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Tracing orbits on conservative maps

by Mário Bessa*

Abstract.—We explore uniform hyperbolicity and its relation with the pseudo orbit tracing property. This property indicates that a sequence of points which is nearly an orbit (affected with a certain error) may be shadowed by a true orbit of the system. We obtain that, when a conservative map has the shadowing property and, moreover, all the conservative maps in a C^1-small neighborhood display the same property, then the map is globally hyperbolic.

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Keywords.—Volume-preserving maps; pseudo-orbits; shadowing; hyperbolicity.

1. Introduction

> "There is strong shadow where there is much light"
> Goethe in Götz von Berlichingen

1.1 The basic framework

In order to start playing with dynamical systems we need a place to play and a given rule acting on it. Once we establish that, we wonder what happens when we repeat the rule ad infinitum. We are mainly interested in two types of playgrounds: volume manifolds and symplectic manifolds. On volume-manifolds the rule is the action of a volume-preserving diffeomomorphism, and on symplectic manifolds the rule is the action of a symplectomorphism. Let us now formalize these concepts.

Let M stands for a closed, connected and C^∞ Riemannian manifold of dimension $d \geq 2$ and let ν be a volume-form on M. Once we equip M with ν we denominate it by a volume-manifold. By a classic result by Moser (see [20]) we know that, in brief terms, there is only one volume-form on M. Actually, in [20] we find and atlas formed by a finite collection of smooth charts $\{\alpha_j \colon U_j \subset M \to \mathbb{R}^d\}_{j=1}^k$ where U_j are open sets and each α_j pullbacks the volume on $\mathbb{R}^d$ into ν. The volume-form allows us to define a measure μ on M which we call Lebesgue measure. A C^r $(r \geq 1)$ diffeomorphism $f \colon M \to M$ is said to be volume-preserving if it keeps invariant the volume structure, say $f^*\nu = \nu$. In other words any Borelian $B \subset M$ is such that $\mu(B) = \mu(f^{-1}(B))$ We denote these maps by $\text{Diff}^r_\mu(M)$. We endow $\text{Diff}^r_\mu(M)$ with the Whitney (or strong) C^r topology (see [1]). In broad terms, two diffeomorphisms f and g are C^r-close if they are uniformly close as well as their first C^r derivatives computed in any point $x \in M$. Such systems emerges quite naturally when considering the time-one map of incompressible flows which are a fundamental object in fluid mechanics (see e.g. [14]).

Denote by $\mathbf{M}$ a $2d$-dimensional $(d \geq 1)$ manifold with a Riemaniann structure and endowed with a closed and nondegenerate 2-form ω called symplectic form. Let μ stands for the volume measure associated to the volume form wedging ω d-times, i.e., $\nu = \omega^d = \omega \wedge \ldots \wedge \omega$. By the

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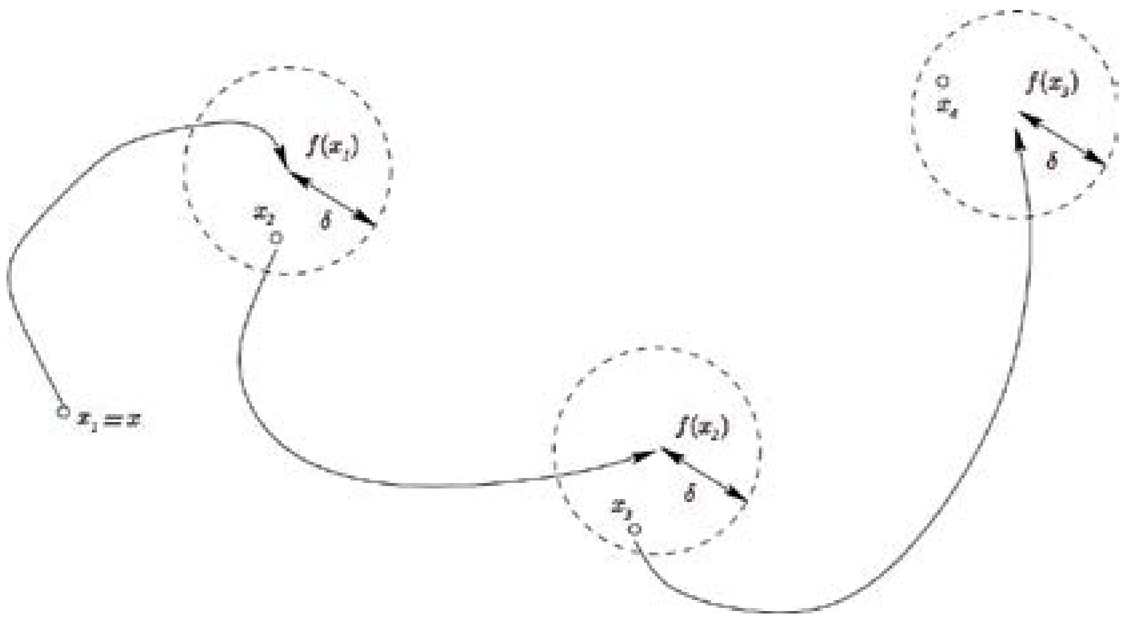

Figure 1. Illustration of a δ-pseudo-orbit

theorem of Darboux (see e.g. [21, Theorem 1.18]) there exists an atlas $\{\varphi_j : U_j \to \mathbb{R}^{2d}\}$, where U_j is an open subset of **M**, satisfying $\varphi_j^*\omega_0 = \omega$ with $\omega_0 = \sum_{i=1}^{d} dy_i \wedge dy_{d+i}$ being the canonical symplectic form. A diffeomorphism $f : \mathbf{M} \to \mathbf{M}$ is called a symplectomorphism if it leaves invariant the symplectic structure, say $f^*\omega = \omega$. Observe that, since $f^*\omega^d = \omega^d$, a symplectomorphism $f : \mathbf{M} \to \mathbf{M}$ preserves the volume measure μ. Moreover, in surfaces, area-preserving diffeomorphisms are symplectomorphisms since the volume-form equals the symplectic form. Symplectomorphisms arise in the classical and rational mechanics formalism as the first return Poincaré maps of hamiltonian flows. For this reason, it has long been one of the most interesting research fields in mathematical physics. We suggest the reference [21] for more details on general hamiltonian and symplectic theories. Let $\text{Symp}^r_\omega(\mathbf{M})$ denote the set of all symplectomorphisms of class C^r defined on **M**. We also endow $\text{Symp}^r_\omega(\mathbf{M})$ with the C^r Whitney topology.

The Riemannian structure induces a norm $\|\cdot\|$ on the tangent bundle TM and also on $T\mathbf{M}$. Denote the Riemannian distance by $d(\cdot,\cdot)$. We will use the canonical norm of a bounded linear map A given by $\|A\| = \sup_{\|v\|=1} \|A \cdot v\|$.

Given a diffeomorphism f, we denote $f^n(x) = f \circ f \circ f \circ \cdots \circ f(x)$ by composing f n-times. We say that a point p on a manifold is periodic of period $n \in \mathbb{N}$ for the diffeomorphism f if $f^n(p) = p$ and n is the minimum positive integer such that previous equality holds.

1.2 Tracing orbits and the shadowing property

The notion of shadowing in dynamical systems is inspired by the numerical computational idea of estimating differences between exact and approximate solutions along orbits and to understand the influence of the errors that we commit and allow on each iterate. We may ask if it is possible to obtain shadowing of approximate trajectories in a given dynamical system by exact ones. Nevertheless, the computational estimates, fitted with a certain error of orbits, are meaningless if they are not able to be realized by true orbits of the original system, and thus, are mere pixel imprecisions which are characteristic of the computational setup. We refer Pilyugin's book [23] for a completed description on shadowing on dynamical systems.

There are, of course, considerable limitations to the amount of information we can extract from a given specific system that exhibits the shadowing property, since a C^1-close system may be absent of that property. For this reason it is of great utility and natural to consider that a selected model can be slightly perturbed in order to obtain the same property—the stably shadowable dynamical systems.

For $\delta > 0$ and $a, b \in \mathbb{R}$ such that $a < b$, the sequence of points $\{x_i\}_{i=a}^{b}$ in M is called a δ-pseudo orbit for f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$ (see Figure 1).

The diffeomorphism f is said to have the shadowing property if for all $\epsilon > 0$, there exists $\delta > 0$, such that for any δ-pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$, there is a point x which ϵ-shadows $\{x_n\}_{n\in\mathbb{Z}}$, i.e. $d(f^i(x), x_i) < \epsilon$.

Let $f \in \text{Diff}^1_\mu(M)$ (respectively, $f \in \text{Symp}^1_\omega(\mathbf{M})$) we say that f is C^1-stably (or robustly) shadowable if there exists a neighborhood $\mathscr{V}$ of f in $\text{Diff}^1_\mu(M)$ (respectively $f \in \text{Symp}^1_\omega(\mathbf{M})$) such that any $g \in \mathscr{V}$ has the shadowing property.

We point out that f has the shadowing property if and only if f^n has the shadowing property for every $n \in \mathbb{Z}$ (see [23]).

1.3 Hyperbolicity and statement of the results

Let us recall that a periodic point p of period π is said to be hyperbolic if the tangent map $Df^\pi(p)$ has no norm one eigenvalues. Being hyperbolic is stable under small C^r perturbations. The notion of hyperbolicity can be generalized to sets rather than periodic orbits.

We say that any element f in the set $\text{Diff}^1_\mu(M)$ is Anosov (or globally hyperbolic) if, there exists $\lambda \in (0, 1)$ such that the tangent vector bundle over M splits into two Df-invariant subbundles $TM = E^u \oplus E^s$, with $\|Df^n|_{E^s}\| \le \lambda^n$ and $\|Df^{-n}|_{E^u}\| \le \lambda^n$. A completely analog definition for symplectomorphisms can be given. We observe that there are plenty Anosov diffeomorphisms which are not volume-preserving and there are plenty Anosov volume-preserving diffeomorphisms which are not symplectic. Anosov was the first one to study these kind if systems when considering the geodesic flow on closed Riemannian manifolds displaying negative curvature ([3]).

Example 1.1 [Arnold's cat map].—The map on the two-torus M, $f : M \to M$ defined by

$$f(x, y) = (2x + y, x + y)(\text{mod } 1)$$

is an area-preserving diffeomorphism thus, since the manifold is two dimensional also symplectomorphism, on the torus which is Anosov.

It is well-known that Anosov diffeomorphisms display the shadowing property (see e.g. [24]). However, the shadowing property itself do not assure hyperbolicity. Notwithstanding, the stability of the shadowing property allows us to conclude hyperbolicity (cf. Theorem A and Theorem B).

The concept of *structural stability* was introduced in the mid 1930s by Andronov and Pontrjagin ([2]), it led to the construction of uniformly hyperbolic theory, and characterizing, along a tour de force program culminated in the works by Mañé ([16, 17, 18]), structural stability as being essentially equivalent to uniform hyperbolicity. In brief terms it means that under small perturbations the dynamics are topologically equivalent: a dynamical system is C^r-structurally stable if it is topologically conjugated to any other system in a C^r neighbourhood.

Being an Anosov map is very rigid and imposes stringent topological constraints on the manifold. Actually, in the late sixties, Franks proved that the only surfaces that support hyperbolic diffeomorphisms are the tori (see [12]).

Given $f \in \text{Diff}^1_\mu(M)$ (respectively $f \in \text{Symp}^1_\omega(\mathbf{M})$) we say that f is in $\mathscr{F}^1_\mu(M)$ (respectively $\mathscr{F}^1_\omega(\mathbf{M})$) if there exists a neighborhood $\mathscr{V}$ of f in $f \in \text{Diff}^1_\mu(M)$ (respectively f in $f \in \text{Symp}^1_\omega(\mathbf{M})$) such that any $g \in \mathscr{V}$, has all the periodic orbits of hyperbolic type.

Our results ([7]) can be seen as a generalization of the result in [25] for symplectomorphisms and volume-preserving diffeomorphisms. Let us state our first result.

Theorem A.—If $f \in \text{Symp}^1_\omega(\mathbf{M})$ is C^1-stably shadowable, then f is Anosov.

Furthermore, we obtain the analogous version for volume-preserving maps.

Theorem B.—If $f \in \text{Diff}^1_\mu(M)$ is C^1-stably shadowable, then f is Anosov.

As we already said Anosov diffeomorphisms impose severe topological restrictions to the manifold where they are supported. Thus, we present a simple but startling consequence of previous theorems that shows how topological conditions on the phase space imposes numerical restrictions to a given dynamical system.

Corollary 1.2.—If the manifold do not support an Anosov diffeomorphisms, then there are no C^1-stably shadowable symplectomorphisms neither C^1-stably shadowable volume-preserving diffeomorphisms.

We end this introduction by recalling a result in the vein of ours; C^1-robust topologically stable symplectomorphisms are Anosov (see [10]). Another result which relates C^1-robust properties with hyperbolicity is the Horita and Tahzibi theorem (see [13]) which states that C^1-robust transitive symplectomorphisms are partially hyperbolic. We also mention the results in [8, 9] where it is obtained that the stable weak shadowing property implies weak hyperbolicity. Informally speaking weakly shadowing allows that the pseudo-orbits may be approximated by true orbits if one forgets the time parametrization and consider only the distance between the orbit and the pseudo-orbit as two sets in the ambient space.

Moreover, weak hyperbolicity allows the existence of subbundles with neutral behavior.

2. Proof of Theorem A

Theorem A is a direct consequence of the following two propositions. The following result, due to Newhouse, can be found in [22].

Proposition 2.1 ([22]).—If $f \in \mathcal{F}^1_\omega(\mathbf{M})$, then f is Anosov.

Proposition 2.2 is a symplectic version of [19, Proposition 1]. Actually, Moriyasu, while working in the dissipative context, considered the shadowing property in the non-wandering set, which, in the symplectic setting, and due to Poincaré recurrence, is the whole manifold **M**. Let us explain with detail this last step: we say that a point x is non-wandering if any open neighborhood U of x is such that $f^n(U) \cap U \neq \emptyset$ for some $n \in \mathbb{N}$. A point x is said to be recurrent if for any open neighborhood U of x we have $f^n(x) \in U$ for some $n \in \mathbb{N}$. Clearly, every recurrent point is non-wandering. It follows from Poincaré recurrence theorem (see e.g. [15]) that, in our conservative context, we have that μ-a.e. point x is recurrent. Since μ is the Lebesgue measure and the set of non-wandering points is closed, we have that the non-wandering points are the whole manifold **M**.

Proposition 2.2.—If f is a C^1-stably shadowable symplectomorphism, then $f \in \mathcal{F}^1_\omega(\mathbf{M})$.

Proof.—The proof is by reductio ad absurdum; let us assume that there exists a C^1-stably shadowable symplectomorphism f having a non-hyperbolic closed orbit p of period π.

In order to go on with the argument we need to C^1-approximate the symplectomorphism f by a new one, f_1, which, in the local coordinates given by Darboux's theorem, is linear in a neighborhood of the periodic orbit p. To perform this task, in the sympletic setting, and taking into account [5, Lemma 3.9], it is required higher smoothness of the symplectomorphism.

Thus, if f is of class C^∞, take $g = f$, otherwise we use Zehnder's smoothing theorem ([26]) in order to obtain a C^∞ C^1-stably shadowable symplectomorphism h, arbitrarily C^1-close to f, and such that h has a periodic orbit q, close to p, with period π. We observe that q may not be the analytic continuation of p and this is precisely the case when 1 is an eigenvalue of the tangent map $Df^\pi(p)$.

If q is not hyperbolic take $g = h$. If q is hyperbolic for $Dh^\pi(q)$, then, since h is C^1-arbitrarily close to f, the distance between the spectrum of $Dh^\pi(q)$ and the unitary circle can be taken arbitrarily close to zero. This means that we are in the presence of a quite feeble hyperbolicity, thus in a position to apply [12, Lemma 5.1]to obtain a new C^1-stably shadowable symplectomorphism $g \in \mathrm{Symp}^\infty_\omega(M)$, C^1-close to h and such that q is a non-hyperbolic periodic orbit.

At this point, we use the *weak pasting lemma* ([5, Lemma 3.9]) in order to obtain a C^1-stably shadowable symplectomorphism f_1 such that, in local canonical coordinates, f_1 is linear and equal to Dg in a neighborhood of the periodic non-hyperbolic orbit, q. Moreover, the existence of an eigenvalue, σ, with modulus equal to one is associated to a symplectic invariant two-dimensional subspace contained in the subspace $E^c_q \subseteq T_q\mathbf{M}$ associated to norm-one eigenvalues. Furthermore, up to a perturbation using again [12, Lemma 5.1], σ can be taken rational. This fact assures the existence of periodic orbits arbitrarily close to the f_1-orbit of q. Thus, there exists $m \in \mathbb{N}$ such that $f_1^{m\pi}(q)|_{E^c_q} = (Dg^{m\pi})_q|_{E^c_q} = id$ holds, say in an η-neighborhood of q. Recall that, since f_1 has the shadowing property $f_1^{m\pi}$ also has. Therefore, fixing $\epsilon \in (0, \eta/4)$, there exists $\delta \in (0, \epsilon)$ such that every δ-pseudo $f_1^{m\pi}$-orbit $\{x_n\}_n$ is ϵ-traced by some point in **M**. Take y such that $d(y, q) = 3\eta/4$ and a closed δ-pseudo $f_1^{m\pi}$-orbit $\{x_n\}_n$ such that any ball centered in x_i and with radius ϵ is still contained in the η-neighborhood of q, moreover, take $x_0 = q$ and $x_s = y$.

By the shadowing property there exists $z \in \mathbf{M}$ such that $d(f_1^{m\pi i}(z), x_i) < \epsilon$ for any $i \in \mathbb{Z}$. Moreover, we observe that $d(f_1^{m\pi i}(z), q) < \eta$ for every $i \in \mathbb{Z}$. Therefore, $z \in E^c_q$. Finally, we reach a contradiction by noting that

$$d(q,z) \geq d(q,x_s) - d(x_s,z) = \\ = d(q,y) - d(x_s, f_1^{m\pi s}(z)) \geq \frac{3\eta}{4} - \epsilon > \frac{\eta}{2} > \epsilon. \qquad \dashv$$

3. Volume-preserving diffeomorphisms

Theorem A also holds on the broader context of volume-preserving diffeomorphisms. Its proof follows the same steps as the one before. The version of Proposition 2.1 for volume-preserving diffeomorphisms was proved in a recent paper by Arbieto and Catalan.

Proposition 3.1 ([4, Theorem 1.1]).—If $f \in \mathcal{F}^1_\mu(M)$, then f is Anosov.

The proof of Theorem B is now reduced to the proof of the following result:

Proposition 3.2.—If f is a C^1-stably shadowable volume-preserving diffeomorphism, then $f \in \mathcal{F}^1_\mu(M)$.

Proof.—Assume that there exists a C^1-stably shadowable $f \in \mathrm{Diff}^1_\mu(M)$ having a non-hyperbolic closed orbit p of period π. Once again we need to C^1-approximate f by a new one, f_1, which, in the local coordinates given by Moser's theorem ([20]), is linear in a neighborhood of the periodic orbit p. Taking into account [5, Theorem 3.6], it is required higher smoothness of the volume-preserving diffeomorphism.

Thus, if f is of class C^∞, take $g = f$, otherwise we use Avila's recent proved smoothing theorem ([6]) in order to obtain a C^∞ C^1-stably shadowable volume-preserving diffeomorphism h, arbitrarily C^1-close to f, and such that h has a periodic orbit q, close to p, with period π.

If q is not hyperbolic take $g = h$. If q is hyperbolic for $Dh^\pi(q)$, then, its weak hyperbolicity allows us to use Franks' lemma proved in [11, Proposition 7.4] for volume-preserving diffeomorphisms and thus obtain a new C^1-stably shadowable volume-preserving diffeomorphism $g \in \mathrm{Diff}^\infty_\mu(M)$, C^1-close to h and such that q is a non-hyperbolic periodic orbit.

Now we use [5, Theorem 3.6] in order to obtain a C^1-stably shadowable volume-preserving diffeomorphism f_1 such that, in local canonical coordinates, f_1 is linear and equal to Dg in a neighborhood of the periodic non-hyperbolic orbit, q. Moreover, the existence of an eigenvalue, σ, with modulus equal to one is associated to an invariant one or two-dimensional subspace contained in the subspace $E^c_q \subseteq T_qM$ associated to norm-one eigenvalues. If its eigendirection is two-dimensional, up to a perturbation using again [11, Proposition 7.4], σ can be taken rational. This fact assures the existence of periodic orbits arbitrarily close to the f_1-orbit of q. Thus, there exists $m \in \mathbb{N}$ such that $f_1^{m\pi}(q)|_{E^c_q} = (Dg^{m\pi})_q|_{E^c_q} = id$ holds, say in a η-neighborhood of q. Finally, we reach a contradiction by arguing exactly as we did in the proof of Theorem A. $\dashv$

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