EXPLOSION OF DIFFERENTIABILITY FOR EQUIVALENCIES BETWEEN ANOSOV FLOWS ON 3-MANIFOLDS

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ABSTRACT. For Anosov flows obtained by suspensions of Anosov diffeomorphisms on surfaces, we show the following type of rigidity result: if a topological conjugacy between them is differentiable at a point, then the conjugacy has a smooth extension to the suspended 3-manifold. This result generalizes the similar ones of Sullivan and Ferreira-Pinto for 1-dimensional expanding dynamics and also a result of Ferreira-Pinto for 2-dimensional hyperbolic dynamics.

1. INTRODUCTION, PRELIMINARY DEFINITIONS AND STATEMENT OF THE RESULTS

1.1. **Introduction.** There is an established theory in hyperbolic dynamics that studies properties of the dynamics and of the topological conjugacies that lead to additional regularity for the conjugacies. In the early seventies Mostow (see [18]) proved that if $\mathbb{H}/\Gamma_X$ and $\mathbb{H}/\Gamma_Y$ are two closed hyperbolic Riemann surfaces covered by finitely generated Fuchsian groups Γ_X and Γ_Y of finite analytic type, and $\phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ induces the isomorphism $i(\gamma) = \phi \circ \gamma \circ \phi^{-1}$, then ϕ is a Möbius transformation if, and only if, ϕ is absolutely continuous. Later, in [21], Shub and Sullivan proved that for any two analytic orientation preserving circle expanding endomorphisms f and g of the same degree, the conjugacy is analytic if, and only if, the conjugacy is absolutely continuous. Furthermore, they proved that if f and g have the same set of eigenvalues, then the conjugacy is analytic. After these results, de la Llave [13] and Marco and Moriyón [15, 16] proved that if Anosov diffeomorphisms have the same set of eigenvalues, then the conjugacy is smooth. For maps with critical points, Lyubich (see [14]) proved that C^2 unimodal maps

with Fibonacci combinatorics and the same eigenvalues are C^1 conjugate. Later on, de Melo and Martens [17] proved that if topological conjugate unimodal maps, whose attractors are cycles of intervals, have the same set of eigenvalues, then the conjugacy is smooth. More recently, Dobbs (see [3]) proved that if a multimodal map f has an absolutely continuous invariant measure, with a positive Lyapunov exponent, and f is absolutely continuous conjugate to another multimodal map, then the conjugacy is C^r in the domain of some induced Markov map of f.

In the present paper we study the explosion of smoothness for topological conjugacies, i.e. the conditions under which the smoothness of the conjugacy in a single point extends to the whole manifold. Tukia, in [24], extended the aforementioned result of Mostow proving that if $\mathbb{H}/\Gamma_X$ and $\mathbb{H}/\Gamma_Y$ are two closed hyperbolic Riemann surfaces covered by finitely generated Fuchsian groups Γ_X and Γ_Y of finite analytic type, and $\phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ induces the isomorphism $i(\gamma) = \phi \circ \gamma \circ \phi^{-1}$, then ϕ is a Möbius transformation if, and only if, ϕ is differentiable at one radial limit point with non-zero derivative. Sullivan [23] proved that if a topological conjugacy between analytic orientation preserving circle expanding endomorphisms of the same degree is differentiable at a point with non-zero derivative, then the conjugacy is analytic. Extensions of these results for Markov maps and hyperbolic basic sets on surfaces were developed by de Faria [4], Jiang [11, 12] and Pinto, Rand and Ferreira [6, 20], among others. For maps with critical points, Jiang [7–10] proved that quasi-hyperbolic 1-dimensional maps are smooth conjugated in an open set with full Lebesgue measure if the conjugacy is differentiable at a point with uniform bound. Very recently (see [1]), Alves, Pinheiro and Pinto proved that if a topological conjugacy between multimodal maps is C^1 at a point in the nearby expanding set of f, then the conjugacy is a smooth diffeomorphism in the basin of attraction of a renormalization interval.

In the present work we begin the generalization of these types of results for continuous-time dynamical systems by proving the corresponding result for Anosov flows obtained by suspensions of Anosov diffeomorphisms on surfaces. More precisely, we prove that if a topological conjugacy between two Anosov flows, obtained from the suspension of Anosov maps in surfaces, is differentiable at a point, then the conjugacy has a smooth extension to the suspended 3-manifold.

1.2. Statement of the results.
Let M be a d-dimensional closed and connected C^∞ Riemannian manifold. In this paper $d = 2$ when we consider diffeomorphisms and $d = 3$ when considering vector fields/flows. Any C^1 vector field $X: M \to TM$ can be integrated into a flow $X_t: M \to M$ which is a time-parameter group of diffeomorphisms. A flow is said to be *Anosov* if the tangent bundle TM splits into three continuous DX_t-invariant non-trivial subbundles $E^0 \oplus E^u \oplus E^s$ where E^0 is the flow direction, the subbundle E^s is uniformly contracted by DX_t and the subbundle E^u is uniformly contracted by DX_{-t} for all $t > 0$. Of course, for an Anosov flow, we have $Sing(X) = \emptyset$ which follows from the fact that the dimensions of the subbundles are constant on the whole manifold. The first example was obtained studying the geodesic flow of surfaces with negative curvature (cf. [2]). Anosov systems for discrete dynamical systems are defined in an analogous way and the prototypical example is given by hyperbolic linear automorphisms of tori.

Our main result is the following (see §2 for detailed definitions):

Theorem 1. *Let $f : M \longrightarrow M$ and $g : N \longrightarrow N$ be two C^∞ surface Anosov diffeomorphisms. Assume that there exists a topological conjugacy $h : M \longrightarrow N$*

between them and, moreover, h is differentiable in a single point. Let c_f and c_g be two ceiling functions over M and N, respectively. If c_f and c_g are differentiable, then the function $\hat{h} : M_{c_f} \longrightarrow N_{c_g}$ defined in (2.2) is differentiable.

In [19] Plante showed that codimension-1 Anosov flows on compact and connected manifolds M are known to admit global cross sections, provided the fundamental group of M is solvable. Clearly, Anosov flows on 3-dimensional manifolds are codimensional. In [19] it is also obtained that when M is a bundle over $\mathbb{S}^1$ fibered by a 2-torus $\mathbb{T}^2$, then any Anosov flow on M is topologically equivalent to the suspension of a hyperbolic automorphism on $\mathbb{T}^2$.

As a direct consequence of [6] and Theorem 1 we obtain:

Corollary 1. *Let $X_t : M \to M$ and $Y_t : M \to M$ be Anosov flows on a closed 3-manifold M which is a bundle over $\mathbb{S}^1$ with fiber bundle $\mathbb{T}^2$ and denote $M = \mathbb{S}^1 \times \mathbb{T}^2$. If $h : M \to M$ is an equivalence between the two flows which is differentiable in $x \in \mathbb{T}^2$, then h is differentiable in the whole M.*

2. Proof of Theorem 1

2.1. Suspension flows.

Let $f : M \longrightarrow M$ be a diffeomorphism and $c : M \longrightarrow \mathbb{R}^+$ a continuous function such that $c(x) \geq a > 0$ for all $x \in M$. We consider also the subspace of $M \times \mathbb{R}^+$ defined by:

$$\tilde{M} = \{(x, t) \in M \times \mathbb{R}^+ : x \in M \ , \ 0 \leq t \leq c(x)\}.$$

Let M_c stand for the quotient space $M_c = \tilde{M}/\sim$, where $\sim$ is an equivalence relation in $\tilde{M}$ defined by $(x, c(x)) \sim (f(x), 0)$. The *suspension of f* with ceiling (or roof) function c is the flow

$$\varphi_t : M_c \longrightarrow M_c$$
$$(x, s) \longmapsto \left(f^n(x), s'\right)$$

where n is univocally determined by

$$(2.1) \qquad \sum_{i=0}^{n-1} c(f^i(x)) \leq t + s < \sum_{i=0}^{n} c(f^i(x))$$

when $t + s \geq c(x)$; in this case we define $s' = s + t - \sum_{i=0}^{n-1} c(f^i(x))$. If $t + s < c(x)$, we take $n = 0$ and $s' = s + t$. In brief, we travel with velocity equal to one and along $\{x\} \times [0, c(x)]$; then we jump to $(f(x), 0)$ and travel through $\{f(x)\} \times [0, c(f(x))]$ and so on until we spend the time t. We observe that the flow is well defined because (2.1) implies

$$0 \leq \underbrace{s + t - \sum_{i=0}^{n-1} c(f^i(x))}_{s'} \leq c(f^n(x)).$$

Remark 2.1. Let us show that φ defined above is a flow; $\varphi_0(x, s) = (x, s)$, since $n = 0$. For the group property we have:

$$\varphi_t\big(\varphi_r(x, s)\big) = \varphi_t\left(f^n(x), r + s - \sum_{i=0}^{n-1} c\big(f^i(x)\big)\right)$$

$$= \left(f^m\big(f^n(x)\big), t + r + s - \sum_{i=0}^{n-1} c\big(f^i(x)\big) - \sum_{i=0}^{m-1} c\big(f^i(f^n(x))\big)\right)$$

$$= \left(f^{n+m}(x), r + s + t - \sum_{i=0}^{n+m-1} c\big(f^i(x)\big)\right)$$

$$= \varphi_{t+r}(x, s)$$

where n and m are such that

$$\sum_{i=0}^{n-1} c\big(f^i(x)\big)) \leq r + s < \sum_{i=0}^{n} c\big(f^i(x)\big) \text{ and}$$

$$\sum_{i=0}^{m-1} c\big(f^i(f^n(x))\big) \leq r + s + t - \sum_{i=0}^{n-1} c\big(f^i(x)\big) < \sum_{i=0}^{m} c\big(f^i(f^n(x))\big).$$

The last inequality follows from

$$\sum_{i=0}^{m-1} c\big(f^i(f^n(x))\big) \leq r + s + t - \sum_{i=0}^{n-1} c\big(f^i(x)\big) < \sum_{i=0}^{m} c\big(f^i(f^n(x))\big)$$

$$\Downarrow$$

$$\sum_{i=0}^{n+m-1} c\big(f^i(x)\big) \leq r + s + t < \sum_{i=0}^{n+m} c\big(f^i(x)\big).$$

2.2. Topological equivalences and conjugacies.

Two flows $\varphi : \mathbb{R} \times M \longrightarrow M$ and $\psi : \mathbb{R} \times N \longrightarrow N$ are said to be *topologically equivalent* if there exists a homeomorphism $h : M \longrightarrow N$ such that h sends orbits of φ into orbits of ψ, and preserves the orientation. Two flows φ and ψ are said to be *topologically conjugated* if there exists a homeomorphism $h : M \longrightarrow N$ sending orbits of φ into orbits of ψ, preserving the orientation and also the time parametrization. Clearly, if φ and ψ are conjugated, then they are also equivalent, just take $\tau_x(t) = t$.

Let there be given two diffeomorphisms $f : M \longrightarrow M$ and $g : N \longrightarrow N$. We consider the suspensions of f and g associated to ceiling functions $c_f : M \longrightarrow \mathbb{R}^+$ and $c_g : N \longrightarrow \mathbb{R}^+$, respectively. Let φ_t and ψ_t be the respective suspension flows. Assume that f and g are topologically conjugated, i.e., there exists a homeomorphism $h : M \longrightarrow N$ such that $g \circ h(x) = h \circ f(x)$ for all $x \in M$.

A natural question is to ask if φ_t and ψ_t are still topologically conjugated. The answer, in general, is negative. However, we will see that the flows are topologically equivalent. In order to prove it we must define a homeomorphism $\hat{h} : M_{c_f} \longrightarrow N_{c_g}$

which conjugates both flows and preserves the fixed orientation. We define $\hat{h}$: $M_{c_f} \longrightarrow N_{c_g}$ in the following way:

$$(2.2) \qquad \hat{h}(x,s) = \psi_{s\frac{c_g(h(x))}{c_f(x)}} \, h \circ \underbrace{\underbrace{\underbrace{\varphi_{-s}(x,s)}_{(x,0)}}_{(h(x),0)}}_{\left(h(x),s\frac{c_g(h(x))}{c_f(x)}\right)}$$

where we consider that $h : M \longrightarrow N$ extends to $h : M \times \{0\} \longrightarrow N \times \{0\}$ and we abuse and keep the same notation. Roughly, we travel by $\{x\} \times [0, c_f(x)]$ until $(x, 0)$, apply h and travel along the segment $\{h(x)\} \times [0, c_g(h(x))]$ the corresponding time. Given $s, t \in \mathbb{R}_0^+$, we consider the following function:

$$n_{s,t} : M \longrightarrow \mathbb{N}_0$$
$$x \longmapsto n_{s,t}(x)$$

where $n_{s,t}(x)$ is the only integer such that

$$(2.3) \qquad \sum_{i=0}^{n_{s,t}(x)-1} c_f\big(f^i(x)\big) \leq t + s < \sum_{i=0}^{n_{s,t}(x)} c_f\big(f^i(x)\big)$$

or $n_{s,t}(x) = 0$, when $s + t < c_f(x)$. The map $t \mapsto n_{s,t}(x)$ is piecewise constant and increasing, for s and x fixed. We would like to find $\tau_{(x,s)} : \mathbb{R} \longrightarrow \mathbb{R}$ strictly increasing such that, for all $t \in \mathbb{R}$,

$$(2.4) \qquad \hat{h}\big(\varphi_t(x,s)\big) = \psi_{\tau_{(x,s)}(t)}\big(\hat{h}(x,s)\big).$$

Let $t' = \tau_{(x,s)}(t)$. The equation (2.4) is equivalent to

$$\hat{h}\left(f^{n_{s,t}(x)}(x), s + t - \sum_{i=0}^{n_{s,t}(x)-1} c_f\left(f^i(x)\right)\right) = \psi_{t'}\left(h(x), s\frac{c_g(h(x))}{c_f(x)}\right)$$

$$\Updownarrow$$

$$\left(\underbrace{h\left(f^{n_{s,t}(x)}(x)\right)}_{g^{n_{s,t}(x)}(h(x))}, \left[s + t - \sum_{i=0}^{n_{s,t}(x)-1} c_f\left(f^i(x)\right)\right]\frac{c_g\left(h\left(f^{n_{s,t}(x)}(x)\right)\right)}{c_f\left(f^{n_{s,t}(x)}(x)\right)}\right)$$
$$= \psi_{t'}\left(h(x), s\frac{c_g(h(x))}{c_f(x)}\right).$$

The equality holds if and only if t' is such that

$$(2.5) \qquad \sum_{i=0}^{n_{s,t}(x)-1} c_g\big(g^i(h(x))\big) \leq t' + s\frac{c_g(h(x))}{c_f(x)} < \sum_{i=0}^{n_{s,t}(x)} c_g\big(g^i(h(x))\big)$$

and

$$t' + s\frac{c_g\left(h(x)\right)}{c_f(x)} - \sum_{i=0}^{n_{s,t}(x)-1} c_g\left(g^i\left(h(x)\right)\right)$$

$$= \left[s + t - \sum_{i=0}^{n_{s,t}(x)-1} c_f\left(f^i(x)\right)\right]\frac{c_g\left(h\left(f^{n_{s,t}(x)}(x)\right)\right)}{c_f\left(f^{n_{s,t}(x)}(x)\right)}.$$

From the previous equality it follows that

$$\tau_{(x,s)}(t) = \left[s + t - \sum_{i=0}^{n_{s,t}(x)-1} c_f\left(f^i(x)\right)\right]\frac{c_g\left(h\left(f^{n_{s,t}(x)}(x)\right)\right)}{c_f\left(f^{n_{s,t}(x)}(x)\right)}$$

$$- s\frac{c_g\left(h(x)\right)}{c_f(x)} + \sum_{i=0}^{n_{s,t}(x)-1} c_g\left(g^i\left(h(x)\right)\right).$$

From (2.3), we get

$$0 \le s + t - \sum_{i=0}^{n_{s,t}(x)-1} c_f\left(f^i(x)\right) < c_f\left(f^{n_{s,t}(x)}(x)\right),$$

thence

$$0 \le \frac{c_g\left(h\left(f^{n_{s,t}(x)}(x)\right)\right)}{c_f\left(f^{n_{s,t}(x)}(x)\right)}\left[s + t - \sum_{i=0}^{n_{s,t}(x)-1} c_f\left(f^i(x)\right)\right]$$

$$< c_g\left(h\left(f^{n_{s,t}(x)}(x)\right)\right) = c_g(g^{n_{s,t}(x)}\left(h(x)\right)).$$

Overall, $\sum_{i=0}^{n_{s,t}(x)-1} c_g\left(g^i\left(h(x)\right)\right)$ is less than or equal to

$$\underbrace{\frac{c_g\left(h\left(f^{n_{s,t}(x)}(x)\right)\right)}{c_f\left(f^{n_{s,t}(x)}(x)\right)}\left[s + t - \sum_{i=0}^{n_{s,t}(x)-1} c_f\left(f^i(x)\right)\right] + \sum_{i=0}^{n_{s,t}(x)-1} c_g\left(g^i\left(h(x)\right)\right)}_{t'+s\frac{c_g(h(x))}{c_f(x)}}$$

and this expression is less than $\sum_{i=0}^{n_{s,t}(x)} c_g\left(g^i\left(h(x)\right)\right)$, where the inequalities on (2.5) follow. Then,

$$n_{s\frac{c_g(h(x))}{c_f(x)},t'}\left(h(x)\right) = n_{s,t}(x),$$

like we wanted.

Clearly, τ is continuous and strictly increasing because for all $(x, s) \in M_{c_f}$ we have

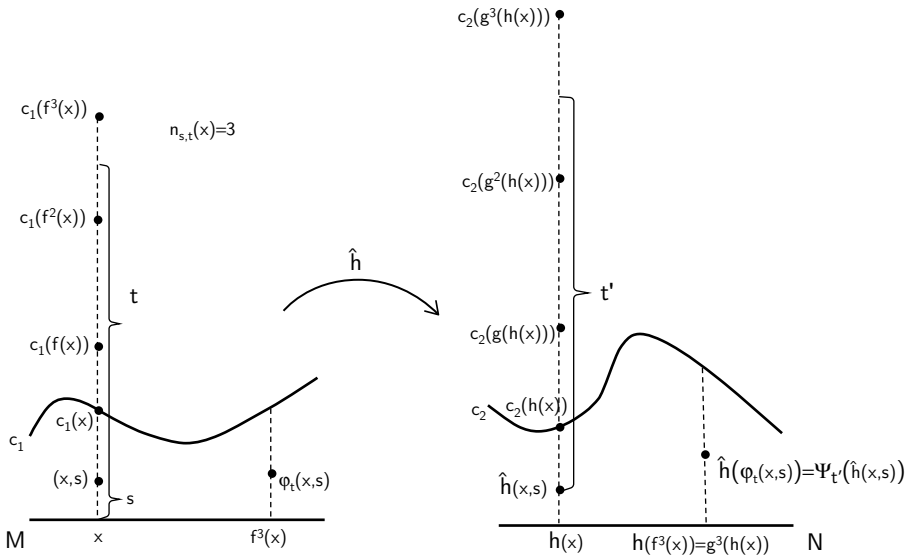$$\frac{d\tau_{(x,s)}(t)}{dt} = \frac{c_g\left(h\left(f^{n_{s,t}(x)}(x)\right)\right)}{c_f\left(f^{n_{s,t}(x)}(x)\right)} > 0.$$

FIGURE 1. Topological equivalence between suspension flows.

If $\hat{h} : M_{c_f} \longrightarrow N_{c_g}$ is a homeomorphism, then φ and ψ are topologically equivalent flows. In fact:

- $\hat{h}$ is continuous because it is the composition of continuous functions;
- $\hat{h}$ is invertible with inverse

$$\hat{h}^{-1}(y,t) := \varphi_{t \frac{c_f(h^{-1}(y))}{c_g(y)}} \circ h^{-1} \circ \psi_{-t}(y,t) = \left(h^{-1}(y), t \frac{c_f\left(h^{-1}(y)\right)}{c_g(y)} \right);$$

- $\hat{h}^{-1}$ is continuous.

In conclusion we have just proved the following:

Proposition 2.2. *The flows φ and ψ are topologically equivalent, $\hat{h}$ being the equivalency between them.*

We begin by obtaining a preliminary result on piecewise differentiability:

Proposition 2.3. *Let $f : M \longrightarrow M$ and $g : N \longrightarrow N$ be two C^∞ surface Anosov diffeomorphisms. Assume that there exists a topological conjugacy $h : M \longrightarrow N$ between them and, moreover, h is differentiable in a single point. Let c_f and c_g be two ceiling functions over M and N, respectively. If c_f and c_g are differentiable, then the function $\hat{h} : M_{c_f} \longrightarrow N_{c_g}$ described above is piecewise differentiable.*

Proof. The homeomorphism $\hat{h}$ is given by

$$\hat{h}(x,s) = \left(h(x), s \frac{c_g(h(x))}{c_f(x)} \right),$$

for all $(x,s) \in M_{c_f}$. Assume that $(x,s) \notin M \times \{0\}$. Then,

$$D\hat{h}_{(x,s)} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{Dh_x} & \mathbf{0} \\ \hline * & \diamond & \star \end{array} \right)$$

where

$$* = s\frac{\partial\left(\frac{c_g(h(x))}{c_f(x)}\right)}{\partial x_1}, \qquad \diamond = s\frac{\partial\left(\frac{c_g(h(x))}{c_f(x)}\right)}{\partial x_2} \qquad \text{and} \qquad \star = \frac{c_g\left(h(x)\right)}{c_f(x)},$$

being $x = (x_1, x_2)$. But, Dh_x exists by the main theorem in [6] $*$ and $\diamond$ exist since c_f and c_g are both differentiable. $\qquad\square$

The lack of differentiability in our construction lies in the way $\hat{h}$ acts in the sections M and N. Next, we carefully reparametrize $\hat{h}$ in order to achieve the differentiability of $\hat{h}$ in the whole suspension manifold. We begin by proving a useful and elementary result about bump functions.

Lemma 2.4. *Given $a < b$ and $c \in \mathbb{R}$, there is a C^∞ function $F_{a,b}^c : \mathbb{R} \longrightarrow \mathbb{R}$ such that $F_{a,b}^c(t) = 0$ for all $t \notin (a,b)$ and $\int_{-\infty}^{\infty} F_{a,b}^c(s)\,ds = c$.*

Proof. Consider the map

$$f_{a,b}(t) = \begin{cases} e^{\frac{-1}{(a-t)(b-t)}}, & t \in (a,b) \\ 0, & \text{otherwise} \end{cases}$$

and now define

$$F_{a,b}^c(t) = \frac{c \cdot f_{a,b}(t)}{\int_a^b f_{a,b}(s)\,ds}.$$

Then, $F_{a,b}^c$ is C^∞ and, since $\int_{-\infty}^{\infty} F_{a,b}^c(s)\,ds = \int_a^b F_{a,b}^c(s)\,ds$, we are done. $\qquad\square$

Recall that $c_f : M \longrightarrow \mathbb{R}_0^+$ and $c_g : N \longrightarrow \mathbb{R}_0^+$ are differentiable functions such that $c_f(x), c_g(y) \geq \alpha > 0$, for all $x \in M$ and $y \in N$. Let $\varepsilon = \alpha/3$. Fix $x \in M$ and let $\varphi_x : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ be a C^∞ map such that

  i. $\varphi_x(t) = 0$ for all $t \in [0, \varepsilon]$;
  ii. $\varphi_x(t) = 0$ for all $t \geq c_f(x) - \varepsilon$ and
  iii. $\int_\varepsilon^{c_f(x)-\varepsilon} \varphi_x(s)\,ds = c_g\left(h(x)\right) - c_f(x)$.

The existence of such a function is guaranteed by Lemma 2.4.

Now, consider $\phi_x(t) = \int_0^t \varphi_x(s) + 1\,ds$. We have:

  i. $\phi_x(t) = \int_0^t 1\,ds = t$, for all $t \in [0, \varepsilon]$;
  ii. for all $t \geq c_f(x)$ we have

$$\phi_x(t) = \int_0^\varepsilon (\varphi_x(s) + 1)\,ds + \int_\varepsilon^{c_f(x)-\varepsilon} (\varphi_x(s) + 1)\,ds + \int_{c_f(x)-\varepsilon}^t (\varphi_x(s) + 1)\,ds$$

$$= \varepsilon + c_g\left(h(x)\right) - c_f(x) + c_f(x) - \varepsilon - \varepsilon + t - c_f(x) + \varepsilon$$

$$= c_g\left(h(x)\right) - c_f(x) + t;$$

  iii. $\phi_x \in C^\infty$.

Let $\hat{M} = \{(x,s) \in M \times \mathbb{R} : s \in [0, c_f(x)]\}$ and define the following equivalence relation: $\left(x, c_f(x)\right) \sim \left(f(x), 0\right)$. Let $M_{c_f} = \hat{M}/\sim$. Similarly, we define N_{c_g}. Now, consider $\hat{h} : M_{c_f} \longrightarrow N_{c_g}$ defined by $\hat{h}(x,s) = \left(h(x), \phi_x(s)\right)$. If h is differentiable at a point, then $\hat{h}$ is differentiable.
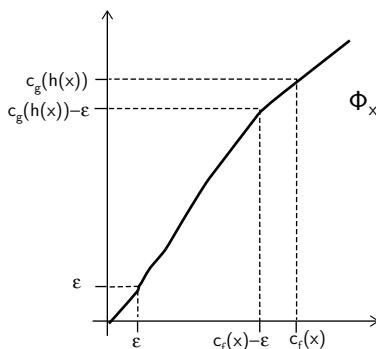
FIGURE 2. The reparametrization ϕ_x on each fiber at x.

Acknowledgement

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