Tight path, what is it (Ramsey-) good for? Absolutely (almost) nothing!

Boyadzhiyska, Simona; Lo, Allan

DOI:
10.5817/CZ.MUNI.EUROCOMB23-026

Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'23
Prague, August 28 - September 1, 2023

Tight path, what is it (Ramsey-)good for? Absolutely (almost) nothing!

(Extended abstract)

Simona Boyadzhiyska* Allan Lo*

Abstract

Given a pair of k-uniform hypergraphs (G, H), the *Ramsey number* of (G, H), denoted by $R(G, H)$, is the smallest integer n such that in every red/blue-colouring of the edges of $K_n^{(k)}$ there exists a red copy of G or a blue copy of H. Burr showed that, for any pair of graphs (G, H), where G is large and connected, the Ramsey number $R(G, H)$ is bounded below by $(v(G) - 1)(\chi(H) - 1) + \sigma(H)$, where $\sigma(H)$ stands for the minimum size of a colour class over all proper $\chi(H)$-colourings of H. Together with Erdős, he then asked when this lower bound is attained, introducing the notion of Ramsey goodness and its systematic study. We say that G is H*-good* if the Ramsey number of (G, H) is equal to the general lower bound. Among other results, it was shown by Burr that, for any graph H, every sufficiently long path is H-good.

Our goal is to explore the notion of Ramsey goodness in the setting of 3-uniform hypergraphs. Motivated by Burr's result concerning paths and a recent result of Balogh, Clemen, Skokan, and Wagner, we ask: what 3-graphs H is a (long) tight path good for? We demonstrate that, in stark contrast to the graph case, long tight paths are generally not H-good for various types of 3-graphs H. Even more, we show that the ratio $R(P_n, H)/n$ for a pair (P_n, H) consisting of a tight path on n vertices and a 3-graph H cannot in general be bounded above by *any* function depending only on $\chi(H)$. We complement these negative results with a positive one, determining the Ramsey number asymptotically for pairs (P_n, H) when H belongs to a certain family of hypergraphs.

DOI: `https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-026`

*School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK. E-mail: `s.s.boyadzhiyska@bham.ac.uk`, `s.a.lo@bham.ac.uk`. The research leading to these results was supported by EPSRC, grant no. EP/V002279/1 (A. Lo) and EP/V048287/1 (A. Lo and S. Boyadzhiyska). There are no additional data beyond that contained within the main manuscript.

1 Introduction

A k-uniform hypergraph H, or a k-graph for short, consists of a (finite) set $V(H)$ of vertices and a set $E(H)$ of k-element subsets of $V(H)$, called (hyper)edges. Given k-graphs G and H, the *Ramsey number* of the pair (G, H), denoted by $R(G, H)$, is the smallest integer n such that, in every red/blue-colouring of the edges of the complete k-graph $K_n^{(k)}$, we can find a red copy of G or a blue copy of H. Ramsey's seminal result [14] implies that $R(G, H)$ is finite for any pair of k-graphs G and H. Since then, the study of Ramsey numbers has become a prominent area of research in combinatorics and has inspired the development of many powerful tools in the field (see for example [9, 13] and the references therein).

Even in the simplest setting, when the uniformity is two, Ramsey numbers are often notoriously difficult to understand. The most well-studied case is when $G = H = K_t$. It is known from the early work of Erdős [10] and Erdős and Szekeres [11] that, up to lower order terms, $2^{t/2} \leq R(K_t, K_t) \leq 2^{2t}$ as $t \to \infty$; these bounds remained essentially best possible for several decades, until very recently Campos, Griffiths, Morris, and Sahasrabudhe [6] announced the first exponential improvement in the upper bound.

Apart from demonstrating the difficulty of understanding Ramsey numbers, this example shows that Ramsey numbers can grow very quickly compared to $v(G)$ and $v(H)$. It is then natural to ask: how *small* can Ramsey numbers be? Here we will always assume that G is connected. As shown by Burr [4], following a slightly weaker observation by Chvátal and Harary [8], for any G and H with $v(G) \geq \sigma(H)$[1], we have

$$R(G, H) \geq (v(G) - 1)(\chi(H) - 1) + \sigma(H). \tag{1}$$

Indeed, colour the complete graph of order $(v(G) - 1)(\chi(H) - 1) + \sigma(H) - 1$ so that the red edges form $\chi(H)$ cliques, one of order $\sigma(H) - 1$ and the rest of order $v(G) - 1$; it is not difficult to check that there is neither a red copy of G nor a blue copy of H in this colouring. A classic result of Chvátal [7] shows that the bound in (1) is attained with equality when the pair consists of a tree and a complete graph. Motivated by this result, Burr [4] and Burr and Erdős [5] investigated what other pairs have this property, introducing the notion of *Ramsey goodness*. More precisely, a graph G is said to be H*-good* if the lower bound in (1) is attained for the pair (G, H). Since its introduction this notion has received considerable attention (see [9, Section 2.5] and the references therein for some history and results). Typically in this line of research H is thought of as a fixed graph and the task is to identify what properties make a (sufficiently large) graph H-good. Several conjectures were made (for example, by Burr [4] and Burr and Erdős [5]), suggesting that, for a fixed graph H, every sufficiently sparse large graph G should be H-good. These conjectures turned out to be false in general, as shown by Brandt [3]. On the other hand, it is known that there are some families of graphs such that every sufficiently large member is H-good for every H. In particular, Burr [4] showed that, for any graph H, any sufficiently long path is H-good.

[1]Here $\sigma(H)$ is the smallest possible size of a colour class in a proper colouring of H using $\chi(H)$ colours.

More generally, Allen, Brightwell, and Skokan [1] showed that, for every fixed H, every large graph with bounded bandwidth is H-good.

We are interested in exploring the notion of Ramsey goodness for hypergraphs. Again, if G is connected[2], the lower bound in (1)[3] holds for all k-graphs H with $v(G) \geq \sigma(H)$. We say that G is *H-good* if equality holds in (1).

The study of Ramsey goodness in hypergraphs was first undertaken by Balogh, Clemen, Skokan, and Wagner [2] and was motivated by a question of Conlon. The n-vertex 3-uniform *tight path* P_n consists of n vertices $v_1, \ldots, v_n$ and hyperedges given by $v_i v_{i+1} v_{i+2}$ for all $i \in [n-2]$. Letting $\mathbb{F}$ denote the *Fano plane*, that is, the unique 3-graph on seven vertices in which every pair of vertices is contained in a unique edge, Conlon asked what 3-graphs are $\mathbb{F}$-good. Balogh, Clemen, Skokan, and Wagner [2] made progress towards answering this question by showing that any sufficiently long tight path is $\mathbb{F}$-good. In light of their work and Burr's result for paths in the graph case [4], we seek to identify what hypergraphs a tight path is good for. We focus specifically on 3-uniform hypergraphs.

2 Results

For a 3-graph H, we say that H is *Ramsey-good for tight paths* if every sufficiently long tight path is H-good. Perhaps surprisingly, it turns out that there are plenty of classes of 3-graphs which are not Ramsey-good for tight paths. It is not difficult to check that the Fano plane $\mathbb{F}$ can be properly 3-coloured so that each hyperedge intersects precisely two different colour classes and that $\chi(\mathbb{F}) = 3$ and $\sigma(\mathbb{F}) = 1$. The first property is crucial, as demonstrated by the following proposition.

Proposition 2.1. *Let H be a 3-graph with $\chi(H) = 3$ and $n \geq 3\sigma(H) + 3$. Assume that in every proper 3-colouring of H, there exists an edge intersecting all three colour classes. Then $R(P_n, H) \geq 2(n-1) + \lfloor \frac{1}{3}n \rfloor > 2(n-1) + \sigma(H)$.*

Proof. Let $N = 2(n-1) + \lfloor \frac{1}{3}n \rfloor - 1$. We first partition the vertex set of $K = K_N^{(3)}$ into sets V_1, V_2, V_3 satisfying $|V_1| = n - 1 = |V_2|$ and $|V_3| = \lfloor \frac{1}{3}n \rfloor - 1$. We then colour every hyperedge intersecting exactly two different sets V_i blue and every other hyperedge red.

Suppose there exists a red tight path P on n vertices. Then P contains a matching of size $\lfloor \frac{1}{3}n \rfloor > |V_3|$, so one of the matching edges does not intersect V_3. This edge must then be fully contained in some V_i for $i \in [2]$, which in turn implies that P is fully contained in this V_i. Hence $v(P) \leq |V_i| < n$, a contradiction.

To see why there is no blue copy of H, note that, since $\chi(H) = 3$, any blue copy of H in K must intersect all three sets V_i. But in every proper 3-colouring of H some edge intersects all three colour classes. Since all edges intersecting all three sets V_i are red, there cannot exist a blue copy of H in K. □

[2]We say that G is connected if G is not a disjoint union of two smaller hypergraphs.

[3]As usual, a proper colouring of a hypergraph H is a colouring of the vertices of H such that no edge of H is monochromatic; $\chi(H)$ is the minimum number of colours in a proper colouring of H, and $\sigma(H)$ is the smallest possible size of a colour class in a proper colouring of H using $\chi(H)$ colours.

It is possible to obtain a result similar to Proposition 2.1 also when $\chi(H) > 3$. Thus, from now on, we concentrate on hypergraphs H that have at least one $\chi(H)$-colouring in which every edge intersects precisely two different colour classes. In fact, we restrict our attention to a special subclass of hypergraphs of this kind, which we define below.

Definition 1. Let $\chi \geq 1$ be an integer and T_χ be a tournament on $[\chi]$. We say that a 3-graph H is a *tournament hypergraph associated to* T_χ if $V(H)$ can be partitioned into sets $A_1 \cup \cdots \cup A_\chi$ so that $E(H) = \{xyz : x, y \in A_i, z \in A_j, (i,j) \in E(T_\chi)\}$, that is, the edge set of H consists of precisely those triples containing two vertices from some set A_i and a third vertex from some set A_j, where (i,j) is an arc of T_χ. For an integer $m \geq 1$, we write $H(T_\chi, m)$ for a tournament hypergraph associated to T_χ in which each vertex class A_i has size m.

Let $\chi \geq 1$ be an integer, T_χ be a non-transitive tournament on $[\chi]$, and $H = H(T_\chi, m)$. It turns out that, in this case, not only is H not Ramsey-good for tight paths, but in fact the ratio $R(P_n, H)/n$ cannot be bounded above by any function depending only on χ. This is the content of the next proposition.

Proposition 2.2. *Let $\chi \geq 3$ and $m \geq 2$ be integers and T_χ be a non-transitive tournament on $[\chi]$. Let $n, t \geq 1$ be integers such that $\left\lfloor \frac{3t}{2} \right\rfloor + 1 < n$. Then $R(P_n, H(T_\chi, m)) \geq (m-1)t + 1$.*

Proof. Let $N = (m-1)t$. We partition the vertex set of $K = K_N^{(3)}$ into sets $V_1, \ldots, V_{m-1}$ satisfying $|V_i| = t$ for all $i \in [m-1]$. We then colour every hyperedge xyz with $x, y \in V_i$ and $z \in V_j$ for $1 \leq i \leq j \leq m-1$ red and every other hyperedge blue.

It is not difficult to see that a red tight path in this colouring has at most $n-1$ vertices. Indeed, any red tight path must contain either vertices from a single V_i, in which case it has at most $t < n$ vertices, or b vertices from a set V_i and at most $\left\lfloor \frac{b}{2} \right\rfloor + 1$ vertices from $V_{i+1} \cup \cdots \cup V_{m-1}$, in which case its number of vertices cannot exceed $t + \left\lfloor \frac{t}{2} \right\rfloor + 1 < n$.

Now suppose there is a blue copy H' of H in K with vertex classes $W_1, \ldots, W_\chi$. For each $j \in [\chi]$, we have $|W_j| = m$, and thus there exists an index $k_j \in [m-1]$ such that $|W_j \cap V_{k_j}| \geq 2$. Note that, since the edges fully contained in a single set V_i are red, for every arc (j, ℓ) of T_χ, no set V_i can contain three vertices x, y, z with $x, y \in W_j$ and $z \in W_\ell$. Therefore, all k_j are distinct. But then by the definition of our colouring $H'\left[\bigcup_{j \in [\chi]} V_{k_j}\right]$ is a tournament hypergraph associated to a transitive tournament, which contradicts the fact that H is associated to a non-transitive tournament. □

Observe that the proof of Proposition 2.2 shows that the same result holds if H is associated (in a similar way as in Definition 1) to any digraph containing a cycle.

The situation is fairly different when H is a tournament hypergraph associated to a *transitive* tournament. We write TT_ℓ for the transitive tournament on $[\ell]$. Once again, H is generally not Ramsey-good for tight paths, but as we will soon see, in this case $R(P_n, H)/n$ *can* be bounded above by a function depending only on $\chi(H)$. Given an integer $\ell \geq 1$, let $\vec{R}(\ell)$ denote the smallest integer N such that any tournament on at least N vertices contains a copy of TT_ℓ. It is well known that $\vec{R}(\ell)$ is finite for any $\ell \geq 1$.

Proposition 2.3. *Let $\chi \geq 3$ be an integer, $R = \vec{R}(\chi)$, and $m \geq R$. Then $H = H(TT_\chi, m)$ satisfies $R(P_n, H) \geq \left(\frac{2}{3}n - 6\right)(R-1) + 1 = (1 + o(1))\frac{2}{3}(R-1)n$ as $n \to \infty$.*

Proof. Let T_{R-1} be a tournament on vertex set $[R-1]$ that does not contain a copy of TT_χ, which exists by the definition of R. Let $N = \left(\lfloor\frac{2}{3}n\rfloor - 5\right)(R-1) \geq \left(\frac{2}{3}n - 6\right)(R-1)$ and $K = K_N^{(3)}$. Partition the vertex set of K into sets $V_1, \dots, V_{R-1}$ with $|V_i| = \lfloor\frac{2}{3}n\rfloor - 5$ for all $i \in [R-1]$. We now assign the colour red to all edges that are fully contained in a single set V_i and all edges of the form xyz for $x \in V_i$ and $y, z \in V_j$, where (i, j) is an arc of T_{R-1}. All remaining edges are coloured blue. Note in particular that the blue edges intersecting precisely two vertex classes form a copy of $H(T_{R-1}, \lfloor\frac{2}{3}n\rfloor - 5)$.

Using a similar argument as in the proof of Proposition 2.2, we conclude that there is no red tight path on n vertices. Suppose there exists a blue copy H' of H with vertex classes $W_1, \dots, W_\chi$. Since $|W_j| \geq R$ for each $j \in [\chi]$, there exists an integer $k_j \in [R-1]$ such that $|W_j \cap V_{k_j}| \geq 2$. As before, all of these k_j are distinct. But then the hypergraph $H'\left[\bigcup_{j\in[\chi]} V_{k_j}\right]$ is a tournament hypergraph associated to TT_χ. But T_{R-1} does not contain a copy of TT_χ, a contradiction. □

It turns out that the lower bound in Proposition 2.3 is asymptotically tight as $n \to \infty$. More precisely, we are able to prove the following theorem.

Theorem 2.4. *Given integers $\chi \geq 2$ and $m \geq 2$ and a real number $\varepsilon > 0$, there exists an integer $n_0 = n_0(\chi, m, \varepsilon)$ such that, for all $n \geq n_0$,*

$$R(P_n, H(TT_\chi, m)) \leq \begin{cases} (1+\varepsilon)n & \text{if } \chi = 2, \\ \left(\frac{2}{3} + \varepsilon\right)(\vec{R}(\chi) - 1)n & \text{if } \chi \geq 3. \end{cases}$$

Since $\vec{R}(3) = 4$, Proposition 2.3 and Theorem 2.4 imply that $R(P_n, H(TT_3, m)) = (2 + o(1))n$ as $n \to \infty$. This means that P_n is *asymptotically* $H(TT_3, m)$-good as $n \to \infty$. In particular, since the Fano plane is a subhypergraph of $H(TT_3, 4)$, Theorem 2.4 extends the result of Balogh, Clemen, Skokan, and Wagner [2] asymptotically to a large family of 3-graphs.

We provide a brief sketch of the proof of Theorem 2.4. Some of the ideas resemble those used in [2]. The proof uses induction on the chromatic number χ. We outline the induction step. Suppose $\chi \geq 3$ and that the theorem holds for $\chi - 1$. Let $\varepsilon > 0$ and $m \geq 2$ be given and $H = H(TT_\chi, m)$. Set $N = \left(\frac{2}{3} + \varepsilon\right)(\vec{R}(\chi) - 1)n$ and suppose there is a colouring of $K_N^{(3)}$ with no red copy of P_n and no blue copy of H.

We first find a red tight path P of length approximately $\frac{2}{3}n$ with a special property: there exist disjoint intervals $I_1, \dots, I_c$ covering most vertices of P such that the vertices of each interval induce a red clique. Our task is then to absorb more vertices from the rest of $K_N^{(3)}$ in between the vertices of each interval I_j. A key idea here is that, since the vertices of each I_j form a clique, we can change the order in which they appear on the path. We then go through the intervals I_j in turn and repeatedly apply the induction hypothesis to $K_N^{(3)} \setminus V(P)$ to find copies of $H(TT_{\chi-1}, m')$ for some appropriately chosen large constant m'. For each such copy H' of $H(TT_{\chi-1}, m')$, either there will be a lot of blue edges with two vertices in I_j and a third vertex in $V(H')$, in which case we can embed a copy of H, or we will find enough red edges of this kind to allow us to absorb a number of vertices from $V(H')$ into the interval I_j (after possibly rearranging the vertices of I_j). Eventually, unless we find a blue copy of H, we will be able to absorb approximately $\frac{1}{2}|I_j|$ vertices into each interval I_j, resulting in a tight path of total length at least n.

3 Conclusion and open problems

A number of natural questions arise from our work.

First of all, it would be interesting to determine the Ramsey numbers of more pairs of the form (P_n, H), for instance, those discussed in Propositions 2.1 and 2.2, at least asymptotically. Similarly, a natural way to improve Theorem 2.4 and Proposition 2.3 is to remove the error term and prove a precise result.

In a slightly different direction, in the examples given in Propositions 2.2 and 2.3, our tournament hypergraphs are fairly dense. It would be interesting to consider subhypergraphs of tournament hypergraphs and investigate how sparse such a subgraph H can be made before (P_n, H) meets the lower bound. We are able to find reasonably sparse hypergraphs H, albeit not subgraphs of tournament hypergraphs, such that (P_n, H) exceeds the general lower bound.

A third possible direction for further research is to consider higher uniformities. Do long k-uniform tight paths behave similarly to 2-uniform paths or 3-uniform tight paths as k increases? We note here that we tried to generalise the result of Balogh, Clemen, Skokan, and Wagner in a different direction, by replacing the Fano plane by a higher-order projective plane $\mathbb{F}^q$ for some prime power q. Surprisingly, long tight paths are generally not $\mathbb{F}^q$-good. It is a simple exercise to show that, when $q \geq 3$, we have $\chi(\mathbb{F}^q) = 2$. Then a result of Keevash and Zhao [12] allows us to build colourings showing that $\mathbb{F}^q$ is not Ramsey-good for tight paths for an infinite sequence of values of q.

References

[1] P. Allen, G. Brightwell, and J. Skokan. Ramsey-goodness – and otherwise. *Combinatorica*, 33:125–160, 2013.

[2] J. Balogh, F. C. Clemen, J. Skokan, and A. Z. Wagner. The Ramsey number of the Fano plane versus the tight path. *Electron. J. Combin.*, 27(1):P1.60, 2020.

[3] S. Brandt. *Expanding graphs and Ramsey numbers.* Freie Univ., Fachbereich Mathematik und Informatik, 1996.

[4] S. A. Burr. Ramsey numbers involving graphs with long suspended paths. *J. London Math. Soc.*, 2(3):405–413, 1981.

[5] S. A. Burr and P. Erdős. Generalizations of a Ramsey-theoretic result of Chvátal. *J. Graph Theory*, 7(1):39–51, 1983.

[6] M. Campos, S. Griffiths, R. Morris, and J. Sahasrabudhe. An exponential improvement for diagonal Ramsey. *arXiv:2303.09521*, 2023.

[7] V. Chvátal. Tree-complete graph Ramsey numbers. *J. Graph Theory*, 1(1):93, 1977.

[8] V. Chvátal and F. Harary. Generalized Ramsey theory for graphs. III. Small off-diagonal numbers. *Pacific J. Math.*, 41(2):335–345, 1972.

[9] D. Conlon, J. Fox, and B. Sudakov. Recent developments in graph Ramsey theory. In *Surveys in combinatorics 2015*, volume 424 of *London Math. Soc. Lecture Note Ser.*, pages 49–118. Cambridge Univ. Press, Cambridge, 2015.

[10] P. Erdős. Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.*, 53:292–294, 1947.

[11] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[12] P. Keevash and Y. Zhao. Codegree problems for projective geometries. *J. Combin. Theory Ser. B*, 97(6):919–928, 2007.

[13] D. Mubayi and A. Suk. A survey of hypergraph Ramsey problems. *Discrete Math. Appl.*, 165:405–428, 2020.

[14] F. P. Ramsey. On a problem in formal logic. *Proc. Lond. Math. Soc.*, 30:264–286, 1930.