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Qualitative Theory of Dynamical Systems

On C^1-Generic Chaotic Systems in Three-Manifolds

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Abstract Let M be a closed 3-dimensional Riemannian manifold. We exhibit a C^1-residual subset of the set of volume-preserving 3-dimensional flows defined on very general manifolds M such that, any flow in this residual has zero metric entropy, has zero Lyapunov exponents and, nevertheless, is *strongly* chaotic in Devaney's sense. Moreover, we also prove a corresponding version for the discrete-time case.

1 Introduction

What is chaos? confusion, lots of periodic motions and inability to predict what might happen (since small errors in the initial states imply large deviations in the future) are the common definitions for this phenomenon. As far as we know the first time the nomenclature *chaos* appeared with the purely mathematical focus was in Li-York's mid 1970s article *Period Three Implies Chaos* [29]. After that, the interest in the matter exploded and we have a wide variety of definitions for this concept. Unfortunately, due to the excessive and abusive use along recent years in all types of strange applications in science and literature, the term chaos became dubious. Actually, the magic word chaos can be used almost for everything, for instance, one can prove how a complex fern is created just by picking the right rule and then do a few iterations. You will get a pretty fern, well, sort of...

Indeed, considering two different definitions of chaos is a very interesting task to try to find examples that meet a definition, but not the other.

In this work we are interested in discussing two of the most readily accepted definitions of chaos: Chaos in the sense of Devaney (see [23, Definition 8.5]) and existence of

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chaos in the sense that the metric entropy is positive. By metric (or measure-theoretic) entropy we mean Kolmogorov-Sinai's entropy (see [22]). Moreover, we establish the link between two, a priori, unrelated concepts—topological constraints on the manifold and chaoticity of flows on those manifolds.

We would like to find an example of a C^1 volume-preserving dynamical system in a 3-dimensional closed manifold M such that (see next section for full details on the definitions):

(a) periodic orbits are dense in M;
(b) it is sensitive to initial conditions;
(c) it has a dense orbit;
(d) the metric entropy is zero and
(e) the Lyapunov exponents are all equal to zero.

In conclusion, this example would be chaotic in Devaney's sense but, nevertheless displays zero entropy and zero Lyapunov exponents.

Here, despite not presenting any example, we show that this task has many possibilities to be successful and we explain where are the adequate manifolds to find these examples. Actually, in Theorem 1, we will prove that most volume-preserving flows in certain (very general) 3-dimensional closed manifolds do not satisfy both definitions simultaneously which is quite counterintuitive.

Our result, although it seems simple and direct, is a consequence of several deep recent and old results in C^1-generic theory of volume-preserving flows. Because of this we will spend some time with the basic settings so that the reader can easily follow our proof.

Finally, in Theorem 2, we also present an analogous result for volume-preserving diffeomorphisms on 3-dimensional manifolds.

2 Volume-Preserving Flows on 3-Manifolds

2.1 Notation and Basic Definitions

Let M be a 3-dimensional closed and connected C^∞ Riemannian manifold and we endowed it with a volume-form ω. Let μ denote the measure associated to ω and call μ the *Lebesgue measure*. We say that a vector field $X\colon M \to TM$ is *divergence-free* if $\nabla \cdot X = 0$ or equivalently if the measure μ is invariant for the associated flow, $X^t\colon M \to M$, $t \in \mathbb{R}$. In this case we say that the flow is *incompressible* or *volume-preserving*. Incompressible flows have plenty of applications, namely to fluid dynamics (see e.g. [30,24]). We denote by $\mathfrak{X}^r_\mu(M)$ $(r \geq 1)$ the space of C^r divergence-free vector fields on M and we endow this set with the usual C^r Whitney topology. Denote by $dist(\cdot,\cdot)$ the distance in M inherited by the Riemannian structure.

Given $X \in \mathfrak{X}^1_\mu(M)$ let $Sing(X)$ denote the set of *singularities* of X and $\mathcal{R} := M \setminus Sing(X)$ the set of *regular* points. Given $x \in M$, if there exists $\tau > 0$ such that $X^\tau(x) = x$ and τ is the minimum number with this property, then the *orbit of* x, denoted by $\mathcal{O}(x) := \cup_{t\in\mathbb{R}} X^t(x)$, is said to be *closed* or *periodic*.

2.2 Hyperbolicity for the Linear Poincaré Flow

The vector field $X\colon M \to TM$ induces a decomposition of the tangent bundle $T_{\mathcal{R}}M$ in a way that each fiber T_xM has a splitting $N_x \oplus \mathbb{R}X(x)$ where $N_x = \mathbb{R}X(x)^{\perp}$ is the normal 2-dimensional subbundle for $x \in \mathcal{R}$.

Consider the automorphism of vector bundles $DX^t : T_{\mathcal{R}}M \longrightarrow T_{\mathcal{R}}M$ such that we have $DX^t(x, v) = (X^t(x), DX^t(x) \cdot v)$ and $\Pi_{X^t(x)}$ the canonical projection on $N_{X^t(x)}$. The linear map $P_X^t(x)\colon N_x \longrightarrow N_{X^t(x)}$ defined by $P_X^t(x) = \Pi_{X^t(x)} \circ DX^t(x)$ is called the *linear Poincaré flow* at x associated to the vector field X. The map P_X^t is the differential of the standard *Poincaré map* $\mathcal{P}_X^t(x) : \mathcal{V}_x \subset \mathcal{N}_x \to \mathcal{N}_{X^t(x)}$, where $\mathcal{N}_{X^s(x)}$, for $s = 0, t$, is a surface contained in M whose tangent space at $X^s(x)$ is $N_{X^s(x)}$ for $s = 0, t$ and $\mathcal{V}_x$ is a small neighborhood of x. By using the implicit function theorem we can guarantee the existence of a continuous time-t arrival function $\tau(x, t)(\cdot)$ from $\mathcal{V}_x$ into $\mathcal{N}_{X^t(x)}$. Of course that, due to the presence of singularities, $\mathcal{V}_x$ may be very small.

Let Λ be a X^t-invariant subset of M. The splitting $N^1 \oplus N^2$ of the normal bundle N is an *m-hyperbolic splitting* for the linear Poincaré flow if it is P_X^t-invariant and there is a *uniform* $m \in \mathbb{N}$ such that for any point $x \in \Lambda$ the following inequalities hold:

$$\|P_X^{-m}(x)|_{N_x^1}\| \leq \frac{1}{2} \quad \text{and} \quad \|P_X^{m}(x)|_{N_x^2}\| \leq \frac{1}{2}. \tag{1}$$

2.3 Anosov Flows and Topological Restrictions on the Manifolds

A flow is said to be *Anosov* if the tangent bundle TM splits into three continuous DX^t-invariant nontrivial subbundles $E^0 \oplus E^1 \oplus E^2$ where E^0 is the flow direction, the sub-bundle E^2 is uniformly contracted by DX^t and the sub-bundle E^1 is uniformly contracted by DX^{-t} for all $t > 0$. Of course that, for an Anosov flow, we have $Sing(X) = \emptyset$ which follows from the fact that the dimensions of the subbundles are constant on the whole manifold. It is well-known that, on compact sets, the hyperbolicity for the linear Poincaré flow is equivalent to the hyperbolicity of the tangent map DX^t. Thus, to prove that a flow is Anosov it is sufficient to prove that M is hyperbolic for the linear Poincaré flow, i.e., (1) holds for all $x \in M$.

Since, in our context, the stable (or unstable) manifold is one-dimensional we can apply the results of Plante and Thurston (see [34]) to conclude that if M supports an Anosov flow, then its fundamental group $\pi_1(M)$ must have exponential growth.

In rough terms a finitely generated fundamental group $\pi_1(M)$ has *exponential growth*, if for any given system of generators of $\pi_1(M)$, the number of elements $\alpha \in \pi_1(M)$ that are represented by words of length at most n grows exponentially with n. More formally, we define the functions:

$$\Gamma_n = \{\text{distinct group elements which words have length} \leq n\},$$

where the length of the word is computed using the (finite number of) generators and

$$\Sigma_r = \{\text{homotopically distinct curves in } M \text{ of length} \leq r\}.$$

We say that a finitely generated group $\pi_1(M)$ has *exponential growth* if there exist constants $C, D > 0$ such that $\#\Gamma_n \geq Ce^{Dn}$. Equivalently, $\pi_1(M)$ has *exponential growth* if there exist constants $A, B > 0$ such that $\#\Sigma_r \geq Ae^{Br}$.

Remark 2.1 Given a compact manifold M which admits a metric of negative sectional curvature, then $\pi_1(M)$ has exponential growth (see [31, Theorem 2]). Surfaces of genus 2 or higher admit also exponential growth (see examples on [31, page 5]). See also Remark 4.1.

2.4 Lyapunov Exponents, Entropy and Chaoticity in the Metric Sense

The next result, due to Oseledets [32], is a cornerstone in smooth ergodic theory. We state here Oseledets' theorem for the linear Poincaré flow of 3-dimensional flows.

Theorem 2.1 (Oseledets) *Let $X \in \mathfrak{X}^1_\mu(M)$. For μ-a.e. $x \in M$ there exists the upper Lyapunov exponent $\lambda^+(X, x)$ defined by the limit $\lim_{t\to+\infty} \frac{1}{t} \log \|P^t_X(x)\|$ that is a non-negative measurable function of x. For μ-a.e. point x with a positive exponent there is a splitting of the normal bundle $N_x = N^u_x \oplus N^s_x$ which varies measurably with x and is such that:*

- *If $v \in N^u_x \setminus \{\vec{0}\}$, then $\lim_{t\to\pm\infty} \frac{1}{t} \log \|P^t_X(x) \cdot v\| = \lambda^+(X, x)$.*
- *If $v \in N^s_x \setminus \{\vec{0}\}$, then $\lim_{t\to\pm\infty} \frac{1}{t} \log \|P^t_X(x) \cdot v\| = -\lambda^+(X, x)$.*
- *If $\vec{0} \neq v \notin N^u_x, N^s_x$, then*

  (i) $\lim_{t\to+\infty} \frac{1}{t} \log \|P^t_X(x) \cdot v\| = \lambda^+(X, x)$ *and*
  (ii) $\lim_{t\to-\infty} \frac{1}{t} \log \|P^t_X(x) \cdot v\| = -\lambda^+(X, x)$.

Given $X \in \mathfrak{X}^1_\mu(M)$ the number $h_\mu(X)$ stands for the *metric entropy* (see [28] for a detailed exposition on this concept) of X and is defined by $h_\mu(X^1)$, where X^1 is the time-one of its associated flow. By Abramov's formula [3] we know that the metric entropy of the time-t map X^t is $|t|h_\mu(X^1)$ for any $t \in \mathbb{R}$.

Definition 2.1 A flow X^t is said to be **chaotic in the measure-theoretic sense** if $h_\mu(X) > 0$.

2.5 Devaney's Definition of Chaos

The *forward orbit of x* is defined by $\mathcal{O}^+(x) = \cup_{t>0} X^t(x)$ and we say that X^t has a *dense orbit* if, for some $x \in M$, we have $M = \overline{\cup_{t>0} X^t(x)}$, where $\overline{A}$ stands for the closure of the set A. In this case we say that the flow X^t is *transitive*. An equivalent definition for a transitive flow is the following: given any nonempty open sets $U, V \subseteq M$, there exists $\tau > 0$ such that $X^\tau(U) \cap V \neq \emptyset$. Now we consider a less general definition. We say that a flow X^t is *topologically mixing* if given any nonempty open sets $U, V \subseteq M$, there exists $\tau > 0$ such that, for all $t \geq \tau$ we have $X^\tau(U) \cap V \neq \emptyset$.

We recall the classic definition of chaos due to Devaney [23] and here we adapted it to the continuous-time context.

Definition 2.2 A flow X^t is said to be **chaotic in the sense of Devaney** if:

(a) X^t is transitive;
(b) the closed orbits are dense in the whole manifold and
(c) X^t is *sensitive to the initial conditions*, i.e., there exists $\delta > 0$ such that for all $x \in M$ and all neighborhood of x, V_x, there exists $y \in V_x$ and $t > 0$ where $d(X^t(y), X^t(x)) > \delta$.

In this case we also say that X^t is *chaotic in the topological sense*. If we switch (a) by "X^t is topologically mixing" then we say that X^t is *strongly chaotic in the topological sense* or X^t exhibits *stronger Devaney chaos*.

It was proved in [10] that condition (c) follows from conditions (a) and (b), and so, in order to be (strongly) chaotic in the sense of Devaney, the system only has to satisfy the (topologically mixing) transitivity property and the density of closed orbits.

2.6 Examples

Example 1 (Volume-preserving C^2 Anosov flows) Let X^t be a volume-preserving C^2 Anosov flow. Recall that, in [4], Anosov proved that the set of closed orbits of an Anosov flow is dense in the non-wandering set. Moreover, by Poincaré's recurrence theorem the non-wandering set equals the whole manifold. Hence, condition (b) in Definition 2.2 is true. We know that there exists non-transitive Anosov flows (see e.g. [25]). However, also in [4], its is proved that, within the volume-preserving class, the Anosov flows are ergodic, thus transitive.[1] Hence, volume-preserving C^2 Anosov flows are chaotic in the topological sense. Observe also that they form an open class. Since a volume-preserving 3-dimensional flow is Anosov if and only if it is structurally stable (see [14, Theorem 1.3]) their metric entropy is locally constant. Since, by Pesin's formula and ergodicity, the entropy equals the positive Lyapunov exponents we get that these flows are chaotic in the measure-theoretic sense.

Example 2 (Suspension flows) Given a measure space Σ, a map $f : \Sigma \to \Sigma$ and a *ceiling function* $h : \Sigma \to \mathbb{R}^+$ satisfying $h(x) \geq \beta > 0$ for all $x \in \Sigma$ we consider the space $M_h \subseteq \Sigma \times \mathbb{R}^+$ defined by

$$M_h = \{(x, t) \in \Sigma \times \mathbb{R}^+ : 0 \leq t \leq h(x)\}$$

with the identification between the pairs $(x, h(x))$ and $(f(x), 0)$. The semiflow defined on M_h by $S^s(x, r) = (f^n(x), r + s - \sum_{i=0}^{n-1} h(f^i(x)))$, where $n \in \mathbb{N}_0$ is uniquely defined by

$$\sum_{i=0}^{n-1} h(f^i(x)) \leq r + s < \sum_{i=0}^{n} h(f^i(x))$$

is called a *suspension semiflow*. Actually, if f is invertible, then $(S^t)_{t \in \mathbb{R}}$ is a flow.

[1] We observe if a measure that gives positive measure to non-empty open sets is ergodic, then the system is transitive.

If we choose $f(x) = 1$, then the suspension flow cannot be topologically mixing. To see this just observe that the integer iterates of $\Sigma \times (0, 1/2)$ are disjoint from $\Sigma \times (1/2, 1)$. However, our choice for f is very restrict and generically we obtain that suspension flows are topologically mixing. A suspension (with a generic ceiling bounded function) over an Anosov area-preserving diffeomorphism is strongly chaotic in the topological sense. Moreover, the chaoticity in the measure-theoretical sense is direct (see e.g. [16, §1.3]).

It is interesting also to point out that the suspension flow of the Anosov linear automorphism with matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and with constant ceiling function equal to 1 is not defined on a 3-torus. Moreover, the quotiented manifold where the suspension evolves has, in fact, a fundamental group with exponential growth.[2]

Example 3 (Incompressible flows with positive Lyapunov exponents) The next construction, suggested to me by Jairo Bochi, allows us to obtain that C^1-dense incompressible three-dimensional flows have subsets with positive Lebesgue measure with nonzero Lyapunov exponents (except, of course, the flow-direction). See also [21].

By, [12], we know that C^1-denselly we have an Anosov flow or else we have dense elliptic closed orbits. If the flow is Anosov (Example 1), then a.e. point in M has nonzero exponents. Otherwise, we consider an elliptic closed orbit p with period π and using the pasting lemma [6] we can perturb it in order to obtain that $P_X^{\pi}(p)$ is a rotation with rational angle and the action of the vector field in linear. Clearly $P_X^{\ell\pi}(p) = id$ for some $\ell \in \mathbb{N}$ and Abramov's formula [3] the metric entropy of the time-t map P_X^t is $|t| h_\mu(P_X^1)$ for any $t \in \mathbb{R}$. Now, we glue, inside the small invariant disc in the transversal section the Hu–Pesin–Talitskaya continuous-time Katok's map (cf. [26]) which is a small perturbation diffeotopic to the identity map.

2.7 Two-Dimensional Area-Preserving Flows

For area-preserving flows on certain surfaces the scenario is quite well understood (see the book [30]). For example, the classification of the limit set of regular orbits of divergence-free vector fields in the two-sphere is given by the celebrated Poincaré–Bendixson theorem ([30, Theorem 1.1.5.]). This type of strong results can only be achieved due to topological arguments typical of the two-dimensional case. We also point out the result by Ulcigrai [37] where it is proved that typical (in the measure-theoretical sense) area-preserving flows are not mixing (also in the measure-theoretical sense).

[2] Recall that, by Plante and Thurston theorem (see [34]), we know that Anosov flows on three-manifolds are necessarily defined on manifolds with fundamental group displaying exponential growth (see also the Appendix by Margulis, *Y-flows on three-dimensional manifolds* on [5]).

3 Generic Results for Flows

In this section we give a brief description of some C^1-generic results to be used in the sequel. Since it is not reasonable to give a complete description of *all* dynamical systems we usually look for properties which hold for open and dense subsets of all the systems, or, properties which hold for residual subsets. Residual subsets are sets that contain a countable intersection of open and dense sets (i.e. Baire's second category sets). Properties that hold residually are *typical* from a topological viewpoint. The set $\mathfrak{X}^1_\mu(M)$ endowed with the C^1 topology is a Baire space, hence C^1-residual subsets are dense.

In [11] was proved that there exists a C^1-residual subset of the set $\mathfrak{X}^1_\mu(M)$ (but *without singularities*), such that any vector field in this residual is Anosov or else, for Lebesgue almost every point, its Lyapunov exponents are equal to zero. In [7] was obtained the same statement for $\mathfrak{X}^1_\mu(M)$. Recently, in [16], was proved that, for the 3-dimensional setting, there exists a C^1-residual set $\mathcal{R} \subset \mathfrak{X}^1_\mu(M)$ such that any vector field $X \in \mathcal{R}$, satisfy the Pesin's entropy formula, that is the metric entropy is equal to the integral, over M, of the positive Lyapunov exponent.

It is a consequence of the celebrated C^1-closing lemma (see [35] the version for volume-preserving flows) that, for a C^1-residual subset of $\mathfrak{X}^1_\mu(M)$, the non-wandering set is equal to the closure of the closed orbits. This result is known as the Pugh–Robinson's general density theorem.

Finally, inspired by [1] and [17], it was obtained in [13] that there exists a C^1-residual subset of the set $\mathcal{R} \subset \mathfrak{X}^1_\mu(M)$ such that any vector field inside this residual is topologically mixing.

We observe that all the results stated above hold for the setting of volume-preserving diffeomorphisms.

4 Rare Coexistence of Different Definitions of Chaos

Let us now prove the following result.

Theorem 1 *There exists a residual $\mathcal{R} \subset \mathfrak{X}^1_\mu(M)$ such that, if $\pi_1(M)$ does not have exponential growth, then any $X \in \mathcal{R}$*

(a) *has zero metric entropy;*
(b) *has zero Lyapunov exponents and*
(c) *is strongly chaotic in Devaney's sense.*

Proof Let $\mathcal{R}_1$ be the residual subset of $\mathfrak{X}^1_\mu(M)$ formed by those vector fields such that, if $X \in \mathcal{R}_1$, then X is Anosov or else Lebesgue almost every point has zero Lyapunov exponent (cf. [11,7]). Since $\pi_1(M)$ does not have exponential growth we conclude that M cannot support Anosov flows and so Lebesgue almost every point in M has zero Lyapunov exponents.

We use [16] and pick $\mathcal{R}_2 \subset \mathfrak{X}^1_\mu(M)$ defined by the residual set of vector fields such that Pesin's entropy formula holds, i.e.,

$$h_\mu(X) = \int_M \lambda^+(X, x)\, d\mu(x),$$

for any $X \in \mathcal{R}_2$. Of course that if $X \in \mathcal{R}_1 \cap \mathcal{R}_2$, then conditions (a) and (b) of the theorem hold (recall that the intersection of residual sets is itself a residual).

Moreover, by Pugh-Robinson's general density theorem (see [35]), we get that there exists a residual subset $\mathcal{R}_3 \subset \mathfrak{X}^1_\mu(M)$ such that if $X \in \mathcal{R}_3$, then the closed orbits of X are dense in the nonwandering set, hence in the whole manifold M.

Now, it was proved in [13], that there exists a residual $\mathcal{R}_4 \subset \mathfrak{X}^1_\mu(M)$ such that any $X \in \mathcal{R}_4$ is topologically mixing.

Finally, using [10] we conclude that any $X \in \mathcal{R}_3 \cap \mathcal{R}_4$ is sensitive to the initial conditions, thus strongly chaotic in Devaney's sense.

The theorem is proved once we define

$$\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \cap \mathcal{R}_4.$$

□

Remark 4.1 We observe that manifolds like the 3-tori and the 3-spheres are in the hypotheses of Theorem 1 (see examples on [31, page 4]). Confront with the suspension flow of the linear Anosov automorphism in Example 2.

Remark 4.2 It was proved in [6] that a volume-preserving flow on three-manifolds is robust transitive if and only of it is an Anosov flow. Thence, the property (c) in Theorem 1 is not C^1-stable. Therefore, if a flow is strongly chaotic in Devaney's sense and, moreover, any flow C^1-close to it is also strongly chaotic in Devaney's sense, then the flow must be Anosov. Thus, this flow cannot live in a manifold M where $\pi_1(M)$ does not have exponential growth.

We recall the definition of chaos in the sense of Auslander and Yorke [8]: the dynamical system must be transitive and sensitive to the initial conditions. We easily obtain the following result.

Proposition 4.1 *There exists a residual subset of $\mathcal{R} \subset \mathfrak{X}^1_\mu(M)$ where the definitions of chaotic in the sense of Auslander and Yorke and in the sense of Devaney coincide.*

Proof The proof is straightforward. Just consider the residual subset $\mathcal{R}$ obtained by the intersection of both residual subsets given by [13] and [35]. Any vector field in $\mathcal{R}$ is transitive (actually topologically mixing) and the closed orbits are dense in M. Once again we recall that in [10] we obtain that condition (c), of the definition chaoticity in the sence of Devaney, follows from conditions (a) and (b). The proposition is proved. □

5 Towards to Generalizations and Some Open Questions

5.1 Volume-Preserving Flows on 4-Dimensional Manifolds

We start by understanding how would the corresponding statement could be for conservative flows defined in 4-dimensional manifolds. First, we observe that Pugh-Robinson's general density theorem is true for higher-dimensions. The result in [10] is abstract and also valid regardless of the dimension. Second, due to recent results by Sun and Tian (see [36]) the result in [16] can be extended to the n-dimensional flow setting.

Proposition 5.1 *Pesin's entropy formula holds for C^1-generic volume-preserving flows in n-dimensional manifolds ($n \geq 4$).*

We make an interlude to introduce the definition of dominated splitting. Take a X^t-invariant set Λ and fix $m \in \mathbb{N}$. A nontrivial P_X^t-invariant and continuous splitting $N_\Lambda = U_\Lambda \oplus S_\Lambda$ is said to have an *m-dominated splitting* for the linear Poincaré flow of X over Λ if the following inequality holds for every $x \in \Lambda$:

$$\frac{\|P_X^m(x)|_{S_x}\|}{\mathfrak{m}\left(P_X^m(x)|_{U_x}\right)} \leq \frac{1}{2}, \tag{2}$$

where $\mathfrak{m}$ stands for the co-norm of the operator, i.e., $\mathfrak{m}(A) = \|A^{-1}\|^{-1}$.

Now, with respect to the Anosov *versus* zero Lyapunov exponents dichotomy in [11,7], the best we have for n-dimensional volume-preserving flows is the (non-global) result in [15]. In that paper it is proved that there exists a residual subset of n-dimensional volume-preserving flows ($n \geq 4$) such that for any element in this residual we have, for almost every point x in the manifold, that x has zero Lyapunov exponents or else the orbit of x is dominated. Unfortunately, these two properties may coexist and the whole manifold may be decomposed in regions with zero Lyapunov exponents and regions with dominated splitting. Even worst, the constant m associated to the domination may vary from orbit to orbit.

Nevertheless, the biggest challenge for the extension of the Theorem 1 is not the difficulty described in the last paragraph. In fact, we might even assume the most favorable circumstances, i.e., there exists a global dichotomy (zero exponents or else dominated splitting in M). The problem is that there is a total lack of knowledge about the topological constraints on the manifolds if we assume that some flow has a dominated splitting over M. Below we will return to this issue (Questions 2 and 3).

We say that $X \in \mathfrak{X}_\mu^1(M)$ is *nonuniformly Anosov* (adapting the definition in [9, pp. 4]) if the system is nonuniformly hyperbolic (all Lyapunov exponents are different from zero) and with a global (i.e. over M) dominated splitting separating the positive exponents from the negative ones. Let $\mathcal{A}(M) \subset \mathfrak{X}_\mu^1(M)$ stands for the subset of nonuniformly Anosov and ergodic volume-preserving vector fields and by $\overline{\mathcal{A}(M)}$ its C^1-closure.

Recently (see [27]), it was proved a conjecture stated in [9, Conjecture p. 2887], namely that C^1-generically 3-dimensional volume-preserving diffeomorphisms have zero Lyapunov exponents at Lebesgue almost every point or else the system is nonuniformly Anosov and ergodic (the definitions are the analogous obvious couterpart for the discrete case).

To obtain a correspondent version for volume-preserving flows there is a non-trivial extra work to do and related to this we present the following question.

Question 1 Given a 4-dimensional manifold M, is there a residual $\mathcal{R} \subset \mathfrak{X}_\mu^1(M)$ such that any $X \in \mathcal{R}$ is in $\mathcal{A}(M)$ or else Lebesgue almost every point has zero Lyapunov exponents?

We say that a flow in M is (uniformly) *partially hyperbolic for the linear Poincaré flow* if there exists an P_t^X-invariant dominated splitting $N = N^u \oplus N^c \oplus N^s$ in M such that N^u is hyperbolic expanding, N^s is hyperbolic contracting, N^u dominates $N^c \oplus N^s$ and N^s is dominated by $N^u \oplus N^c$.

Although there are know some deep results about the topological constraints on the manifolds which support partial hyperbolic diffeomorphisms [19,20,33] nothing is known when we refer to the continuous-time counterpart. To be more precise, we may ask:

Question 2 What are the topological obstructions on a closed 4-dimensional manifold if it supports some partially hyperbolic (volume-preserving) flow?[3]

Question 3 What are the topological obstructions on a closed 4-dimensional manifold if it supports some flow in $\mathcal{A}(M)$?[4]

If we answer positively to Question 1, then, using also Proposition 5.1, we have proved the following:

Conjecture 1 Let M be a closed Riemannian smooth 4-dimensional manifold. There exists a residual $\mathcal{R} \subset \mathfrak{X}^1_\mu(M) \setminus \overline{\mathcal{A}(M)}$ such that any $X \in \mathcal{R}$,

(a) has zero metric entropy;
(b) has zero Lyapunov exponents and
(c) is strongly chaotic in Devaney's sense.

5.2 Volume-Preserving Diffeomorphisms on 3-Dimensional Manifolds

Let us denote by $\text{Diff}^1_\mu(M)$ the set of C^1 volume-preserving diffeomorphisms defined in a 3-dimensional manifold M. Observe that $\text{Diff}^1_\mu(M)$ endowed with the C^1-topology is a Baire space.

We say that a diffeomorphism $f: M \to M$ is (uniformly) *partially hyperbolic* if there exists an Df-invariant dominated splitting $TM = E^u \oplus E^c \oplus E^s$ in M such that E^u is hyperbolic expanding, E^s is hyperbolic contracting, E^u dominates $E^c \oplus E^s$ and E^s is dominated by $E^u \oplus E^c$.

Remark 5.1 If there exists a dominated splitting $E^1 \oplus E^2 \oplus E^3$ over M, (where each E^i is, of course, 1-dimensional), such that E^1 dominates E^2, and E^2 dominates E^3. Then, using [18, Lemma 7.10], we conclude that the splitting $E^1 \oplus E^2 \oplus E^3$ is partially hyperbolic, i.e., E^1 is uniformly expanding and E^3 is uniformly contracting.

Theorem 2 *There exists a residual $\mathcal{R} \subset \text{Diff}^1_\mu(M)$ such that, if the universal cover of M is not homeomorphic to $\mathbb{R}^3$, then any $f \in \mathcal{R}$*

[3] Since, partially hyperbolic flows cannot have singularities [38], one obvious conclusion is that the Euler characteristic of M is equal to zero.

[4] The restrictions should come from the dominated splitting hypothesis instead of the nonuniformly property because it is well-known that, due to Hu–Pesin–Talitskaya's theorem [26], any compact manifold supports a nonuniformly hyperbolic flow (eventually without any domination).

(a) *has zero metric entropy;*
(b) *has zero Lyapunov exponents and*
(c) *is strongly chaotic in Devaney's sense.*

Proof Let $\mathcal{R}_1$ be the residual subset of $\text{Diff}^1_\mu(M)$ formed by those diffeomorphisms such that, if $f \in \mathcal{R}_1$, then f is nonuniformly Anosov (and ergodic) or else Lebesgue almost every point has zero Lyapunov exponent (cf. [27]). Since the universal cover of M is not homeomorphic to $\mathbb{R}^3$ we conclude, using [33, Theorem 1.14], that M cannot support partially hyperbolic diffeomorphisms and so Lebesgue almost every point in M has zero Lyapunov exponents.

Using [36] we consider the residual subset $\mathcal{R}_2 \subset \text{Diff}^1_\mu(M)$ such that Pesin's entropy formula holds and, for any $f \in \mathcal{R}_2$, we have $h_\mu(f) = 0$. Then, for $f \in \mathcal{R}_1 \cap \mathcal{R}_2$, the conditions (a) and (b) of the theorem hold.

By Pugh-Robinson's general density theorem (see [35]), we get that there exists a residual subset $\mathcal{R}_3 \subset \text{Diff}^1_\mu(M)$ such that if $f \in \mathcal{R}_3$, then the periodic orbits of f are dense in the whole manifold M.

Now, we use [2], and obtain that there exists a residual $\mathcal{R}_4 \subset \text{Diff}^1_\mu(M)$ such that any $f \in \mathcal{R}_4$ is topologically mixing.

Finally, using [10] we conclude that any $f \in \mathcal{R}_3 \cap \mathcal{R}_4$ is sensitive to the initial conditions, thus chaotic in Devaney's sense. We define $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \cap \mathcal{R}_4$ and the proof is completed. □

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