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Interfaces with Other Disciplines

A defined benefit pension plan game with Brownian and Poisson jumps uncertainty☆

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ABSTRACT

In this paper, we study the optimal management of an aggregated pension fund of defined benefit type by means of a differential game with two players, the firm and the participants. We assume that the fund wealth is greater than the actuarial liability and then the manager builds a pension fund surplus. In order to contemplate sudden changes in the financial market, the surplus can be invested in a portfolio with a bond and several risky assets where the uncertainty comes from Brownian motions and Poisson processes. The aim of the participants is to maximize a utility of the extra benefits. The game is analyzed in three scenarios. In the first, the aim of the firm is to maximize a utility of the fund surplus, in the second, to minimize the probability that the fund surplus reaches a low level, and in the third, to minimize the expected time of reaching a benchmark surplus. An infinite horizon is considered, and the game is solved by means of the dynamic programming approach. The influence of the jumps of the financial market on the Nash equilibrium strategies and the fund surplus is studied by means of a numerical illustration.

1. Introduction

The role of pension plans in the economy has become more relevant in this time due to the importance for the manager and the plan participants to guarantee future benefits that allow them to have a reasonable standard of living. In the case of defined benefit (DB henceforth) pension plans, where benefits are fixed in advance, the manager has to comply with the liabilities, then it is the firm that takes the financial risk. But at the same time, the manager will be rewarded with the returns of the optimal investment of the fund, so there is an intrinsic motivation to find an optimal investment strategy. In this way, participants will receive the promised benefits at the end of the plan. When the fund assets achieve high values, both manager and participants can obtain extra benefits, even in a non-cooperative scenario. In this paper we consider a game to distribute the fund surplus between both. Thus, the firm is not the only agent and its aim is not to default the promised payments as in other pension models.

According to the Thinking Ahead Institute, in 2021, DB funds account for 63.5% of the total assets of the world's leading pension funds, as shown in Hodgson et al. (2022). The reason for this is the advantage of the participant in making the manager concerned about the fund's solvency instead themselves, and having ensured a final benefit in advance. On the other hand, the DB pension plans continue to be important in the OCDE countries because their pension systems are of DB type, as described Table 2 in Urbano et al. (2021). These are some reasons why it is still pertinent to analyze this type of pension plan.

The optimal management of pension plans using techniques of dynamic programming has been studied in the specialized literature. The first papers date back to Haberman & Sung (1994), where dynamic programming in discrete time was applied to a DB pension plan model. Later, Boulier et al. (1995); Cairns (2000) and Josa-Fombellida & Rincón-Zapatero (2001), used dynamic programming in continuous time by means of the accommodation of the portfolio selection problems, presented in Merton (1971), to pension plans, both DB and defined contribution type. Other relevant papers are Baltas et al. (2022); Boulier et al. (2001); Chang et al. (2003); Gao (2009); Gerrard et al. (2004); Hainaut (2014); Hainaut & Deelstra (2011); Josa-Fombellida & Rincón-Zapatero (2004); Le Courtois & Menoncin (2015); Li et al. (2021) and Zhao & Wang (2022).

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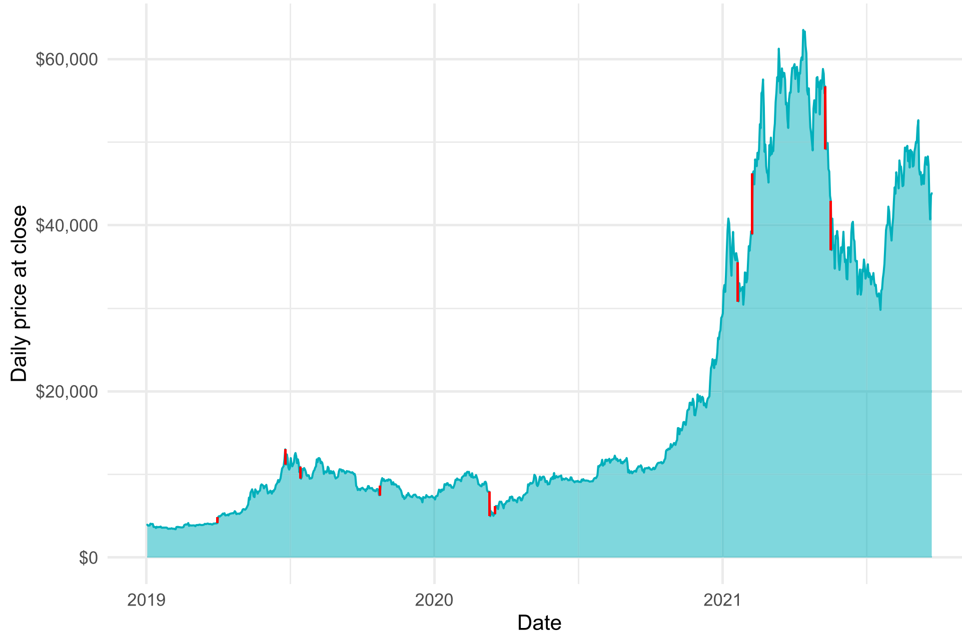

Fig. 1. Bitcoin evolution from January 2019 to September 2021.

Attending to the gap between liabilities and fund assets, there are two different types of defined benefit pension plans. When liabilities are more than assets, then we are referring to an underfunded pension plan. However, when assets are over liabilities, then the pension plan is overfunded. Although most of the DB pension plans are underfunded, a good investment or a bull market can result in a plan being overfunded. Silverblatt (2019), Exhibit 2, showed that some companies of the S&P 500 had overfunded pension plans, concretely 43 firms in 2017 and 46 in 2018. In an overfunded plan, the manager can decide how to invest a fund surplus in the stock market in order to give the possibility of providing extra benefits to the participants. There is no standard rule to define the best strategy to follow in these cases, but the extra benefits obtained due to the surplus investment can be a good incentive to participants. Josa-Fombellida & Rincón-Zapatero (2019) have studied this approach by means of a non-cooperative overfunded pension plan game, where the investment of the surplus (to provide additional benefits to the participants) is realized in a standard stock market where the uncertainty comes from Brownian motions. We propose, however, a new scenario of investment where the financial market is more realistic, allowing jumps.

The appearance of jumps is observed in the financial markets in periods that not necessarily coincides with a global economic or health crisis. Wu (2003) analyzed and calibrated a jump model with U.S. stock market index (S&P 500), concluding that jumps are an inherent part of the asset price movement. Several authors have taken into account jumps to model the risky asset price coming from this and other markets. More intensity is observed in new emerging markets as Nasdaq and cryptocurrencies. News or events can have a big impact on financial market values, specially concerning these new markets. As an example to demonstrate this, we show the evolution of Bitcoin over time because it is visualized more clearly. Although it is not the main market for investing pension funds, there have recently been pension funds that have partially invested in the cryptocurrency market. Figure 1 shows the evolution of the Bitcoin price from January 2019 to September 2021. Some remarkable facts in the period studied, during which the price fell or rose suddenly around 25% are: the lockdown and the vaccine discoveries by the worldwide pandemic of COVID-19, the Tesla investment announcement in Bitcoin or the sale of its shares, and the new regulations of the Chinese government with a crackdown on cryptocurrency. For all these reasons, it is not strange to see sudden shocks in the time series of the stock market and it is important to be prepared to deal with this type of events. At the same time, it may also be possible to take advantage of the higher benefits that can be obtained from them. Jump diffusion processes, where uncertainty comes from Brownian motions and Poisson processes, can then be a great option to model these sudden changes in the series. Although we have now considered the Bitcoin example, in the numerical illustration of this paper we will work with the S&P500.

There has been literature referring to jumps since the dynamic programming approach in continuous time. The first one was Merton (1971), describing a model composed of a riskless bond and several risky assets, whose uncertainty is modeled separately by a Brownian motion and a Poisson process. Later, Wu (2003) considered that the risky asset is a jump diffusion process in a dynamic asset allocation problem, while Guo & Xu (2004) studied a mean-variance portfolio selection problem where the prices of stocks follow a jump-diffusion process with Poisson jumps. In addition, Ngwira & Gerrard (2007) introduced the Poisson jumps in the risky asset of a dynamic defined benefit stochastic pension plan model. In Josa-Fombellida & Rincón-Zapatero (2012), the management of a defined benefit pension plan model is also considered, but one where both benefits and risky assets are jump diffusion processes. Finally, Zhang & Guo (2020) consider the management of a defined contribution pension plan where the salary and the risky asset are both jump diffusion processes.

We assume an interaction between the two agents, the firm (through the manager) and the participants (through the union of these workers), as a dynamic game in a pension plan with jumps. Leong & Huang (2010) applied this concept to the dividend payments of a government, modeling this as a stochastic differential game; as well as Guan & Liang (2016), where the interaction between two different pension plans is fitted as a dynamic game. Later, Josa-Fombellida & Rincón-Zapatero (2019) considered a game for a unique pension plan where the firm and the participants are the players. Guan et al. (2022) have recently analyzed a robust stochastic game built from the defined benefit pension plan model of Josa-Fombellida & Rincón-Zapatero (2019), but without jumps. Nevertheless, in this paper, we are considering the game for a unique pension plan extending the model of Josa-Fombellida & Rincón-Zapatero (2019), so the challenge is to find the equilibrium strategies for both the manager and the participants with the addition of Poisson uncertainty. This increases the technical difficulties to apply the Hamilton–Jacobi–Bellman (HJB henceforth) system and get totally explicit solutions. Three scenarios of the game are considered according to the preferences of the firm. In the first, the firm maximizes a utility of the surplus, in the second, minimizes the probability that the surplus reaches a low value, and in the third, minimizes the expected time of reaching an objective surplus value. The main novelties of the current paper with respect to Josa-Fombellida & Rincón-Zapatero (2019) are: 1) the risky assets are stochastic processes with Poisson jumps in addition to the Brownian motion; 2) the effects of the sign and intensity of the jumps on the Nash equilibrium strategies and on the fund surplus are studied; 3) the time evolution of the fund surplus is analyzed; and 4) the study of the new third game scenario.

With this modeling, we found that small or big jumps in the risky assets have an impact on the time evolution of the fund surplus. Moreover, jumps give relevance to the effect of the risk aversion on the returns of the investment. The same conclusion was obtained for the worker's extra benefit ratio, where the interaction between risk aversion and Poisson uncertainty determines whether or not benefits increase. The model illustrates that, with a bull economic regime, the benefits increase for low values of the risk aversion parameters and decrease for high values. In addition, with a bull market and upward jumps, a greater intensity of the jumps produces an increase in surplus and extra benefits, but at the cost of slightly increasing the investment. Upward jumps in the bear market bring about a decrease in the investment with respect to the risk aversion, but with downward jumps it increases and the manager should short sell.

Likewise, in the first game scenario, the jump intensity has a higher impact on the benefits in a bull regime than in a bear regime, especially when jumps are negative. This makes sense since, when the market declines prominently, benefits will be negatively affected at the retirement time. In this same scenario, with a bull market, the investment is decreasing in line with the risk aversion, but independently of the jump type, and borrowing is necessary with low risk aversion. However, borrowing can be necessary with a bear market and downward jumps. In the second game, with a bull regime, the investment is a concave function of the risk aversion (increases until a moderate value and then decreases), and neither borrowing nor shortselling are necessary. In the third game, the optimal growth portfolio of Merton's model is the investment equilibrium strategy when there are no jumps. More fund surplus is obtained than with the other scenarios, because when a low value is reached, this fact is not taken into account.

The paper is structured as follows. Section 2 establishes the basis of the DB pension plan, describes the financial market where the surplus operates and obtains its evolution. At the end of the section, the admissible strategies and the Markov perfect Nash equilibrium concepts are presented in a general framework for this model. Three scenarios for the game in an infinite horizon are considered. The aim of the participants is to maximize a CRRA utility function of the benefits in all scenarios. In the first game scenario, the firm maximizes a CRRA utility of the surplus, in the second, minimizes the probability that the surplus reaches a low value and, in the third, minimizes the expected time for reaching a high aim surplus value. The Nash equilibrium strategies for both players are shown and analyzed for the first game scenario in Section 3 and for the second and third in Section 4. Section 5 contains a sensitivity analysis of the equilibrium strategies and surplus with respect to the jump and risk aversion parameters. In Section 6, we establish some conclusions and propose further related research.

2. The pension game

Consider an overfunded defined benefit pension plan, that is, there are more fund assets than liabilities. The firm creates a fund surplus and from it decides to negotiate dynamically extra benefits between the firm and the representatives of the participants. This excess pension benefit is added to the agreed benefits at the moment of retirement. We model this conflict between the firm and the union as a non-cooperative dynamic game with two agents, the fund manager or the owner of the firm and the union that represents the workers or participants. The main objective of the firm is to keep the fund surplus at an acceptable level and the aim of the workers is to increase the extra benefits or premium benefits as much as possible.

In this section, we describe the first elements of the game. The fund surplus is invested in a financial market composed of one riskless asset and several risky assets. In order to include the sudden variations of the market, the uncertainty is modeled by Brownian motions and Poisson processes. The players' payoffs and the Nash equilibrium strategies are defined for a general framework, which are specified in three game scenarios in the next sections. The game is an extension of the pension game analyzed in Josa-Fombellida & Rincón-Zapatero (2019) that includes Poisson jumps in the model.

2.1. The financial market

Following Josa-Fombellida & Rincón-Zapatero (2012), we suppose that the risky assets are jump diffusion processes where the uncertainty is given by Brownian motions and Poisson processes. To model the pension game, we consider a probability space $(\Omega^w, \mathscr{F}^w, \mathbb{P}^w)$, where $\mathbb{P}^w$ is a probability measure on Ω^w and $\mathscr{F}^w = \{\mathscr{F}_t^w\}_{t\geq 0}$ is a complete and right continuous filtration generated by the l-dimensional standard Brownian motion $w = (w_1, \dots, w_l)^\top$, that is to say, $\mathscr{F}_t^w = \sigma\{(w_1(s), \dots, w_l(s));\ 0 \leq s \leq t\}$, $t \geq 0$. We also consider an m-dimensional Poisson process $N = (N_1, \dots, N_m)^\top$ with constant intensity $\lambda = (\lambda_1, \dots, \lambda_m)^\top$, $\lambda_1, \dots, \lambda_m \in \mathbb{R}_+$, defined on a complete probability space $(\Omega^N, \mathscr{F}^N, \mathbb{P}^N)$, where $\mathscr{F}_t^N = \sigma\{(N_1(s), \dots, N_m(s));\ 0 \leq s \leq t\}$, $t \geq 0$. Note that the process $H_i(t) = N_i(t) - \lambda_i t$, $i = 1, \dots, m$, is an $\mathscr{F}^N$-martingale, which is called the compensated Poisson process; see Jeanblanc-Picqué & Pontier (1990) and García & Griego (1994). This fact facilitates the stochastic calculus and the use of the dynamic programming method. Let $(\Omega, \mathscr{F}, \mathbb{P}) = (\Omega^w \times \Omega^N, \mathscr{F}^w \otimes \mathscr{F}^N, \mathbb{P}^w \otimes \mathbb{P}^N)$ denote the product probabilistic space. We suppose w and N are independent processes on this space.

The plan sponsor manages the fund in an unbounded planning horizon by means of a portfolio formed by n risky assets $S^1, \dots, S^n$, which are extended geometric Brownian motions (GBM henceforth, stochastic processes extending the deterministic exponential function),[1] and a riskless asset or bond S^0 (its price is an exponential

[1] The extented GBM with Poisson jumps is a particular case of the exponential Lévy process where the jump process is a Poisson process. Following, for instance, Øksendal & Sulem (2005) or Hanson (2007), in the scalar case, the risky asset S can be expressed in the exponential form as follows: $S(t) = se^{(b-\sigma^2/2)t+\sigma w(t)}(1+\varphi)^{N(t)}$, and then, it is positive. This property will be observed for the equilibrium fund surplus in the three scenarios considered along the paper. Other papers, such as Bihary et al. (2020), have used and analyzed a class of exponential Lévy processes to model holding stocks.

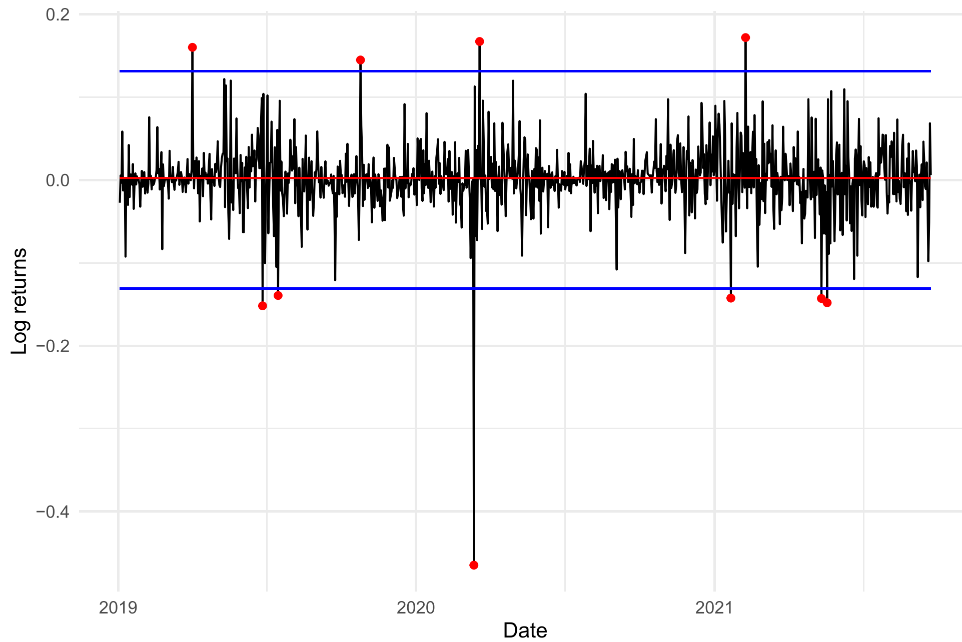

Fig. 2. Jumps of the Bitcoin prices from January 2019 to September 2021.

function), as proposed Guo & Xu (2004), that is, whose evolutions are given by the equations:

$$dS^0(t) = rS^0(t)dt, \quad S^0(0) = 1, \tag{1}$$

$$dS^i(t) = S^i(t-)\Big(b_i dt + \sum_{j=1}^{l}\sigma_{ij}dw_j(t) + \sum_{k=1}^{m}\varphi_{ik}dN_k(t)\Big), \quad S^i(0) = s_i > 0, \quad i = 1,\dots,n. \tag{2}$$

Here $r > 0$ denotes the short risk-free rate of interest and $b_i > 0$ the mean rate of return of the risky asset S^i. The uncertainty parameters are the volatility coefficients $\sigma_{ij} > 0$ and the jump magnitude coefficients $\varphi_{ik} > -1$. It is usual to assume that $b_i + \sum_{k=1}^{m}\lambda_k\varphi_{ik} > r$, for each $i = 1,\dots,n$, so the manager has incentives to invest with risk. The matrix (σ_{ij}) is denoted by σ, the matrix (φ_{ik}) is denoted by φ, b is the (column) vector $(b_1,\dots,b_n)^\top$ and $\bar{1}$ is the (column) vector of 1's. We will suppose that the symmetric matrix $\Sigma = \sigma\sigma^\top$ and the matrix $(\sigma|\varphi)(\sigma|\varphi)^\top$ are positive definite.

Consider again the Bitcoin example as an illustration. In order to estimate the value of the parameters of the jump diffusion process, we use the logarithms of the returns and a variation is considered as a jump when its absolute value is higher than the quantile 0.9995. Figure 2 shows the downward and upward jumps of the Bitcoin price from January 2019 to September 2021.

A standard estimation of the values of the parameters was carried out considering one Brownian motion and two Poisson processes. The parameters b and σ are estimated as in Josa-Fombellida & Rincón-Zapatero (2019), and the intensity jump parameters as the mean of the jumps over the total number of days and the diffusion jump parameters as the mean of the jump amplitudes. Table 1 shows the estimated values of the parameters.

2.2. The fund surplus

Since the pension plan is overfunded, the union claims extra benefits P over the excess fund, or fund surplus X, to be distributed among retired workers. The fund surplus $X > 0$ is invested

Table 1
Estimated values of the parameters.

b	σ	λ_1	φ_1	λ_2	φ_2
3.199279	1.255706	0.004032258	0.160932	0.006048387	−0.1982548

in the riskless asset S^0 and in the n risky assets $S^1,\dots,S^n$. Let $\Pi = (\pi_1,\dots,\pi_n)^\top$, where each π_i is the proportion of surplus to be invested in S^i, so that $1 - \sum_{i=1}^{n}\pi_i$ is invested in S^0. Borrowing and shortselling are allowed. A negative value of π_i means that the sponsor sells a part of the risky asset S^i short while, if π_i is larger than 1, he or she then gets into debt to purchase the corresponding stock, borrowing money at the riskless interest rate r.

The dynamics of the surplus X is driven by

$$dX(t) = \sum_{i=1}^{n}\pi_i(t)X(t)\frac{dS^i(t)}{S^i(t)} + \Big(1 - \sum_{i=1}^{n}\pi_i(t)\Big)X(t)\frac{dS^0(t)}{S^0(t)} - P(t)dt, \tag{3}$$

with $X(0) = x > 0$. By substituting (1) and (2) in (3), the dynamic surplus evolution under the investment policy Π is

$$dX(t) = \Big(rX(t) + \Pi^\top(t)(b - r\bar{1})X(t) - P(t)\Big)dt + \Pi^\top(t)X(t)\sigma\, dw(t) + \Pi^\top(t)X(t)\varphi\, dN(t), \tag{4}$$

with the initial condition $X(0) = x > 0$.

2.3. The players' strategies

The firm chooses the portfolio and the union chooses the benefits. A strategic profile (P, Π) is called admissible if the extra benefits strategy $\{P(t) : t \ge 0\}$ and the investment strategy $\{\Pi(t) : t \ge 0\}$ are Markovian processes and stationary, $P = P(X)$ and $\Pi = \Pi(X)$, adapted to filtration $\{\mathcal{F}_t\}_{t\ge0}$, and $P(t)$ and $\Pi(t)$ are $\mathcal{F}_t$-measurable, $\forall t > 0$, and such that they satisfy the integrability condition

$$\mathbb{E}\int_0^T P(t)dt + \mathbb{E}\int_0^T \Pi^\top(t)\Pi(t)dt < \infty, \; \forall T > 0.$$

Thus, the stochastic differential equation (SDE henceforth) (4) admits a unique solution for every initial condition $X(0)=x$. We denote by $\mathcal{A}^U\times\mathcal{A}^F$ the set of admissible strategy profiles.

Given an initial condition x, we denote the payoff of the union with the admissible strategy (P,Π) as $J_U(x;P,\Pi)$ and the payoff of the firm $J_F(x;P,\Pi)$. As introduced in Josa-Fombellida & Rincón-Zapatero (2019), in a dynamic non-cooperative setting, the relevant solution concept is the Markov perfect Nash equilibrium (MPNE henceforth).[2] An MPNE of the pension game is a pair of admissible strategies $(P^*,\Pi^*)\in\mathcal{A}^U\times\mathcal{A}^F$, such that, for any $(P,\Pi)\in\mathcal{A}^U\times\mathcal{A}^F$ for any $x>0$

$$J_U(x;P^*,\Pi^*)\ge J_U(x;P,\Pi^*),$$

$$J_F(x;P^*,\Pi^*)\ge J_F(x;P^*,\Pi).$$

The value functions of the union V_U and the firm V_F are, respectively,

$$V_U(x)=\max_{P\in\mathcal{A}^U}\left\{J_U(x;P,\Pi^*)\ :\ \text{s.t. (4) and } X(0)=x\right\},$$

$$V_F(x)=\max_{\Pi\in\mathcal{A}^F}\left\{J_F(x;P^*,\Pi)\ :\ \text{s.t. (4) and } X(0)=x\right\},$$

that is,

$$V_U(x)=J_U(x;P^*,\Pi^*),$$

$$V_F(x)=J_F(x;P^*,\Pi^*).$$

The next sections describe the players' payoffs in three situations. In all of them, the aim of the union is to maximize a utility of the extra benefits. In the first, the firm wants to maximize a utility of the fund surplus, while in the others, the firm minimizes the probability that the fund surplus will reach a low level, before an objective high level or minimizes the expected time that it takes to reach a benchmark level. The Nash equilibrium strategies are analyzed in all of them.

3. Nash equilibrium strategies when the firm maximizes a utility of the fund surplus

In this section, we consider the first game scenario where the firm maximizes a utility function of the surplus. This objective makes the DB plan attractive to the participants.

As said before, the workers union controls P. The payoff of the union, to be maximized on the class of the admissible controls $\mathcal{A}^U$, is:

$$J_U(x;P,\Pi)=\mathbb{E}_x\int_0^\infty e^{-\alpha t}u(P(t))dt, \tag{5}$$

where u is a utility function of benefits and $\alpha>0$ is the time preference of the union. Here $\mathbb{E}_x$ denotes the conditional expectation, given that $X(0)=x$. The firm controls Π. The payoff of the firm, to be maximized on the class of the admissible controls $\mathcal{A}^F$, is:

$$J_F(x;P,\Pi)=\mathbb{E}_x\int_0^\infty e^{-\beta t}v(X(t))dt, \tag{6}$$

where v is a utility function of the surplus and $\beta>0$ is the time preference of the manager. We consider CRRA utility functions:[3]

$$u(P)=\frac{P^{1-\gamma}-1}{1-\gamma},\quad \gamma>0,\ \gamma\neq 1,$$

$$v(X)=\frac{X^{1-\delta}-1}{1-\delta},\quad \delta>0,\ \delta\neq 1.$$

[2] References as Başar & Olsder (1999) and Dockner et al. (2000) can be consulted.

[3] When $\gamma=1$ or $\delta=1$, the utility functions considered are of logarithmic type: $u(P)=\ln P$, $v(X)=\ln X$.

These functions are both increasing and strictly concave. We are assuming both players are risk-averse, $\gamma,\delta>0$. When $0<\gamma,\delta<1$, they are low risk-averse, when $\gamma=\delta=1$, they are moderate risk-averse, and when $\gamma,\delta>1$, they are high risk-averse.

In order to solve the game with the dynamic programming approach, we obtain the HJB equations. We extend the arguments followed in Josa-Fombellida & Rincón-Zapatero (2012) to game theory in the sense of Dockner et al. (2000). Let V_U and V_F be the value function of the players, respectively; then, the HJB system of PDEs is

$$\begin{aligned}\alpha V_U(x)&=\max_P\left\{\frac{P^{1-\gamma}-1}{1-\gamma}+\left(rx+\Pi^\top(b-r\bar{1})x-P\right)V_U'(x)\right\}\\&\quad+\frac{1}{2}\Pi^\top\Sigma\Pi x^2V_U''(x)+\sum_{k=1}^m\lambda_k\left(V_U\left(x+\sum_{i=1}^n\pi_ix\varphi_{ik}\right)-V_U(x)\right),\\ \beta V_F(x)&=\max_\Pi\left\{\frac{x^{1-\delta}-1}{1-\delta}+\left(rx+\Pi^\top(b-r\bar{1})x-P\right)V_F'(x)\right.\\&\quad\left.+\frac{1}{2}\Pi^\top\Sigma\Pi x^2V_F''(x)+\sum_{k=1}^m\lambda_k\left(V_F\left(x+\sum_{i=1}^n\pi_ix\varphi_{ik}\right)-V_F(x)\right)\right\}.\end{aligned}$$

From the optimality conditions, one gets

$$P^{-\gamma}-V_U'(x)=0\Rightarrow P=(V_U'(x))^{-1/\gamma}, \tag{7}$$

$$\begin{aligned}&(b_i-r)xV_F'(x)+\sum_{j=1}^n a_{ij}\pi_jx^2V_F''(x)\\&\quad+\sum_{k=1}^m\lambda_kV_F'\left(x+\sum_{j=1}^n\pi_jx\varphi_{jk}\right)\varphi_{ik}x=0,\end{aligned} \tag{8}$$

for all $i=1,\dots,n$, where $a_{ij}=\sum_{p=1}^l\sigma_{ip}\sigma_{jp}$, that is to say, the element (i,j) of matrix $\Sigma=\sigma\sigma^\top$. Plugging into the HJB system,

$$\begin{aligned}\alpha V_U(x)&=\frac{-1}{1-\gamma}+\frac{\gamma}{1-\gamma}\left(V_U'(x)\right)^{1-1/\gamma}+rxV_U'(x)+\Pi^\top(b-r\bar{1})xV_U'(x)\\&\quad+\frac{1}{2}\Pi^\top\Sigma\Pi x^2V_U''(x)+\sum_{k=1}^m\lambda_k\left(V_U\left(x+\sum_{i=1}^n\pi_ix\varphi_{ik}\right)-V_U(x)\right),\end{aligned} \tag{9}$$

$$\begin{aligned}\beta V_F(x)&=\frac{x^{1-\delta}-1}{1-\delta}+\left(rx+\Pi^\top(b-r\bar{1})x-\left(V_U'(x)\right)^{-1/\gamma}\right)V_F'(x)\\&\quad+\frac{1}{2}\Pi^\top\Sigma\Pi x^2V_F''(x)+\sum_{k=1}^m\lambda_k\left(V_F\left(x+\sum_{i=1}^n\pi_ix\varphi_{ik}\right)-V_F(x)\right).\end{aligned} \tag{10}$$

The following result shows the Nash equilibrium benefit and investment strategies and the equilibrium fund surplus for the first game scenario.

In order to simplify the length of some equations along the paper, we denote

$$\begin{aligned}\Psi(\Pi,\mu):=\ &r+\Pi^\top(b-r\bar{1})-\frac{1}{2}\mu\Pi^\top\Sigma\Pi\\&+\frac{1}{1-\mu}\sum_{k=1}^m\lambda_k\left((1+\Pi^\top\varphi_k)^{1-\mu}-1\right),\end{aligned} \tag{11}$$

where $\mu>0$, $\mu\neq 1$ and φ_k is the column k of the matrix φ, $k=1,\dots,m$. When $\mu=1$, applying the L'Hôpital rule, we have

$$\begin{aligned}\Psi(\Pi,1)\ &=\lim_{\mu\to 1}\Psi(\Pi,\mu)=r+\Pi^\top(b-r\bar{1})\\&\quad-\frac{1}{2}\Pi^\top\Sigma\Pi+\sum_{k=1}^m\lambda_k\ln(1+\Pi^\top\varphi_k).\end{aligned} \tag{12}$$

We assume the technical conditions: $1+\Pi^\top\varphi_k>0$, for all k.

Proposition 3.1. *Consider the system of algebraic equations*

$$b_i - r - \delta\sum_{j=1}^{n} a_{ij}\pi_j + \sum_{k=1}^{m}\lambda_k\Big(1+\Pi^\top\varphi_k\Big)^{-\delta}\varphi_{ik} = 0, \quad i=1,\ldots,n. \tag{13}$$

Assume that the constants A and B, determined by

$$\Big(\frac{\alpha}{1-\gamma} - \Psi(\Pi,\gamma)\Big)A^{1/\gamma} = \frac{\gamma}{1-\gamma}, \tag{14}$$

$$\Big(\frac{\beta}{1-\delta} - \Psi(\Pi,\delta) + A^{-1/\gamma}\Big)B = \frac{1}{1-\delta}, \tag{15}$$

where Π is the vector of solutions of (13) and $\Psi(\Pi,.)$ is given by (11), are both positive and finite. Then the value functions for the game (5), (6), (4) are

$$V_U(x) = A\frac{x^{1-\gamma}}{1-\gamma} - \frac{1}{\alpha(1-\gamma)},$$
$$V_F(x) = B\frac{x^{1-\delta}}{1-\delta} - \frac{1}{\beta(1-\delta)},$$

and the Nash equilibrium in feedback form is (P^*,Π^*), where

$$P^*(x) = A^{-1/\gamma}x, \tag{16}$$

Π^* is the constant determined by (13), and the equilibrium fund surplus is the extended GBM with Poisson jumps given by

$$dX^*(t) = \Big(r + \Pi^{*\top}(b - r\bar{1}) - A^{-1/\gamma}\Big)X^*(t)dt + \Pi^{*\top}\sigma X^*(t)\,dw(t) + \Pi^{*\top}\varphi X^*(t)\,dN(t), \tag{17}$$

with $X^*(0)=x>0$.

Proof. Let us complete the previous arguments. Because it is not possible to obtain Π explicitly from (8), we first try $V_U(x) = A\frac{x^{1-\gamma}}{1-\gamma} - \frac{M}{1-\gamma}$, $V_F(x) = B\frac{x^{1-\delta}}{1-\delta} - \frac{N}{1-\delta}$, with A,B,M,N suitable constants, in the optimality conditions. From (7), we get that the benefit P is explicitly found in terms of the surplus X, $P=A^{-1/\gamma}x$, that is to say (16), where the constant A must be determined with the HJB equation. From (8), we get that the vector of investments Π is the constant proportion of surplus that solves the algebraic system (13).

We now insert the expressions for V_U and V_F into the HJB equations above (9), (10). We obtain that $M=1/\alpha$ and $N=1/\beta$, while A and B are also the positive solutions to (14) and (15), where Π is the vector of solutions of (13). Note that $V_U''<0$ because $A>0$ and $V_F''<0$ because $B>0$. Thus Π^* and P^* are the maximizers of the HJB system.

Substituting (P^*,Π^*) in the SDE (3) of the surplus X^* we get (17). By Theorem 8.5 of Dockner et al. (2000), the proof concludes when the transversality conditions

$$\lim_{t\to\infty} e^{-\alpha t}\mathbb{E}_x V_U(X^*(t)) = \lim_{t\to\infty} e^{-\beta t}\mathbb{E}_x V_F(X^*(t)) = 0 \tag{18}$$

are checked. It is straightforward that the surplus evolution (4), under the optimal strategies, is given by (17). It is an extended GBM because the optimal investment Π^* is constant in X. Thus, adapting Arnold (1974), p. 139, to Poisson jumps, we obtain that for a real number p,

$$\mathbb{E}_x(X^*(t))^p = x^p\exp\Big\{p\Big(r + \Pi^{*\top}(b-r\bar{1}) - A^{-1/\gamma} - \frac{1}{2}\Pi^{*\top}\Sigma\Pi^*\Big)t + \sum_{k=1}^{m}\lambda_k\Big(\big(1+\Pi^{*\top}\varphi_k\big)^p - 1\Big)t + \frac{p^2}{2}\Pi^{*\top}\Sigma\Pi^* t\Big\}.$$

Replacing p by $(1-\gamma)$ and later by $(1-\delta)$, we obtain that the transversality conditions (18) are

$$(1-\gamma)\Big(\Psi(\Pi^*,\gamma) - A^{-1/\gamma}\Big) < \alpha,$$
$$(1-\delta)\Big(\Psi(\Pi^*,\delta) - A^{-1/\gamma}\Big) < \beta.$$

These two inequalities are equivalent to the non-negative assumption of the constants A and B, respectively. □

As we have already discussed in Section 2.1, the fund surplus can be expressed as follows

$$X^*(t) = x\exp\Big\{\Big((r + \Pi^{*\top}(b-r\bar{1}) - A^{-1/\gamma} - \frac{1}{2}\Pi^{*\top}\Sigma\Pi^*\Big)t + \Pi^{*\top}\sigma\,w(t)\Big\}\Pi_{k=1}^{m}(1+\Pi^{*\top}\varphi_k)^{N_k(t)}$$

and then it is positive almost surely because $x>0$ and $1+\Pi^{*\top}\varphi_k>0$, for all k.

Note that, in the scalar case, where $n=l=m=1$, it is possible to check if a solution to (13) exists. A necessary condition for a solution to exist is $1+\pi\varphi>0$, that is, the uncertainty of the Poisson processes and the investment strategies are positively compensated. If we define $f(\pi) = b - r - \sigma^2\pi\delta + \lambda(1+\pi\varphi)^{-\delta}\varphi$, then we have $\lim_{\pi\to-\infty} f(\pi)=\infty$ and $\lim_{\pi\to\infty} f(\pi) = -\infty$, because $\delta>0$. Thus, applying Bolzano's Theorem, an investment strategy π such that $f(\pi)=0$ exists, that is, it is a solution of (13). On the other hand, $f'(\pi) = -\delta(\sigma^2+\lambda\varphi^2(1+\pi\varphi)^{-\delta-1})<0$, then $f(\pi)$ is strictly decreasing and this implies uniqueness. If we assume $f(0)=b-r+\lambda\varphi$ to be positive, then we have a unique positive investment strategy if the condition $1+\pi\varphi>0$ holds (for instance, when $\varphi\geq 0$). Note that negative investments, that is, allowing shortselling, can be found for negative diffusion jump parameters. Given a solution π of (13), it is straightforward to obtain A and B from (14), (15).

Remark 3.1. It is not difficult to check that for logarithmic utility functions $u(P)=\ln P$ and $v(X)=\ln X$, the value functions of the union and the firm are

$$V_U(x) = \frac{1}{\alpha}\ln x + \frac{1}{\alpha}(\ln\alpha - 1) + \frac{1}{\alpha^2}\Psi(\Pi^*,1),$$
$$V_F(x) = \frac{1}{\beta}\ln x - \frac{\alpha}{\beta^2} + \frac{1}{\beta^2}\Psi(\Pi^*,1),$$

where $\Psi(\Pi^*,1)$ is given by (12), the MPNE is (P^*,Π^*), where $P^*(x)=\alpha x$ and Π^* is the constant solution of (13), but for $\delta=1$, and the fund surplus evolution is

$$dX^*(t) = \Big(r + \Pi^{*\top}(b-r\bar{1}) - \alpha\Big)X^*(t)dt + \Pi^{*\top}\sigma X^*(t)\,dw(t) + \Pi^{*\top}\varphi X^*(t)\,dN(t).$$

Remark 3.2. For $\lambda=0$, we are considering the pension game without jumps. This case is also achieved for $\varphi=0$. We obtain the same value functions and equations (13)–(15), for A,B,π, as in Josa-Fombellida & Rincón-Zapatero (2019), which allows the game to be explicitly solved. In the particular case where $\sigma=0$, uncertainty comes only from Poisson jumps.

The jump parameters intervene in the implicit expressions of the constants A, B and the investment strategies Π^*. So they also influence the value functions V_U and V_F, the extra benefits P^* and the equilibrium surplus X^*. From (14), the extra benefit relative to surplus is the constant

$$\frac{P^*}{X} = A^{-1/\gamma} = \frac{\alpha}{\gamma} - \frac{1-\gamma}{\gamma}\Psi(\Pi^*,\gamma),$$

where $\Psi(\Pi^*, \gamma)$ is given by (11), and then its first order derivative with respect to the jump intensity λ_k is

$$\frac{\partial}{\partial \lambda_k} A^{-1/\gamma} = -\frac{1}{\gamma}\Big(\big(1+\Pi^{*\top}\varphi_k\big)^{1-\gamma} - 1\Big),$$

for each k. Note that it is positive when $\gamma > 1$ and $\Pi^{*\top}\varphi_k > 0$ and when $\gamma < 1$ and $\Pi^{*\top}\varphi_k < 0$. In this case, the extra benefit increases with the jump intensity λ_k. It is negative when $\gamma > 1$ and $\Pi^{*\top}\varphi_k < 0$ and when $\gamma < 1$ and $\Pi^{*\top}\varphi_k > 0$ (the extra benefit decreases with the jump intensity). We can observe that the extra benefit $\frac{P^*}{X}$ converges to $-\infty$ when the risk aversion parameter tends to infinite. It is difficult to analytically examine the influence of the parameters on the equilibrium investment strategies because it is not possible to obtain the investment explicitly from (13). For this reason, we carry out a numerical analysis in Section 5. However, first of all, we show a previous approximation to an explicit form of the solutions in the following Remark.

Remark 3.3. For small jumps we can approximate the value of Π by Taylor's series. The most complicated terms to clear in (13) are $\big(1+\Pi^\top\varphi_k\big)^{-\delta}$, which can be approximated by $1-\delta\Pi^\top\varphi_k$. Now, after the approximation, the system (13)–(15) can be solved explicitly. In order to simplify, we show the solution in the scalar case, where $n = l = m = 1$. The equilibrium investment strategy is

$$\pi \simeq \frac{1}{\delta}\frac{b-r+\lambda\varphi}{\sigma^2+\lambda\varphi^2} = \frac{1}{\delta}\frac{\theta}{\sqrt{\sigma^2+\lambda\varphi^2}}, \tag{19}$$

where $\theta = \frac{b-r+\lambda\varphi}{\sqrt{\sigma^2+\lambda\varphi^2}}$ is the Sharpe ratio or market price of risk of the portfolio,[4] and it is assumed that $b+\lambda\varphi > r$, thus the manager has more incentives to invest in the risky asset. Taking the approximation to be good, the first derivative $\pi'(\delta) = -\frac{1}{\delta^2}\frac{\theta}{\sqrt{\sigma^2+\lambda\varphi^2}} < 0$, and then a greater risk aversion of the manager δ implies a lower investment π. Analogously, investment π decreases when the Brownian uncertainty σ^2 increases. However $\pi'(\lambda) = -\frac{(b-r)\varphi^2-\sigma^2\varphi}{\delta(\sigma^2+\lambda\varphi^2)^2} > 0$ when $\varphi > 0$ and $(b-r)\varphi < \sigma^2$, thus the investment increases with the jump intensity when the diffusion jump parameter is positive and a technical condition on the parameters of the assets holds. The opposite property holds, that is, investment is decreasing with the jump intensity when $\varphi < 0$ and $(b-r)\varphi > \sigma^2$. Finally the investment increases with the Sharpe ratio. It is possible to obtain explicitly $A^{-1/\gamma} = P^*/X$ from (14) to (19), but is not very tractable for analyzing the sensitivity of the equilibrium strategies and the surplus with respect to the parameters from a theoretical point of view. For this reason, we decided to make a numerical illustration without the use of this Taylor approximation.

Following the footnote of Section 2.1, the fund surplus equilibrium X^* is positive independently of the jump. In order to study the expected surplus evolution, from (17) we obtain

$$\mathbb{E}_x X^*(t) = x\exp\left\{\big(r+\Pi^\top(b-r\bar{1}) - A^{-1/\gamma} + \Pi^\top\varphi\lambda\big)t\right\},$$

that converges to ∞ if and only if $r+\Pi^\top(b-r\bar{1})+\Pi^\top\varphi\lambda > A^{-1/\gamma}$. Jumps with high intensity favor this behavior. In the opposite case, where the exponent is negative, the expected surplus converges to 0.

We also can obtain the evolution of the equilibrium extra benefits. From (17), and using $P^* = A^{-1/\gamma}X$, it satisfies the SDE:

$$dP(t) = \Big(rP(t) + \Pi^\top(b-r\bar{1})P(t) - A^{-1/\gamma}P(t)\Big)dt + \Pi^\top\sigma P(t)\,dw(t) + \Pi^\top\varphi P(t)\,dN(t),$$

where $P(0) = A^{-1/\gamma}x$, thus the sources of uncertainty are the Brownian motions and the Poisson processes. Therefore, the benefits have the same convergency properties as the surplus.

The manager of the plan is also interested in the study of the equilibrium fund assets F. As the plan is overfunded, that is, the fund F exceeds the actuarial liability AL, the manager considers a fund surplus X equal to a proportion of the overfunded actuarial liability $F-AL$, $X = k(F-AL)$, with k being a constant fixed by the manager, $0 < k < 1$. Then the fund is $F = \frac{1}{k}X + AL$, and we can find the SDE that it satisfies if we know the evolution of AL.

A sensitivity analysis of the parameters can be very difficult because of the implicit solutions found for the system (13)–(15). Therefore, we draw the main conclusions numerically in Section 5.

Remark 3.4. When there is a hierarchical interaction between the agents, the firm and the union, it is possible to find and analyze Stackelberg[5] equilibrium strategies of the non-cooperative differential game, depending which agent assumes the leader role. It is not difficult to check that, if the manager is the leader and decides to use constant feedback investment strategies, the investment and benefit Stackelberg equilibrium strategies coincide with the Nash equilibrium strategies. We obtain the same conclusion when the union is the leader and decides to use a linear feedback benefit strategy. Thus, the Nash equilibrium strategies solve the leader-follower problem in the pension plan game, maximizing the utilities.

4. Nash equilibrium strategies behind boundary conditions

In this section, we consider the other two scenario games where the firm minimizes: 1) the probability that the fund surplus reaches a low level, or 2) the expected time of reaching a bechmark fund surplus. The first objective supports the sustainability of the plan. For this purpose, we consider a variant of the previous game, adding two boundary levels $\nu > 0$ (for the upper level) and $\ell > 0$ (for the lower level), where $\nu > l$ and a starting point of the surplus $X(0) = x \in (\ell, \nu)$.

Both objectives fit a general form of the payoff of the firm

$$J_F(x; P, \pi) = \mathbb{E}_x\Big(\int_0^T f(X^{(P,\Pi)}(t))dt + h(X^{(P,\Pi)}(T))\Big), \ \ \ell < x < \nu, \tag{20}$$

where f and h are general utility and bequest functions, respectively, $P \in A_U$ is fixed and $\Pi \in A_F$ is the proportion of the surplus that the firm chooses. We denote by T_z the first time that X hits the value $z \ge 0$, that is to say, $T_z = \inf\{t > 0 | X^{(P,\Pi)}(t) = z\}$, and $T = \min\{T_\ell, T_\nu\}$.

On the other hand, the payoff of the union is

$$J_U(x; P, \pi) = \mathbb{E}_x\Big(\int_0^T e^{-\alpha t}u(P(t))dt + e^{-\alpha T}g(X^{(P,\pi)}(T))\Big),$$

where g is a bequest function and $X^{(P,\pi)}$ is the fund surplus under profile (P,π) satisfying (4). We assume that $u(P)$ is the same CRRA utility function as in Section 3.

The game would be specified with two expressions for the payoffs and the evolution of X given by (4). Finally, the boundary conditions are

$$V_U(\ell) = g(\ell), \quad V_U(\nu) = g(\nu), \quad V_F(\ell) = h(\ell), \quad V_F(\nu) = h(\nu).$$

The difficulty of finding a solution for any function g was covered in Josa-Fombellida & Rincón-Zapatero (2006). So, following Josa-Fombellida & Rincón-Zapatero (2019), the problem is reformulated

[4] Björk & Slinko (2006), in Appendix A, define the Sharpe ratio for a jump diffusion process, in particular for a Wiener–Poisson process.

[5] References as Başar & Olsder (1999) and Dockner et al. (2000) can be consulted.

as an approximation of the true game, where the payoff of the union turns is

$$J_U(x; P, \pi) = \mathbb{E}_x \int_0^\infty e^{-\alpha t} u(P(t))dt, \tag{21}$$

the transversality condition being the boundary condition for the value function V_U. So now the boundary conditions are

$$\lim_{t\to\infty} e^{-\alpha t}\mathbb{E}_x V_U(X^{P,\Pi}(t)) = 0, \qquad V_F(\ell) = h(\ell), \qquad V_F(\nu) = h(\nu). \tag{22}$$

This stochastic control problem of Dirichlet type has been analyzed in Krylov (1980). Following, Josa-Fombellida & Rincón-Zapatero (2006), the HJB system for this game is

$$\begin{aligned}\alpha V_U(x) = \max_P &\left\{ \frac{P^{1-\gamma}-1}{1-\gamma} + \left(rx + \Pi^\top(b - r\bar{1})x - P\right)V_U'(x)\right\}\\ &+ \frac{1}{2}\Pi^\top\Sigma\Pi x^2 V_U''(x) + \sum_{k=1}^m \lambda_k\left(V_U\left(x+\sum_{i=1}^n \pi_i x\varphi_{ik}\right) - V_U(x)\right),\\ 0 = \max_\Pi &\left\{ \left(rx + \Pi^\top(b - r\bar{1})x - P\right)V_F'(x)\right.\\ &\left.+ \frac{1}{2}\Pi^\top\Sigma\Pi x^2 V_F''(x) + \sum_{k=1}^m \lambda_k\left(V_F\left(x+\sum_{i=1}^n \pi_i x\varphi_{ik}\right) - V_F(x)\right)\right\} + f(x).\end{aligned}$$

From the optimality conditions, one gets

$$P^{-\gamma} - V_U'(x) = 0 \Rightarrow P = (V_U'(x))^{-1/\gamma}, \tag{23}$$

$$\begin{aligned}&(b_i - r)xV_F'(x) + \sum_{j=1}^n a_{ij}\pi_j x^2 V_F''(x)\\ &+ \sum_{k=1}^m \lambda_k V_F'\left(x + \sum_{j=1}^n \pi_j x\varphi_{jk}\right)\varphi_{ik}x = 0,\end{aligned} \tag{24}$$

for all $i = 1, \dots, n$. Plugging into the HJB system,

$$\begin{aligned}\alpha V_U(x) = &\frac{-1}{1-\gamma} + \frac{\gamma}{1-\gamma}\left(V_U'(x)\right)^{1-1/\gamma} + rxV_U'(x) + \Pi^\top(b - r\bar{1})xV_U'(x)\\ &+ \frac{1}{2}\Pi^\top\Sigma\Pi x^2 V_U''(x) + \sum_{k=1}^m \lambda_k\left(V_U\left(x+\sum_{i=1}^n \pi_i x\varphi_{ik}\right) - V_U(x)\right),\end{aligned} \tag{25}$$

$$\begin{aligned}0 = &\left(rx + \Pi^\top(b - r\bar{1})x - \left(V_U'(x)\right)^{-1/\gamma}\right)V_F'(x) + \frac{1}{2}\Pi^\top\Sigma\Pi x^2 V_F''(x)\\ &+ \sum_{k=1}^m \lambda_k\left(V_F\left(x+\sum_{i=1}^n \pi_i x\varphi_{ik}\right) - V_F(x)\right) + f(x).\end{aligned} \tag{26}$$

In the following, we analyze the two game scenarios depending on the aim of the firm.

4.1. Minimizing the probability of reaching a low surplus

The firm's aim is to maximize the probability of reaching the objective value ν before the ruin value ℓ. The payoffs of the firm can be defined as

$$J_F(x; P, \pi) = \mathbb{P}_x(T_\nu < T_\ell), \;\; \ell < x < \nu, \tag{27}$$

where $P \in A_U$ is fixed and $\pi \in A_F$ is the proportion of the surplus that the firm chooses. Here $\mathbb{P}_x$ denotes the conditional probability, given that $X(0) = x$. Recall that we denote by T_z the first time that X hits the value $z \geq 0$. With this specification of the objective functional of the firm, $f = 0$, $h(\ell) = 0$ and $h(\nu) = 1$ in (20). The union's aim is given by (21), that is to say, to maximize the expected utility along an unbounded time horizon.

The following result shows the Nash equilibrium benefit, the investment strategies and the equilibrium fund surplus for this second game scenario.

Proposition 4.1. *Assume that the system*

$$b_i - r - \eta\sum_{j=1}^n a_{ij}\pi_j + \sum_{k=1}^m \lambda_k\left(1 + \Pi^\top\varphi_k\right)^{-\eta}\varphi_{ik} = 0, \qquad i = 1, \dots, n, \tag{28}$$

$$\frac{\alpha}{1-\gamma} = D^{-1/\gamma}\frac{\gamma}{1-\gamma} + \Psi(\Pi, \gamma), \tag{29}$$

$$0 = \Psi(\Pi, \eta) - D^{-1/\gamma}, \tag{30}$$

where $\Psi(\Pi, .)$ is given by (11), has the constants D, η, Π, with D positive and finite, and $1 + \Pi^\top\varphi_k > 0$, for all k, as a solution. Then the union and firm value functions of the pension game (4), (27), (21) are

$$\begin{aligned}V_U(x) &= D\frac{x^{1-\gamma}}{1-\gamma} - \frac{1}{\alpha(1-\gamma)},\\ V_F(x) &= \frac{x^{1-\eta} - \ell^{1-\eta}}{\nu^{1-\eta} - \ell^{1-\eta}},\end{aligned}$$

the MPNE is (P^r, Π^r), where

$$P^r(x) = D^{-1/\gamma}x, \tag{31}$$

and Π^r is a constant solution of (28), while the equilibrium fund surplus is the extended GBM with Poisson jumps given by

$$\begin{aligned}dX^r(t) = &\left(r + \Pi^{r\top}(b - r\bar{1}) - D^{-1/\gamma}\right)X^r(t)dt\\ &+ \Pi^{r\top}\sigma X^r(t)\,dw(t) + \Pi^{r\top}\varphi X^r(t)\,dN(t).\end{aligned}$$

Proof. Trying with the solutions

$$\begin{aligned}V_U(x) &= D\frac{x^{1-\gamma}}{1-\gamma} - \frac{M}{1-\gamma},\\ V_F(x) &= B\frac{x^{1-\eta}}{1-\eta} - \frac{N}{1-\eta},\end{aligned}$$

where D, M, B, N and η are constants to determine, the optimality conditions (23) and (24) let us obtain Π^r as the solution of (28) and the expression (31) for P^r.

After substitution, the first equation of HJB (25) turns to

$$\frac{\alpha D x^{1-\gamma}}{1-\gamma} - \frac{\alpha M}{1-\gamma} = \left(\frac{\gamma D^{-1/\gamma}}{1-\gamma} + \Psi(\Pi, \gamma)\right)Dx^{1-\gamma} - \frac{1}{1-\gamma},$$

for any initial condition x. We deduce that $M = \frac{1}{\alpha}$ and (29). From the second HJB equation (26) we obtain (30). By the boundary conditions we obtain: $B\ell^{1-\eta} = N$ and $B\nu^{1-\eta} - N = 1 - \eta$, which allows us obtain V_F. Note that $V_U'' < 0$ because $D > 0$ and $V_F'' < 0$ for all η. Thus Π^r and P^r are the maximizers of the HJB system.

Finally, we prove that the transversality condition (22) holds, following Theorem 8.5 of Dockner et al. (2000), as in the previous Proposition 3.1. By Arnold (1974), p. 139, for a real number p,

$$\begin{aligned}\mathbb{E}_x(X^r(t))^p = x^p \exp&\left\{\; p\left(r + \Pi^{r\top}(b - r\bar{1}) - D^{-1/\gamma} - \frac{1}{2}\Pi^{r\top}\Sigma\Pi^r\right)t\right.\\ &\left.+ \sum_{k=1}^m \lambda_k\left(\left(\left(1 + \Pi^{r\top}\varphi_k\right)^p - 1\right)t + \frac{p^2}{2}\Pi^{r\top}\Sigma\Pi^r t\right\},\end{aligned}$$

After replacing p by $(1-\gamma)$, (22) holds if and only if

$$(1-\gamma)\left(\Psi(\Pi^r, \gamma) - D^{-1/\gamma}\right) < \alpha. \tag{32}$$

It is easy to prove that this inequality is equivalent to the nonnegative assumption of D, replacing (29) in (32). □

The equilibrium fund surplus X^r is positive a.s. as in the first game scenario. The extra benefit strategy is proportional to the surplus and the investment strategy is constant. Neither strategy depends on ℓ or ν. The expected surplus evolution evolves according to the SDE

$$\mathbb{E}_x X^r(t) = x\exp\left\{\left(r+\Pi^{r\top}(b-r\bar{1})-D^{-1/\gamma}+\Pi^{r\top}\varphi\lambda\right)t\right\},$$

that converges to ∞ if and only if $r+\Pi^{r\top}(b-r\bar{1})+\Pi^{r\top}\varphi\lambda > D^{-1/\gamma}$.

In a similar way to the first game scenario, the jump parameters influence the equilibrium strategies and surplus. From (29), the extra benefit relative to surplus

$$\frac{P^r}{X} = D^{-1/\gamma} = \frac{\alpha}{\gamma} - \frac{1-\gamma}{\gamma}\Psi(\Pi^r,\gamma),$$

increases with the intensity of the jump λ_k if $\gamma > 1$ and $\Pi^{r\top}\varphi_k > 0$ and decreases with λ_k if $\gamma > 1$ and $\Pi^{r\top}\varphi_k < 0$. We numerically analyze the sensitivity of the equilibrium strategies with respect to some parameters in Section 5, as with the first game scenario.

Remark 4.1. For $\lambda = 0$ or $\varphi = 0$, we consider the pension game without jumps. We obtain the corresponding equations to (28), (29), (30), for Π^r, D, η, as in Josa-Fombellida & Rincón-Zapatero (2019), which allows the game to be solved explicitly obtaining two MPNE when $0 < \gamma < 1$ and one when $\gamma > 1$.

The logarithmic case, where $\gamma = 1$, can be analyzed easily, as in the first game scenario. Parameter γ only affects equation (29), that where $D = 1/\alpha$ when $\gamma = 1$, and then the equilibrium extra benefit is $P^r(x) = \alpha x$.

The Taylor approximation of Π can also be considered here, but we prefer to analyze the solutions numerically.

If we assume a hierarchical interaction between the firm and the union, it is possible to check that, if the manager is the leader and decides to use constant feedback investment strategies, the investment and benefit Stackelberg equilibrium strategies coincide with the Nash equilibrium strategies. We obtain the same conclusion when the union is the leader and decides to use a linear feedback benefit strategy.

4.2. Minimizing the expected time to reach a benchmark surplus

The firm's aim is to minimize the expected time to reach a good benchmark fund surplus ν. The payoff of the firm can be defined as

$$J_F(x;P,\pi) = \mathbb{E}_x T_\nu,\ \ x<\nu, \tag{33}$$

where $P \in A_U$ is fixed and $\pi \in A_F$ is the proportion of the surplus that the firm chooses. Here $\mathbb{E}_x$ denotes the conditional expectation, given that $X(0) = x$. Recall that we denote by T_ν the first time that X hits the value $\nu \ge 0$. With this specification of the objective functional of the firm, $f = 1$ and $h(\nu) = 0$ in (20). The union's aim is still given by (21), that is to say, to maximize the expected utility along an unbounded time horizon.

The following result shows the Nash equilibrium benefit, the investment strategies and the equilibrium fund surplus for the third game scenario.

Proposition 4.2. *Consider the system*

$$b_i - r - \sum_{j=1}^{n} a_{ij}\pi_j + \sum_{k=1}^{m}\lambda_k\left(1+\Pi^\top\varphi_k\right)^{-1}\varphi_{ik} = 0, \qquad i=1,\dots,n, \tag{34}$$

and, from its vector of solutions Π, with $1+\Pi^\top\varphi_k > 0$, for all k, define the constants K and R, as follows:

$$\frac{\alpha}{1-\gamma} = \frac{\gamma}{1-\gamma}K^{-1/\gamma} + \Psi(\Pi,\gamma), \tag{35}$$

$$R = \left(\Psi(\Pi,1) - K^{-1/\gamma}\right)^{-1}, \tag{36}$$

where $\Psi(\Pi,.)$ is given by (11) and (12), with K and R positive and finite. Then the union and firm value functions of the pension game (4), (33), (21) are

$$V_U(x) = K\frac{x^{1-\gamma}}{1-\gamma} - \frac{1}{\alpha(1-\gamma)},$$
$$V_F(x) = R\ln\left(\frac{\nu}{x}\right),$$

the MPNE is (P^b,Π^b), where

$$P^b(x) = K^{-1/\gamma}x, \tag{37}$$

and Π^b is a constant solution of (34), while the equilibrium fund surplus is the extended GBM with Poisson jumps given by

$$dX^b(t) = \left(r+\Pi^{b\top}(b-r\bar{1})-K^{-1/\gamma}\right)X^r(t)dt + \Pi^{b\top}\sigma X^r(t)\,dw(t) + \Pi^{b\top}\varphi X^b(t)\,dN(t).$$

Proof. The proof is very similar to the proof of Proposition 4.1. □

The equilibrium fund surplus X^b is positive a.s. as in the other two scenarios. The extra benefit strategy is proportional to the surplus and the investment strategy is constant and does not depend on γ. Neither strategy depends on ν. The expected surplus evolution evolves according to the SDE

$$\mathbb{E}_x X^b(t) = x\exp\left\{\left(r+\Pi^{b\top}(b-r\bar{1})-K^{-1/\gamma}+\Pi^{b\top}\varphi\lambda\right)t\right\},$$

which converges to ∞ if and only if $r+\Pi^{b\top}(b-r\bar{1})+\Pi^{b\top}\varphi\lambda > K^{-1/\gamma}$.

In a similar way to the previous game scenarios, the jump parameters influence the equilibrium strategies and surplus. From (35), the extra benefit relative to surplus

$$\frac{P^b}{X} = K^{-1/\gamma} = \frac{\alpha}{\gamma} - \frac{1-\gamma}{\gamma}\Psi(\Pi^b,\gamma),$$

increases with the intensity of the jump λ_k if $\gamma > 1$ and $\Pi^{b\top}\varphi_k > 0$ and decreases with λ_k if $\gamma > 1$ and $\Pi^{b\top}\varphi_k < 0$. We numerically analyze the sensitivity of the equilibrium strategies with respect to some parameters in Section 5.

Remark 4.2. The pension game without jumps, where $\lambda = 0$ or $\varphi = 0$, was not analyzed in Josa-Fombellida & Rincón-Zapatero (2019). It is very easy to obtain the corresponding equations to (28), (29), (30), for Π^b, K, R. The famous optimal growth portfolio strategy $\Pi^b(x) = \Sigma^{-1}(b-r\bar{1})$ of Merton's model is the equilibrium investment strategy of the game where the firm minimizes the expected time to reach a benchmark fund surplus. The relative benefit is

$$\frac{P^b(x)}{x} = K^{-1/\gamma} = \frac{\alpha}{\gamma} - \frac{1-\gamma}{\gamma}\left(r+\theta^\top\theta - \frac{1}{2}\gamma\theta^\top\theta\right),$$

where $\theta = \sigma^{-1}(b-r\bar{1})$ is the market price of risk.

Analogous comments to Remark 4.1 with respect to the logarithmic case, the Taylor approximation and the Stackelberg equilibrium can be made.

5. Illustrations

In this section, a sensitivity analysis of the jump parameters and the risk aversion parameters is performed. Our aim is to study the equilibrium investment strategy, the extra benefits and the

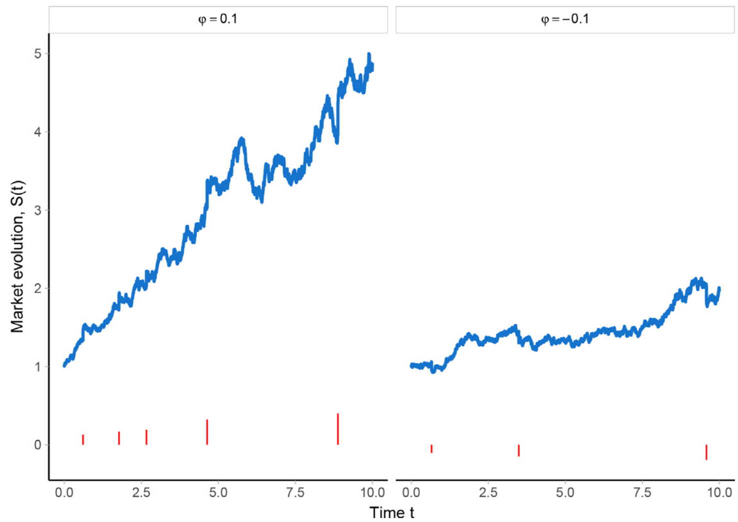

Fig. 3. Risky asset for $\lambda = 0.25$ and $\varphi = \pm 0.1$.

fund surplus for the three game scenarios. We begin the exposition by considering a bull regime, and we later include a bear regime[6]

In order to simplify the development, we consider the scalar case, where[7] $l = m = n = 1$. The parameters used to illustrate the simulations in a bull regime are $b = 0.144604$, $r = 0.01$, $\sigma = 0.10748$, $\alpha = \beta = 0.02$, with a final time $T = 10$ years and initial asset price $S_0 = 1$, taken from Josa-Fombellida & Rincón-Zapatero (2019). The parameters were estimated from S&P 500 index data. Consider an initial surplus $X(0) = x = 0.1$. We vary the values of the jump intensity $\lambda = 0, 0.25, 0.5$ along the graphical analysis. The case without jumps is covered for $\lambda = 0$. In order to cover two types of jump, upward jumps and downward jumps, the uncertainty Poisson process takes two values $\varphi = -0.1, 0.1$. We also consider several values for the risk aversion parameters, $\gamma, \delta \geq 1$, depending on the analysis. More specifically, we focus on risk aversion parameters $\gamma, \delta \in [0, 10]$ which agree with the empirical studies of Azar (2006). The equations are solved numerically with the standard R Stats package.

The time evolution of the risky asset with Poisson jumps is shown in Fig. 3 by means of two paths as an example. The left hand graph shows the upward jumps case and the right shows the downward jumps case. We consider a moderate intensity jump value of the parameter λ=0.25, and $\varphi = \pm 0.1$. The red vertical segments represent the time and the magnitude of the jump on the same axis scale. The price of the risky asset shows a significative increase with the upward jumps, but only a slight increase with the downward jumps. We observe 5 positive jumps at the times 0.6, 1.8, 2.7, 4.6, and 8.9 years, in the left hand graph, and three negative jumps at times 0.7, 3.5 and 9.6, in the right hand graph.

Now, the first game scenario is considered. The time evolution of the equilibrium surplus X in this Brownian–Poisson financial market is drawn in Fig. 4 for a risk aversion $\gamma = \delta = 2$, a jump intensity $\lambda = 0.25$, and the two cases of diffusion jump parameters $\varphi = \pm 0.1$. The equilibrium risky investment strategy and the equilibrium benefit proportion are $\pi^* = 6.236489$ and $P^*/X = 0.2580915$, respectively, with upward jumps, and $\pi^* = 3.366769$ and $P^*/X = 0.1126737$ with downward jumps. In this bull regime, we observe the increase in the left hand path surplus, starting with a small value of $X(0) = 0.1$ and finishing with a value of $X(10) = 8.098480$ at the end of the tenth year. Thus, a small effect of the jump in the risky asset can suppose a big increment in the surplus evolution. Otherwise, with downward jumps, the values of the surplus remain low, specifically $X(10) = 0.1356466$.

After this, the effects of the jump and the risk aversion on the equilibrium expected fund surplus are compared. We consider several values of the parameters, including those used in the previous surplus paths: risk aversion $\gamma = \delta = 2, 5$, jump intensity $\lambda = 0, 0.25, 0.5$, and diffusion jump $\varphi = \pm 0.1$. The expected surplus evolution is represented in Fig. 5. It is strictly increasing and convex with respect to the time, even though the market gives downward jumps. The jump intensity has a great influence on the expected surplus. With a positive diffusion jump parameter, the expected surplus increases with the jump intensity. For instance, for $\gamma = \delta = 2$, the final expected surplus in the case without jumps is $\mathbb{E}X(10) = 34.0887063$, versus $\mathbb{E}X(10) = 175.9474755$ when the market presents a positive jump intensity $\lambda = 0.25$. However, with downward jumps, the expected surplus decreases with the jump intensity. In fact, as could be seen in the previous figure with a path, the expected surplus values are much higher with positive diffusion jumps than with negative ones. In particular, upward (downward) jumps provide more (less) surplus than if there are no jumps. On the other hand, the risk aversion negatively influences the expected surplus.

Figure 6 shows the investment strategy of the firm in the first game scenario as a function of the risk aversion parameter for several values of λ and φ. We consider only the case where $\gamma = \delta \geq 1$. The investment decreases when the risk aversion $\gamma = \delta$ increases. This decrease is significant because, for instance with $\lambda = 0.25$, $\varphi = 0.1$, the investment begins for $\gamma = \delta = 1$ at $\pi^* = 12.60925$ and finishes for $\gamma = \delta = 10$ at $\pi^* = 1.232873$. We can observe that, with the positive jump diffusion parameter, the risky investment must increase slightly when the jump intensity increases. Thus, the investment without jumps is less. The behavior is similar with a negative diffusion jump parameter, $\varphi = -0.1$. The investment strategy is lower when the intensity of the jump in-

[6] In the bull regime, the economy is booming, and in the bear regime, it is in recession. We select the data characterizing both regimes, following the recommendations in Zou & Cadenillas (2017). The risk premium is greater in boom periods than in recession periods, $\mu_1 - r_1 > \mu_2 - r_2$, the stock volatility is greater when the economy is in recession, $\sigma_2 > \sigma_1$, and we assume that the risk premium by unit of volatility is higher in the boom periods than under recession, $\frac{\mu_1 - r_1}{\sigma_1^2} > \frac{\mu_2 - r_2}{\sigma_2^2}$. We have denoted the bull regime with subscript 1 and the bear regime with 2.

[7] Another alternative for the illustration is to include two Poisson processes, that is to say, $m = 2$, with different signs for φ, as in Josa-Fombellida & Rincón-Zapatero (2012). Similar situations would be reached having four cases of signs of φ_1, φ_2 that lead to two, going up or down.

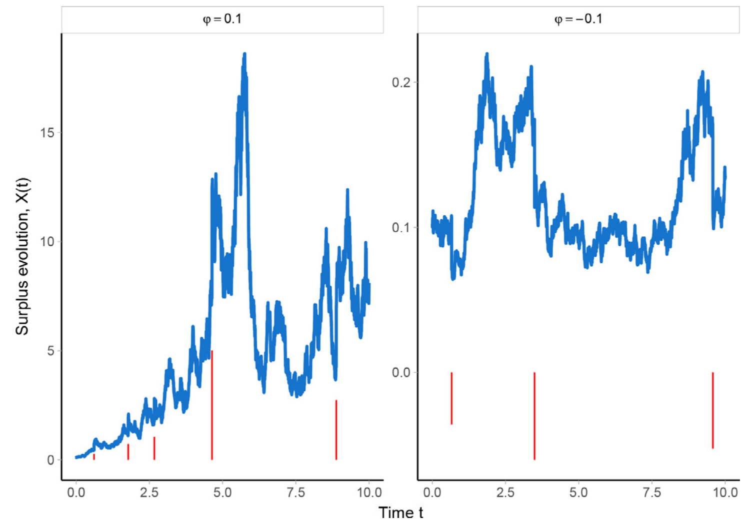

Fig. 4. Surplus time evolution for risk aversion $\gamma = \delta = 2$, Poisson jump parameters $\lambda = 0.25$ and $\varphi = \pm 0.1$. First game scenario. Bull regime.

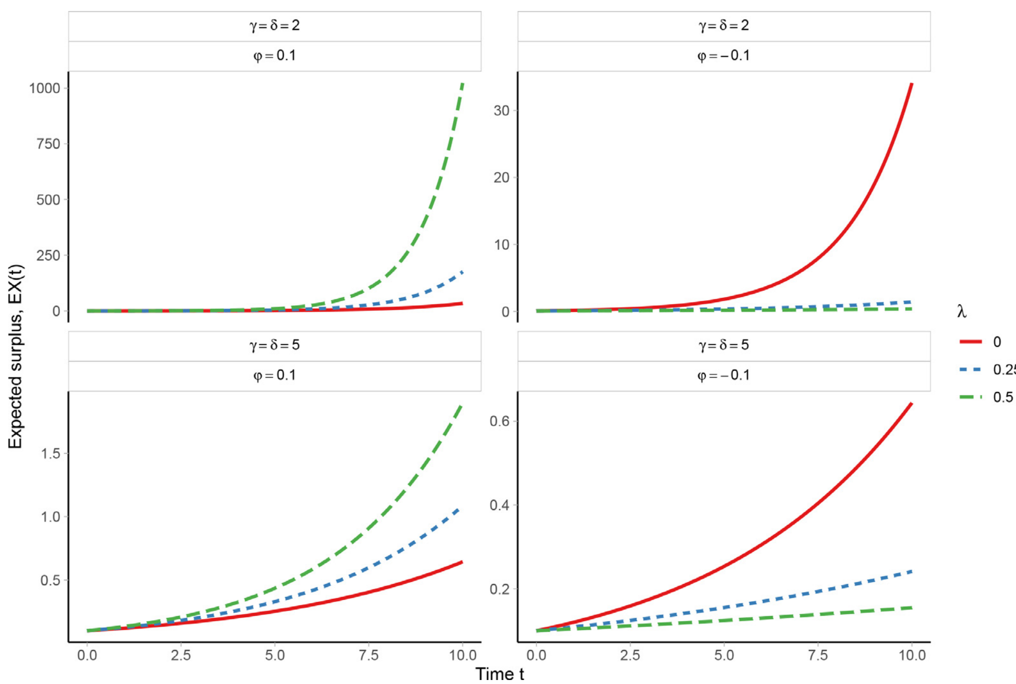

Fig. 5. Expected surplus time evolution with (blue and green, dashed) and without jumps (red, solid) for risk aversion $\gamma = \delta = 2, 5$, and Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. First game scenario. Bull regime. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

creases, but this growth is more intense than with a positive jump magnitude.

The percentage of the workers' extra benefit $A^{-1/\gamma} = P^*/X$ is illustrated in Fig. 7. The extra benefit increases with the risk aversion until $\gamma = \delta = 2.5$ and then decreases. Higher extra benefits are obtained with upward jumps. In the upward (downward) jumps case, the benefits are higher with a higher (lower) jump intensity. In particular, more (less) benefit is obtained with upward (downward) jumps than without jumps. The maximum proportion of benefits from the surplus is reached when the risk aversion is equal to 2.5, which in the upward jumps case is 28% of the fund surplus with a jump intensity of 0.5, 24% with a moderate jump intensity of 0.25 and 20% without jumps. In the downward jumps case, the maximum benefits decrease to 11% for $\lambda = 0.25$ and 7% for $\lambda = 0.5$.

In the following, we analyze the equilibrium strategies in a bear regime of the financial market. We consider that the values of the parameters are $b = 0.014$, $r = 0.01$, $\sigma = 0.2678$, $\alpha = \beta = 0.02$, with a final time $T = 10$ years and initial asset price $S_0 = 1$, taken from Josa-Fombellida & Rincón-Zapatero (2019). We also consider an initial surplus $X(0) = x = 0.1$.

The evolution of the strategies is quite similar to the bull regime case, but some trends change. The risky investment evolution with a bear regime is shown in Fig. 8. With upward jumps, the risky investment decreases with the risk aversion and is higher with more jump intensity. Now borrowing is not necessary. However, with downward jumps, shortselling can be necessary. Note that condition $b + \lambda\varphi > r$ is not satisfied because $\varphi = -0.1$. The jump intensity increases the shortselling. With the bear regime, smaller extra benefits are obtained than with the bull regime. See

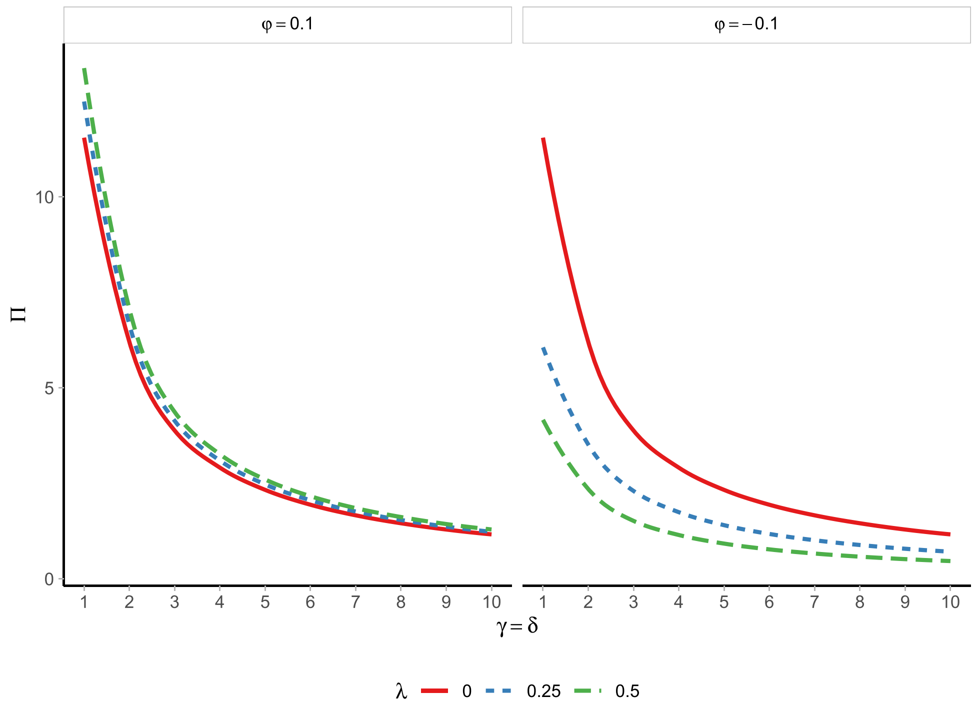

Fig. 6. Proportion of fund surplus invested in the risky asset by the firm as a function of the risk aversion parameters $\gamma = \delta$ for the Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. First game scenario. Bull regime.

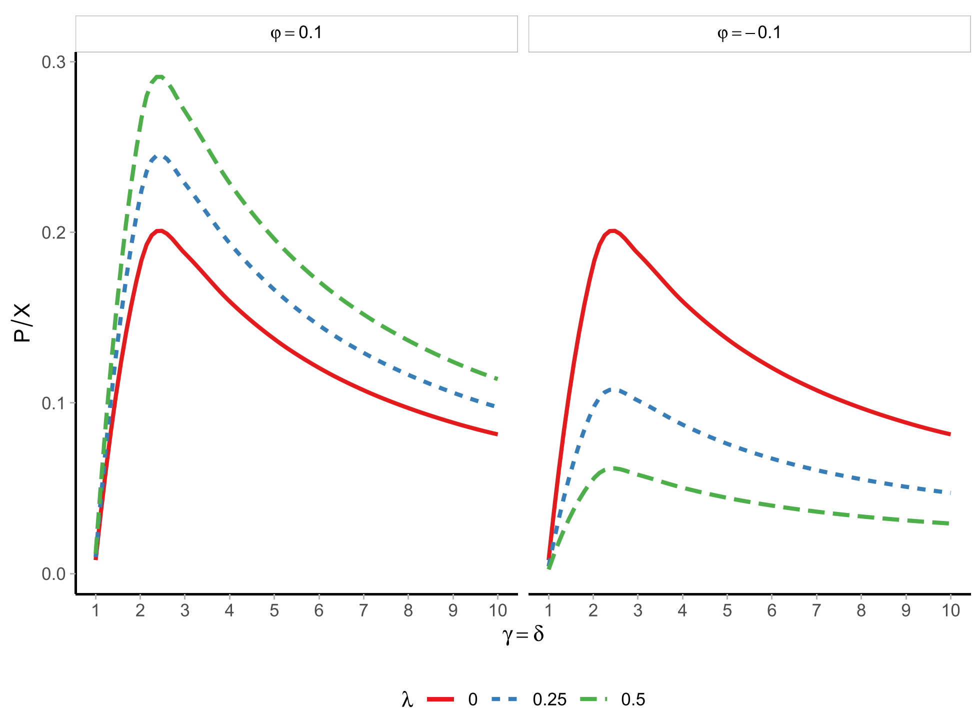

Fig. 7. Percentage of benefits claimed by the union as a function of the risk aversion parameters $\gamma = \delta$ for the Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. First game scenario. Bull regime.

Fig. 9. Note that the maximum values of the proportion of extra benefits with upward jumps are inside the range [1.9%, 2%], that is, around 10 times lower than in the bull regime. The graphical behavior is different to the bull case, because P/X is a convex and decreasing function with respect to the risk aversion $\gamma = \delta$. With upward jumps, more benefits are obtained, also with a greater jump intensity. However, with downward jumps, the benefit is higher when the jump intensity increases, unlike in the bull case. Thus, more benefit is achieved with jumps than without jumps, independently of the type of jump. The variation of the expected surplus with the bear regime is very small, less than 0.015 units. The biggest expected surplus is reached for low risk aver-

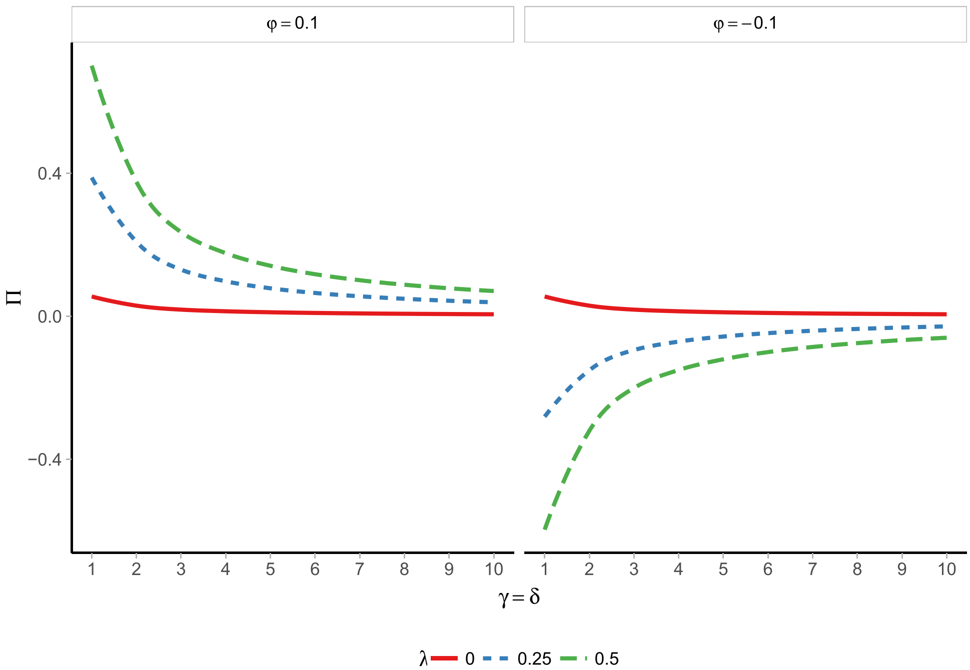

Fig. 8. Proportion of fund surplus invested in the risky asset by the firm as a function of the risk aversion parameters $\gamma = \delta$ for the Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. First game scenario. Bear regime.

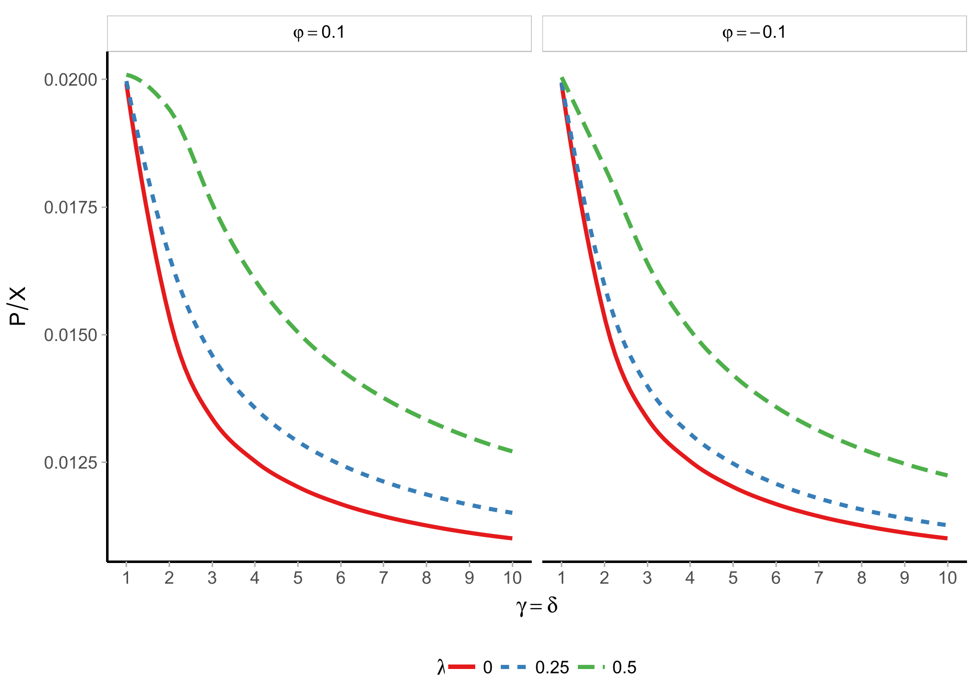

Fig. 9. Percentage of benefits claimed by the union as a function of the risk aversion parameters $\gamma = \delta$ for the Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. First game scenario. Bear regime.

sion, with upward and high intensity jumps: $\mathbb{E}X(10) = 0.11$. It is only time increasing for the high jump intensity, $\lambda = 0.5$, otherwise it is decreasing. The expected surplus is a little bit higher with upward jumps. With upward jumps, the expected surplus is higher when the risk aversion is lower. But with downward jumps, this is not necessarily so. More surplus is obtained with jumps than without jumps. The risk aversion γ influences the expected surplus, but this depend on the intensity of the jump. See Fig. 10.

We now analyze the second game scenario, where the aim of the firm is to maximize the probability that the surplus reaches a good value before a ruin value. In order to simplify, we consider the bull case. We first show two paths of the optimal surplus in the baseline case described for the first game scenario, see Fig. 11.

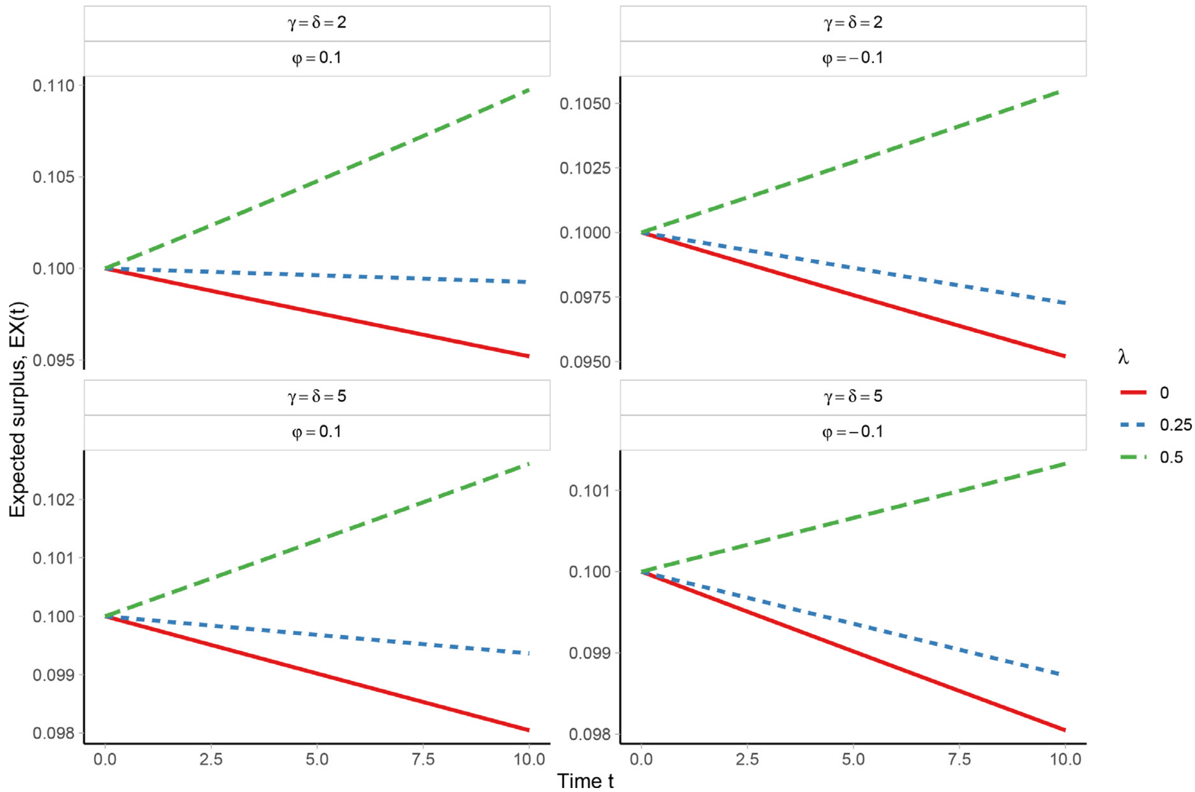

Fig. 10. Expected surplus time evolution with (blue and green, dashed) and without jumps (red, solid) for risk aversion $\gamma = \delta = 2, 5$, and Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. First game scenario. Bear regime. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

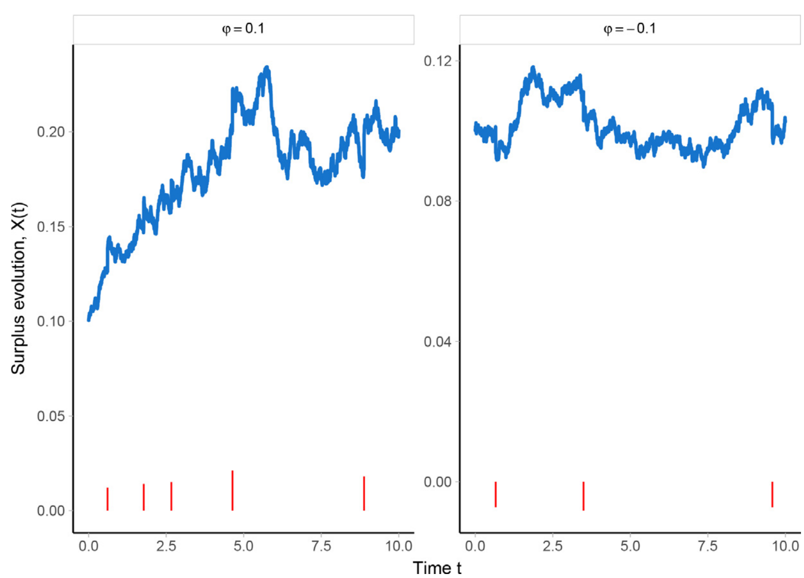

Fig. 11. Surplus time evolution for risk aversion $\gamma = 2$, Poisson jump parameters $\lambda = 0.25$ and $\varphi = \pm 0.1$. Second game scenario. Bull regime.

The variability is less than in the first scenario, and in the downward jumps case than in the upward jumps case.

The expected surplus is smaller in the second than in the first game scenario. With up (down)-ward jumps, the expected surplus increases (decreases) when the jump intensity increases. It is higher with upward jumps than with downward jumps. The behavior is similar to the first game scenario, but it is bigger with high risk than with low risk aversion. See Fig. 12.

We observe that Π^r and P^r/X are lower than Π^* and P^*/X, respectively, as in the case without jumps analyzed in Josa-Fombellida & Rincón-Zapatero (2019). See Figs. 6, 7, 13 and 14. The behavior of the extra benefit P^r/X is similar to P^*/X, but the values

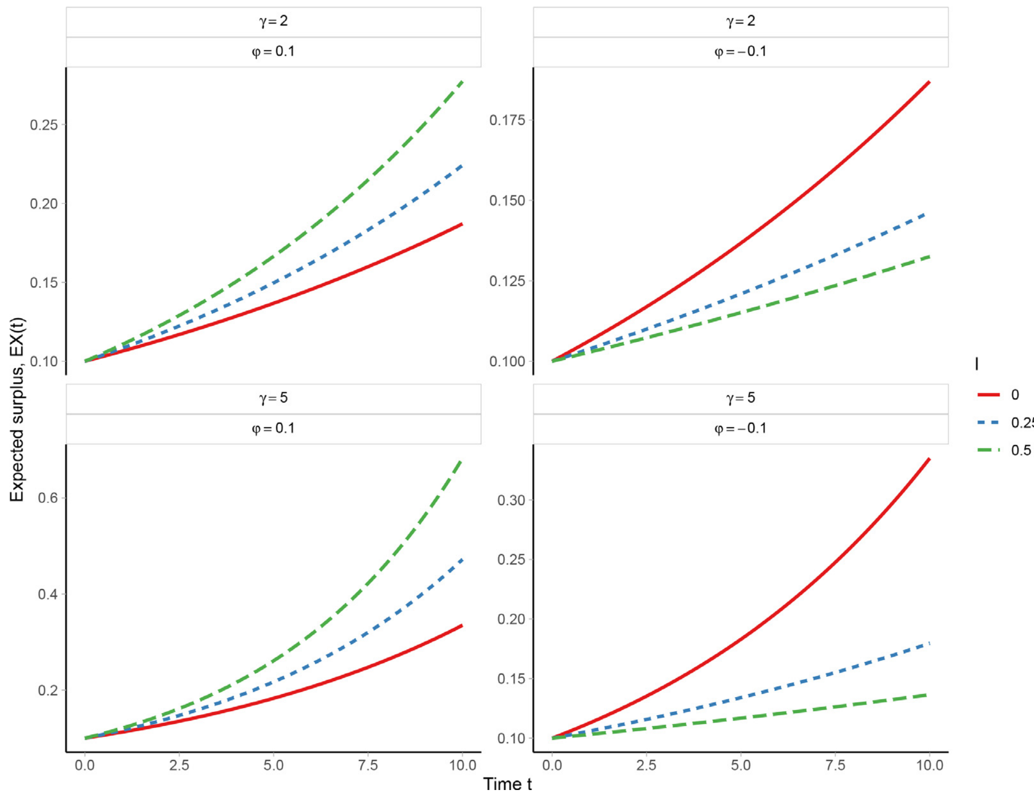

Fig. 12. Expected surplus time evolution with (blue and green, dashed) and without jumps (red, solid) for risk aversion $\gamma = 2, 5$, and Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. Second game scenario. Bull regime. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

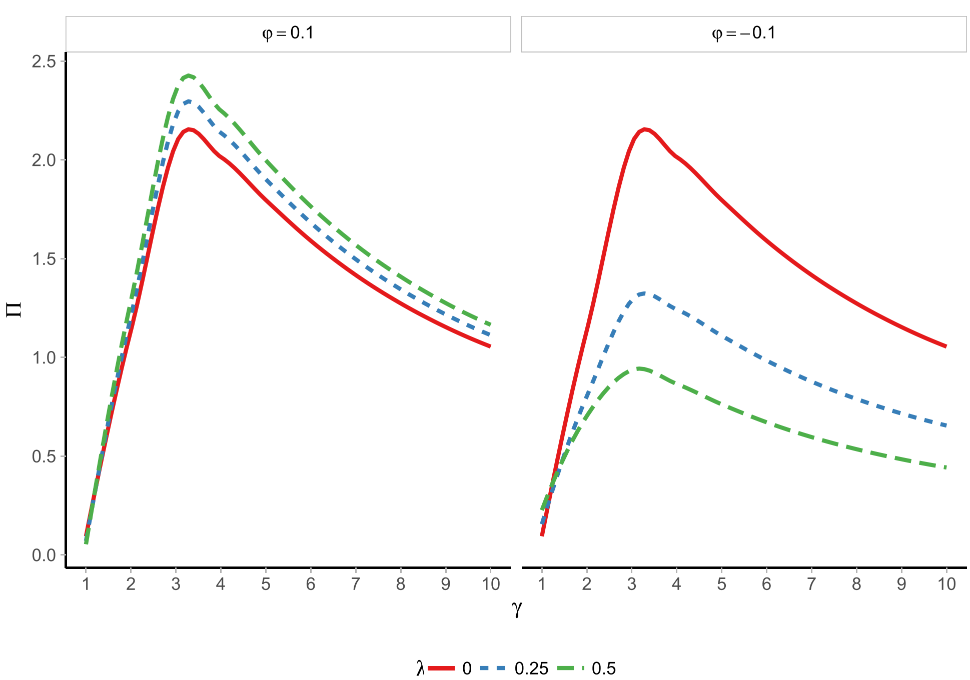

Fig. 13. Proportion of fund surplus invested in the risky asset by the firm as a function of the risk aversion parameter γ for the Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. Second game scenario. Bull regime.

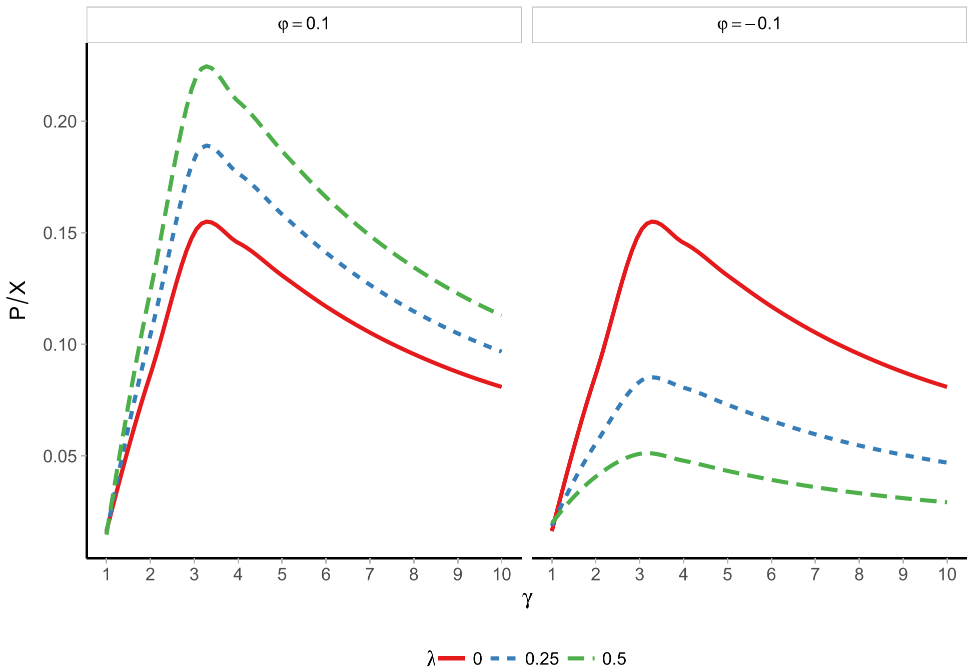

Fig. 14. Percentage of benefits claimed by the union as a function of the risk aversion parameter γ for the Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. Second game scenario. Bull regime.

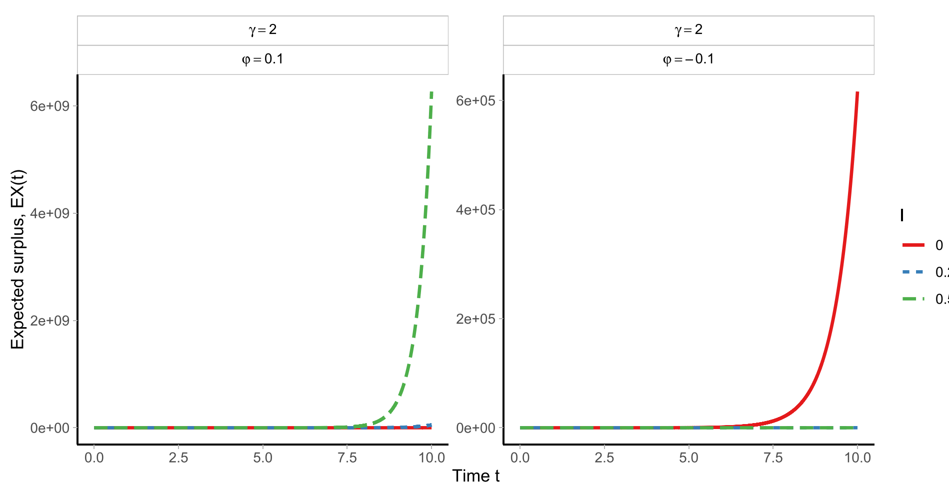

Fig. 15. Expected surplus time evolution with (blue and green, dashed) and without (red, solid) jumps for risk aversion $\gamma = 2$, and Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. Third game scenario. Bull regime. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

are somewhat smaller. It is lower with downward jumps than with upward jumps. With up (down) ward jumps, the extra benefit increases (decreases) when the intensity jump increases. We observe the same property for the investment strategy. However, the investment Π^r has a different behavior from Π^*. Π^r is concave with respect to risk aversion γ and reaches lower values, between 0 and 2.5. The maximum investment proportion is achieved for a moderate risk aversion of around 3.5. Finally, it is interesting to observe that more surplus, investment and benefit are obtained with jumps than without jumps.

With the third game scenario, more surplus is obtained than with the second, even in the bear case, because the probability that the surplus reaches a low value is not minimized, that is, it is only important that it reaches its target value ν. With the very high risk aversion of the union, it is not possible to find an equilibrium strategy for the game. Figures 15–17 show the expected equilibrium fund surplus $\mathbb{E}X^b$ and the equilibrium strategies Π^b and P^b, in the bull case, for some small high risk aversion parameters, respectively. A behavior similar to that of the second game, with respect to the time and the risk aversion, is observed. With upward (downward) jumps, the expected surplus, benefit and investment increase (decrease) with the intensity of the jump. The investment effort must be greater than in the second game. The highest extra benefit, above 20% of the surplus, is achieved with a higher intensity positive jump and a moderate risk aversion of 1.40.

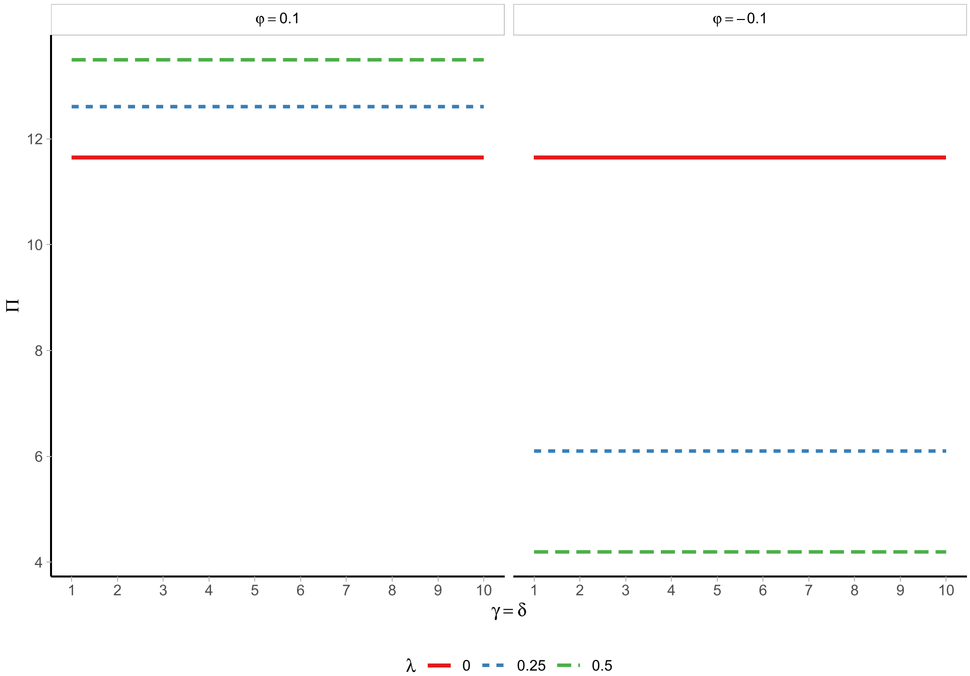

Fig. 16. Proportion of fund surplus invested in the risky asset by the firm as a function of the risk aversion parameter γ for the Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. Third game scenario. Bull regime.

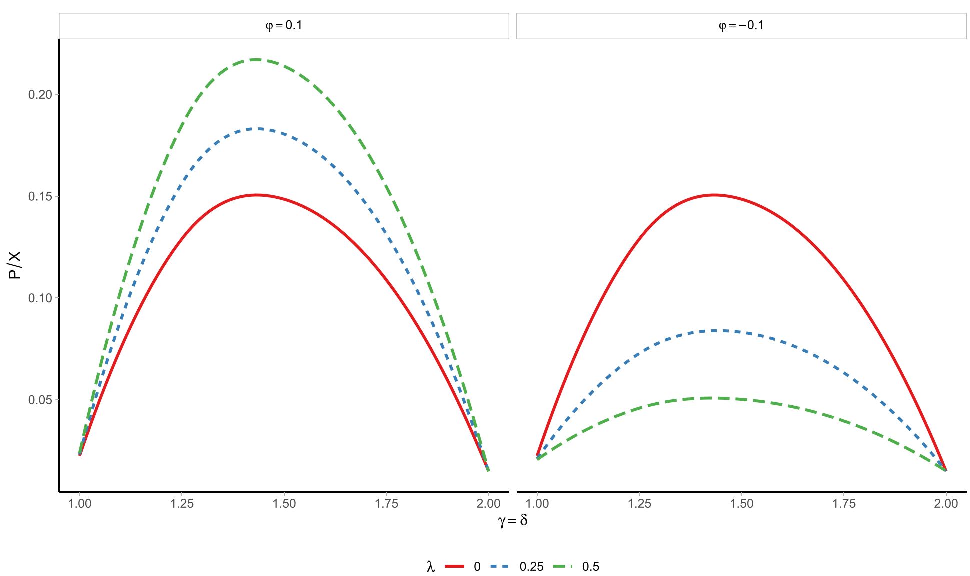

Fig. 17. Percentage of benefits claimed by the union as a function of the risk aversion parameter γ for the Poisson jump parameters $\lambda = 0, 0.25, 0.5$ and $\varphi = \pm 0.1$. Third game scenario. Bull regime.

6. Conclusions

The financial market can be severely affected by unexpected news because of results in sudden changes in the asset prices. For instance, consider the evolution of the prices of the risky assets of the financial market over a time interval with a home lockdown or the first appearance of the vaccine against COVID-19; or consider simply the cryptocurrency market at almost any period. This reinforces the idea of properly modeling unexpected situations, for a defined benefit and overfunded pension plan game that contemplates the interaction between manager and participants. For that, we have considered a model with jumps given by Poisson processes without excluding the randomness of the Brownian motion.

The aim of the participants of the pension plan is to maximize a utility of the extra benefits. Three scenarios have been considered according to firm preferences. The jump process parameters, the risk aversion parameters and the economic regime influence the equilibrium strategies and the fund surplus in all scenarios. By means of a numerical illustration, we have checked that it is possible to obtain more return in the form of surplus and benefit than with the absence of jumps. Generally, with upward jumps, the benefit increases with the jump intensity and it is higher in the first game scenario, where the aim of the firm is to maximize a utility of the fund surplus. A lesser investment effort is necessary in the second game, where the aim of the manager is to minimize the probability that the fund surplus reaches a low level before a high value. In the third game, where the aim of the manager is to maximize the expected time to reach a high value, only equilibrium strategies are found with moderate and little high risk aversion, but with high fund surplus and a reasonable extra benefit value. Two interesting facts are observed in all the scenarios: the equilibrium fund surplus never reaches the ruin value 0 and upward jumps can make the surplus increase along time, even in bear periods.

Future research should be directed at including jumps, in order to contemplate some unstable periods of the financial market in other pension plan models. In this pension game model, it could be interesting to consider other aims for the firm, such as to minimize/maximize the expected discounted penalty/reward obtained upon achievement of a performance surplus goal and to consider, in the true game with boundary conditions, hierarchical interaction between the firm and the union, that is to say, to analyze Stackelberg equilibrium strategies, especially when the union assumes the leader role.

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