $(\forall x)(x \cap m)$ ”, we write “ $(\Pi x^*)(x^* \in m^*)$ ” and somewhere stipulate that m^* picks out m .

This methodology in place, we can proceed to characterize the class ω to be used in the Fregean version of the *Principles* Appendix B paradox. (Here the script letter ω is used instead of a Roman letter, since, for Frege, Roman letters cannot be used as constants.) The definition employs the transform of “ $(\forall x)(x \cap m)$ ” suggested above. It reads as follows:

$$\text{(Df. } \omega) \Vdash \omega = \overset{\cdot}{\varepsilon} (\exists mm^*) [\Delta(m^*, m) \ \& \ \varepsilon = (\Pi x^*)(x^* \in m^*) \ : \& \ : \sim (\varepsilon \cap m)]$$

This definition avoids the defects Frege sees in the Russellian definition. No sign is used ambiguously. The Gothic letter “m” stands for a class, “m*” for a *Sinn* picking out m. The *Sinn* m* is the constituent of $(\Pi x^*)(x^* \in m^*)$; but it is the class m for which the question arises as to whether $(\varepsilon \cap m)$. Now consider the following instance of axiom FC^{+SB}35:

$$\vdash (\exists a^*)\Delta(a^*, \omega)$$

This states that there is at least one *Sinn* picking out the class ω . Let us call the *Sinn* postulated by this instance of FC^{+SB}35 “ ω^* ”, and conclude:

$$(\text{Pp. } \omega^*) \vdash \Delta(\omega^*, \omega)$$

We can then use ω^* in defining the *Gedanke* stating that everything in is class ω . Thus, we characterize \mathfrak{r} as follows:

$$(\text{Df. } \mathfrak{r}) \quad \Vdash \mathfrak{r} = (\Pi x^*)(x^* \in \omega^*)$$

It follows from these definitions that the contradiction appears in the following form:

$$\vdash (\mathfrak{r} \cap \omega) \ \& \ \sim(\mathfrak{r} \cap \omega)$$

Proof.

- | | | |
|-----|---|-------------------------------|
| 1. | $\vdash F(a^*) = F(b^*) \ .\ \supset \ .\ a^* = b^*$ | Axiom FC ^{+SB} 17 |
| 2. | $\vdash (\Pi x^*)(x^* \in \omega^*) = (\Pi x^*)(x^* \in m^*) \ .\ \supset \ .\ \omega^* = m^*$ | 1, <i>ri, ri, ri</i> |
| 3. | $\vdash f(a) \ .\ \supset \ .\ a = b \ .\ \supset \ .\ f(b)$ | <i>fc</i> |
| 4. | $\vdash \Delta(\omega^*, \omega) \ .\ \supset \ .\ \omega^* = m^* \ .\ \supset \ .\ \Delta(m^*, \omega)$ | 3, <i>ri, ri, ri</i> |
| 5. | $\vdash \omega^* = m^* \ .\ \supset \ .\ \Delta(m^*, \omega)$ | 4, Pp. ω^* , <i>mp</i> |
| 6. | $\vdash (\Pi x^*)(x^* \in \omega^*) = (\Pi x^*)(x^* \in m^*) \ .\ \supset \ .\ \Delta(m^*, \omega)$ | 2, 5, <i>syll</i> |
| 7. | $\vdash \Delta(a, b) \ \& \ \Delta(a, c) \ .\ \supset \ .\ b = c$ | Axiom FC ^{+SB} 13 |
| 8. | $\vdash \Delta(m^*, \omega) \ \& \ \Delta(m^*, m) \ .\ \supset \ .\ \omega = m$ | 7, <i>ri, ri, ri</i> |
| 9. | $\vdash p \ \& \ q \ .\ \supset \ .\ r \ .\ \supset \ .\ p \ .\ \supset \ .\ q \ \supset \ r$ | <i>fc</i> |
| 10. | $\vdash \Delta(m^*, \omega) \ \& \ \Delta(m^*, m) \ .\ \supset \ .\ \omega = m \ .\ \supset \ .\ \Delta(m^*, \omega) \ .\ \supset \ .\ \Delta(m^*, m) \ \supset \ (\omega = m)$ | 9, <i>ri, ri, ri</i> |
| 11. | $\vdash \Delta(m^*, \omega) \ .\ \supset \ .\ \Delta(m^*, m) \ \supset \ (\omega = m)$ | 8, 10, <i>mp</i> |
| 12. | $\vdash (\Pi x^*)(x^* \in \omega^*) = (\Pi x^*)(x^* \in m^*) \ .\ \supset \ .\ \Delta(m^*, m) \ \supset \ (\omega = m)$ | 6, 11, <i>syll</i> |
| 13. | $\vdash \mathfrak{r} = (\Pi x^*)(x^* \in m^*) \ .\ \supset \ .\ \Delta(m^*, m) \ \supset \ (\omega = m)$ | 12, <i>df. \mathfrak{r}</i> |
| 14. | $\vdash a = b \ .\ \supset \ .\ f(b) \ \supset \ f(a)$ | <i>fc</i> |
| 15. | $\vdash \omega = m \ .\ \supset \ .\ \sim(\mathfrak{r} \cap m) \ \supset \ \sim(\mathfrak{r} \cap \omega)$ | 14, <i>ri, ri, ri</i> |

16. $\vdash p \supset q \supset r \supset q \& p \supset r$ *fc*
17. $\vdash \mathcal{R} = (\Pi x^*)(x^* \in m^*) \supset \Delta(m^*, m) \supset (\omega = m) \supset \supset$
 $\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \supset \omega = m$ 16, *ri, ri, ri*
18. $\vdash \Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \supset \omega = m$ 13, 17, *mp*
19. $\vdash \Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \supset \sim(\mathcal{R} \cap m) \supset \sim(\mathcal{R} \cap \omega)$ 15, 18, *syll*
20. $\vdash p \supset q \supset r \supset p \& q \supset r$ *fc*
21. $\vdash \Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \supset \sim(\mathcal{R} \cap m) \supset \sim(\mathcal{R} \cap \omega) \supset \supset$
 $\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m) \supset \supset \sim(\mathcal{R} \cap \omega)$ 20, *ri, ri, ri*
22. $\vdash \Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m) \supset \supset \sim(\mathcal{R} \cap \omega)$
 19, 21, *mp*
23. $\vdash (\mathcal{R} \cap \omega) \supset \sim[\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m)]$ 22, *con*
24. $\vdash (\mathcal{R} \cap \omega) \supset (\forall mm^*) \sim[\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m)]$
 23, *gen, gen*
25. $\vdash \sim(\forall mm^*) \sim[\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m)] \supset \sim(\mathcal{R} \cap \omega)$
 24, *con*
26. $\vdash (\exists mm^*) [\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m)] \supset \sim(\mathcal{R} \cap \omega)$
 25, *df. $\exists a, \exists d$*
27. $\vdash f(a) = (a \cap \varepsilon f(\varepsilon))$ (CA) (see p. 56)
28. $\vdash (\exists mm^*) [\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m)] =$
 $\{\mathcal{R} \cap \varepsilon(\exists mm^*) [\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m)]\}$ 27, *ri, ri*
29. $\vdash (\exists mm^*) [\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m)] = (\mathcal{R} \cap \omega)$
 28, *df. ω*
30. $\vdash (\mathcal{R} \cap \omega) \supset \sim(\mathcal{R} \cap \omega)$ 26, 29, *i*
31. $\vdash a \supset a$ *fc*
32. $\vdash \sim(\mathcal{R} \cap \omega) \supset \sim(\mathcal{R} \cap \omega)$ 31, *ri*
33. $\vdash \sim(\mathcal{R} \cap \omega)$ 30, 32, *inev*
34. $\vdash \sim(\exists mm^*) [\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m)]$ 29, 33, *i*
35. $\vdash (\forall m) \sim(\exists m^*) [\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m)]$ 34, *df. $\exists a$*
36. $\vdash (\forall a) f(a) \supset f(a)$ Axiom FC3
37. $\vdash (\forall m) f(m) \supset f(a)$ 36, *cg*
38. $\vdash (\forall m) f(m) \supset f(\omega)$ 37, *ri*
39. $(\forall m) \sim(\exists m^*) [\Delta(m^*, m) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap m)] \supset$
 $\sim(\exists m^*) [\Delta(m^*, \omega) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap \omega)]$ 38, *ri*
40. $\vdash \sim(\exists m^*) [\Delta(m^*, \omega) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap \omega)]$ 35, 39, *mp*
41. $\vdash (\forall m^*) \sim[\Delta(m^*, \omega) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap \omega)]$ 40, *df. $\exists d$*
42. $\vdash (\forall a^*) f(a^*) \supset f(a^*)$ Axiom FC^{SB10}
43. $\vdash (\forall m^*) f(m^*) \supset f(a^*)$ 42, *cg*
44. $\vdash (\forall m^*) f(m^*) \supset f(\omega^*)$ 43, *ri*
45. $\vdash (\forall m^*) \sim[\Delta(m^*, \omega) \& \mathcal{R} = (\Pi x^*)(x^* \in m^*) \& \sim(\mathcal{R} \cap \omega)] \supset$
 $\sim[\Delta(\omega^*, \omega) \& \mathcal{R} = (\Pi x^*)(x^* \in \omega^*) \& \sim(\mathcal{R} \cap \omega)]$ 44, *ri*
46. $\vdash \sim[\Delta(\omega^*, \omega) \& \mathcal{R} = (\Pi x^*)(x^* \in \omega^*) \& \sim(\mathcal{R} \cap \omega)]$ 41, 45, *mp*

47.	$\vdash \sim(a \& b \& \sim c) \supset (a \supset b \supset c)$	<i>fc</i>
48.	$\vdash \sim[\Delta(\omega^*, \omega) \& \mathcal{R} = (\Pi x^*)(x^* \in \omega^*) \& \sim(\mathcal{R} \cap \omega)] \supset$ $[\Delta(\omega^*, \omega) \supset \mathcal{R} = (\Pi x^*)(x^* \in \omega^*) \supset (\mathcal{R} \cap \omega)]$	47, <i>ri, ri, ri</i>
49.	$\vdash \Delta(\omega^*, \omega) \supset \mathcal{R} = (\Pi x^*)(x^* \in \omega^*) \supset (\mathcal{R} \cap \omega)$	46, 48, <i>mp</i>
50.	$\vdash \mathcal{R} = (\Pi x^*)(x^* \in \omega^*) \supset (\mathcal{R} \cap \omega)$	49, Pp. ω^* , <i>mp</i>
51.	$\vdash \mathcal{R} \cap \omega$	50, df. \mathcal{R} , <i>mp</i>
52.	$\vdash p \supset q \supset p \& q$	<i>fc</i>
53.	$\vdash (\mathcal{R} \cap \omega) \supset \sim(\mathcal{R} \cap \omega) \supset (\mathcal{R} \cap \omega) \& \sim(\mathcal{R} \cap \omega)$	52, <i>ri, ri</i>
54.	$\vdash \sim(\mathcal{R} \cap \omega) \supset (\mathcal{R} \cap \omega) \& \sim(\mathcal{R} \cap \omega)$	51, 53, <i>mp</i>
55.	$\vdash (\mathcal{R} \cap \omega) \& \sim(\mathcal{R} \cap \omega)$	33, 54, <i>mp</i>

The *Principles of Mathematics* Appendix B paradox plagues Frege after all. Moreover, it occurs not only in this version. If one attempts to transcribe the Russellian version that involves propositional functions instead of classes into the Fregean logic of *Sinn* and *Bedeutung*, the result is, in effect, the antinomy used to prove system FC^{+SB} inconsistent in the previous chapter.

How did Frege miss it? Apparently, Frege focused too much on the way in which Russell formulated the paradox and not enough on the Cantorian diagonal construction lying behind it. Just as Russell was committed to a class for every proposition and a proposition for every class, in violation of Cantorian principles, likewise Frege was committed to a class for every *Gedanke* and a *Gedanke* for every class. While the formulation Russell gave in terms of propositions would obviously not threaten Frege, Frege seemingly did not thoroughly consider the question of whether a similar antinomy would result if his logical language were expanded to treat truths regarding *Sinne* and *Gedanken*. Of course, answering this question would require actually expanding his language in ways dictated by the theory of *Sinn* and *Bedeutung*. He never did so, and thus was blind to the contradiction.

The *Principles of Mathematics* Appendix B paradox is formulated differently in the Russellian and Fregean cases. These differences are not without significance. The Russellian version, while involving the intensional logic of propositions, is not a *semantical* antinomy. It does not involve notions such as truth or meaning. The case is different with the Fregean version; it cannot be formulated without using the function sign “ Δ ” to express the relationship between *Sinne* and the *Bedeutungen* they pick out. It is more properly called a *semantical* antinomy. It is an importance difference between Fregean *Gedanken* and Russellian propositions that the former are *semantic* entities. This must be taken into account when we consider various possible solutions to these sorts of contradictions.

LESSONS LEARNED FROM RUSSELL

Because Frege was blind to these sorts of contradictions in his philosophy, he never attempted to formulate solutions for them. Russell, however, was not so blind. Despite the differences in formulation, in beginning to look for a broadly Fregean solution to these problems, it is worthwhile to ask whether Frege could have accepted any solution considered by Russell.

Russell's first worked out solution to the *Principles* Appendix B paradox is that of "Les paradoxes de la logique," wherein Russell denied the existence of general propositions. General *Gedanken* are involved in both the formulation of the Appendix B paradox given above and the contradiction discussed in the previous chapter used to prove the inconsistency of system FC^{+SB} . Could Frege hope to eliminate these antinomies in his philosophy by denying the existence of general *Gedanken*? Perhaps, but what would be the philosophical justification for such a move? We noted above that after abandoning the theory of denoting concepts, Russell was left in the dark about the constituents of general propositions. But the constituents of general *Gedanken* are in no way mysterious to Frege. *Gedanken* are composed of the *Sinne* of the expressions making up the propositions expressing the *Gedanken*. Frege understands quantifiers as having higher-level functions as their *Bedeutungen*; the *Sinne* of quantifiers are incomplete *Sinne* picking out these functions. They are no more mysterious than any other incomplete *Sinne*. Without abandoning his theory of quantification as involving higher-level functions, it is not clear what philosophical grounds Frege could have for denying general *Gedanken*. We do not want to make such a move simply as an artificial dodge of the contradictions; we want some philosophical insight into what goes wrong in their deduction. In the context of Fregean philosophy, this solution cannot do this.

The other Russellian solution to the "paradoxes of propositions" mentioned above is the adoption of a ramified type hierarchy. In Chapter 4 we already discussed why orders would be difficult to justify from a Fregean standpoint. Frege's ontology of *Gedanken* as third-realm entities makes it difficult to defend the claim that some version of Poincaré's vicious-circle principle applies to their truth-conditions. Indeed, as we have seen, orders only became justified to Russell when he radically changed his ontology of propositions. In the *Principles*, where Russell's ontology of propositions is perhaps closest to Frege's ontology of *Gedanken*, Russell found the suggestion that propositions come in different orders as "harsh and highly artificial."²⁵ However, we shall discuss this in more detail in the next chapter, where we focus more specifically on possible alterations of Frege's philosophy in light of the difficulties discussed here.

²⁵Russell, *Principles of Mathematics*, §500.

Thus, we might not be able to find an adequately Fregean solution to the difficulties discussed here by examining Russell's own proposed solutions. But this does not mean that there are no lessons to learn from Russell relevant to a Fregean diagnosis of the antinomies. Perhaps most importantly, from the very beginning, Russell was aware of the Cantorian origins of such contradictions. Cantor had shown that the cardinal number of classes of entities of some domain must always be larger than the number of entities in that domain. Thus, the number of classes of propositions must be greater than the number of propositions. However, Russell was committed to a proposition for every class of propositions, such as the proposition stating its logical product. Thus, for Russell, the cardinal number of propositions was as great as the number of classes of propositions, in violation of Cantor's result. The case is similar with Frege. Frege is committed to as many *Gedanken* as classes of *Gedanken*.

It can be concluded that any logical system that contains ontological commitment to intensional entities such as Russell's propositions or Frege's *Gedanken* must be very careful regarding the cardinal number of intensions posited. If the number is sufficiently great, so as to form a 1-1 mapping with classes or functions, contradiction can easily result. This issue is taken up in greater detail in the next chapter.

For both Russell and Frege, the number of intensions to which their systems are committed is great. For Russell, for any formula α , the system is committed to a proposition $\{\alpha\}$. As we have seen, Frege is committed to something similar through the process of "*Sinn*-transformation" described earlier. For any expression of the system FC, we can form a name of the *Sinn* it expresses using our *Sinn* constants. Axioms $FC^{+SB}35$, $FC^{+SB}36$ and $FC^{+SB}37$ similarly commit the system to *Sinne* for everything nameable even in the expanded language. It should be noted that the commitment to an intensional entity for every expression is not in and of itself problematic. It is only problematic if the identity conditions of such intensions are very strict. The stricter the identity conditions of intensions, the more intensions are posited. The *Principles* Appendix B paradox would not arise for Russell if it were possible for " $\{(q)(q \in w \supset q)\}$ " and " $\{(q)(q \in r \supset q)\}$ " to stand for the same proposition without w and r being the same class. Similarly, it would not apply to Frege if it were possible for " $(\Pi x^*)(x^* \in w^*)$ " and " $(\Pi x^*)(x^* \in m^*)$ " to denote the same *Gedanke* without " w^* " and " m^* " denoting the same *Sinn* picking out the same class. Frege and Russell require rather strict identity conditions for *Gedanken* and propositions, respectively. By and large, if two expressions in their logical systems differ by anything more than the choice of letters for bound variables, the intensions posited for the expressions are non-identical.

It might be thought, then, that the sorts of difficulties discussed here might be solved by loosening the identity conditions of the intensional entities posited.

The problem is that very strict identity conditions for intensions are required if those intensions are to be viewed as related in belief and other intentional states. Frege needs strict identity conditions for his *Gedanken* if he is to understand belief states as relations between a believer and a *Gedanke* believed. In previous chapters, we discussed the interpretation that propositions can be understood as expressing the same *Gedanke* when they necessarily have the same truth-value or are logically equivalent. This was concluded to conflict with Frege's doctrine of indirect *Bedeutung*, since it is not true that if A and B are logically equivalent that anyone who believes A must also believe B. The identity conditions for *Gedanken* must be more stringent than logical necessity. One of the very purposes of our development of a logical calculus for the theory of *Sinn* and *Bedeutung* was to allow for the transcription of statements of propositional attitudes and related inferences. If we make the identity criteria for *Gedanken* too weak, then our logical calculus would become incapable of doing so adequately. These issues are taken up in greater detail in the next chapter.

OTHER SYSTEMS OF INTENSIONAL LOGIC

At this point, it would be extremely helpful to examine how other thinkers have addressed this problem. Is it possible to develop a logical calculus that is ontologically committed to intensions, and is able to fashion the identity conditions of such intensions stringent enough for their involvement in propositional attitudes, but that is also not committed to so many intensions that Cantor's theorem is violated? Here, modal logic will be of very little help. In modal logic, formulae can be treated as intensionally equivalent if they are logically equivalent. Even those systems of modal logic that do involve commitment to intensional "entities" such as propositions or *Gedanken* (thoughts) need not employ stringent identity conditions for them.²⁶ Unfortunately, there has been relatively little work on the sort of intensional logic that would really be helpful in this context. Those philosophers who do have theories of meaning and the nature of propositional attitudes that make reference to intensional entities, or "meanings," have not, by and large, developed logical calculi to reflect their views.

²⁶It is also worth noting that those modal logicians who have attempted to proxy a theory of propositions (to serve, e.g., as the contents of mental states) in terms of *classes of possible worlds, etc.*, have encountered Cantorian antinomies very similar to those discussed in the present work. See, e.g., Kaplan, "A Problem in Possible World Semantics," in *Modality, Morality and Belief: Essays in Honor of Ruth Barcan Marcus*, ed. Walter Sinnott-Armstrong, et al. (Cambridge: Cambridge University Press, 1995), 41-52.

There are, however, at least a few philosophers whose theories of meaning make reference to intensional entities at least somewhat similar to Frege's *Sinne*, but whose understanding of the ontology of such entities is sufficiently different that it is worth pausing to consider them here. Firstly, consider the theory of the nature of Intentionality found in the work of John Searle. Searle explicitly identifies himself as someone whose views on the nature of meaning are broadly Fregean, but who abandons the robust metaphysics of *Sinne* as third-realm entities Frege himself held. He writes:

On one interpretation of Frege, my general approach to Intentionality is a matter of revising and extending Frege's conception of "*Sinn*" . . . and my approach to the special problem of reference is in some respects Fregean in spirit, though, of course, not in detail. Specifically, it is possible to distinguish at least two independent strands in Frege's account of the relations between expressions and objects. First, in his account of the *Sinn* and *Bedeutung* of *Eigennamen* [proper names], an expression refers to an object because the object fits or satisfies the *Sinn* associated with the expression. Second, in his fight against psychologism Frege felt it necessary to postulate the existence of a "third realm" of abstract entities: senses [*Sinne*], propositions [*Gedanken*], etc. . . . My own account is Fregean in accepting the first of these strands, but I reject the second . . . it is not necessary to postulate any special metaphysical realms . . . If you think about the Evening Star under the mode of presentation "Evening Star", and I think about the same planet under the same mode of presentation, the sense in which we have an abstract entity in common is the utterly trivial sense in which, if I go for a walk in the Berkeley hills and you go for exactly the same walk, we share an abstract entity, the same walk, in common. The possibility of shared Intentional contents does not require a heavy metaphysical apparatus any more than the possibility of shared walks.²⁷

Searle's philosophy of language centers around discussion of such things as "senses" and "propositions," which are in many ways understood similarly to Frege's *Sinne* and *Gedanken*, respectively. The metaphysics of such entities, however, is understood quite differently. On Searle's account, linguistic meaning is parasitic on the "Intentionality" of the mind, which is taken as fundamental and indefinable in terms of anything more basic.

At first blush, it might be thought that this revised metaphysics of propositions or *Gedanken* might be quite advantageous when it comes to attempting to find solutions to the sorts of antinomies or paradoxes of *Gedanken* considered here. If one holds that propositions (in Searle's sense) exist only

²⁷Searle, *Intentionality*, 197-8.

insofar as there are mental states correlated with them, and denies that such propositions exist in a third realm of abstract semantic entities, then there may be room to be much more conservative about which and how many propositions exist. The remaining details would be worked out by developing a calculus for the intensional logic of propositions in Searle's sense, one that would include axioms governing under what conditions propositions can be assumed to exist as well as their relations to mental states and their identity conditions.

Searle himself, however, has not worked a completely developed system. The closest he has come is work he has done in collaboration with Daniel Vanderveken under the auspices of "illocutionary logic," which Vanderveken has developed somewhat further on his own.²⁸ Illocutionary logic is logic that is able to treat other forms of speech acts besides assertions, such as questions, commands and greetings. For Searle and Vanderveken, almost all speech acts involve both an illocutionary force and a propositional content. The sentences "You are mowing the lawn" and "Are you mowing the lawn?" share a similar propositional content, but differ in illocutionary force, whereas "Are you mowing the lawn?" and "Are you shoveling the sidewalk?" share the same illocutionary force but differ in propositional content.

The illocutionary aspects of Vanderveken's systems are of relatively little interest for our present purposes. What is interesting is that the systems he develops involve a notion of a proposition in Searle's sense, and even go so far as to attempt to treat their identity conditions and to give an analysis of propositional attitudes in virtue of them. Given that Searle's account of propositions is based somewhat on Frege's understanding of *Gedanken*, in some ways Vanderveken's systems share important features with the systems developed in the previous chapter of the present work. For example, Vanderveken identifies propositions as complex "senses" (*Sinne*), and even employs Frege's composition principle.²⁹ Moreover, as propositions are taken as related in propositional attitudes, Vanderveken agrees that their identity conditions must be more stringent than logical equivalence or strict implication.³⁰ Unfortunately, however, it is not at all clear that Vanderveken's systems make good on the prospect considered earlier that Searle's metaphysics of propositions might help solve the propositional paradoxes. Vanderveken's systems do not reflect a sort of conservatism about the number of propositions

²⁸Daniel Vanderveken and John R. Searle, *Foundations of Illocutionary Logic* (Cambridge: Cambridge University Press, 1985); Daniel Vanderveken, *Meaning and Speech Acts*, 2 vols. (Cambridge: Cambridge University Press, 1990-1).

²⁹Vanderveken, *Meaning and Speech Acts*, 1:47.

³⁰Vanderveken, *Meaning and Speech Acts*, 1:84-98, "Some Philosophical Remarks on the Theory of Types in Intensional Logic," *Erkenntnis* 17 (1982): 111n.

that could be posited. Indeed, in his independent work, Vanderveken even allows that there may be an uncountably infinite number of propositions, considering that there are uncountably many real numbers, and a different proposition can be formed for each real number.³¹

As a result, Vanderveken's systems do not help us see a way towards solving the paradoxes of *Gedanken*. Indeed, Vanderveken's systems only avoid these antinomies because they are not developed sufficiently in order to deal with higher-order propositions. Vanderveken develops his systems only far enough to deal with propositions of the first-order, propositions that are not in any way about other propositions or involve quantification over ranges involving other propositions. In a footnote in which Vanderveken briefly discusses expanding his approach to be able to treat higher-order propositions, he advocates dividing propositions into different ramified types, i.e., he recommends adopting *ramified type-theory*.³² But there is no discussion of the philosophical justification for ramification nor whether it is philosophically consistent with his own understanding of the metaphysics of propositions. While ramification, as explained earlier, does block the sorts of antinomies plaguing Frege's understanding of the logic of *Sinne* and *Gedanken*, Vanderveken's work does not represent what we had hoped: a source for finding a philosophically viable solution to the paradoxes of *Gedanken* based on Searle's revised metaphysics.

Moreover, it is not entirely clear that having a philosophy of the nature of intensional entities and meanings such that meaning is taken to be an irreducible feature of mental states would actually suffice to eliminate the sorts of problems under discussion. To see this, consider the philosophy of Gustav Bergmann. For Bergmann as well, linguistic meaning is parasitic on the meaning or intentionality of mental states, and the latter is taken as irreducible and fundamental. Rather than understanding "propositions" or "thoughts," *pace* Frege and Russell, as *individuals* or *objects* existing independently of minds, Bergmann understands them as being simple and indefinable *properties* (or, as he prefers to say, "characters") of mental awarenesses. Indeed, Bergmann sometimes even calls these properties "thoughts" or "propositions". In the discussion that follows, I shall call them "thoughts". If we suppose that *f* is the property a mental awareness has just in case that awareness is of Socrates's being bald, and *g* is the property a mental act has just in case it is a belief, then Bergmann would analyze my belief that Socrates is bald thusly: there is some

³¹Vanderveken, *Meaning and Speech Acts*, 1:95.

³²Vanderveken, *Meaning and Speech Acts*, 2:77.

mental act a , had by me, which has both properties f and g .³³ Bergmann believed that by treating thoughts as characters of mental awarenesses that he avoided some of the difficult questions that philosophers such as Frege faced. He writes:

Awarenesses are just individuals among individuals. Thoughts or meanings are just simple characters among simple characters. They are not, like Frege's nonmental meanings [*Sinne*], nebulous entities of odd ontological kinds.³⁴

However, it is not entirely clear that Bergmann is right. Putting thoughts in the type of properties of individuals does not completely excuse him from having to at least address the sorts of worries here under discussion.

Bergmann, a self-admitted "ideal-language philosopher," made certain suggestions as to how his views regarding the nature of meaning could be captured in an intensional logical calculus. In such works as "Intentionality" and "Analyticity," Bergmann describes a logical language he dubs " L ".³⁵ The language is built upon the system of Russell and Whitehead's *Principia Mathematica*, except with ramification "suppressed" in accord with Ramsey's suggestions.³⁶ Bergmann then adds two new primitive symbols, the quoting operator, ' \dots ', and the sign for "intentional sense of *meaning*," M . The first transforms a sentence in the logical language into a sign for a thought, where again, these are taken to be *properties* of individual mental states. Bergmann disallows quantification into contexts involving the quoting operator. The sign " M " is flanked on the left side by a sign for such a property, and on the right by a formula of the language. The whole formed, e.g., " $f M p$ " is regarded as true just in case f is the thought that p , or in other words, that f is a property a mental state just in case its content is p . Thus, ' $(x)(x = x)$ ' is the property a mental state has just in case it is about everything's being self-identical. The sentence:

$$'(x)(x = x)' M (x)(x = x)$$

³³Gustav Bergmann, "Intentionality," in *Meaning and Existence* (Madison: University of Wisconsin Press, 1959), 3-38, "Acts," in *Logic and Reality* (Madison: University of Wisconsin Press, 1964), 3-44.

³⁴Gustav Bergmann, "Sameness, Meaning and Identity," in *Meaning and Existence*, 138.

³⁵Bergmann, "Intentionality," 18-38, "Analyticity," in *Meaning and Existence*, 73-90.

³⁶Ramsey, Frank P., "The Foundations of Mathematics," in *Foundations: Essays in Philosophy, Logic, Mathematics and Economics* (Atlantic Highlands, New Jersey: Humanities Press, 1978), 152-212.

is to be regarded as the analytic truism that the thought that everything is self-identical means that everything is self-identical.

Bergmann does not go so far as to give axioms and inference rules for this expanded logical language, though he does go on to describe an expanded account of what sentences in the language are to be regarded as analytically true such that one gets a sense of what sorts of axioms Bergmann would have added. He writes for example:

Every simple clause of ' M ' is either analytic or it is contradictory, i.e., its negation is analytic. It is analytic if and only if the predicate to the left of ' M ' is formed by the quoting operator from the sentence to the right of ' M '.³⁷

If we use α and β schematically for any (closed) formulae of the language, if α and β are the same, then a sentence of the form " α M β " is to be regarded as an analytic truth; otherwise, its negation, viz., " $\sim(\alpha$ M $\beta)$ " is to be regarded as an analytic truth. As Bergmann sometimes puts it, "a thought . . . is uniquely determined by its text."³⁸ Although Bergmann sometimes suggests that treating thoughts as properties absolves him from having to specify identity conditions for them,³⁹ it is clear that he is actually committed to thoughts having rather *stringent* identity conditions. In fact, the identity conditions he requires for thoughts are even stronger than those taken under Church's Alternative (0), as it can be inferred from the passage above that Bergmann would not even treat sentences differing only by alphabetic change of bound variable as synonymous. Consider the following:

' $(x)(x = x)$ ' M $(x)(x = x)$

' $(z)(z = z)$ ' M $(x)(x = x)$

For Bergmann, while the first is an analytic truth, the second is not only *not* an analytic truth, it is actually a *self-contradiction!*

This alone should be enough to convince us that Bergmann would also have to face challenges regarding Cantorian paradoxes of thoughts or propositions. Indeed, even from the small portion of the rules of L we have considered, it appears that a version of the *Principles* Appendix B paradox does plague Bergmann. Because thoughts are regarded as *properties*, the paradox has to be "pushed up" a type, but it still seems formulable. Because it is built upon a

³⁷Bergmann, "Intentionality," 32. See also "Analyticity," 75.

³⁸Bergmann, "Sameness, Meaning and Identity," 138.

³⁹Bergmann, "Sameness, Meaning and Identity," 138.

“Ramseyfied” version of *Principia Mathematica*, the system does employ a simple theory of types. In what follows, I shall use lowercase letters x, y, z , etc., for the type of individuals. I use lowercase letters in the middle of the alphabet, f, g, h , etc., for the type of properties of individuals. Predicates formed by means of the quoting operator fall into this type. Lastly, I use uppercase letters F, G, H for the type of properties of properties of individuals. This makes it possible to do without type-subscripts in what follows.

For each property of properties G there will be a thought that every property of individuals has G , viz., ‘ $(g)Gg$ ’. Some of these thoughts will be such as to fall under the property of properties for which they state the universal generalization, some will not. Now consider the property of properties F , which a property has just in case it is a thought of this form but does *not* have the property of properties for which it states the universal generalization. Then consider h , the thought that every property has F . The question then arises as to whether h has F . It does just in case it does not.

Let us briefly sketch the proof of the contradiction informally. First, consider the following instance of *Principia Mathematica*’s comprehension principle (which, given the suppression of ramification, is now indifferent to impredicativity):

$$(1) (f)[Ff \equiv (\exists G)(f M (g)Gg \ .\& \sim Gf)]^{40}$$

This in effect defines F as suggested above. Let us now proceed to define h , also as described above, thusly:

$$(2) h = '(g)Fg'$$

Instantiating (1) to h , we get:

$$(3) Fh \equiv (\exists G)(h M (g)Gg \ .\& \sim Gh)$$

We can now prove that from either the assumption Fh or the assumption $\sim Fh$, the opposite follows. First, assume:

$$(4) Fh$$

⁴⁰The parallelism between this formulation and the one given for Russellian logic above would be more clear if we formulated this instead as “ $(f)[Ff \equiv (\exists G)(f = '(g)Gg' \ .\& \sim Gf)]$ ”. However, this would involve quantifying into a context involving the quoting operator, which is illegitimate.

By (3) and (4), we can then conclude:

$$(5) (\exists G)(h M (g)Gg \cdot \&. \sim Gh)$$

Substituting the definition of h in (2) for its first occurrence in (5), we get:

$$(6) (\exists G)(' (g)Fg' M (g)Gg \cdot \&. \sim Gh)$$

However, the only property of properties for which this could be true is F itself. To see this, consider that for any G different from F , the first conjunct, ' $(g)Fg$ ' $M (g)Gg$, is self-contradictory. Then, let us instantiate (6) to F itself, arriving at:

$$(7) ' (g)Fg' M (g)Fg \cdot \&. \sim Fh$$

The second conjunct of (7) contradicts our assumption in (4). We can then conclude that our assumption at (4) was incorrect. However, if we assume instead:

$$(8) \sim Fh$$

Then, by (8) and (3) we get:

$$(9) \sim(\exists G)(h M (g)Gg \cdot \&. \sim Gh)$$

By the rules of the quantifiers and propositional logic, this becomes:

$$(10) (G)(h M (g)Gg \supset. Gh)$$

Instantiating this to F itself, we get:

$$(11) h M (g)Fg \supset. Fh$$

Substituting the definition of h in (2) in its first occurrence in (11), we get:

$$(12) ' (g)Fg' M (g)Fg \supset. Fh$$

The antecedent of this conditional is an analytic truth. Therefore, by *modus ponens*:

$$(13) Fh$$

From either assumption Fh or $\sim Fh$, the opposite follows. This seems to show that Bergmann's intensional logic is formally inconsistent.

Treating thoughts or propositions as properties or characters of mental states does not solve our problems. This is not to say that there are not possible solutions one could take consistent with Bergmann's philosophy. In fact, it is not entirely clear that the language L we have been considering adequately reflects Bergmann's own views. For instance, at least by 1963, Bergmann had adopted what he calls "the principle of exemplification," which is the view that a universal (property or relation) exists only if instantiated.⁴¹ Moreover, Bergmann also subscribed to a view he called "elementarism," by which he meant the thesis that there are no *simple* universals of higher types (properties of properties, etc.)⁴² These views are not reflected in L , a higher-order logic with a very strong comprehension principle borrowed from *Principia Mathematica*. It is not entirely clear whether Bergmann could use such views to solve the contradiction. He might insist that elementarism rules out such a property as F , though this is not entirely clear, considering that F might be regarded as a *defined*, not a *simple*, property.⁴³ He might also insist that the principle of exemplification makes it impermissible simply to define h as ' $(g)Fg$ ', because it would have to first be proven that this thought exists, i.e., that some mental states exhibit it. Here, too, it is not entirely clear that the problem is solved, considering that I myself seem to have had mental states exhibiting this thought while writing these passages.

Bergmann does not seem to have considered the sorts of antinomies of thoughts, meanings or propositions that have loomed large in this and the preceding chapter. Thus, we do not know what sort of response he would have given. Examining Bergmann's work then will not likely provide us with a new solution to the difficulties in the Fregean logic of the theory of *Sinn* and *Bedeutung* that would be seen as adequate from the perspective of Fregean philosophy. What it does teach us is that the same sorts of problems and challenges are at least a concern for a variety of views that bear similarities to Frege's own. The conclusion of our examination of rival philosophical systems in this chapter seems clear: *any* theory of meaning that incorporates commitment to finely-grained intensional entities, whether they be Russellian or Searlean propositions, Fregean *Gedanken*, Bergmannian thoughts, or something similar, must at least address the possibility that commitment to such entities might give

⁴¹Gustav Bergmann, "Stenius on the *Tractatus*," in *Logic and Reality*, 245-6, "Synthetic *A Priori*," in *Logic and Reality*, 291-2.

⁴²Gustav Bergmann, "Elementarism," in *Meaning and Existence*, 115-23.

⁴³This would depend exactly on how we are to understanding the semantics and justification of the comprehension principle. I cannot, however, delve into this here.

rise to certain sorts of antinomies and other paradoxes. However, we are no closer to finding a genuinely *Fregean* solution to the problems in the logic of the theory of *Sinn* and *Bedeutung*. From Russell, however, we have at least learned a diagnostic method: that such antinomies are often the result of a violation of Cantor's theorem. In the next chapter, we turn to a more detailed Cantorian analysis of the problems facing Frege, and starting from this perspective, we consider what options are available to someone with broadly similar philosophical views for escaping these difficulties.

Possible Revisions to Frege's Philosophy

ADEQUATE VERSUS *AD HOC* SOLUTIONS

One of the major themes of the present work has been that Frege's logical theory is incomplete because his extant Begriffsschrift does not reflect the theory of *Sinn* and *Bedeutung*. The systems developed in Chapter 5, FC^{+SB} and FC^{+SB+V} , eliminate this missing part of Frege's work. However, we also found that the development of these systems reveals flaws in Frege's overall philosophy that have been scarcely addressed or discussed by Frege's philosophical progeny. In this chapter, we consider whether or not solutions to these difficulties can be found that are consistent with a broadly Fregean philosophical system. The systems described in Chapter 5 were meant to represent fully Frege's own commitments. Since these systems are subject to the paradoxes described in the preceding chapters, obviously we cannot expect to find solutions compatible with *strict* adherence to Frege's own doctrines. The interesting question, however, is whether or not Frege's philosophy could be revised in such a way that its spirit and overall form are maintained while the problems are avoided in a philosophically adequate way.

First, however, we must say a little about what would constitute an adequate solution. A solution to a philosophical puzzle can be adequate only if it is philosophically well-grounded. If changes are to be made to Frege's philosophy or logical calculus, we require some explanation for what the philosophical reasons are for the changes, and how they fit within a coherent philosophy of language, theory of meaning, metaphysics and understanding of logic. We do not simply want to make arbitrary changes to Frege's views on an *ad hoc* basis simply to avoid the paradoxes and antinomies. This would be to abandon our task as philosophers: to provide philosophical answers to philosophical

questions. In my own eyes, it would also detract from the integrity of Frege's philosophical project and vision.

However, it might be thought that the paradoxes and antinomies *themselves* provide all the rationale that is required for adopting a certain proposed solution. Avoiding contradiction is of course something required of any satisfactory theory. One might argue that avoiding contradiction itself provides us with sufficient justification for adopting a change to Frege's views that succeeds in doing so. However, on a little reflection, it is clear that matters are a bit more complicated. It is true that the discovery of contradictions forces us to make *some change or another* to a philosophical theory. However, it alone does not tell us what changes to make. It is perhaps possible to avoid the contradictions simply by placing arbitrary restrictions on our inference rules. For example, we could avoid the deduction of a certain contradiction by limiting the rule of *modus ponens* such that it applies in every case except for some special case involved in the deduction. Such arbitrary moves could be made to formally block all of the antinomies and paradoxes discussed in the preceding chapters. Whether or not such moves could be made to block all *possible* contradictions is not quite so clear. However, surely, these moves do not constitute *genuine* solutions to the paradoxes.

For any solution to be genuine, we need some explanation for why it is selected as opposed to any of a multitude of other arbitrary moves that could be made simply to dodge the difficulties. The only adequate such explanations would be philosophical ones: ones that demonstrate that the changes made are actually *demande*d for solid philosophical reasons. Quite likely, the reasons will be *independent* of the ability of the changes to solve the particular antinomies. Ideally, the solutions to the antinomies should *naturally* follow from independently reasonable philosophical precepts. This must be kept in mind in what follows.

A CANTORIAN ANALYSIS OF THE DIFFICULTIES

In the preceding two chapters, four paradoxes that arise in the logic of the theory of *Sinn* and *Bedeutung* were discussed. There are doubtless others. However, it is quite likely that any possible modifications that could be made to Frege's philosophy that would solve these paradoxes would also solve the others. Let us first remind ourselves what those paradoxes are. Firstly, there is a rather simple antinomy involving classes and *Sinne*. Consider the class ♠ of all those *Sinne* that pick out classes in which they are not included. Then consider any *Sinn* ♠* picking out this class. The *Sinn* ♠* seems to be in the class ♠ just in case it is not. In what follows, let us call this the *class/Sinn antinomy*. Next we considered the (contingent) paradox that results from assuming that Church's favorite *Gedanke* is the *Gedanke* that Church's favorite *Gedanke* does not pick out the True. From

this assumption, it follows that Church's favorite *Gedanke*, Υ , picks out the True just in case it does not. Following the practice employed in Chapters 4 and 5, let us call this the *modified Epimenides paradox*. Lastly, we also considered two modified Fregean versions of Russell's *Principles of Mathematics* Appendix B paradox. In the last chapter, we considered a version closest to Russell's original formulation. Consider for each class, the *Gedanke* that everything is in that class. Now, consider the class ω of all *Gedanken* of this form that are not in their associated class, as well as the *Gedanke* ρ , that everything is in ω . The *Gedanke* ρ is itself in ω just in case it is not. In what follows, this will be referred to as the *class/Gedanke antinomy*. Lastly, in Chapter 5, we considered an antinomy very similar to this, save focusing on concepts rather than classes. For each concept, consider the *Gedanke* that everything falls under the concept. Next, consider the concept, $F(\xi)$, of *being a Gedanke for which there is a concept such that the Gedanke is the Gedanke that everything falls under the concept, but such that the Gedanke itself does not fall under the concept*. Then, we consider the *Gedanke*, $(\Pi a^*)\forall(a^*)$, that everything falls under this concept. The *Gedanke* $(\Pi a^*)\forall(a^*)$ itself falls under the concept $F(\xi)$ if and only if it does not. This will be called the *concept/Gedanke antinomy*.

Our discussion of Church's Logic of Sense and Denotation in Chapter 4 and our comparison of Frege with other philosophers in Chapter 6 revealed that similar sorts of paradoxes and antinomies are a concern for others as well. Our discussions there did not provide us with a solution consistent with the Fregean mindset, but the examination of Russell's work did at least provide us with a diagnostic tool: that such antinomies are often the result of a violation of Cantor's power-class theorem. This is worth discussing in more detail here. As noted in the previous chapter, Cantor's theorem is the mathematical result that the cardinal number of entities in a certain domain must always be smaller than the number of classes of those entities. Let us briefly remind ourselves how it is that Cantor proved this result. Certainly, there must be at least as many classes of entities in the domain as there are entities in the domain given that for each entity, one class will be the class containing only that entity. However, Cantor proved that there also cannot be the *same* number of entities as there are classes. If there were the same number, there would have to be a 1-1 function f mapping entities in the domain on to classes of entities in the domain. However, this can be proven to be impossible. Some entities in the domain would be mapped by f onto classes that contain them, whereas others would not. However, consider the class of entities in the domain that are not in the classes on to which f maps them. This is itself a class of entities of the domain, and thus, f would have to map it on to some particular entity in the domain. The problem is that the question then arises as to whether this entity is in the class onto which f maps it. Given the class in question, it does just in case it does not. Therefore, there can

be no such function f , and hence, the number of entities in any domain cannot be equal to the number of classes of those entities. Since the number cannot be greater either, we can infer that there must be *more* classes.¹

The sort of reasoning used in this proof is, of course, familiar to anyone acquainted with logical and philosophical paradoxes. Cantor's proof was the inspiration for Russell's discovery of both the *Principles* Appendix B paradox as well as the simpler Russell's paradox. Cantor's result relates *directly* to at least two of the paradoxes mentioned earlier. Firstly, are there more *Sinne* or classes of *Sinne*? According to Cantor's theorem, there must be more classes of *Sinne*. However, the system FC^{+SB+V} is committed to the existence of a *different Sinn* for every class of *Sinne*, since every class will have at least one *Sinn* picking it out, by axiom FC^{+SB} 35. Therefore, FC^{+SB+V} is committed to as many *Sinne* as classes of *Sinne*, in violation of Cantor's theorem. To arrive at the class/*Sinn* antinomy, we need only consider a mapping from classes of *Sinne* onto *Sinne* picking them out, and then, consider the Cantorian class of all *Sinne* not in the classes onto which they are mapped, i.e., the class of *Sinne* picking out classes they are not in, viz., \clubsuit . The case is similar with the class/*Gedanke* antinomy. Here, the relevant question is whether there are more *Gedanken* or classes of *Gedanken*. Again, by Cantor's theorem, there must be more classes of *Gedanken*. However, as we have seen, the system FC^{+SB+V} is committed to the existence of a *different Gedanke* for each class, such as the *Gedanke* that everything is in the class. One could then simply consider a mapping from each class of *Gedanken* to the *Gedanke* that states that everything is in that class. We then consider the class of all such *Gedanken* not in their associated class, and we arrive at the class ω from the class/*Gedanke* antinomy. These two antinomies seem to result from *direct* violations of Cantor's theorem.

Since the concept/*Gedanke* antinomy does not involve classes, it is not quite as obviously a violation of Cantor's theorem construed as a theorem about classes. However, it is quite clear that the same sort of reasoning Cantor would have used to prove that the number of classes of *Sinne* or of *Gedanken* must be greater than the number of *Sinne* or *Gedanken* themselves could also be used to prove that the number of concepts under which *Gedanken* might fall must be greater than the number of *Gedanken* themselves. The system FC^{+SB} , however, is committed to as many *Gedanken* as concepts. For each concept, we can form a *different Gedanke*: the *Gedanke* that every object falls under the concept. Now, instead of considering the *class* of all *Gedanken* of this form that do not fall under their corresponding concept, we simply consider the *concept* of being such a *Gedanke*, viz., $\bar{F}(\xi)$. The concept/*Gedanke* antinomy then results. For the

¹Georg Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers*, trans. Philip E.B. Jourdain (New York: Dover, 1955).

same reasons it is problematic for there to be as many *Sinne* or *Gedanken* as classes thereof, it is likewise problematic that there be as many *Gedanken* as concepts under which they may or may not fall.

The primary significance of the Cantorian analysis of the genesis of these three antinomies is that it helps make clear what sorts of modifications to Frege's philosophy might have the potential to solve the difficulties. From a Cantorian perspective, the difficulties arise from Frege's commitment to such a large number of logical entities: not only is Frege committed to the objective existence of so many classes and concepts, but he is also committed to just as many entities in the realm of *Sinn*, including at least one *Sinn* for every class, and at least one *Gedanke* for every concept. Any adequate solution to any one of these antinomies must necessarily involve taking one of a limited number of possible strategies.

Firstly, we might abandon metaphysical commitment to a certain subset of logical entities altogether. For example, we could consider completely abandoning the commitment to classes as genuine entities, or alternatively, we could consider completely abandoning the commitment to *Sinne*, etc. Let us dub approaches that involve complete abandonment of commitment to some realm of logical entities as "strategy one."

Secondly, we could take steps to limit the cardinal number of such entities posited. Frege himself explicitly argued that there are an infinite number of both classes and *Gedanken* (FA §84-6, PW 136n). However, contrary to the system FC^{+SB+V} , according to which there are as many *Gedanken* as classes and concepts, we might try to work out some theory according to which while there is a non-denumerably infinite number of classes and concepts, there is only a denumerably infinite number of *Sinne* and *Gedanken*. Let us dub approaches that attempt to curtail the *number* of logical entities posited in such a fashion as "strategy two."

The only other possible approach would be to make certain moves to undermine the intelligibility of the Cantorian constructions themselves. This would be to maintain commitment to such entities as *Sinne*, concepts and/or classes, but modify our understanding of these entities in such a way that the problems are avoided. For example, we might develop an understanding of *Sinne* and classes according to which the question does not arise for a *Sinn* picking out a class whether that *Sinn* itself is in that class. Similarly, we could insist that for every concept, the *Gedanke* that every object falls under that concept is not of the right sort of logical "order" for the question even to arise as to whether that *Gedanke* falls under that concept. Let us dub *ramified type-theory* and similar approaches that make this sort of move as "strategy three." The identification of these three categories of possible solutions provides a starting point for looking for possible revisions to Frege's philosophy.

Unfortunately, however, the same sort of Cantorian analysis cannot be applied to the modified Epimenides paradox. This paradox does not in any straightforward way derive from a violation of Cantor's theorem. It is, rather, the paradigm of a semantical paradox: a paradox that arises from certain theories of the nature of meaning or reference and puzzles regarding *self-reference*. It involves a *Gedanke* that makes reference to itself, or, to put the point more precisely, a *Gedanke* that includes as a constituent a *Sinn* that picks out that very *Gedanke*. However, it is not completely unrelated to the other paradoxes, all of which themselves involve semantic entities such as *Sinne* and *Gedanken*. Thus, while considering possible responses to the Cantorian antinomies, it might reasonably be hoped that such responses might also provide clues as to the correct resolution of this paradox. Let us then move on to considering possible responses.

DROPPING CLASSES AND/OR CONCEPTS

By and large, strategy-one approaches to solving the paradoxes would take us too far from the Fregean mindset really to be an option if we are committed to maintaining even the spirit of Frege's own views. For example, in the way many philosophers speak, a theory of meaning is said to be "Fregean" or "Neo-Fregean" if and only if it makes references to meaning-entities such as Frege's *Sinne*. If we took the strategy-one approach of wholly abandoning *Sinne*, our philosophy of language would no longer be Fregean or even Neo-Fregean.

However, the strategy-one approach of abandoning classes is perhaps an exception to this rule. As discussed in Chapter 2, Frege understands classes as value-ranges of concepts. It is not entirely clear that very much is lost if we simply deny the existence of value-ranges as objectively existing objects, provided we maintain commitment to the concepts themselves. Indeed, as noted earlier, towards the end of his life Frege himself apparently became convinced that his belief in such entities was a bewitchment of language. Therefore, abandoning the commitment to classes and class theory represents a possible move for someone with a broadly Fregean perspective. It is of course true that Frege himself defined numbers as particular sorts of classes. It might be thought, then, that abandoning classes would require abandoning also Frege's approach to logicism—and indeed, Frege's own renunciation of classes seemed to go hand in hand with his final abandonment of logicism. However, while it is true that some of the details of Frege's demonstrations of the basic laws of number theory could not be maintained without commitment to classes, it is not entirely clear that taking this approach would require renouncing logicism entirely. In his substitutional theory, Russell showed that it was possible to proxy class-theory in a logical language that did not actually contain commitment to classes as objectively existing entities. Russell himself wrote to Frege that, "I believe I

have discovered that classes are entirely superfluous.”² If we allow ourselves to make certain other changes to Frege's philosophy, it might be possible that Russell's strategies with regard to doing away with classes could be modified for Fregean logic while maintaining the core of Frege's approach to logicism. We could do away with commitment to value-ranges or extensions of concepts and make due with the concepts themselves. Given that, for Frege, concepts are, like classes, entirely extensional, the changes that would have to be made might not even be very great. Of course, it is also open to us to simply abandon Frege's logicism.

It might be hoped that giving up classes would bring us a long way towards solving the paradoxes under discussion. Normally, the presence of Russell's paradox in Frege's extant logical system is blamed on Frege's Basic Law V (axiom FC^{+V7}). This axiom would simply be dropped if we dropped commitment to classes. Russell's paradox is thus solved. It is tempting to think that many of the other antinomies we have been discussing are simply intensional byproducts of the same flaw in Frege's logic. And indeed, it is true that by dropping commitment to classes, which amounts to adopting system FC^{+SB} instead of system FC^{+SB+V} , we solve not only Russell's paradox, but also the class/*Sinn* and class/*Gedanke* antinomies.

The real drawback to abandoning classes, however, is that it does not solve all of the problems. We noted above that the core of Frege's approach to logicism could perhaps be salvaged even if classes were abandoned, because it is possible to proxy much of class-theory using concepts. This is a double-edged sword. As we have seen, it is possible to develop versions of Cantorian paradoxes that do not actually make use of classes. The concept/*Gedanke* antinomy results simply by modifying the class/*Gedanke* antinomy to make use of concepts rather than classes. Although not every antinomy demonstrable in FC^{+SB+V} is also demonstrable in FC^{+SB} , FC^{+SB} is nevertheless inconsistent. FC^{+SB} is subject not only to the concept/*Gedanke* antinomy, but is also subject to other paradoxes, such as the modified Epimenides paradox. Even if we do drop commitment to classes, we still must look for other ways of solving the problems of system FC^{+SB} .

Given the success of dropping commitment to classes in solving some of the antinomies, it might be thought that one ought to take the parallel strategy-one solution for dealing with the concept/*Gedanke* antinomy: that is, to simply concepts altogether. This, however, would require modifying Frege's philosophy nearly beyond recognition. As clear from works such as “Funktion und Begriff” and “Über Begriff und Gegenstand,” logical analysis in terms of

²Russell to Frege, Churt, 24 May 1903, *Philosophical and Mathematical Correspondence*, by Gottlob Frege, 158.

functions and their arguments and treating concepts as a sort of function are central to Frege's approach to logic. They can be seen as representing the fundamental break between his approach to logic and the approach taken in the older Aristotelian tradition. Frege's Begriffsschrift, as we have seen, is a function calculus, and commitment to concepts understood as a sort of function is central to its very conception. Without commitment to concepts, one would have to abandon Frege's entire approach to logic and replace his Begriffsschrift with some completely different form of logical calculus. This approach could only have any plausibility, moreover, if some replacement for Fregean concepts were offered in their place. However, the most plausible such replacements—things such as properties or attributes—would very likely face similar challenges (as seen, e.g., in the work of Russell and Bergmann).

If we want to maintain the core of Frege's logical theory, the strategy-one approach of abandoning concepts altogether is probably not the most promising avenue. However, it may be worth considering the strategy-two response of attempting to limit or modify the number or sorts of concepts posited. As we saw in Chapter 2, Frege's logic contains quite powerful assumptions regarding the existence of functions, which are enshrined in the Roman instantiation principle for function variables. As we saw there, it leads to results as strong as a system including an impredicative comprehension principle for functions taking the following form:

(CP) $\vdash (\exists f)(\forall a)(f(a) = A(a))$, where $A(a)$ is any Begriffsschrift expression containing "a" but not containing "f", and such that the whole is a wfe.

Frege is committed to the existence of a function for every open expression of the language, regardless of the complexity of the expression. Given that concepts are a type of function, this entails the existence of a concept for any specifiable set of conditions, no matter how complex. It is this feature of Frege's logic that allows us to define signs for such concepts as *being a Gedanke that picks out the True* (as in the modified Epimenides paradox) or *being a Gedanke for which there is a concept such that the Gedanke is the Gedanke that everything falls under the concept, but such that the Gedanke itself does not fall under the concept* (as in the concept/Gedanke antinomy), and to instantiate function-variables to these defined concept-signs.

It might be hoped then, that short of dropping concepts altogether, a way could be found of reducing Frege's commitment to *so many* concepts that would solve the antinomies. We noted in Chapter 2 that certain contemporary writers such as Cocchiarella and Heck have devised modifications of Frege's extant system that involve placing restrictions on what functions are posited to exist,

or, equivalently, that involve restricting functional comprehension or the Roman instantiation rule.

Let us consider Cocchiarella's work first. Cocchiarella has devised a number of related systems based on a language he calls HST* for *homogeneous simple types*. Strictly speaking, HST* and its variants involve a type free language, in which predicate variables and constants are allowed even in subject position.³ However, HST* is able to *proxy* simple type-theory by only postulating properties to exist corresponding to those open formulae that could be transcribed into a language utilizing simple type-theory without violating type-restrictions. Thus, it would postulate a property corresponding to the open formula " $\xi > 7$ ", i.e., the property of being greater than seven, but it would not postulate a property corresponding to the open formula " $\sim\phi(\phi)$ ", i.e., the property of being a property that does not apply to itself. Thus, it becomes impossible to formulate Russell's paradox in any of its variant forms.

Of course, the systems Cocchiarella develops do not include the logic of the theory of *Sinn* and *Bedeutung*. It is natural to ask whether they could be expanded to do so, and if so, whether they would avoid the problems under discussion. While Cocchiarella's systems are amenable to expansion to include certain approaches to intensional logic, it is not at all clear how one would go about expanding Cocchiarella's systems to include a Fregean sort of intensional logic. Cocchiarella fashions his systems in a predicate language, rather than a functional language such as Frege's. There are no complex individual terms in Cocchiarella's language, and without being able to form complex terms standing for the entities of the third realm of *Sinn*, it is not clear how one would go about attempting such an expansion. This is not to say that it would be impossible, but without further investigation it is extremely difficult to imagine what such a system would be like and how well it would fare in terms of solving the sorts of difficulties discussed in the preceding chapters. Likely, however, it would not fare exceedingly well. Cocchiarella's systems, as we have seen, are equiconsistent with simple type-theory. Expansions of Cocchiarella's system into the realm of the logic of *Sinn* and *Bedeutung* would solve only those problems solved by simple type-theory. As we discovered by examining Church's original formulations of the Logic of Sense and Denotation, however, simple type-theory on its own is powerless to prevent both the modified Epimenides Paradox and the variants of the Russell-Myhill antinomy. It is worth noting that Church's Logic of Sense and Denotation does not fall prey to the simple class/*Sinn* antinomy, because a *Sinn* picking out a class is never of the right type to be a member of the class it picks out. But the other antinomies remain to be solved. Expansions of Cocchiarella's systems would likely do no

³For references to Cocchiarella's works, see chap. 2, note 28.

better.⁴ They would solve the class/*Sinn* antinomy but would require other fixes for the remaining problems. Moreover, the restrictions Cocchiarella places on the comprehension principle, i.e., on what properties he posits to exist, are made somewhat on an *ad hoc* basis. His main justification for them is “on pain otherwise of generating Russell’s paradox,”⁵ rather than any independently reasonable philosophical hypothesis.

Let us turn to Heck. Heck has devised a modification of Frege’s original system that restricts Frege’s Roman instantiation rule for functions to “predicative cases,” which, in Chapter 2, I called system PFC^{+V} .⁶ Heck has shown this system to be consistent even with the inclusion of Basic Law V and the commitment to classes. We could imagine making similar modifications to the systems FC^{+SB} or FC^{+SB+V} , which we might dub PFC^{+SB} or PFC^{+SB+V} . Therein, we would disallow instantiating Roman letters for functions or incomplete *Sinne* to “impredicative” complex function or incomplete *Sinn* names, i.e., those that include higher-level quantifiers ranging over functions or incomplete *Sinne* of the same level or ant level higher than themselves. As far as I am able to determine, it may not be possible to prove any contradictions in system PFC^{+SB} . The problematic concept $F(\xi)$ of the concept/*Gedanke* antinomy is impredicative; its definition involves quantification over *both* functions and incomplete *Sinne*.

However, there are difficulties with this approach. Firstly, it does not solve the modified Epimenides paradox. None of the uses of the Roman instantiation rule in the derivation of this paradox involve impredicative function or incomplete *Sinn* names. This paradox is, of course, a contingent one; it involves an assumption about what Church’s favorite *Gedanke* is. Thus, it does not result in an outright contradiction being demonstrable from the basic laws of logic. Even if the system PFC^{+SB} is not inconsistent outright, it still faces difficulties. This itself may be enough to prompt us to look for solutions elsewhere.

Moreover, there are more grave concerns. Firstly, we must look more closely at exactly how weak the systems PFC^{+SB} and PFC^{+SB+V} actually are. Frege’s logical theory was innovative in numerous respects, but perhaps some of his most important contributions were the development of contemporary quantifier notation and the first axiomatization of higher-order logic. However,

⁴Noted added, May 2001. After I wrote this, Cocchiarella has published an article addressing this issue, at least in part. Since this article appeared too late to be discussed here, I recommend that the reader study it carefully. See his “Russell’s Paradox of the Totality of Propositions,” *Nordic Journal of Philosophical Logic* 5 (2000): 25-38.

⁵Cocchiarella, *Logical Studies*, 165.

⁶Heck, “The Consistency of Predicative Fragments of Frege’s *Grundgesetze der Arithmetik*,” 209-20.

in the systems $\text{PFC}^{+\text{SB}}$ and $\text{PFC}^{+\text{SB}+\text{V}}$, certain basic logical operations involving quantifiers cannot even be applied in all cases. The natural deduction inference rule often called “universal instantiation” is captured in Frege’s logic via axioms FC3, FC4 and FC5 and Roman instantiation. However, in $\text{PFC}^{+\text{SB}}$, $\text{PFC}^{+\text{SB}+\text{V}}$ and even simple PFC, this rule cannot even be applied to expressions involving certain sorts of multiple generality. Consider the proposition:

$$\vdash (\forall a)(\exists f)f(a)$$

This says (roughly) that every object falls under at least one concept. If we wanted to infer from this that *the number two* falls under at least one concept, in the system FC, we would first turn to axiom FC3:

$$\vdash (\forall a)f(a) \supset f(a)$$

We would then use Roman instantiation to replace “*a*” with the object name “2”:

$$\vdash (\forall a)f(a) \supset f(2)$$

We would then use Roman instantiation to replace “ $f(\xi)$ ” with the function name “ $(\exists f)f(\xi)$ ”:

$$\vdash (\forall a)(\exists f)f(a) \supset (\exists f)f(2)$$

Then, by *modus ponens*, we could infer:

$$\vdash (\exists f)f(2)$$

However, in the systems built on PFC, this inference would be impossible, because the Roman instantiation from “ $f(\xi)$ ” to “ $(\exists f)f(\xi)$ ” would violate the predicativity requirement. PFC and $\text{PFC}^{+\text{SB}}$ simply fail to capture some basic logical truths and operations. For similar reasons, in PFC and $\text{PFC}^{+\text{SB}}$, the substitutivity of identicals cannot be proven in all cases.⁷ Indeed, these systems are so weak that, in them, one cannot demonstrate such basic logical truths as the symmetry of identity. While it may be true that it is impossible to

⁷The substitutivity of identicals is a consequence of Frege’s Basic Law III (Axiom FC6), but in PFC, the function quantifier in the consequent could only be instantiated to predicative function expressions. This would rule out, for example, inferring “ $\vdash (\exists f)f(4 + 4)$ ” on the basis of “ $\vdash (\exists f)f(2)$ ” and “ $\vdash 2 + 2 = 4$ ”.

demonstrate a contradiction in the system $\text{PFC}^{+\text{SB}}$, it is also impossible to demonstrate many *fundamental* logical truths. It is simply too weak.

Lastly, the changes Heck advocates seem simply *ad hoc*; again, no rationale is given for the predicativity requirement save staving off contradiction. Moreover, the predicativity requirement seems at odds with the very notion of a functional language. Frege's language is really designed with the assumption that any incomplete or open expression is the name of a function. Indeed, this was quite natural for Frege, as he modeled his language on the language of arithmetic. Mathematicians have almost always made the assumption that any mathematical formula involving the variable "x" can be seen as corresponding to a function of x , whether it is something as simple as " x^2 ", or something more complex such as " $x^2 + 6x - 3$ ", or indeed something much more complex still (even something involving different sorts of quantification). This is justified given that mathematicians tend to assume that there exists a function for every distinct, determinate mapping from numbers onto other numbers.

Frege of course wanted to provide a foundation for arithmetic, and thus needed a system that preserved this mathematical assumption. This is why he allowed function variables to be instantiated to any incomplete expression, regardless of complexity. Frege modeled his language on the language of arithmetic, but aimed to create a more general language, one that might prove handy for work in other disciplines. However, since the language was to retain the function/argument structure of mathematical formulae, this required expanding the notion of a function to allow arguments and values that are not numbers. This made it natural enough for Frege to adopt the assumptions made by mathematicians regarding mathematical functions for functions generally, such as the assumption that a function exists for every open expression. For Frege, a function exists for every distinct, determinate mapping from objects on to other objects generally. These assumptions are built into the foundations of his language so deeply that to modify them would require completely restructuring his logical language if even basic logical truths or operations are still to be captured.

It is also worth bearing in mind that Frege's powerful, impredicative, Roman instantiation rule does not by *itself* lead to contradiction or antinomy. The system FC contains this rule, and is wholly consistent. It only leads to contradiction or antinomy in conjunction with other commitments, such as to classes or to a robust realm of *Sinn*. Thus, I do not think that the most fruitful avenue for attempting to find resolutions to the difficulties in the Fregean logic of *Sinn* and *Bedeutung* is to look for modifications to his commitment to concepts and functions. This commitment can be shown to be harmless in itself, and any modification of it would require completely abandoning Frege's approach to the creation of a logical language. If we are to allow *any* functions

whose domain and range include all objects, it simply makes the most sense to posit a function for every different mapping from entities in the domain onto entities in the range. This entails a concept for every mapping of argument objects onto truth-values, regardless of the complexity of the incomplete expression needed to denote such a concept in a logical language.

REDUCING THE NUMBER OF *SINNE* AND *GEDANKEN*

Frege's commitment to such a multitude of functions and concepts does not, in isolation, lead to paradox or antinomy; the system FC becomes inconsistent only when expanded to FC^{+SB} . This makes it natural to conclude that the commitment to such a large third realm of *Sinn* is the true root of the problems. Moreover, the addition of the commitment to meaning-entities such as *Sinne* and the sign, " $\Delta(\xi, \zeta)$ ", introduces *semantic* elements into the logical language, and certainly these moves might be seen as to blame for the introduction of semantical paradoxes such as the modified Epimenides paradox. Moreover, as noted in the previous chapter, even the variants of the *Principles of Mathematics* Appendix B paradox, though not semantical in Russell's logic, can be understood as involving semantic notions in Frege's logic. This makes it natural to look carefully at the expansion of Frege's logic suggested by the theory of *Sinn* and *Bedeutung* and attempt to find a solution by modifying or dropping some of the added axioms.

It was noted earlier that the strategy-one solutions of completely dropping commitment to *Sinne* or *Gedanken* are not feasible if we are committed to maintaining a broadly Fregean theory of meaning. However, the strategy-two approach of attempting to reduce somehow the number of *Sinne* and *Gedanken* requires further examination. The Cantorian analysis of the problems reveals that the class/*Sinn*, class/*Gedanke* and concept/*Gedanke* antinomies seem to result from positing as many *Sinne* and *Gedanken* as classes and concepts. Again, we may hope to find a solution to the antinomies by attempting to find a way to reduce the number of *Sinne* and *Gedanken* posited. Let us focus our discussion on the class/*Sinn* and concept/*Gedanke* antinomies; it should be clear how similar considerations could also be applied to the class/*Gedanke* antinomy.

Consider first the simple class/*Sinn* antinomy. This antinomy seems to result from the commitment to as many *Sinne* as classes. If one looks at the new axioms added to the system FC^{+SB} , the commitment to so many *Sinne* seems to result from axiom $FC^{+SB}35$, which asserts that for every object, there is at least one *Sinn* picking it out. Since classes are objects, this entails that for every class, there is a *Sinn* picking it out. It is worth noting, however, that this axiom does not immediately lead by itself to the violation of Cantor's theorem; at least one axiom governing the identity conditions of *Sinne* is also required. Specifically, while $FC^{+SB}35$ does guarantee a *Sinn* picking out every class, if it were possible

for the same *Sinn* to pick out more than one class, then $FC^{+SB}35$ alone would not violate Cantor's theorem because the *Sinn* it postulates might be the same for different classes. However, axiom $FC^{+SB}13$ rules this out, because it requires that a *Sinn* never pick out more than one thing. Axiom $FC^{+SB}13$ is, however, probably too central to the very notion of a *Sinn* to seriously consider giving up. Allowing a *Sinn* to pick out more than one object as *Bedeutung* would amount to giving up the determinacy of meaning. If the *Sinn* in question is a *Gedanke*, it could even have the result that a *Gedanke* could be both true and false (i.e., could pick out both the True and the False). Let us focus instead on axiom $FC^{+SB}35$.

We have already discussed some of the perils of allowing there to be some classes or other objects that are not picked out by any *Sinn*. As we noted in Chapter 3, given a Fregean account of *Sinne*, the identity of indiscernibles seems to require that every object be picked out by at least one *Sinn*. If the identity of indiscernibles holds, then there is at least one set of descriptive information uniquely satisfied by any given object, and thus, if a *Sinn* is understood as picking out its *Bedeutung* in virtue of it alone satisfying some set of descriptive information, there would have to be at least one *Sinn* for every object. Moreover, a *Gedanke* is only about a certain object if it contains a constituent *Sinn* picking out that object, and a proposition is only about what it is in virtue of what *Gedanke* it expresses. If we hold that there are some objects not picked out by any *Sinn*, then there must be some objects about which it is *impossible* to form *Gedanken* or name using language. Of course, since there can only be denumerably many expressions in a language with a finite number of primitive signs, if there are more than denumerably many objects, there must be some objects that *are not* nameable in any *given* language. But the stronger claim that there are some objects that *could not* be named in *any* language because there are no *Sinne* picking them out is an odd result. Moreover, on Frege's own view, a *fact* is simply a true *Gedanke* (CP 368). Thus, because a *Gedanke* is about a given object only if it contains a constituent *Sinn* picking out that object, denying that all objects have *Sinne* would, *on this view*, mean that some objects exist *about which there are no facts*. Surely, this is an intolerable result. Nevertheless, perhaps these difficulties could be overcome by adopting a different understanding of facts, and by either denying the identity of indiscernibles (a controversial hypothesis anyway) or reinterpreting the relationship between a *Sinn* and its *Bedeutung*.

However, there are larger obstacles to attempting to take this solution. As seen in Chapter 5, axiom $FC^{+SB}35$ is used in the derivation of theorem 13.1, the theorem to the effect that an incomplete *Sinn* can pick out only one function as *Bedeutung*:

Theorem $FC^{+SB}13.1 \vdash (\forall a a^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \&$
 $(\forall a a^*)[\Delta(a^*, a) \supset \Delta(F(a^*), g(a))] \therefore (\forall a)(f(a) = g(a))$

Even if we dropped axiom $FC^{+SB}35$, it would be desirable to maintain this theorem to preserve the determinacy of the meaning with regard to functions as *Bedeutungen*. However, if we were to take theorem 13.1 as an additional primitive axiom (or add other axioms from which it could be derived), it would be possible to derive $FC^{+SB}35$ as a theorem.⁸ It is difficult to escape including $FC^{+SB}35$, one way or another.

Even if we managed, somehow, to avoid this difficulty, the approach of dropping axiom $FC^{+SB}35$ is, for another reason, unsuited to solve the class/*Sinn* antinomy. The class/*Sinn* antinomy centers around the class \clubsuit , consisting of all *Sinne* picking out classes they are not in, and \clubsuit^* , a *Sinn* picking out \clubsuit . The only place in the demonstration of the contradiction at which axiom $FC^{+SB}35$ appears is in providing justification for the introduction of \clubsuit^* . However, it would be possible to introduce \clubsuit^* without appealing to this axiom at all! All that is important for the deduction of the contradiction is that there is some *Sinn* \clubsuit^* , picking out \clubsuit . We could arrive at this result simply by *defining* " \clubsuit^* " as standing for the *Sinn* of the sign " \clubsuit ". Because " \clubsuit " is itself a defined sign, we could define " \clubsuit^* " simply by using the *Sinn*-transform of the definiens of " \clubsuit ". Recall that the definition of " \clubsuit " was:

$$\Vdash \clubsuit = \varepsilon^2(\exists a)[\Delta(\varepsilon, a) \& \sim(\varepsilon \cap a)]$$

Given the definitions of the existential quantifier and conjunction signs, this is equivalent to:

$$\Vdash \clubsuit = \varepsilon^2(\forall a)[\Delta(\varepsilon, a) \supset (\varepsilon \cap a)]$$

⁸As an informal sketch of the proof, consider the case in which we have $FC^{+SB}13.1$ as an axiom or established result. Now assume there is some object o that is not picked out by any *Sinne*. Then consider the concepts $\xi = \xi$ and $\xi \neq o$. These concepts have the same truth-value as value for every argument except for o itself. It follows from axiom $FC^{+SB}42$ that $\vdash (\forall a a^*)[\Delta(a^*, a) \supset \Delta(a^* \approx a^*, a = a)]$. Given our assumption, it also holds that $\vdash (\forall a a^*)[\Delta(a^*, a) \supset \Delta(a^* \approx a^*, a \neq o)]$, because the antecedent is false for all values of a^* when a is o , and the consequent is always true for all other values of a . Then, by $FC^{+SB}13.1$, $\vdash (\forall a)[(a = a) = (a \neq o)]$, which leads immediately to contradiction when instantiated to o . Hence, there is no such o , and $FC^{+SB}35$ would hold. This follows closely a similar proof given by Anderson for Church's early systems. See Anderson, "Some New Axioms," 221.

Using the sign “ ε ” introduced in Chapter 6, we could simply *define* “ \clubsuit^* ”:

$$\Vdash \clubsuit^* = \overset{\circ}{\varepsilon} \neg(\Pi a^*)[\blacktriangle(\varepsilon, a^*) \rightarrow (\varepsilon \in a^*)]$$

From these definitions, the theorem “ $\vdash \Delta(\clubsuit^*, \clubsuit)$ ” follows straightforwardly from axioms FC^{+SB}39 through FC^{+SB}45, even *without* FC^{+SB}35. Axiom FC^{+SB}35 is not even necessary for the deduction of the contradiction.

What is to be learned from this? As it turns out, it is not sufficient simply to deny that all classes have *Sinne* picking them out. To avoid the contradiction, we would have to hold that \clubsuit is itself one such class. It is, however, extremely difficult to hold that \clubsuit that is one such class given that we seem to be able to define a name for it simply using logical primitives. So long as the expression used in the definition has a *Sinn*, there must be at least one *Sinn* picking out \clubsuit , because surely the *Sinn* of the definiens denotes \clubsuit if it denotes at all. The only way to block this result would be to suggest that one or more of the logical constants used in the definition does not express a *Sinn*. However, on any Fregean theory of meaning, any significant bit of language must express a *Sinn* if it is to have a *Bedeutung*. To claim that some sign could be defined in a logical language with a *Bedeutung* but no *Sinn* would be in effect to abandon the core idea of Frege’s theory of meaning: that expressions have their *Bedeutungen* only in virtue of their *Sinne*. If it is legitimate to define a sign such as “ \clubsuit ”, that sign must have a *Sinn*, and this *Sinn* must pick out the *Bedeutung* of the sign, viz., \clubsuit itself.

To sum up, the only way of taking this approach to solving the class/*Sinn* antinomy would require us to give up even the claim that a *Sinn* exists for every class we can *name* in our logical language. This surely is going too far. We can certainly imagine more conservative ontologies of *Sinne* such that *Sinne* are understood as proto-linguistic entities—that no more *Sinne* exist than could be expressed in an ideally expressive language. However, as we have seen, this is not sufficient to solve the antinomies. Provided we require that every expression in our language expresses a *Sinn*, then we are committed to at least one *Sinn* picking out any class that we can name in the language. Since \clubsuit is one such class, this approach cannot be used to solve the class/*Sinn* antinomy. At least in this case, the antinomy can be solved as discussed earlier by the alternative approach of dropping commitment to classes.⁹

⁹There is perhaps another route worth considering. For Frege, a class abstract formed by the smooth-breathing is to be regarded as a name in the full sense. However, in some logical treatments of sets or classes, e.g., Quine’s *Set Theory and Its Logic*, a class abstract such as “ $\hat{x}\Phi x$ ” is not treated as a genuine name. Rather, when it appears in a proposition such as “ $y \in \hat{x}\Phi x$ ”, the proposition is to be regarded as shorthand for an

Let us then turn to discussion of the concept/*Gedanke* antinomy, which cannot be solved by dropping commitment to classes. Is the strategy-two approach of reducing the number of *Gedanken* posited any more successful here? The relevant considerations are very similar. The concept/*Gedanke* antinomy seems to result from positing as many *Gedanken* as concepts, in violation of Cantorian principles. This result was obtained by noting that for each concept, there is at least one *Gedanke*: the *Gedanke* that every object falls under that concept, and that the resulting *Gedanke* is different for every concept. A *Gedanke* involves a certain concept only if the *Gedanke* includes an incomplete *Sinn* picking out the concept. This might seem to point a finger at axiom FC^{+SB}36, as it guarantees that every concept has at least one incomplete *Sinn* picking it out. Once again, however, this axiom alone does not lead to a violation of Cantorian principles. One also needs certain axioms governing the identity conditions of *Gedanken*. More on this later.

Can a solution to the concept/*Gedanke* antinomy be found by denying axiom FC^{+SB}36? At first, it might be thought possible. The antinomy centers around the concept $\mathcal{F}(\xi)$, the concept of *being a Gedanke for which there is a concept such that the Gedanke is the Gedanke that everything falls under the concept, but such that the Gedanke itself does not fall under the concept*. However, in order to arrive at the antinomy, we need to form a *Gedanke* that everything falls under *this* concept. To do so, we introduced an incomplete *Sinn*, $\mathfrak{A}(\delta)$, picking out $\mathcal{F}(\xi)$. The justification given for the introduction of $\mathfrak{A}(\xi)$ in Chapter 5 was axiom FC^{+SB}36. The problematic *Gedanke*, $(\Pi\alpha^*)\mathfrak{A}(\alpha^*)$ can only be formed if we have an incomplete *Sinn* $\mathfrak{A}(\delta)$, picking out $\mathcal{F}(\xi)$. It might be hoped, then, that the antinomy could be blocked by denying that all concepts have incomplete *Sinne* picking them out.

However, this solution does not work. For the same reason we cannot solve the class/*Sinn* antinomy by denying axiom FC^{+SB}35, we cannot solve the concept/*Gedanke* antinomy by denying axiom FC^{+SB}36. Because the concept sign " $\mathcal{F}(\xi)$ " is defined purely in terms of logical constants, we could simply

existential proposition such as " $(\exists z)[(x)(x \in z \equiv \Phi x) \ \& \ y \in z]$ ", in which no class abstract appears. In a language such as this, there would be no *genuine name* for the class \clubsuit , and hence, one could not argue that there must be at least one *Sinn* picking it out simply because the name of the class must express a *Sinn*. However, it can surely be wondered whether this approach is sufficiently Fregean. It seems to cohere much better with a Russellian theory of naming, in which so-called "names" of classes are treated like disguised definite descriptions. Indeed, one must wonder whether, on this view, any *Sinne* picking out classes would be posited to exist at all. In any case, there is no parallel approach to solving the concept/*Gedanke* antinomy, at least short of giving up on a functional language altogether.

define the sign “ $\forall(\delta)$ ”, standing for its *Sinn* by using the *Sinn*-transform of its definiens,¹⁰ just as it is possible in the case of the class/*Sinn* antinomy to define “ \clubsuit^* ” using the *Sinn*-transform of the definiens of the sign “ \clubsuit ”. Indeed, this very approach was actually taken in the formulation given for the modified Epimenides paradox in Chapter 5. There, rather than introducing the incomplete *Sinn* sign “ $\neg(\delta)$ ” by appealing to an instance of $FC^{+SB}36$ for the concept $T(\xi)$, which it picks out, “ $\neg(\delta)$ ” was defined using the *Sinn*-transform of the definiens in the definition of “ $T(\xi)$ ”. Taking the same approach for the concept/*Gedanke* antinomy would obviate the need for appealing to axiom $FC^{+SB}36$. As before, it turns out that we cannot solve the concept/*Gedanke* antinomy simply by claiming just that *some* concepts do not have incomplete *Sinne* picking them out. We would have to insist that $F(\xi)$ is one such concept. However, this is extraordinarily difficult to maintain, given that we can define the sign “ $F(\xi)$ ” using logical constants only. Unless we are willing to go so far as to say that certain expressions in our logical language are without *Sinn*, this strategy for solving the antinomy will not work.

However, there is another possible way of reducing the number of *Gedanken* posited besides denying that all concepts have incomplete *Sinne* picking them out. This would be to adopt looser identity conditions for *Sinne*. As noted earlier, positing an incomplete *Sinn* for every concept does not in itself guarantee as many *Gedanken* as concepts. Because it is easy to form a *Gedanke* given an incomplete *Sinn*, it does guarantee that one can form a *Gedanke* for each concept, e.g., the *Gedanke* that everything falls under the concept (as presented through that incomplete *Sinn*). However, if it were possible for the *Gedanken* so-formed to be the *same* for different concepts, one could still maintain that there were fewer *Gedanken*.

If one looks at the derivation of the contradiction, it involves theorem $FC^{+SB}48.1$:

$$\vdash (\Pi a^*)F(a^*) = (\Pi a^*)G(a^*) \therefore (\forall a^*)(F(a^*) = G(a^*))$$

Put in terms of ordinary language, this suggests that the *Gedanke* expressed by “every object is an *X*” can be the same as the *Gedanke* expressed by “every object is a *Y*” only if the concept names “() is an *X*” and “() is a *Y*” express the

¹⁰Actually, the system as developed in Chapter 5 is not quite rich enough to provide such a definition, because we would also need to add a constant standing for the *Sinn* of the sign “ $(\Pi a^*)\phi(a^*)$ ”, or, like Church, add a hierarchy of such signs, since this sign is used in the definition of “ $F(\xi)$ ”. Obviously, if the sign has meaning, it must have a *Sinn*, and so one cannot hope to avoid the antinomy simply by suggesting that there is no such *Sinn*.

same incomplete *Sinn*. This principle is justified in the system FC^{+SB} by its understanding of *Gedanken* as wholes consisting of component *Sinne*. If the *Gedanke* expressed by “every object is an *X*” is composed of the *Sinne* of its component expressions, because the other component of the *Gedanke* is the same as that in the *Gedanke* expressed by “every object is a *Y*,” the whole will be the same if and only if “() is an *X*” and “() is a *Y*” have the same *Sinn*.

However, if we deny that *Gedanken* are composite in this way, and adopt looser identity conditions for them, then there may be room for denying theorem $FC^{+SB}48.1$. For example, suppose we held that propositions express the same *Gedanke* whenever they are logically equivalent or true in all and only the same possible worlds. This would allow propositions that have components that express different *Sinne* to nevertheless express the same *Gedanke*. Consider, for example, the propositions, “every object is something that is either a cat or not a cat” and “every object is something that is either a table or not a table.” Since these two propositions are both tautologies, they are logically equivalent and true in all the same possible worlds. However, it certainly is not clear that all of the corresponding component parts of these propositions express the same *Sinne*. Adopting logical equivalence as sufficient for the identity of *Gedanken* may give us room for denying theorem $FC^{+SB}48.1$, certain of the axioms from which it was derived (e.g., $FC^{+SB}19$) and some of the other axioms of system FC^{+SB} that require identity conditions for *Gedanken* stricter than logical equivalence.

However, there are problems with this approach. In Chapter 3, we concluded that the times at which Frege seems to suggest that something similar to logical equivalence is sufficient for the identity of *Gedanken* must be seen as mere lapses and not reflective of his considered views. Looser identity conditions for *Sinne* seem to conflict with both his logicist views and his account of *oratio obliqua*. If Frege held that all necessarily equivalent propositions express the same *Gedanke*, he would be forced to conclude that all logical and mathematical truths express the same *Gedanke*. He would lose his ability to respond to the belief paradoxes by appealing to differences in *Sinn*, since it certainly seems possible for someone to believe a certain proposition without believing all propositions logically equivalent to it. Solving the belief paradoxes, and explaining the difference in informativity between propositions such as “ $\neg 12 = 12$ ” and “ $\neg 5 + 7 = 12$ ” were some of the original motivations for adopting the theory *Sinn* and *Bedeutung* in the first place. This strategy would seem to throw out the baby with the bath-water.

Moreover, it is not even entirely clear that adopting logical equivalence is sufficient to solve the antinomy. Here we have to take a very close look at the details of the Cantorian derivation of the contradiction. It is true that by adopting looser identity conditions for *Gedanken*, we can show theorem $FC^{+SB}48.1$ to be

false. However, the Cantorian diagonal construction involved in the contradiction does *not* require that “() is an X ” and “() is a Y ” express the same *Sinn* if “every object is an X ” and “every object is a Y ” express the same *Gedanke*: it only requires that they refer to the same concept. Unless it is possible for expressions of the form “every object is an X ” and “every object is a Y ” to express the same *Gedanke* even if “() is an X ” denotes a different concept from “() is a Y ,” then one is still committed to as many *Gedanken* as concepts. Indeed, if one looks at the derivation of the contradiction given in Chapter 5 carefully enough, it is clear that the contradiction will arise provided that any *Gedanke* $(\Pi\alpha^*)G(\alpha^*)$ identical to $(\Pi\alpha^*)\forall(\alpha^*)$ is such that the incomplete *Sinn* $G(\delta)$ picks out the same concept as $\forall(\delta)$, viz., $\overline{F}(\xi)$. It is not actually required that $G(\delta)$ be the same incomplete *Sinn* as $\forall(\delta)$.

It must be remembered that concepts are extensional; the concepts denoted by “() is an X ” and “() is a Y ” are the same if they always yield the same truth-value for the same argument. Therefore, in order for this approach to solve the antinomy, one would have to hold that “every object is an X ” and “every object is a Y ” could be logically equivalent even if “() is an X ” and “() is a Y ” are not coextensive. However, it is difficult to imagine that “every object is an X ” and “every object is a Y ” could be *logically equivalent* unless the concept denoted by “() is an X ” is *necessarily coextensive* with that denoted by “() is a Y ,” in which case the former is the same as the latter. To return to the example given above, the expressions “() is something that is either a cat or not a cat” and “() is something that is either a table or not a table” might not have the same *Sinn*, but they certainly denote the same concept, since both have the True as value for any object as argument (in any possible world).

In order for adopting looser identity conditions for *Gedanke* actually to solve the concept/*Gedanke* antinomy, we might very well have to adopt identity conditions of *Gedanken* even *looser* than necessary equivalence. That is, we would have to allow propositions that are not even necessarily equivalent to express the same *Gedanke*. However, this is extremely implausible. It would entail that there are some possible worlds in which the same *Gedanke* is both true and false. Even for those who do not understand necessity in terms of “possible worlds,” the suggestion that two propositions that might have different truth-values could be synonymous would be extremely difficult to maintain.

There is a genuine quandary surrounding the belief conditions of *Sinne* and *Gedanken*. The number of *Gedanken* posited is inversely proportional to how loose we make their identity conditions; the more *Gedanken* we identify, the fewer we actually posit. However, if we make the identity conditions of *Gedanken* too loose, they are no longer fine-grained enough to do what we want them to do. However, if we make their identity conditions too stringent, we commit ourselves to so many that it is very difficult to avoid Cantorian

paradoxes. This might be enough to make even those of us who are otherwise attracted to a Fregean account of meaning in terms of intensional entities such as *Sinne* to sympathize with Quine's complaint that "intensions are creatures of darkness," ever bewildering us with their elusive identity conditions.¹¹

In conclusion, we seem to have had little success finding any way of taking the strategy-two approach of attempting to solve the antinomies by reducing the number of *Sinne* and *Gedanken* posited. This is perhaps not quite as distressing with regard to the antinomies involving classes, since these can still be solved by the strategy-one approach of eliminating commitment to classes altogether. However, we have yet to find an adequate solution to the concept/*Gedanke* antinomy or modified Epimenides paradox. Strategy three seems our only hope.

RAMIFICATION AS A SOLUTION TO THE PARADOXES

Strategy three involves attempting to find some way of revising Frege's philosophy of the nature of concepts, classes and *Sinne* that undermines the intelligibility of asking the questions that give rise to the contradictions. Consider, for example, that on Frege's ontology of functions, a function of a given level can never take itself or any other function of the same level as argument. The level-division in types of functions is philosophically well-grounded by Frege's view of functions as unsaturated entities. However, this aspect of his philosophy makes it impossible to formulate a certain version of Russell's paradox that Russell himself was concerned with: the question of whether or not the property of being a property that does not apply to itself applies to itself. For Frege, the question never even arises for a given concept whether that concept falls under itself. Thus, this version of Russell's paradox is solved in Frege's philosophy by adopting a theory of the nature of functions that makes the paradoxical question unintelligible (*PMC* 132-3). There may be hope that we could find similar solutions to the antinomies arising in the logic of the theory of *Sinn* and *Bedeutung*. For example, with regard to the class/*Sinn* antinomy, perhaps we could find an account of the nature of *Sinne* and classes such that the question would never even arise for a class and a *Sinn* picking out that class whether or not that *Sinn* is in the class it picks out. On this strategy, one would attempt to solve the antinomies in virtue of some appeal to logical types or orders.

The only type distinction made in the system FC is the distinction between objects and functions of varying levels. It is thus able to block the properties or concepts version of Russell's paradox. The system FC^{+V} treats all classes as falling the same type: the type of objects, and thus falls prey to the classes version of Russell's paradox. In addition to the types of FC and FC^{+V}, the

¹¹Quine, "Quantifiers and Propositional Attitudes," 188, 193.

systems FC^{+SB} and FC^{+SB+V} add types for complete *Sinne* and incomplete *Sinne* of various levels. However, all *Gedanken* and complete *Sinne* fall into the same type, regardless of their constituents or *Bedeutungen*, and all concepts and incomplete *Sinne* of a certain level fall into the same type, regardless of their composition or *Bedeutungen*. Therefore, these systems fall prey to antinomies discussed in the previous two chapters. In order to avoid these problems, one would need to adopt a more complicated type-theory, such that within each level of kinds of *Sinne* there is a distinction between various orders. This would be to adopt a form of *ramified type-theory*. Depending on how the details are worked out, it might be possible to restrict the ramification just to different orders of *Sinne* (as opposed to distinguishing orders within non-*Sinn* objects and functions). We have discussed ramified type-theory briefly in previous chapters. It is worth delving further into it here.

In Chapter 4, we discussed already ways in which ramification could be employed to solve some of the antinomies. *Gedanken*, for example, could be divided into various orders. At the lowest order, we have ordinary non-general *Gedanken* such as those expressed by “Socrates is bald” or “Hypatia is wise.” *Gedanken* that are somehow *about Gedanken* of this order would be in the next highest order, and so on. Of course, there remains to be worked out what it means for one *Gedanke* to be “about” another *Gedanke* or other *Gedanken*. Exactly what would count would depend on the details of the ramification and how it is understood. However, at least these three sorts of cases seem to apply. Firstly, if a *Gedanke* involves quantifying over a range that includes *Gedanken*, the *Gedanke* involving the quantification must be of a higher order than the *Gedanken* quantified over. Secondly, if a *Gedanke* is about a class that itself might include *Gedanken* as members, it must be of a higher order than those *Gedanken*. Lastly, if a *Gedanke* is directly about another *Gedanke* in the sense of including as constituent a *Sinn* whose *Bedeutung* is itself a *Gedanke*, the whole *Gedanke* must be of a higher order than the *Gedanke* that is the *Bedeutung* of the constituent *Sinn*.

For the system of ramification to be complete, we would have to ramify also complete *Sinne* that are not *Gedanken*, as well as incomplete *Sinne*. For example, we would insist that a *Sinn* that has as *Bedeutung* a class that includes *Sinne* must be of a higher order than the *Sinne* that might be included in the class. An incomplete *Sinn* that involves quantification over incomplete *Sinne* must be of a higher order than the incomplete *Sinne* quantified over, even if the incomplete *Sinne* are of the same level. The order of the constituent *Sinne* within a *Gedanke* would then be taken to affect the order of the whole in such a way that the order of the whole *Gedanke* could never be lower than the order of any constituent *Sinn*.

There are actually different forms such a ramified theory could take. Unfortunately, I cannot discuss all of the formal details of the various ways in which ramification could be built into a logical calculus here. It should be clear, however, how ramification could be employed to solve the antinomies. In the case of the class/*Sinn* antinomy, we have insisted that if a *Sinn* picks out a class of *Sinne*, the *Sinn* must be of a higher order than the *Sinne* that could be members of the class it picks out. Thus, the question would never arise as to whether a *Sinn* would be a member of a class it picks out. This would mean that the problematic class in the class/*Sinn* antinomy, \clubsuit , is undefinable. In the case of the class/*Gedanke* antinomy, if we consider the *Gedanke* that everything is in some class, that *Gedanke* could never be of the right order to included in the class it is about. Thus, the class of all such *Gedanke* not in their associated class, ω , would similarly be undefinable. The concept/*Gedanke* antinomy requires careful consideration. If one looks at the definition of the concept-sign " $\mathcal{F}(\xi)$ ", it involves quantification over incomplete *Sinne*. In a ramified type-theory, this quantification would have to be limited to incomplete *Sinne* of a particular order. The incomplete *Sinn* $\forall(\delta)$, because it picks out $\mathcal{F}(\xi)$, must be of a higher order than the incomplete *Sinne* involved in this definition. If so, then, the question would not arise as to whether the *Gedanke* $(\Pi a^*)\forall(a^*)$ falls under $\mathcal{F}(\xi)$, because it would not be of the appropriate order.

Indeed, ramification could even be used to find a solution to the modified Epimenides paradox, because it would suggest exactly what is problematic about the sort of self-reference involved in the paradox. If a *Gedanke* is about another *Gedanke*, i.e., if it asserts that the *Gedanke* does not pick out the True as *Bedeutung*, it must be of a higher order than the *Gedanke* it is about. The *Gedanke* $\neg\mathcal{T}(\Upsilon^*)$ contains as constituent the *Sinn* Υ^* . The paradox entails allowing this *Sinn* to pick out as *Bedeutung* the *Gedanke* $\neg\mathcal{T}(\Upsilon^*)$ —a *Gedanke* of which it itself forms a part. However, if we hold this to be impossible, the antinomy is avoided. To put the point in terms of ordinary language, the *Gedanke* expressed by "Church's favorite *Gedanke* is not true" contains as constituent the *Sinn* of "Church's favorite *Gedanke*." The order of the whole *Gedanke* must, however, be higher than the order of any *Gedanke* that a constituent *Sinn* has as *Bedeutung*. This would rule out the *Sinn* of "Church's favorite *Gedanke*" having as *Bedeutung* the *Gedanke* expressed by the whole proposition. Really, instead of writing "Church's favorite *Gedanke* is not true," it would clearer to write "Church's favorite *Gedanke* of order n is not true," and then it becomes obvious that the *Gedanke* expressed by this proposition cannot itself be the *Gedanke* denoted by the subject clause. This *Gedanke* is about *Gedanken* of order n , and for this reason cannot itself be of order n .

Certainly, it would be possible to modify the systems FC^{+SB+V} and/or FC^{+SB} to create a system with ramified types in which the paradoxes discussed in the

previous chapters do not arise. The difficulty that faces anyone who would take this solution is not, as with some of the previous solutions considered, that it is not sufficient to block the contradictions. The difficulty lies rather with providing a philosophical justification for the adoption of orders consistent with even a *broadly* Fregean theory of meaning. If one cannot find solid philosophical reasons why, for example, it must be impossible for a *Gedanke* to involve quantification over a range that might include the *Gedanke* itself, then adopting order-subscripts on our variables would be just as much an *ad hoc* solution to the difficulties as would simply making arbitrary restrictions to our inference rules to block the antinomies.

THE JUSTIFICATION OF RAMIFICATION

Briefly in Chapter 4, I discussed some reasons why it is very difficult to reconcile ramification with Frege's own views, i.e., the reasons Frege himself would likely not have been able to accept ramification. This might not be seen as overly problematic in this context, however, since we are here considering possible ways of *revising* Frege's own views in order to escape the difficulties they face. Although a strict Fregean might not be able to accept ramification, perhaps a theory of meaning that is still *broadly* Fregean can be found for which things are different. Nevertheless, it is worth looking more closely at why a *strict* Fregean could not accept ramification, because it is only from this perspective that we can gauge exactly how far from Frege's own views it would be necessary to deviate for ramification to be justifiable.

On the interpretation of the nature of *Sinne* given in Chapter 3, a (complete) *Sinn* represents a set of descriptive information that picks out its *Bedeutung* in virtue of it alone satisfying the descriptive information. On this reading, the relation between a *Sinn* and the *Bedeutung* it picks out is, to use Kripke's terminology, "non-rigid."¹² The *Sinn* simply picks out *whatever it is* that uniquely satisfies the descriptive information, assuming there is any such thing. The *Sinn* itself is indifferent to *which* object it is that satisfies this information. In other words, the *Sinn* does not in itself directly presuppose any particular object for its *Bedeutung*; it would remain unchanged if some other entity were to satisfy the descriptive information or even if there were no object at all satisfying it. The *Sinn* simply lays out the descriptive criteria, and at this point, its role ends; the *Bedeutung* is determined by the way the world is. This same *Sinn* can be imagined to pick out different entities in different possible worlds. On this understanding of the relationship between a *Sinn* and the object it picks out, the relationship is rather distant. The existence of the *Sinn* is not logically dependent upon the existence of the *Bedeutung*.

¹²See Kripke, *Naming and Necessity*, 48.

However, if we understand the relationship between a *Sinn* and the object it picks out in this distant way, it makes it very difficult to understand what is incoherent about a *Sinn* picking out itself or something involving itself. The *Sinn* expressed by “Kevin’s favorite object” simply lays out certain descriptive criteria in virtue of which a certain object is presented. The *Sinn* itself is indifferent as to which object satisfies these criteria or whether any object satisfies them at all. Given Frege’s realism about *Sinne*, i.e., his view that they possess timeless, objective existence in a third realm of *Sinn*, there would be nothing scandalous or incoherent then about this *Sinn* picking out *itself* as *Bedeutung*. The *Sinn* itself is indifferent as to which object satisfies the criteria it sets out; thus, it should not be impossible that *it itself* should just so happen to be the object that satisfies those criteria, the object I favor most. If we allow this *Sinn* to pick out itself, it seems that there should similarly be no barrier to the *Sinn* of the expression “Church’s favorite *Gedanke*” picking out a *Gedanke* of which this very *Sinn* forms a part, given that the *Sinn* lays out criteria something must meet in order for it to be its *Bedeutung*, it is indifferent as to which object, or even if any object, satisfies these criteria.

The case is similar if we turn to *Gedanken* that involve quantification over a range that includes the *Gedanke* itself. On Frege’s own understanding, a quantifier is simply a sign for a higher-level function, and thus its *Sinn* is simply a higher-level incomplete *Sinn*. Given the interpretation of the nature of incomplete *Sinne* given in Chapter 3, an incomplete *Sinn* is understood simply as an incomplete set of descriptive information. Here, too, this *Sinn* does not presuppose any particular entities. In Chapter 3 it was decided that the incomplete *Sinn* of a concept expression such as “ ξ has a heart” could be understood as corresponding to the incomplete descriptive information one might describe using the phrase “the truth-value of ()’s having a muscle that pumps blood through the arteries” (though again, one must not confuse the *Sinn* with the phrase itself, as *Sinne* are not linguistic entities). The *Sinn* of a second-level quantifier would be understood as an incomplete *Sinn* that is capable of mutual saturation with this sort of first-level incomplete *Sinn*. Thus, we might understand the *Sinn* of “ $(\forall a)(\dots a \dots)$ ” as corresponding to the incomplete descriptive information of “the truth-value of every complete entity’s being such that ((it)) is the True,” such that when this incomplete *Sinn* mutually saturates with the one just considered, the whole *Gedanke* formed could be described using the descriptive information, “the truth-value of every complete entity’s being such that (the truth-value of (it)s having a muscle that pumps blood through the arteries) is the True.” This *Gedanke* simply lays out a set of descriptive information. It is indifferent as to what complete entities there are; it does not directly presuppose any particular objects as falling within its range. Therefore, there is nothing incoherent about this very *Gedanke* being one of the

complete entities upon which the *Bedeutung* of the whole *Gedanke* depends. Since the *Gedanke* simply *happens to be* a complete object, and since the truth-value of its having a muscle that pumps blood through the arteries is not the True, this *Gedanke* can be one of the entities that makes its *Bedeutung* the False.

Frege's *Sinne* and *Gedanken* are objectively real entities. They are in no way constructed by mind or language. Their relation to their *Bedeutungen* is wholly non-rigid; in and of themselves their existence does not presuppose any particular entities as their *Bedeutungen* or even any *Bedeutungen* at all. As I put the point in Chapter 4, orders are usually justified in virtue of some sort of "vicious circle principle"; however, Frege's own understanding of the nature of *Sinne* and *Gedanken* makes it impossible to understand what makes the circle vicious. This seems to preclude the historical Frege from providing any justification for the adoption of orders. It is not simply enough to say that the antinomies and paradoxes themselves *demand* the adoption of orders of *Sinne* and *Gedanken*. One would also have to give a *different* account of the nature of *Sinne* and *Gedanken* that explains the philosophical grounds of the ramification. Frege's own views provide no clues as to why it should be impossible for a *Sinn* to have itself or something including itself as *Bedeutung*, or why it should be impossible for a *Gedanke* to involve quantification over a range that includes itself. Unless we are able to give a philosophical explanation for orders, their introduction appears only as an *ad hoc* move to dodge the contradictions, and nothing more. This is not to say that one *cannot* give an alternative account of the nature of *Sinne* and *Gedanken* that would provide philosophical justification for orders. But it is crucial to note that doing so would entail deviating substantially from Frege's own views.

Let us then consider briefly how we might think to revise Frege's account of the nature of *Sinne* and *Gedanken* in order to explain the justification for orders. The preceding discussion seems to suggest at least one possible strategy: to adopt an account of the relationship between a *Sinn* and the *Bedeutung* it presents that is not quite as non-rigid. Suppose we were to adopt an account of the nature of *Sinne* such that a *Sinn* is metaphysically dependent in some way upon its *Bedeutung*. Indeed, the description of *Sinne* as "modes of presentation" almost suggests something along the following lines: different *Sinne* for the same *Bedeutung* can be understood as different ways of getting or arriving at the *Bedeutung* cognitively. However, it is impossible for there to be a way of getting at or arriving at what does not exist. First, the existence of the *Bedeutung* is determined, and then, in virtue of its existence, different ways of grasping it are thereby also determined to exist. This might give us room to insist that the *Sinne* could not exist without the *Bedeutung*, for without the thing there to grasp, there could be no ways of grasping it. If we adopt an account like this, we may be able to explain what is metaphysically incoherent about a *Sinn* picking out as

Bedeutung itself or something involving itself. If we suppose that the *Sinn* is metaphysically dependent upon its *Bedeutung*, i.e., that the existence of the *Sinn* is determined by the existence of the *Bedeutung*, then for a *Sinn* to present itself (or something involving itself) as *Bedeutung*, it would have to have determined itself to exist. We might suppose this to be simply metaphysically incoherent. Similarly, we could suggest that a *Gedanke* involving quantification over a range depends metaphysically on the members of the range it quantifies over. Therefore, it cannot quantify over itself because, once again, its existence would depend on itself in some incoherent manner.

This sort of view does not sit well with Frege's own understanding of *Sinne* as metaphysically independent, objective entities. On Frege's understanding, a *Sinn* does not in any way directly presuppose its *Bedeutung*. As we have seen, Frege allows the existence of *Sinne* that have no *Bedeutung* at all. The sort of account we have been giving seems to sit much better with a more deflationary understanding of the metaphysics of *Sinne*. For example, it seems to cohere better with accounts of the nature of *Sinne* or *Gedanken* that make these out to be mental or linguistic entities. If one adopts a mentalistic understanding of the nature of *Gedanken*, for example, one might be able to explain orders of *Gedanken* as involving different levels on which the mind supposedly operates. One might even be able to explain the impossibility of a *Sinn* having itself or something involving itself as *Bedeutung* by suggesting that the intentionality of the mind involves causal relations, and that it is causally impossible for something to be a partial or full cause of itself. Alternatively, if one adopts a linguistic understanding of the nature of *Sinne* and *Gedanken*, one might be able to provide justification for orders in terms of some sort of Tarskian hierarchy of languages. If *Gedanken* are understood as linguistic entities, then their order could be explained in terms of where they fall in the linguistic hierarchy. If a *Gedanke* is about another *Gedanke*, it must always be of a higher order than the one it is about, because, arguably, it is impossible to talk *about* the *Gedanken* in one language save in some language higher up in the hierarchy.

I cannot here spell out any complete theory of meaning consistent with adopting ramification as a solution the paradoxes. It should be clear, however, that any such theory would require substantial deviation from Frege's own views. Indeed, such a theory would likely involve giving up certain attractive features his views seem to have. As we have seen, such an approach may necessitate giving an account of *Sinne* such that the existence of *Sinne* metaphysically depends upon the existence of their *Bedeutungen*. This would entail denying that there are *Sinne* without *Bedeutungen*. Frege himself is able to use such *Sinne* to explain how it is that expressions such as "least rapidly converging series" and "Romulus" can be meaningful even if they have no *Bedeutung*. A different account would have to be given if *Sinne* without

Bedeutungen are ruled out. Frege's view that *Sinne* and *Gedanken* are independent of both language and mind also has its advantages that would be lost on the rival understandings. Frege of course appeals to *Sinne* and *Gedanken* to explain the meaningfulness of language. However, if *Sinne* and *Gedanken* are themselves taken to be linguistic entities, this will obviously be impossible. If *Sinne* and *Gedanken* are taken instead to be mental entities, difficulties might be found in explaining how it is that different people seem to be able to grasp the same *Gedanke* or why it is that the truth-value of a *Gedanke* does not seem to depend in any way on anyone's psychological states. Frege's own view readily provides answers to these questions. However, there may yet be room for working out solutions to these problems.

Substantial revision to Frege's own views would be required if the paradoxes described in the preceding two chapters plaguing the logic of the theory of *Sinn* and *Bedeutung* are to be solved by appeal to ramified type-theory. Such revisions may also require giving up some of the virtues of Frege's own views. However, this strategy may still be the most promising way of salvaging a *broadly* Fregean theory of meaning. We concluded earlier that the strategy-one approach of altogether abandoning *Sinne*, *Gedanken* or functions and concepts are not very promising avenues for those interested in maintaining the core of Frege's approach to meaning. The strategy-one approach of dropping classes was found to deal adequately with certain of the problems, but left others still to be solved. We were also unable to find adequate ways of making good on the strategy-two approaches of attempting to reduce the number of concepts, *Sinne* or *Gedanken* in such a way as to block the Cantorian paradoxes. This leaves us only with taking the strategy-three approach of attempting to undermine the intelligibility of the Cantorian constructions, by attempting to dodge the problematic questions leading to the paradoxes by showing them to be unintelligible given some theory of logical types. We have seen, however, that this requires ramified type-theory. If the only way to justify adopting ramified type-theory is to deviate substantially from Frege's own understanding of the nature of *Sinne*, it seems that we are forced to do so if we are to retain a theory of meaning that is Fregean in *any* degree.

CONCLUSION

Its commitment to naive class theory and the resulting derivation of Russell's paradox are often thought to be the primary, sometimes the only, significant formal problems with Frege's logical theory. However, as we have seen, the theory of *Sinn* and *Bedeutung* is intricately tied to Frege's logical views, and indeed, Frege's extant Begriffsschrift is incomplete because the commitments of the theory of *Sinn* and *Bedeutung* are not axiomatized therein. However, filling this gap by providing a logical calculus for the theory of *Sinn* and *Bedeutung*

based upon a close reading of Frege's own works reveals difficulties and paradoxes, some of which are wholly unrelated to naive class theory. Frege's logical theory thus contains internal flaws that have hitherto been scarcely addressed or discussed by philosophers. Perhaps most importantly, the concept-version of the *Principles of Mathematics* Appendix B paradox, which we have here called the "concept/*Gedanke* antinomy," shows that Frege's Begriffsschrift, even bereft of the axioms governing classes that lead to the previously identified inconsistencies, is inconsistent when expanded to include the commitment of the theory of *Sinn* and *Bedeutung*. A close consideration of the possible ways in which Frege's views might be modified to solve these difficulties reveals that the only hope for salvaging a broadly Fregean theory of meaning requires substantially revising some of the details of Frege's own views, perhaps adopting a quite different account of the nature of *Sinne*. We would not have been able to recognize these difficulties or evaluate possible solutions to them without the development of a logical calculus for the theory of *Sinn* and *Bedeutung*, hence the importance of the present project.

These results are of course quite important for the full appreciation and assessment of Frege's philosophy. However, they can also be seen as important for a variety of views in the philosophy of language. While there are very few, if any, contemporary philosophers who subscribe to Frege's own theory of meaning down to the letter, there are a number of philosophers of language whose theories of meaning are broadly Fregean in thinking that the reference or denotation (in Frege's terminology, *Bedeutung*) of a linguistic expression is determined indirectly by means of some entity that constitutes the sense or meaning (Frege's *Sinn*) of the expression. For example, the views of such thinkers as Michael Dummett, Graeme Forbes, Jerrold Katz, John Searle, Gabriel Segal, Terence Parsons and David Kaplan could all be said to be "Neo-Fregean" to some extent or another. As we have seen, similar sorts of challenges face a variety of views resembling Frege's. One can only recommend to such thinkers that they attempt to develop their views in sufficient detail to determine to what extent such paradoxes and antinomies apply to them, and what routes to resolving these difficulties they would advocate. In order to develop a logical calculus for Frege's theory of meaning, we were forced to clear up some of the interpretative details of Frege's views, and indeed, speculate what his views would have been on certain issues that he did not explicitly discuss. If nothing else, developing a logical calculus for a theory of meaning requires one to fill in the details and resolve the ambiguities of that theory, which is beneficial for its own sake. Fregean theories of meaning indeed have many attractions; the challenge is working out the details of a Fregean theory that escapes the persistent danger of collapsing under the heavy weight of a robust metaphysics of intensional entities.

In Frege's own case, while it is true that the difficulties and problems revealed by expanding his Begriffsschrift in line with his theory of meaning show that the latter must be revised or abandoned, this should not convince us that there is nothing positive to be learned from Frege about the proper development of intensional logic. The successes of the logic of *Sinn* and *Bedeutung* in such areas as solving philosophical puzzles regarding identity, propositional attitudes and "quantifying in" testify to the power of Frege's views. These successes may prove helpful in the attempt to work out the details of a revised calculus for intensional logic. Indeed, were it not for the paradoxes, Fregean intensional logic would have many attractions. The paradoxes themselves are quite surprising and unexpected. To be sure, their presence does force us to revise Frege's views. However, in thinking through them and considering how such difficulties could be avoided, those of us interested in the philosophy of meaning can make significant strides in the direction of the truth. Here, it is worth remembering Frege's own reaction to Russell's discovery of the inconsistency of his logical system. He wrote:

Your discovery is at any rate a very remarkable one, and it may perhaps lead to a great advance in logic, undesirable as it may seem at first sight. (*PMC* 132)

Summary of Definitions, Axioms and Inference Rules

DEFINITIONS

- (Df. \neq) $\Vdash (x \neq y) = \sim(x = y)$
 (Df. \vee) $\Vdash (x \vee y) = (\sim x \supset y)$
 (Df. $\&$) $\Vdash (x \& y) = \sim(x \supset \sim y)$
 (Df. \equiv) $\Vdash (x \equiv y) = (\dashv x = \dashv y)$
 (Df. \cap) $\Vdash (x \cap y) = \lambda \alpha \sim(\forall f)[(y = \varepsilon f(\varepsilon)) \supset \sim(f(x) = \alpha)]$
 (Df. $\exists a$) $\Vdash (\exists a)f(a) = \sim(\forall a)\sim f(a)$
 (Df. $\exists \beta$) $\Vdash (\exists \beta)M_\beta(f(\beta)) = \sim(\forall \beta)\sim M_\beta(f(\beta))$
 (Df. $\exists \beta, \gamma$) $\Vdash (\exists \beta)M_{\beta\gamma}(f(\beta, \gamma)) = \sim(\forall \beta)\sim M_{\beta\gamma}(f(\beta, \gamma))$
 (Df. $\exists a^*$) $\Vdash (\exists a^*)f(a^*) = \sim(\forall a^*)\sim f(a^*)$
 (Df. $\exists \mathfrak{F}$) $\Vdash (\exists \mathfrak{F})M_\beta(\mathfrak{F}(\beta)) = \sim(\forall \mathfrak{F})\sim M_\beta(\mathfrak{F}(\beta))$
 (Df. $\exists \mathfrak{F}, \gamma$) $\Vdash (\exists \mathfrak{F})M_{\beta\gamma}(\mathfrak{F}(\beta, \gamma)) = \sim(\forall \mathfrak{F})\sim M_{\beta\gamma}(\mathfrak{F}(\beta, \gamma))$
 (Df. \spadesuit) $\Vdash \spadesuit = \varepsilon(\exists a)[\Delta(\varepsilon, a) \& \sim(\varepsilon \cap a)]$
 (Df. S) $\Vdash S(x) = (\exists a^*)(x = a^*)$
 (Df. T) $\Vdash T(x) = \Delta(x, (\forall a)(a \supset a))$
 (Df. \beth) $\Vdash \beth(x^*) = \blacktriangle(x^*, (\Pi a^*)(a^* \rightarrow a^*))$
 (Df. F) $\Vdash F(x) = (\exists f\mathfrak{F})\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{F}(a^*), f(a))] \&$
 $(x = (\Pi a^*)\mathfrak{F}(a^*)) \& \sim f(x)\}$
 (Df. \in) $\Vdash (x^* \in y^*) = \lambda \alpha \neg(\Pi \mathfrak{F})[(y^* \approx \varepsilon \mathfrak{F}(\varepsilon)) \rightarrow \neg(\mathfrak{F}(x^*) \approx \alpha)]$
 (Df. ω) $\Vdash \omega = \varepsilon(\exists mm^*)[\Delta(m^*, m) \& \varepsilon = (\Pi x^*)(x^* \in m^*) : \& : \sim(\varepsilon \cap m)]$
 (Df. $\boldsymbol{\nu}$) $\Vdash \boldsymbol{\nu} = (\Pi x^*)(x^* \in \boldsymbol{\omega}^*)$

AXIOMS

System FC consists only of axioms FC1 through FC6. System FC^{+V} consists of axioms FC1 through FC6 and FC^{+V}7 through FC^{+V}9. System FC^{+SB} consists of axioms FC1 through FC6 and axioms FC^{+SB}10 through FC^{+SB}53. System FC^{+SB+V} consists of all the axioms below.

- Axiom FC1. $\vdash a \supset (b \supset a)$
 Axiom FC2. $\vdash \sim(\sim a = \sim b) \supset (\sim a = \sim b)$
 Axiom FC3. $\vdash (\forall a)f(a) \supset f(a)$
 Axiom FC4. $\vdash (\forall f)M_\beta(f(\beta)) \supset M_\beta(f(\beta))$
 Axiom FC5. $\vdash (\forall f)M_{\beta\gamma}(f(\beta, \gamma)) \supset M_{\beta\gamma}(f(\beta, \gamma))$
 Axiom FC6. $\vdash g(a = b) \supset g((\forall f)(f(b) \supset f(a)))$
 Axiom FC^{+V}7. $\vdash (\varepsilon f \varepsilon) = \alpha g(\alpha) = (\forall a)(f(a) = g(a))$
 Axiom FC^{+V}8. $\vdash a = \varepsilon(a = \varepsilon)$
 Axiom FC^{+V}9. $\vdash (\forall b)\sim[a = \varepsilon(b = \varepsilon)] \supset (\wedge a = a)$
 Axiom FC^{+SB}10. $\vdash (\forall a^*)f(a^*) \supset f(a^*)$
 Axiom FC^{+SB}11. $\vdash (\forall \mathfrak{F})M_\beta(\mathfrak{F}(\beta)) \supset M_\beta(F(\beta))$
 Axiom FC^{+SB}12. $\vdash (\forall \mathfrak{F})M_{\beta\gamma}(\mathfrak{F}(\beta, \gamma)) \supset M_{\beta\gamma}(F(\beta, \gamma))$
 Axiom FC^{+SB}13. $\vdash \Delta(a, b) \ \& \ \Delta(a, c) \ \therefore (b = c)$
 Axiom FC^{+SB}14. $\vdash \Delta(a, b) \supset \mathcal{S}(a)$
 Axiom FC^{+SB}15. $\vdash \sim \mathcal{S}(\varepsilon f \varepsilon)$
 Axiom FC^{+SB}16. $\vdash \sim \mathcal{S}(\sim a)$
 Axiom FC^{+SB}17. $\vdash F(a^*) = F(b^*) \ \therefore a^* = b^*$
 Axiom FC^{+SB}18. $\vdash F(a^*, b^*) = F(c^*, d^*) \ \therefore (a^* = c^*) \ \& \ (b^* = d^*)$
 Axiom FC^{+SB}19. $\vdash [M_\beta^*(F(\beta)) = M_\beta^*(G(\beta))] \ \& \ (\forall a^*\mathfrak{G})[M_\beta^*(\mathfrak{F}(\beta)) \neq \mathfrak{G}(\mathfrak{F}(a^*))] \ \therefore (\forall a^*)(F(a^*) = G(a^*))$
 Axiom FC^{+SB}20. $\vdash [M_{\beta\gamma}^*(F(\beta, \gamma)) = M_{\beta\gamma}^*(G(\beta, \gamma))] \ \& \ (\forall a^*b^*\mathfrak{G})[M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)) \neq \mathfrak{G}(\mathfrak{F}(a^*, b^*))] \ \therefore (\forall a^*b^*)(F(a^*, b^*) = G(a^*, b^*))$
 Axiom FC^{+SB}21. $\vdash [F(a^*) = G(a^*) \ \& \ F(b^*) = G(b^*) \ \& \ a^* \neq b^*] \supset (\forall a^*)(F(a^*) = G(a^*))$
 Axiom FC^{+SB}22. $\vdash [F(a^*, b^*) = G(a^*, b^*) \ \& \ F(c^*, d^*) = G(c^*, d^*) \ \& \ (a^* \neq b^*) \ \& \ (c^* \neq d^*) \ \& \ (a^* \neq c^*) \vee (b^* \neq d^*)] \supset (\forall a^*b^*)(F(a^*, b^*) = G(a^*, b^*))$
 Axiom FC^{+SB}23. $\vdash F(a^*) = G(a^*) \ \& \ (\forall \mathfrak{H})(\mathfrak{H}(a^*) \neq F(b^*)) \ \& \ (\forall \mathfrak{H})(\mathfrak{H}(a^*) \neq G(c^*)) \ \therefore (\forall a^*)(F(a^*) = G(a^*))$
 Axiom FC^{+SB}24. $\vdash F(a^*, b^*) = G(a^*, b^*) \ \& \ (a^* \neq b^*) \ \& \ (\forall \mathfrak{H})(\mathfrak{H}(a^*) \neq F(c^*, d^*)) \ \& \ (\forall \mathfrak{H})(\mathfrak{H}(b^*) \neq F(x^*, y^*)) \ \& \ (\forall \mathfrak{H})(\mathfrak{H}(a^*) \neq G(m^*, n^*)) \ \& \ (\forall \mathfrak{H})(\mathfrak{H}(b^*) \neq G(w^*, z^*)) \ \therefore (\forall a^*b^*)(F(a^*, b^*) = G(a^*, b^*))$
 Axiom FC^{+SB}25. $\vdash [M_\beta^*(F(\beta)) = N_\beta^*(F(\beta)) \ \& \ M_\beta^*(G(\beta)) = N_\beta^*(G(\beta)) \ \& \ F(a^*) \neq G(a^*)] \ \therefore (\forall \mathfrak{F})[M_\beta^*(\mathfrak{F}(\beta)) = N_\beta^*(\mathfrak{F}(\beta))]$

- Axiom FC^{+SB}26. $\vdash [M_{\beta\gamma}^*(F(\beta, \gamma)) = N_{\beta\gamma}^*(F(\beta, \gamma)) \ \&. \\ M_{\beta\gamma}^*(G(\beta, \gamma)) = N_{\beta\gamma}^*(G(\beta, \gamma)) \ \&. F(a^*, b^*) \neq G(a^*, b^*)] \ \supset. \\ (\forall \mathfrak{F})[M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)) = N_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma))]$
- Axiom FC^{+SB}27. $\vdash F(a^*) = a^* \ \supset. (\forall b^*)(F(b^*) = b^*)$
- Axiom FC^{+SB}28. $\vdash F(a^*, b^*) \neq a^*$
- Axiom FC^{+SB}29. $\vdash M_{\beta}^*(F(\beta)) = F(a^*) \ \supset. (\forall \mathfrak{G})(M_{\beta}^*(\mathfrak{G}(\beta)) = \mathfrak{G}(a^*))$
- Axiom FC^{+SB}30. $\vdash M_{\beta\gamma}^*(F(\beta, \gamma)) = F(a^*, b^*) \ \supset. \\ (\forall \mathfrak{G})(M_{\beta\gamma}^*(\mathfrak{G}(\beta, \gamma)) = \mathfrak{G}(a^*, b^*))$
- Axiom FC^{+SB}31. $\vdash F(a^*) = G(b^*) \ \&. a^* \neq b^* \ \supset: (\forall c^*)(\exists \mathfrak{H})(\mathfrak{H}(b^*) = F(c^*))$
- Axiom FC^{+SB}32. $\vdash \Delta(F(a^*), b) \supset (\exists a)\Delta(a^*, a)$
- Axiom FC^{+SB}33. $\vdash \Delta(M_{\beta}^*(F(\beta)), b) \supset (\exists f)(\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))]$
- Axiom FC^{+SB}34. $\vdash \Delta(M_{\beta\gamma}^*(F(\beta, \gamma)), b) \supset \\ (\exists f)(\forall aa^*bb^*)[\Delta(a^*, a) \ \& \ \Delta(b^*, b) \ \supset. \Delta(F(a^*, b^*), f(a, b))]$
- Axiom FC^{+SB}35. $\vdash (\exists a^*)\Delta(a^*, b)$
- Axiom FC^{+SB}36. $\vdash (\exists \mathfrak{F})(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{F}(a^*), f(a))]$
- Axiom FC^{+SB}37. $\vdash (\exists \mathfrak{F})(\forall aa^*bb^*)[\Delta(a^*, a) \ \& \ \Delta(b^*, b) \ \supset. \Delta(\mathfrak{F}(a^*, b^*), f(a, b))]$
- Axiom FC^{+SB}38. $\vdash \Delta(a^*, a) \supset \Delta(\dots a^*, \text{---} a)$
- Axiom FC^{+SB}39. $\vdash \Delta(a^*, a) \supset \Delta(\neg a^*, \sim a)$
- Axiom FC^{+SB}40. $\vdash \Delta(a^*, a) \supset \Delta(\uparrow a^*, \setminus a)$
- Axiom FC^{+SB}41. $\vdash \Delta(a^*, a) \ \& \ \Delta(b^*, b) \ \supset. \Delta(a^* \rightarrow b^*, a \supset b)$
- Axiom FC^{+SB}42. $\vdash \Delta(a^*, a) \ \& \ \Delta(b^*, b) \ \supset. \Delta(a^* \approx b^*, a = b)$
- Axiom FC^{+SB}43. $\vdash \Delta(a^*, a) \ \& \ \Delta(b^*, b) \ \supset. \Delta(\blacktriangle(a^*, b^*), \Delta(a, b))$
- Axiom FC^{+SB}44. $\vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \supset \Delta((\Pi a^*)F(a^*), (\forall a)f(a))$
- Axiom FC^{+SB}45. $\vdash (\forall aa^*)[\Delta(a^*, a) \supset \Delta(F(a^*), f(a))] \supset \Delta(\varepsilon F(\varepsilon), \varepsilon f(\varepsilon))$
- Axiom FC^{+SB}46. $\vdash (\forall \mathfrak{F})\{(\forall aa^*)[\Delta(a^*, a) \supset \Delta(\mathfrak{F}(a^*), f(a))] \supset \\ \Delta(M_{\beta}^*(\mathfrak{F}(\beta)), M_{\beta}(f(\beta)))\} \supset \Delta((\Pi \mathfrak{F})M_{\beta}^*(\mathfrak{F}(\beta)), (\forall f)M_{\beta}(f(\beta)))$
- Axiom FC^{+SB}47. $\vdash (\forall \mathfrak{F})\{(\forall aa^*bb^*)[\Delta(a^*, a) \ \& \ \Delta(b^*, b^*) \ \supset. \Delta(\mathfrak{F}(a^*, b^*), f(a, b))] \\ \supset \Delta(M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)), M_{\beta\gamma}(f(\beta, \gamma)))\} \supset \Delta((\Pi \mathfrak{F})M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)), (\forall f)M_{\beta\gamma}(f(\beta, \gamma)))$
- Axiom FC^{+SB}48. $\vdash (\Pi a^*)F(a^*) \neq G(F(a^*))$
- Axiom FC^{+SB}49. $\vdash \varepsilon F(\varepsilon) \neq G(F(a^*))$
- Axiom FC^{+SB}50. $\vdash (\Pi \mathfrak{F})M_{\beta}^*(\mathfrak{F}(\beta)) \neq G(M_{\beta}^*(F(\beta)))$
- Axiom FC^{+SB}51. $\vdash (\Pi \mathfrak{F})M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)) \neq G(M_{\beta\gamma}^*(F(\beta, \gamma)))$
- Axiom FC^{+SB}52. $\vdash (\Pi \mathfrak{F})M_{\beta}^*(\mathfrak{F}(\beta)) = (\Pi \mathfrak{F})N_{\beta}^*(\mathfrak{F}(\beta)) \ \supset. \\ (\forall \mathfrak{F})[M_{\beta}^*(\mathfrak{F}(\beta)) = N_{\beta}^*(\mathfrak{F}(\beta))]$
- Axiom FC^{+SB}53. $\vdash (\Pi \mathfrak{F})M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)) = (\Pi \mathfrak{F})N_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)) \ \supset. \\ (\forall \mathfrak{F})[M_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma)) = N_{\beta\gamma}^*(\mathfrak{F}(\beta, \gamma))]$

INFERENCE RULES

Horizontal amalgamation rules (hor):

The following pairs of Begriffsschrift expressions are interchangeable, i.e., wherever one of these expressions occurs within a proposition, one can infer a proposition in which the other replaces it, where A and B are any wfes, C any Gothic letter, and D(C) any expression containing Gothic letter C such that $(\forall C)D(C)$ would be a wfe.

$\lceil \text{---} A \rceil$::	$\lceil \text{---} A \rceil$
$\lceil \sim \text{---} A \rceil$::	$\lceil \sim A \rceil$
$\lceil \text{---} \sim A \rceil$::	$\lceil \sim A \rceil$
$\lceil \text{---} (A \supset B) \rceil$::	$\lceil (A \supset B) \rceil$
$\lceil (\text{---} A \supset B) \rceil$::	$\lceil (A \supset B) \rceil$
$\lceil (A \supset \text{---} B) \rceil$::	$\lceil (A \supset B) \rceil$
$\lceil \text{---} (A = B) \rceil$::	$\lceil (A = B) \rceil$
$\lceil \text{---} (\forall C)D(C) \rceil$::	$\lceil (\forall C)D(C) \rceil$
$\lceil (\forall C) \text{---} D(C) \rceil$::	$\lceil (\forall C)D(C) \rceil$
$\lceil \text{---} \Delta(A, B) \rceil$::	$\lceil \Delta(A, B) \rceil$

Interchange (int):

Where A, B, C are wfes, from $\lceil \vdash A \supset (B \supset C) \rceil$, infer $\lceil \vdash B \supset (A \supset C) \rceil$.

Contraposition (con):

Where A and B are wfes, from $\lceil \vdash A \supset B \rceil$, infer $\lceil \vdash \sim B \supset \sim A \rceil$.

Antecedent amalgamation (amal):

Where A and B are wfes, from $\lceil \vdash A \supset (A \supset B) \rceil$, infer $\lceil \vdash A \supset B \rceil$.

Detachment (mp):

Where A and B are wfes, from $\lceil \vdash A \supset B \rceil$ and $\lceil \vdash A \rceil$, infer $\lceil \vdash B \rceil$.

Hypothetical syllogism (syll):

Where A, B and C are wfes, from $\lceil \vdash A \supset B \rceil$ and $\lceil \vdash B \supset C \rceil$, infer $\lceil \vdash A \supset C \rceil$.

Inevitability (inev):

Where A and B are wfes, from $\lceil \vdash A \supset B \rceil$ and $\lceil \vdash \sim A \supset B \rceil$, infer $\lceil \vdash B \rceil$.

Change in generality notation (gen):

- a) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a Roman object letter, and C is a Gothic object letter not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.
- b) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a Roman object letter, C is a Gothic object letter not contained in $\lceil A(B) \rceil$, and D is a wfe not containing B, then from $\lceil \vdash D \supset A(B) \rceil$, infer $\lceil \vdash D \supset (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.
- c) Where $\lceil A(B) \rceil$ is a wfe containing B, a first-level Roman function letter, and C is a Gothic function letter with the same number of argument places as B that is not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.
- d) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a first-level Roman function letter, C is a Gothic function letter with the same number of argument-places as B that is not contained in $\lceil A(B) \rceil$, and D is a wfe not containing B, then from $\lceil \vdash D \supset A(B) \rceil$, infer $\lceil \vdash D \supset (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.
- e) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a Roman complete *Sinn* letter, and C is a Gothic complete *Sinn* letter not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.
- f) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a Roman complete *Sinn* letter, C is a Gothic complete *Sinn* letter not contained in $\lceil A(B) \rceil$, and D is a wfe not containing B, then from $\lceil \vdash D \supset A(B) \rceil$, infer $\lceil \vdash D \supset (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.
- g) Where $\lceil A(B) \rceil$ is a wfe containing B, a first-level Roman incomplete *Sinn* letter, and C is a Gothic incomplete *Sinn* letter with the same number of argument places as B that is not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.
- h) Where $\lceil A(B) \rceil$ is a wfe containing B, where B is a first-level Roman incomplete *Sinn* letter, C is a Gothic incomplete *Sinn* letter with the same number of argument places as B that is not contained in $\lceil A(B) \rceil$, and D is a wfe not containing B, then from $\lceil \vdash D \supset A(B) \rceil$, infer $\lceil \vdash D \supset (\forall C)A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$.

Roman instantiation (ri):

- a) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of Roman object letter B, and C is any wfe containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, where C replaces every occurrence of B in $\lceil A(B) \rceil$, *provided that $\lceil \vdash A(C) \rceil$ is a wfe.*
- b) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of one-place first-level Roman function letter B, and C is any one-place first-level function name containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty places ξ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$, *provided that $\lceil \vdash A(C) \rceil$ is a wfe.*
- c) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of two-place first-level Roman function letter B, and C is any two-place first-level function name containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty places ξ and ζ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$, *provided that $\lceil \vdash A(C) \rceil$ is a wfe.*
- d) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of B, where B is a second-level Roman function letter with one-place argument, and C is any second-level function name with one-place argument containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty place ϕ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$, and whatever sign used for mutual saturation in C replacing all occurrences of β in $\lceil A(B) \rceil$, *provided that $\lceil \vdash A(C) \rceil$ is a wfe.*
- e) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of B, where B is a second-level Roman function letter with two-place argument, and C is any second-level function name with two-place argument containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty place ϕ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$, and whatever signs used for mutual saturation in C replacing all occurrences of β and γ in $\lceil A(B) \rceil$, *provided that $\lceil \vdash A(C) \rceil$ is a wfe.*
- f) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of Roman complete *Sinn* letter B, and C is any complete *Sinn* expression containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, where C replaces every occurrence of B in $\lceil A(B) \rceil$.
- g) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of one-place first-level incomplete *Sinn* letter B, and C is any one-place first-level incomplete *Sinn* expression containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from

$\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty places δ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$.

h) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of two-place first-level incomplete *Sinn* letter B, and C is any two-place first-level incomplete *Sinn* expression containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty places δ and χ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$.

i) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of B, where B is a second-level incomplete *Sinn* letter with one-place argument, and C is any second-level incomplete *Sinn* expression with one-place argument containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty place θ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$. and whatever sign used for mutual saturation in C replacing all occurrences of β in $\lceil A(B) \rceil$.

j) Where $\lceil A(B) \rceil$ is a wfe containing one or more occurrences of B, where B is a second-level incomplete *Sinn* letter with two-place argument, and C is any second-level incomplete *Sinn* expression with two-place argument containing no Gothic or Greek letters contained in $\lceil A(B) \rceil$, from $\lceil \vdash A(B) \rceil$, infer $\lceil \vdash A(C) \rceil$, with C replacing every occurrence of B from $\lceil A(B) \rceil$, the empty place θ of C being filled with the corresponding arguments to B in $\lceil A(B) \rceil$. and whatever signs used for mutual saturation in C replacing all occurrences of β and γ in $\lceil A(B) \rceil$.

Change of Gothic or Greek letter (cg):

a) Where $\lceil D(A(B)) \rceil$ is a wfe containing $\lceil A(B) \rceil$, itself a wfe containing Gothic object letter B, and C is a different Gothic object letter not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash D(A(B)) \rceil$, infer $\lceil \vdash D(A(C)) \rceil$, with C replacing all occurrences of B in $\lceil A(B) \rceil$.

b) Where $\lceil D(A(B)) \rceil$ is a wfe containing $\lceil A(B) \rceil$, itself a wfe containing Greek object letter B, and C is a different Greek object letter not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash D(A(B)) \rceil$, infer $\lceil \vdash D(A(C)) \rceil$, with C replacing all occurrences of B in $\lceil A(B) \rceil$.

c) Where $\lceil D(A(B)) \rceil$ is a wfe containing $\lceil A(B) \rceil$, itself a wfe containing Gothic function letter B, and C is a different Gothic function letter with the same number of argument-places as B that is not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash D(A(B)) \rceil$, infer $\lceil \vdash D(A(C)) \rceil$, with C replacing all occurrences of B in $\lceil A(B) \rceil$.

d) Where $\lceil D(A(B)) \rceil$ is a wfe containing $\lceil A(B) \rceil$, itself a wfe containing Gothic complete *Sinn* letter B, and C is a different Gothic complete *Sinn* letter not

contained in $\lceil A(B) \rceil$, then from $\lceil \vdash D(A(B)) \rceil$, infer $\lceil \vdash D(A(C)) \rceil$, with C replacing all occurrences of B in $\lceil A(B) \rceil$.

e) Where $\lceil D(A(B)) \rceil$ is a wfe containing $\lceil A(B) \rceil$, itself a wfe containing Gothic incomplete *Sinn* letter B, and C is a different Gothic incomplete *Sinn* letter with the same number of argument places as B that is not contained in $\lceil A(B) \rceil$, then from $\lceil \vdash D(A(B)) \rceil$, infer $\lceil \vdash D(A(C)) \rceil$, with C replacing all occurrences of B in $\lceil A(B) \rceil$.

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