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Stretching Generic Pesin's Entropy Formula

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Abstract
We prove that Pesin's entropy formula holds generically within a broad subset of volume-preserving bi-Lipschitz homeomorphisms with respect to the Lipschitz–Whitney topology.

Keywords Nonuniform hyperbolicity · Lyapunov exponents · Metric entropy · Pesin formula · Bi-Lipschitz

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1 Introduction

The *metric entropy* is a central concept in ergodic theory and has connections with several areas of physics (like thermodynamic formalism and statistic mechanics) and computer science (information theory), among others. The concept of metric entropy requires a measure in a given state space and a measurable map keeping that measure invariant and, some sense, it measures the complexity of this system. A very complete survey on entropy can be found in [16].

Lyapunov exponents are key objects in smooth ergodic theory. In rough terms these exponents measure the asymptotic growth rate of the tangent map of a diffeomorphism along orbits and restricted to certain fixed directions. Positive or negative exponents ensures, respectively, exponential divergence or convergence of nearby orbits and zero exponents warrant no exponential behavior. Since orbits drifting away to the future or to the past is a topological property it is expected to find several ways to define a similar notion for maps that are only continuous. On the other way, considering maps which are only measurable, as we did above when considering metric entropy, seems to be very poor to develop an interesting theory of Lyapunov

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exponents. Furthermore, as a quite completed theory for C^1 maps is available since the late sixties [22], it is common to consider smooth maps when dealing with Lyapunov exponents.

The problem of relating these two concepts is an old question in dynamical systems. In the late seventies Ruelle [26] obtained an upper bound to the metric entropy of a C^1 measurable map preserving a probability measure. This upper bound is precisely the integral, with respect to the given measure, of the positive Lyapunov exponents. Indeed, Pesin [23] has already proved that the metric entropy is precisely the integral of the positive Lyapunov exponents as long as the map is C^2 (actually $C^{1+\alpha}$ where $\alpha > 0$ is the Hölder exponent is enough) and the invariant measure is equivalent to the Lebesgue measure. In order to obtain this equality, nowadays called *Pesin's entropy formula*, Pesin developed an important invariant manifold theory which is central in modern dynamics and the basis of *nonuniform hyperbolicity*. Clearly, the Morse-Smale systems are examples which attest that Ruelle inequality may be strict. We notice that in [18] Mañé cleverly was able to bypass the Pesin invariant manifold theory approach to prove Pesin's entropy formula. The literature connecting this topic with Statistical Physics is immense (see e.g., [27,28] and references therein).

Despite the fact that Pesin's entropy formula requires a C^2 hypothesis on the map and also the fact that C^1-generically (meaning a dense G_δ) C^1 maps are not of class C^2, Tahzibi obtained in [30] a simple but somehow alluring result: a C^1-generic area-preserving diffeomorphism satisfy Pesin's entropy formula. Later, Tahzibi theorem was generalized to any dimension by Sun and Tian [29] and to volume-preserving flows and low dimension Hamiltonians by the first author and Varandas [5]. In [29] the authors were able to follow Mañé arguments in [18] on the proof of Pesin's entropy formula using only C^1 regularity of the diffeomorphisms, but with an additional hypothesis of a dominated splitting for almost every point. See also [32] for a proof of this formula for a class of maps between C^1 and $C^{1+\alpha}$.

In overall, we saw that metric entropy requires only measurable regularity, Lyapunov exponents need C^1 assumptions on the map and, finally, Pesin's entropy formula demands for $C^{1+\alpha}$ regularity. What about the Lipschitz class which lies in between the measurable maps and the C^1 ones? Is there any chance to stretch out the generic Pesin's entropy formula up to the broader class of volume-preserving bi-Lipschitz homeomorphisms but with the coarse Lipschitz–Whitney topology? In the present paper we discuss this problem (Theorem 1). We point out that when working within spaces of Hölder maps we run serious risks with respect to the regularity of the composition operator (see [12]). Moreover, in general, bi-Lipschitz maps are not approximated (with respect to a Lipschitz–Whitney topology) by diffeomorphisms and this represent an obstacle to obtain several results. For example, contrary to the C^1 case, in the bi-Lipschitz class we cannot deduce that Pesin's entropy formula holds Lipschitz–Whitney-densely through approximating those maps by C^2-diffeomorphisms. To overcome the crucial use of the continuity of the derivative map in previous strategies we introduce instead an extra regularity property (the ℓ-property) in order to obtain Pesin's formula in a subset of bi-Lipschitz maps.

Altogether, our main result broadly widened the scope of this very important formula in dynamical systems and we expect to have applications in contexts where we have lack of differentiability. There are several examples of applications and activity in Lipschitz dynamics: (i) When working with differential equations in order to have Picard–Lindelöf uniqueness of integrability into a flow we impose only Lipschitz regularity. So is is natural to ask if C^1-type results also work in the broader regularity class of Lipschitz dynamical systems. (ii) The interest on 'real-world' systems, usually non smooth, lead to the study of Lipschitz mappings on compact sets and a generalization of the notion of hyperbolicity (cf. [13, §3.1.3]). (iii) The structural stability issues which are a cornerstone in the whole theory of dynamical systems are still valid for Lipschitz dynamics (see [3,33]). (iv) A new theory of Lyapunov exponents in this class is available [6,20] shedding a light on the phenomenon of having, contrary to the classical theory, the Lyapunov spectrum with cardinality above the dimension of the manifold (see [6, Example 1]). This can be somehow related to the aforementioned semi-hyperbolicity. (v) Also the expansive maps which are smooth except at finitely many points (aka pseudo-Anosov maps) and which were introduced by Thurston [31] are included in our setting. (vi) From a perspective of topological dynamical systems we recall that global hyperbolicity for homeomorphisms was defined in [1, §11.3] to be expansive maps displaying also the shadowing property. These are the so called *Anosov homeomorphisms* which contain the Lipschitz homeomorphisms and it would be quite interesting to figure out about the values of its Lyapunov exponents and its entropy in the subclass of volume-preserving Lipschitz ones.

Besides conceptual topological and analytical issues concerning the class of bi-Lipschitz maps which will be treated in Sects. 2.2 and 2.3 respectively, the main ingredients to prove Theorem 1 are stated at Sect. 2.5 and they are the Lipschitz version of Ruelle's inequality (Proposition 1; Sect. 3), the Lipschitz version of Bochi–Mañé–Viana dichotomy (Theorem 2; Sect. 5) and also the Sun and Tian strategy for the particular case of dominated splitting (Theorem 3; Sect. 6). The proof of main Theorem 1 is given in Sect. 7. We finally remark that for the main Theorem 1 we consider a measure induced by a volume form. This follows from the use of the Lipschitz version of Bochi-Mañé-Viana dichotomy (Theorem 2) as intermediate step, which is stated under this condition. However, Proposition 1 and Theorem 3 hold for any smooth measure.

2 Definitions and Main Results

2.1 Preliminaries

Consider a compact connected boundaryless C^∞ manifold M of dimension $d \geq 2$ and a reference Lebesgue probability measure μ on M. Let $\mathrm{C}(M)$ be the set of continuous maps on M endowed with the usual metric defined by

$$d_0^+(f,g) = \max_{x\in M} \{d(f(x), g(x)) \colon x \in M\}.$$

We also consider the space of homeomorphisms on M, denoted by $\mathrm{Homeo}(M)$, with the metric

$$d_0(f,g) = d_0^+(f,g) + d_0^+\left(f^{-1}, g^{-1}\right).$$

Notice that $(\mathrm{C}(M), d_0^+)$ and $(\mathrm{Homeo}(M), d_0)$ are complete metric spaces. We say that $f \in \mathrm{C}(M)$ is *Lipschitz* if there exists a constant $L \geq 0$ such that

$$d(f(x), f(y)) \leq L\, d(x,y),$$

for all $x, y \in M$. The infimum of such constants is called the *Lipschitz constant* of f, that we denote by $\mathrm{lip}(f)$:

$$\mathrm{lip}(f) = \sup_{x\neq y} \frac{d(f(x), f(y))}{d(x,y)}.$$

We say that a homeomorphism $f\colon M \to M$ is *bi-Lipschitz* or a *lipeomorphism* if both f and its inverse f^{-1} are Lipschitz maps. We denote by $\mathrm{Lips}(M)$ the set of all Lipschitz maps on M and by $\mathrm{Lipeo}(M)$ the set of all lipeomorphisms on M. Finally, we denote by $\mathrm{Diff}^1(M)$ the set of C^1 diffeomorphisms on M. Clearly, $\mathrm{Diff}^1(M) \subset \mathrm{Lipeo}(M) \subset \mathrm{Homeo}(M) \subset C(M)$. We are interested in conservative dynamics, so that we denote by $\mathrm{Diff}^1_\mu(M)$, $\mathrm{Homeo}_\mu(M)$ and $\mathrm{Lipeo}_\mu(M)$ the set of maps $f \in \mathrm{Diff}^1(M)$, $f \in \mathrm{Homeo}(M)$ or $f \in \mathrm{Lipeo}(M)$, respectively, that preserves the measure μ.

We recall that by Rademacher's theorem [25] (see also [14]) the derivative of a Lipschitz map f is defined for a full μ-measure subset $\tilde{M}_f \subseteq M$. In particular, $\mu(M \setminus \tilde{M}_f) = 0$, and, since f is μ-invariant we conclude that $\mu(f^n(M \setminus \tilde{M}_f)) = 0$ for all $n \in \mathbb{Z}$. Therefore the set of orbits with points where f is not differentiable is the zero measure set $\cup_{n\in\mathbb{Z}} f^n(M \setminus \tilde{M}_f)$:

$$\mu\left(\cup_{n\in\mathbb{Z}}\, f^n\left(M \setminus \tilde{M}_f\right)\right) \leq \sum_{n\in\mathbb{Z}} \mu\left(f^n\left(M \setminus \tilde{M}_f\right)\right) = 0.$$

In view of this we may consider $\tilde{M}_f \subset M$ invariant by f, and the existence of Df_x implies the existence of $Df_{f^n(x)}$ for all $n \in \mathbb{Z}$.

From Moser's theorem (see [21]) we may use charts (U, φ) such that the induced measure $\varphi_*\mu$ coincides with the Lebesgue measure in $\varphi(U) \subset \mathbb{R}^d$. Moreover, given a point x we will use once and for all preferred charts (U, φ) and (V, φ), with $x \in U$ and $f(x) \in V$, to define the chart representative map $\hat{f} = \varphi^{-1} \circ f \circ \psi$. For instance we may enumerate the (finite) charts and take always the chart with lowest index.

If $A : T_x M \to T_y M$ is a linear map, the norm $\|A\|$ is defined in the usual way

$$\|A\| = \sup_{0\neq v\in T_x M} \frac{\|Av\|}{\|v\|}.$$

Using charts we may consider A as a linear map $A : \mathbb{R}^d \to \mathbb{R}^d$. In particular, letting (U_i, φ_i), be the established charts associated to $x_i \in U_i$, $i = 1, \ldots, 4$, for linear maps $A\colon T_{x_1}M \to T_{x_2}M$ and $B\colon T_{x_3}M \to T_{x_4}M$ we have

$$\|A - B\| = \|D\varphi_2 A(D\varphi_1)^{-1} - D\varphi_4 B(D\varphi_3)^{-1}\|.$$

2.2 The Λ-topology for Lipeomorphisms

We are going to introduce the Lipschitz–Whitney topology on the set of lipeomorphisms in M. Given a map $h : W \subset \mathbb{R}^d \to \mathbb{R}^d$ we write $\mathrm{lip}_W(h)$ for the corresponding euclidean Lipschitz constant:

$$\mathrm{lip}_W(h) = \sup_{u\neq v, u,v\in W} \frac{\|h(u) - h(v)\|}{\|u - v\|}.$$

Let $\|\cdot\|_{C^0(\mathbb{R}^d)}$ denote the usual uniform norm on the space $C^0(\mathbb{R}^d)$ of continuous maps $f\colon \mathbb{R}^d \to \mathbb{R}^d$. We notice that the space $\mathrm{Lips}(\mathbb{R}^d)$ of Lipschitz maps from $\mathbb{R}^d$ to $\mathbb{R}^d$ has a linear structure, and that $\|f\|_{\mathrm{Lips}} := \|f\|_{C^0(\mathbb{R}^d)} + \mathrm{lip}_{\mathbb{R}^d}(f)$ defines a complete norm. Similarly, $\|f\|_{\mathrm{Lipeo}} := \max\{\|f\|_{\mathrm{Lips}}, \|f^{-1}\|_{\mathrm{Lips}}\}$ define a complete norm on the space $\mathrm{Lipeo}(\mathbb{R}^d)$ of bi-Lipschitz maps from $\mathbb{R}^d$ to $\mathbb{R}^d$.

We will set now a topology for $\mathrm{Lipeo}(M)$. Since any smooth metric is Lipschitz equivalent to the Euclidean metric, f is a Lipschitz map if and only if for any pair of charts (U, φ) and (V, ψ) the map $\hat{f} = \psi \circ f \circ \varphi^{-1}$ is Lipschitz on the Euclidean domain $W = \varphi(U \cap f^{-1}(V))$, that is $\mathrm{lip}_W(\hat{f}) < \infty$. We notice that if for $x \in U \cap f^{-1}(V)$ the derivative Df_x is well defined, then also is $\|D\hat{f}_{\varphi(x)}\|$, and moreover we have $\|D\hat{f}_{\varphi(x)}\| \leq \mathrm{lip}_W(\hat{f}) < \infty$.

Given $f \in \mathrm{Lipeo}(M)$ consider: charts (U, φ), (V, ψ) on M such that $f(U) \cap V \neq \emptyset$; compact sets $K \subset U \cap f^{-1}(V)$ and $L \subset f(U) \cap V$; and $0 < \varepsilon \leq \infty$. Define a (weak) subbasic neighborhood

$$\mathcal{N}(f; (U, \varphi), (V, \psi), K, L, \varepsilon) \tag{1}$$

to be the set of maps $g \in \mathrm{Lipeo}(M)$ such that $g(K) \subset V$, $g^{-1}(L) \subset U$, $d_0(f, g) < \varepsilon$, and moreover

$$\mathrm{lip}_{\varphi,\psi}(f - g) := \mathrm{lip}_{\varphi(K)}(\psi \circ f \circ \varphi^{-1} - \psi \circ g \circ \varphi^{-1}) < \varepsilon,$$

and

$$\mathrm{lip}_{\psi,\varphi}(f^{-1} - g^{-1}) := \mathrm{lip}_{\psi(L)}(\varphi \circ f^{-1} \circ \psi^{-1} - \varphi \circ g^{-1} \circ \psi^{-1}) < \varepsilon.$$

We stress that, unless M is a linear space, the expression $f - g$ has no meaning, and its use is only to remind the relation between the corresponding charts representatives of f and g. Let Λ be the topology generated by the (weak) subbasic neighborhoods (1). This topology is similar to the (weak) C^r-Whitney topology, with the distance between the derivatives on each chart being here replaced by the control of the Lipschitz constant of the difference of the chart representatives of the maps; see [15]. In view of this, we also call Λ the (weak) Lipschitz–Whitney topology. This topology was also stated in [11] that we also refer for some comments on this subject.

Consider a finite family $\{\mathcal{N}_i\}_{i\in I} = \{\mathcal{N}(f; (U_i, \varphi_i), (V_i, \psi_i), K_i, L_i, \varepsilon_i)\}_{i\in I}$ of subbasic neighborhoods such that both families $\{K_i\}_{i\in I}$ and $\{L_i\}_{i\in I}$ cover M, and all ε_i are smaller than some ε_0 for which $d(f(K_i), \partial V_i) > 2\varepsilon_0$, for all $i \in I$. Let N be a neighborhood of f obtained as the intersection of the subbasic sets $\mathcal{N}_i$. There is a metric d_Λ compatible with its topology that for each $f, g \in N$ assigns

$$d_\Lambda(f, g) = \max\left\{d_0(f, g), \max_{i\in I}\left\{\mathrm{lip}_{\varphi_i,\psi_i}(f - g), \mathrm{lip}_{\psi_i,\varphi_i}\left(f^{-1} - g^{-1}\right)\right\}\right\}.$$

The closure $\bar{N}$ of N is then a complete metric space and thus a Baire space. From a well know fact, since each $f \in \mathrm{Lipeo}(M)$ has a neighborhood which is a Baire space then $(\mathrm{Lipeo}(M), \Lambda)$ is itself a Baire space.

2.3 The ℓ-Property

Given $x_0 \in \tilde{M}_f$, the linearization of $\hat{f} = \psi \circ f \circ \varphi^{-1}$ at $u_0 = \varphi(x_0)$ is given for $u \in B(u_0, r)$, $r > 0$, by

$$\hat{f}(u) = \hat{f}(u_0) + D\hat{f}_{u_0}(u - u_0) + \rho(u),$$

and the remainder $\rho = \rho(f, x_0, r)\colon B(u_0, r) \to \mathbb{R}^d$ varies Lipschitz continuously on the variable $u \in B(u_0, r)$.

We are interested in lipeomorphisms such that the Lipschitz constant of the remainder vanishes as we consider smaller radius for the corresponding domain. We say that $f \in \mathrm{Lipeo}_\mu(M)$ satisfies the *ℓ-property* at $x_0 \in \tilde{M}$ if

$$\lim_{r\to 0} \mathrm{lip}_{B(u_0,r)}\left(\rho\left(f^{\pm 1}, u_0, r\right)\right) = 0. \tag{2}$$

We denote by $\mathrm{Lipeo}_\mu^\ell(M) \subset \mathrm{Lipeo}_\mu(M)$ the set of lipeomorphisms such that for μ almost every (a.e.) $x_0 \in M$ the ℓ-property holds. Notice that the ℓ-property is algebraically closed with respect to the composition.

We say that $f \in \mathrm{Lipeo}_\mu(M)$ is *almost* C^1 (with respect to μ) if both $Df^{\pm 1}_{(\cdot)}$ are continuous when restricted to $\tilde{M}_f$. Notice that this is weaker then saying that $Df^{\pm 1}_x$ is continuous μ-a.e. x on M, because $Df^{\pm 1}_x$ could not be even defined in some points x of M in the almost C^1 case.

Lemma 2.1 *If* $f \in \mathrm{Lipeo}^\ell_\mu(M)$ *then* f *is almost* C^1 *(w.r.t.* μ*).*

Proof We want to show that for $x \in \tilde{M}_f$, $\|Df_x - Df_y\|$ goes to zero as y goes to x, with $y \in \tilde{M}_f$. Assume already that x, y are close enough such that both lie in the same chart domain U and $\varphi(y) \in B(\varphi(x), r)$ for small $r > 0$. For $v \in B(\varphi(x), r)$ we have, writing $\hat{f}$ for $\psi \circ f \circ \varphi^{-1}$,

$$\hat{f}(v) = \hat{f}(\varphi(x)) + D\hat{f}_{\varphi(x)}(v - \varphi(x)) + \rho(v),$$

where $\rho = \rho(f, x, r)$, and then

$$D\hat{f}_{\varphi(y)} = D\hat{f}_{\varphi(x)} + D\rho_{\varphi(y)}.$$

Thus,

$$\left\| D\hat{f}_{\varphi(x)} - D\hat{f}_{\varphi(y)} \right\| \leq \left\| D\rho_{\varphi(y)} \right\| \leq \mathrm{lip}_{B(\varphi(x),r)}(\rho),$$

which goes to 0 as $r \to 0$. This implies that $\|Df_x - Df_y\|$ also goes to to 0 as $r \to 0$. The computations for f^{-1} are similar. $\square$

As we notice in the following example, the ℓ-property everywhere is not universal among lipeomorphisms.

Example 1 Define an area-preserving lipeomorphism in $\mathbb{R}^2$ by $F(x, y) = (x, y + x^2 \sin(1/x))$ for $(x, y) \neq (0, 0)$ and $F(0, 0) = (0, 0)$. Notice that F is differentiable everywhere; $DF_{(0,0)} = Id$, $DF_{(x,y)} \cdot (u, v) = (u, v + [2x \sin(1/x) - \cos(1/x)]u)$ and $\det DF = 1$. Clearly, $F \notin C^1$ and $F \in \mathrm{Lipeo}_\mu(\mathbb{R}^2)$ but $F \notin \mathrm{Lipeo}^\ell_\mu(\mathbb{R}^2)$. More precisely,

$$F(x, y) = (0, 0) + DF_{(0,0)}(x, y) + \rho(x, y) = (x, y) + (0, x^2 \sin(1/x)),$$

being $\rho(x, y) = (0, x^2 \sin(1/x))$, which does not satisfy property (2).

Given $f \in \mathrm{Lipeo}(M)$ we recall the neighborhood N of f obtained as the intersection of the subbasic sets $\{\mathcal{N}_i\}_{i \in I}$, and which closure $\bar{N}$ has a complete metric d_Λ.

Lemma 2.2 $\mathrm{Lipeo}^\ell_\mu(M) \cap \bar{N}$ *is a* Λ*-closed subset of* $\bar{N}$*.*

Proof By Lebesgue's lemma, there exists $\delta_0 > 0$ such that for all $x \in M$, the ball $B(x, \delta_0)$ is contained at least in one of the domain U_i of the charts, and similarly its image $f(B(x, \delta_0))$, just by taking δ_0 to be the minimal of the two Lebesgue numbers of the covers $\{U_i\}$ and $\{f^{-1}(U_i)\}$.

Set $\bar{N}^\ell = \mathrm{Lipeo}^\ell_\mu(M) \cap \bar{N}$. Let (f_n) be a sequence of maps in $\bar{N}^\ell$ converging to $f \in \bar{N}$. For each n there is a μ-full measure subset $\tilde{M}_n \subset M$ of points a where $D(f_n)_a$ is defined. Similarly, there is a μ-full measure subset $\tilde{M}_f \subset M$ of points a where Df_a is defined. Let a be any point in the μ-full measure subset $\tilde{M} = \tilde{M}_f \bigcap_n \tilde{M}_n$. There are suitable $i, j \in I$ such that for all $v \in B(\varphi_i(a), r)$, $r > 0$, with $B(\varphi_i(a), r) \subset \varphi_i(U_i)$, and writing $\hat{f}$ for $\psi_j \circ f \circ \varphi_i^{-1}$, we have

$$\hat{f}(v) = f(\varphi_i(a)) + \left(D\hat{f}\right)_{\varphi_i(a)} (v - \varphi_i(a)) + \rho(v)$$

and, for each n, writing $\hat{f}_n$ for $\psi_j \circ f_n \circ \varphi_i^{-1}$,

$$\hat{f}_n(v) = \hat{f}_n(\varphi_i(a)) + (D\hat{f}_n)_{\varphi_i(a)}(v - \varphi_i(a)) + \rho_n(v),$$

where $\rho = \rho(f, a, r)$ and $\rho_n = \rho(f_n, a, r)$.

Since $f_n \in \mathrm{Lipeo}^{\ell}_{\mu}(M)$ we have for all n that $\lim_{r\to 0} \mathrm{lip}_{B(\varphi_i(a),r)}(\rho_n) = 0$. We are going to prove that $f \in \mathrm{Lipeo}^{\ell}_{\mu}(M)$, that is, for any given $\epsilon > 0$ there is $R > 0$ such that $\mathrm{lip}_{B(\varphi_i(a),R)}(\rho) < \epsilon$ (and consequently the same holds for all $0 < r < R$). Let $\epsilon > 0$ be given. We may found $N_0 > 0$ such that $d_\Lambda(f_{N_0}, f) < \epsilon/3$ and R_0 such that $\mathrm{lip}_{B(\varphi_i(a),R_0)}(\rho_{N_0}) < \epsilon/3$. Hence

$$\begin{aligned}
\mathrm{lip}_{B(\varphi_i(a),R)}(\rho) &\le \mathrm{lip}_{B(\varphi_i(a),R_0)}(\rho_{N_0}) + \mathrm{lip}_{B(\varphi_i(a),R_0)}(\rho - \rho_{N_0}) \\
&< \frac{\epsilon}{3} + \mathrm{lip}_{\varphi_i(U_i)}(\hat{f} - \hat{f}_{N_0}) + \left\| D(\hat{f} - \hat{f}_{N_0})_{\varphi_i(a)} \right\| \\
&\le \frac{\epsilon}{3} + 2\,\mathrm{lip}_{\varphi_i(U_i)}(f - f_{N_0}) < \epsilon.
\end{aligned}$$

In an analogous way we perform the same computation for f^{-1}. The closure of μ-invariant maps follows straightforward. We conclude that $f \in \bar{N}^{\ell}$. □

Since $\bar{N}$ is a complete metric space, from Lemma 2.2 we conclude that the same holds for $\bar{N}^{\ell}$. We then have the following.

Corollary 2.3 $(\mathrm{Lipeo}^{\ell}_{\mu}(M), \Lambda)$ *is a Baire space.*

In the following simple example we show that the Λ-closure of $\mathrm{Diff}^1_{\mu}(\mathbb{S}^2)$ is not equal to $\mathrm{Lipeo}^{\ell}_{\mu}(\mathbb{S}^2)$. Thus, any attempt of using Λ-approximation by smooth maps to study volume-preserving lipeomorphisms is doomed to failure.

Example 2 Take the area-preserving lipeomorphism $f(x, y) = (x + |y|, y)$ on the annulus $\mathbb{S}^1 \times (-1, 1)$ and assume, by contradiction, that exists a C^1 area-preserving map $g(x, y) = (g_1(x, y), g_2(x, y))$ such that $d_\Lambda(g, f) < 1$. Let us define $\alpha(y) = g_1(0, y)$, $\beta(y) = f_1(0, y) = |y|$ and

$$\Delta_y := \frac{|\alpha(y) - \beta(y) - (\alpha(0) - \beta(0))|}{|y - 0|}.$$

Clearly

$$\Delta_y = \left| \frac{\alpha(y) - \alpha(0)}{|y|} - \left(\frac{\beta(y) - \beta(0)}{|y|} \right) \right| = \left| \frac{\alpha(y) - \alpha(0)}{|y|} - 1 \right|.$$

We observe that

$$\lim_{y\to 0^+} \frac{\alpha(y) - \alpha(0)}{|y|} = \alpha'(0) = \frac{\partial g_1}{\partial y}|_{(0,0)} \quad \text{and} \quad \lim_{y\to 0^-} \frac{\alpha(y) - \alpha(0)}{|y|} = -\alpha'(0) = -\frac{\partial g_1}{\partial y}|_{(0,0)}.$$

Hence one of these numbers $\alpha'(0)$ or $-\alpha'(0)$ is ≤ 0, contradicting $d_\Lambda(g, f) < 1$. Finally, we extend in a volume-preserving fashion to $\mathbb{S}^2$ by considering two centers $(-1, 0, 0)$ and $(0, 0, 1)$ obtaining a smooth map on $\mathbb{S}^2$ except along the 'equator' on which is only Lipschitz but still satisfying ℓ-property.

Remark 2.1 Despite the fact that $\mathrm{Lipeo}^{\ell}_{\mu}(M)$ is a Baire space it is not separable. Indeed, it is easy to show that existence of countable and dense subsets cannot be achievable within our context. For $\hat{y} \in [0, 1]$ take the family of area-preserving lipeomorphisms $f_{\hat{y}}(x, y) = (x, |y-$

$\hat{y}|)$ on $\mathbb{R}^2$. For $\hat{y}_1 \neq \hat{y}_2$ we have $d_\Lambda(f_{\hat{y}_1}, f_{\hat{y}_2}) \geq 2 + |\hat{y}_2 - \hat{y}_1|$. Thus we obtain a discrete subset of area-preserving lipeomorphisms satisfying the ℓ-property but with cardinality equal to the cardinality of $\mathbb{R}$, thus cannot be separable.

2.4 Lyapunov Exponents and Domination

Since for all $x \in \tilde{M}_f \cap \tilde{M}_{f^{-1}}$ we have $\|Df_x^{\pm 1}\| \leq \text{lip}\left(f^{\pm 1}\right) < \infty$, we have

$$\int_M \log^+ \left\|Df_x^{\pm 1}\right\| \, d\mu(x) < +\infty.$$

Furthermore, the map $Df : \tilde{M}_f \times \mathbb{Z} \to \mathrm{SL}(d, \mathbb{R})$, given by $Df(x, n) = Df_x^n$ is a cocycle, where $\mathrm{SL}(d, \mathbb{R})$ stand for the d-dimensional special linear group with entries in $\mathbb{R}$. Indeed, $Df(x, 0) = Id$ for all $x \in \tilde{M}_f$ and, by using the chain rule for Lipschitz maps (see [17]), we get $Df(x, n+m) = Df(f^n(x), m) \cdot Df(x, n)$ for all $n, m \in \mathbb{Z}$ and $x \in \tilde{M}_f$. Under these assumptions we can may apply Oseledets' theorem to the dynamical cocycle Df (see e.g., [4]).

Theorem 2.4 (Oseledets) *Let $f \in Lipeo_\mu(M)$, then there is a μ full measure subset $\mathcal{O}_f \subset M$ such that, for all $x \in \mathcal{O}_f$ there exist*

(1) (Oseledets' splitting) *a Df-invariant splitting of the fiber $T_xM = E_x^1 \oplus \cdots \oplus E_x^{k(x)}$ along the orbit of x, and*
(2) (Lyapunov exponents) *real numbers $\hat{\lambda}_1(f, x) > \cdots > \hat{\lambda}_{k(x)}(f, x)$, with $1 \leq k(x) \leq d$, such that*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df_x^n \cdot v\| = \hat{\lambda}_i(f, x),$$

for any $v \in E_x^i \setminus \{0\}$ and $i = 1, \ldots, k(x)$.

Let $\lambda_1(f, x) \geq \cdots \geq \lambda_d(f, x)$ denote de d Lyapunov exponents counted with multiplicity and set $\lambda_i^+(f, x) = \max\{\lambda_i(f, x), 0\}_{i=1}^d$. The set $\{\lambda_i(f, x)\}_{i=1,\ldots,d}$ is called the *Lyapunov spectrum* of f at x. When all Lyapunov exponents are equal (in our setting this means all equal to zero), we say that the Lyapunov spectrum is *trivial*.

We recall now the definition of a dominated splitting associated with $f \in \mathrm{Lipeo}_\mu(M)$. Let $\mathfrak{m}(A) = \|A^{-1}\|^{-1}$ denotes the co-norm of a linear map A. Consider $m \in \mathbb{N}$. A nontrivial Df-invariant μ-measurable splitting $T_{\mathcal{D}}M = E_{\mathcal{D}} \oplus F_{\mathcal{D}}$, with $\mathcal{D} \subset \tilde{M}_f$, is said to be an *m-dominated splitting* for Df over $\mathcal{D}$ if the following inequality holds for any $x \in \mathcal{D}$:

$$\frac{\|Df_x^m|_{E_x}\|}{\mathfrak{m}(Df_x^m|_{F_x})} \leq \frac{1}{2}. \tag{3}$$

Sometimes we write $\mathcal{D}_m$ to emphasize that the rate of time needed to observe (3) is m.

2.5 Main Results

In this section we establish our main result that Pesin's entropy formula holds generically in $\mathrm{Lipeo}_\mu^\ell(M)$ and give also the major results evolved. Let $h_\mu(f)$ stands for the metric entropy of f with respect to the measure μ (for full details see e.g., [19, Chapter IV]).

Theorem 1 *There exists a* Λ*-residual subset* $\mathcal{R} \subset \mathrm{Lipeo}^{\ell}_{\mu}(M)$ *such that for all* $f \in \mathcal{R}$

$$h_\mu(f) = \int_M \sum_{i=1}^{d} \lambda_i^+(f, x)\, d\mu(x). \tag{4}$$

To prove (4) we start by obtaining Ruelle's inequality for Lipschitz maps which is proved in Sect. 3.

Proposition 1 *For all* $f \in \mathrm{Lips}(M)$ *preserving the measure* μ *we have*

$$h_\mu(f) \leq \int_M \sum_{i=0}^{d} \lambda_i^+(f, x)\, d\mu(x).$$

In order to obtain the inverse inequality we start with a Lipschitz version of the celebrated Bochi-Mañé-Viana theorem [9, Theorem 1].

Theorem 2 *There exists a* Λ*-residual subset* $\mathcal{R} \subset \mathrm{Lipeo}^{\ell}_{\mu}(M)$ *such that for all* $f \in \mathcal{R}$ *there exists a* μ *full measure subset* $N = \mathcal{Z} \cup \mathcal{D} \subset \tilde{M}_f$ *such that* $\lambda(f, x) = 0$ *for all* $x \in \mathcal{Z}$*, and for all* $x \in \mathcal{D}$ *the Oseledets splitting is an* m_x*-dominated splitting for* Df *along the orbit of* x*, for some* $m_x \in \mathbb{N}$.

The proof of this dichotomy is given in Sect. 5 and relies on the upper semicontinuity of the *integrated Lyapunov exponents map* with respect to the Lipschitz–Whitney topology (Proposition 2 in Sect. 4.2) and that a continuity point of the integrated Lyapunov exponents have either an Oseledets' dominated splitting or a trivial spectrum (Theorem 4 in Sect. 5). To deal with the dominated component $\mathcal{D}$ on the previous result we obtain a Lipschitz version of a result by Sun and Tian ([29, Theorem 2.2]) that we prove in Sect. 6.

Theorem 3 *Let* $f \in \mathrm{Lipeo}^{\ell}_{\mu}(M)$ *and* $m\colon M \to \mathbb{N}$ *to be an* f*-invariant measurable function. If for* μ*-a.e.* $x \in M$ *there is an* $m(x)$*-dominated splitting* $E \oplus F$ *for* Df *on the orbit of* x*, then*

$$h_\mu(f) \geq \int_M \sum_{i=1}^{\dim(F)} \lambda_i(f, x)\, d\mu(x).$$

Observe that in Theorem 3 if F is associated to the non-negative Lyapunov exponents and E is associated to the negative Lyapunov exponents (or F is associated to the positive Lyapunov exponents and E is associated to the non-positive Lyapunov exponents), then (4) holds for $f \in \mathrm{Lipeo}^{\ell}_{\mu}(M)$ with dominated splitting. This theorem is proven in Sect. 6.

Remark 2.2 We stress that Proposition 1 and Theorem 3 hold if we replace the Lebesgue measure by a smooth measure.

The proof of Theorem 1 is given in Sect. 7.

3 Ruelle's Inequality for Lipeomorphisms

Margulis stated in a unpublished work an inequality relating the metric entropy with the integral of positive Lyapunov exponents for C^1 diffeomorphisms and for smooth measures. Later this result was extended to arbitrary C^1 maps by Ruelle [26]. Several proofs can be found in the literature, as well as some variations and extensions of the statement. In particular, a version for piecewise Lipschitz interval maps can be found at [7, Theorem 7.1]. In the following we write down a proof of the Ruelle inequality for Lipschitz maps on a manifold. Besides the mentioned work of Ruelle, we also borrow some ideas from [19] and [24]. Also, the Rademacher theorem is vital in order to obtain this inequality in the present setting.

Proof of Proposition 1 We recall that by Rademacher's theorem we may assume that the derivative Df_x is defined in a full μ-measure subset $\tilde{M}_f \subset M$. Since M is compact, there is $\zeta > 0$ such that for every $x \in M$ the exponential map

$$\exp_x : B(0, \zeta) \subset T_x M \to B(x, \zeta) \subset M$$

is a C^∞ diffeomorphism. Fix $n \in \mathbb{N}$. There is $0 < \epsilon < \zeta/2$ such that $f^n(B(x, \epsilon)) \subset B(f^n(x), \zeta/2)$ and, for any $x \in \tilde{M}_f$, $y \in M$ with $d(x, y) \leq \epsilon$ we have

$$d(f^n(y), \exp_{f^n(x)} \circ Df_x^n \circ \exp_x^{-1}(y)) \leq d(x, y). \tag{5}$$

For each $m \in \mathbb{N}$ let S_m be a maximal ϵ/m-separated set of M. We define a finite partition $\mathcal{P}_m = \{P_m(x_0) : x_0 \in S_m\}$ of M such that $P_m(x_0) \subset \overline{\operatorname{int}(P_m(x_0))}$ and

$$\operatorname{int}(P_m(x_0)) = \left\{y \in M : d(y, x_0) < d(y, x_0'),\ \forall x_0' \in S_m \setminus \{x_0\}\right\}.$$

For any $x \in M$, let $P_m(x)$ be the element of $\mathcal{P}_m$ containing x. We have $B(x_0, \epsilon/2m) \subset P_m(x) \subset B(x, 2\epsilon/m)$ for some $x_0 \in S_m$ and $\operatorname{diam}(P_m(x)) \leq 2\epsilon/m$ which goes to 0 as $m \to \infty$. Set now

$$\nu_{n,m}(x) = \#\{P \in \mathcal{P}_m : f^n(P_m(x)) \cap P \neq \emptyset\} \quad \text{and} \quad \nu_n(x) = \limsup_{m \to \infty} \nu_{n,m}(x).$$

Note first that for every $n, m \geq 1$,

$$\nu_{n,m}(x) \leq \left(\frac{\operatorname{Lips}(f^n) 2\epsilon/m}{\epsilon/2m}\right)^d \leq 4^d \operatorname{Lips}(f)^{nd}. \tag{6}$$

We claim that

$$h_\mu(f^n, \mathcal{P}_m) \leq \int_M \log \nu_{n,m}(x)\, d\nu(x). \tag{7}$$

To obtain the inequality in (7) we notice that

$$\begin{aligned}
h_\mu(f^n, \mathcal{P}_m) &= \lim_{k \to \infty} H_\mu\left(\mathcal{P}_m | \bigvee_{j=1}^{k} f^{-nj}(\mathcal{P}_m)\right) \\
&\leq H_\mu(\mathcal{P}_m | f^{-n}(\mathcal{P}_m)) \ \text{(since we have a decreasing sequence)} \\
&= \sum_{A \in f^{-n}(\mathcal{P}_m)} \mu(A) \left(\sum_{Q \in \mathcal{P}_m : Q \cap A \neq \emptyset} -\frac{\mu(Q \cap A)}{\mu(A)} \log \frac{\mu(Q \cap A)}{\mu(A)}\right) \\
&= \sum_{B \in \mathcal{P}_m} \mu(f^{-n}(B)) \left(\sum_{Q \in \mathcal{P}_m : f^n(Q) \cap B \neq \emptyset} -\frac{\mu(Q \cap f^{-n}(B))}{\mu(f^{-n}(B))} \log \frac{\mu(Q \cap f^{-n}(B))}{\mu(f^{-n}(B))}\right) \\
&\leq \sum_{B \in \mathcal{P}_m} \mu(f^{-n}(B)) \log \#\{Q \in \mathcal{P}_m : f^n(Q) \cap B \neq \emptyset\} \\
&= \int_M \log \nu_{n,m}(x)\, d(f_*^n \mu)(x) \\
&= \int_M \log \nu_{n,m}(x)\, d\mu(x).
\end{aligned}$$

Denote by $V_\delta(A)$ the δ-neighborhood of a set $A \subset M$ (or $A \subset T_{f^n(x)}M$). By (5) we have

$$f^n(P_m(x)) \subset V_{2\epsilon/m}\left(\exp_{f^n(x)}\left(Df_x^n\left(B(0, 2\epsilon/m)\right)\right)\right).$$

If $f^n(P_m(x)) \cap P_m(x_0') \neq \emptyset$ for some $x_0' \in S_m$, then

$$B(x_0', \epsilon/2m) \subset P_m(x_0') \subset V_{4\epsilon/m}\left(\exp_{f^n(x)}(Df_x^n(B(0, 2\epsilon/m)))\right).$$

We recall that there is $b = b(\zeta) \geq 1$ such that for any $x \in M$, if $x, y \in B(x, \zeta)$ then

$$b^{-1}d(y, z) \leq \left|\exp_x^{-1}(y) - \exp_x^{-1}(z)\right| \leq b\, d(y, z).$$

Thus

$$B(\exp_{f^n(x)}^{-1}(x_0'), \epsilon/2bm) \subset V_{4b\epsilon/m}\left(Df_x^n(B(0, 2\epsilon/m))\right).$$

Since $B(x_0', \epsilon/2m)$ are disjoint with $x_0' \in S_m$, also does $B(\exp_{f^n(x)}^{-1}(x_0'), \epsilon/2bm)$. Hence

$$\nu_{n,m}(x) \leq \frac{\operatorname{vol}\left(V_{4b\epsilon/m}\left(Df_x^n(B(0, 2\epsilon/m))\right)\right)}{\min_{x_0' \in S_m} \operatorname{vol}\left(B(\exp_{f^n(x)}^{-1}(x_0'), \epsilon/2bm)\right)}.$$

Given a linear map A, denote by $\chi_i(A)$ the non-negative square root of the i th eigenvalue of A^*A where A^* is the conjugate transpose of A.

Lemma 3.1 *Given a linear map $A : X \to Y$ is between d-dimensional Euclidean spaces and a positive constant ℓ, there exists a constant $C = C(d, \ell)$ such that, for any $r > 0$ we have*

$$\operatorname{vol}(V_{\ell r}(A(B_r(0)))) < Cr^d \prod_{i=1}^{d} \max\{\chi_i(A), 1\}.$$

Lemma 3.1 implies that there is a constant C depending only on d and the geometry of M such that for each $n, m \geq 1$ and $x \in \tilde{M}$

$$\nu_{n,m}(x) \leq C \prod_{i=0}^{d} \max\{\chi_i(Df_x^n), 1\}.$$

In particular, we conclude that for μ-a.e. $x \in M$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu_n(x) \leq \sum_{i=0}^{d} \lambda_i^+(f, x), \tag{8}$$

Now, for every $n \geq 1$ we have

$$h_\mu(f) = \frac{1}{n} h_\mu(f^n).$$

Hence, from (6) and dominated convergence theorem we get

$$\begin{aligned} h_\mu(f) &= \lim_{m\to\infty} \frac{1}{n} h_\mu(f^n, \mathcal{P}_m) \\ &\le \limsup_{m\to\infty} \int_M \frac{1}{n} \log \nu_{n,m}(x)\, d\mu(x) \\ &\le \int_M \limsup_{m\to\infty} \frac{1}{n} \log \nu_{n,m}(x)\, d\mu(x) \\ &= \int_M \frac{1}{n} \log \nu_n(x)\, d\mu(x), \end{aligned}$$

and also

$$h_\mu(f) \le \int_M \limsup_{n\to\infty} \frac{1}{n} \log \nu_n(x)\, d\mu(x).$$

By (8) we conclude the proof of Proposition 1. □

4 Continuity of Domination and Integrated Lyapunov Exponents

4.1 Dominated Splittings

Throughout this section we consider $f \in \text{Lipeo}^\ell_\mu(M)$ and $T_\mathcal{D}M = E_\mathcal{D} \oplus F_\mathcal{D}$ an *m-dominated splitting* for Df over $\mathcal{D} \subset \tilde{M}_f$. The proof of the next result is essentially similar to the smooth case (see e.g. [10, pp. 292]). Note that in the smooth case analogous of Lemma 4.1 below, the continuity of the derivative plays a decisive role. We can reproduce the argument in our case since, by Lemma 2.1, f is almost C^1 with respect to μ. The next result will be used in Sect. 6.

Lemma 4.1 *(Continuous dependence and extension to the closure) The maps $\mathcal{D} \ni x \mapsto E_x$ and $\mathcal{D} \ni x \mapsto F_x$ are continuous for $x \in \mathcal{D}$. Moreover if $x \in M$ is such that its orbit is accumulated by points in $\mathcal{D}$ and Df_y is defined for all $y = f^i(x)$, then the orbit of x has an m-dominated splitting.*

By Lemma 4.1 the set $\mathcal{D}_m$ is closed in the sense that its closure displays an m-dominated splitting in where Df is defined. Within this open/closeness in measure theoretic sense we have that the set $\Gamma_m := M \setminus \mathcal{D}_m$ is open. The *index* of the splitting is the dimension of the bundle $F_\mathcal{D}$. The dominated splitting structure is a ‘weak’ form of uniform hyperbolicity, in fact it behaves like a uniform hyperbolic structure in the projective space RP^{n-1}.

Remark 4.1 (Transversality) There exists $\alpha > 0$ such that $\measuredangle(E_x, F_x) > \alpha$ for all $x \in \mathcal{D}$. Indeed, we prove that there exists $\alpha > 0$ such that $\|u - s\| > \alpha$ for any unitary vectors $u \in F_x$, $s \in E_x$ and $x \in \mathcal{D}$. Otherwise, there exists $x_n \in \mathcal{D}$ such that $u_n \in F_{x_n}$, $s_n \in E_{x_n}$ and $\|u_n - s_n\| \to 0$. As Df is essentially bounded we can choose $m_n \to +\infty$ such that

$$\frac{1}{2} < \frac{\left\| Df^{m_n}_{x_n} \cdot s_n \right\|}{\left\| Df^{m_n}_{x_n} \cdot u_n \right\|} < 2,$$

contradicting the domination.

It is easy to see that given a periodic point $p \in M$ with a dominated splitting for some $f \in \text{Lipeo}^\ell_\mu(M)$, then for any $g \in \text{Lipeo}^\ell_\mu(M)$ with $d_\Lambda(f, g) < \delta$, for small δ, has the periodic point p_g which is the continuation of p. Nevertheless, Dg may no longer exists along the g-orbit of p_g (see also [33, p. 271] which is somehow related to this issues).

4.2 Upper Semicontinuity of the Integral of Lyapunov Exponents

In this section we prove the upper semicontinuity of the integral of the sum of first Lyapunov exponents with respect to the Lipschitz–Whitney topology on $\text{Lipeo}_\mu^\ell(M)$. For $k = 1, \dots, d$ define the k^{th} *integrated Lyapunov exponent* of $f \in \text{Lipeo}_\mu^\ell(M)$ by

$$\mathcal{L}_k(f) = \int_M \sum_{i=1}^{k} \lambda_i(f, x) d\mu(x).$$

It was proved in [6, §4] that when $f \in \text{Homeo}_\mu(M)$ then $\mathcal{L}_k$ cannot be upper semicontinuous w.r.t. the C^0-topology. Moreover, in [8, Proposition 2.1] is was proved that when $f \in \text{Diff}_\mu^1(M)$ then $\mathcal{L}_k$ is upper semicontinuous w.r.t. C^1-topology. In our setting we are able to prove that $\mathcal{L}_k$ is Λ-upper semicontinuous.

Lemma 2.1 is fundamental to overcome the absence of continuity of the derivative and, therefore, to be able to use the general result on the upper semicontinuity for cocycles in the L^p-norm [2, Theorem 2].

Proposition 2 *The function* $\mathcal{L}_k : \text{Lipeo}_\mu^\ell(M) \ni f \to \mathcal{L}_k(f) \in [0, \infty)$ *is upper semicontinuous with respect to the Lipschitz–Whitney topology for all* $k = 1, \dots, d-1$, *that is, for every* $f \in \text{Lipeo}_\mu^\ell(M)$ *and* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $d_\Lambda(f, g) < \delta$ *then* $\mathcal{L}_k(g) < \mathcal{L}_k(f) + \epsilon$. *Moreover,* $\mathcal{L}_d$ *is* Λ*-continuous.*

Proof Consider any $\epsilon > 0$. Lemma 2.1 implies that there exists $0 < \delta < \epsilon/4$ such that if $d_\Lambda(f, g) < \delta$ then $d_0(f, g) < \delta$ and $\|D(g^{-1})_{f(x)} - D(g^{-1})_{g(x)}\| < \epsilon/4$ for μ-a.e. x. Moreover, by similar arguments of that on the proof of Lemma 2.1 one can see that, by reducing δ if necessary, we have $\|Df_z^{\pm 1} - Dg_z^{\pm 1}\| < \epsilon/4$, for all $z \in \tilde{M}_f \cap \tilde{M}_g$. Then

$$\|Df - Dg\|_1 := \int_M \|Df_x - Dg_x\| \, d\mu(x) < \frac{\epsilon}{4},$$

and

$$\begin{aligned} \|(Df)^{-1} - (Dg)^{-1}\|_1 &:= \int_M \|(Df_x)^{-1} - (Dg_x)^{-1}\| \, d\mu(x) \\ &\leq \int_M \left\| D(f^{-1})_{f(x)} - D(g^{-1})_{f(x)} \right\| \, d\mu(x) \\ &\quad + \int_M \left\| D(g^{-1})_{f(x)} - D(g^{-1})_{g(x)} \right\| \, d\mu(x) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}. \end{aligned}$$

Hence $\|Df - Dg\|_1 + \left\|(Df)^{-1} - (Dg)^{-1}\right\|_1 < \epsilon$. In view of this we may apply [2, Theorem 2] that, we recall, do not require the maps to be ergodic. □

5 The Bochi–Mañé–Viana Theorem for the Lipschitz Cass

5.1 Proof of Theorem 2

In the present section we prove Theorem 2. As in [9] we will first prove a result which, together with Proposition 2, directly implies Theorem 2.

Theorem 4 *If the map*

$$f \in \mathrm{Lipeo}^{\ell}_{\mu}(M) \mapsto (\mathcal{L}_1(f), \ldots, \mathcal{L}_{d-1}(f)) \in \mathbb{R}^{d-1}$$

is continuous at f_0 then for μ-almost every $x \in M$, the Oseledets splitting of f_0 is either dominated along the orbit of x or else the Lyapunov spectrum of f at x is trivial.

At first glance, and since the Lipschitz–Whitney topology is coarser than the C^1 topology, it seems that Theorem 4 follows from [9, Theorem 2]. Nevertheless, the naive approach of approximating $f \in \mathrm{Lipeo}^{\ell}_{\mu}(M)$ by $g \in \mathrm{Diff}^1_{\mu}(M)$ and then use [9] directly will not work because $\mathrm{Diff}^1_{\mu}(M)$ is not Λ-dense in $\mathrm{Lipeo}^{\ell}_{\mu}(M)$ (recall Example 1). What we will do in the next few pages is to present a skeleton of the hard, long and highly technical proof made by Bochi and Viana in order to perform the proof in the $\mathrm{Lipeo}^{\ell}_{\mu}(M)$ case.

Like in the smooth case the perturbations of a map f are made by composing it with a map h with small support and close to the identity. However, the composition map $h \mapsto f \circ h$ can be discontinuous (see [12, Example 6.4]) with respect to Λ and then f could be Λ-far from $g := f \circ h$. Fortunately, as our perturbations are always carefully cooked ones we can choose h smooth enough in order to achieve the continuity and this will be crucial to obtain the Bochi-Mañé-Viana theorem for the Lipschitz class.

We begin with a major concept regarding the perturbation framework which is the *realizable sequences*. Observe that our Definition 5.1 differs from [9, Definition 2.10] slightly as we need to adapt it to the non smooth case. For $x \in M$ and small $r > 0$ we define

$$b(x, r) = \varphi^{-1}\big(B(\varphi(x), r)\big).$$

We will always assume that r is small enough so that $\overline{b(x, r)} \subset U$, where, we recall, $U \subset M$ is the domain of φ. The sets $b(x, r)$ will be called *disks*.

Definition 5.1 Given $f \in \mathrm{Lipeo}^{\ell}_{\mu}(M)$, $\epsilon_0 > 0$, $\kappa \in (0, 1)$ and a non periodic point $x \in \tilde{M}_f$, we call a sequence of volume-preserving linear maps $L_i : T_{f^i(x)}M \to T_{f^{i+1}(x)}M$ ($i = 0, \ldots, n-1$) an (ϵ_0, κ)*-realizable sequence of length n at x* if the following holds: for every $\gamma > 0$, there is $r > 0$ such that the iterates $f^j(\overline{b(x, r)})$ are two-by-two disjoint for $j = 0, \ldots, n$ and given any nonempty open set $\mathbf{U} \subset b(x, r)$ there are $g \in \mathrm{Lipeo}^{\ell}_{\mu}(M)$ satisfying $d_{\Lambda}(f, g) < \epsilon_0$ and a measurable set $\mathbf{K} \subset \mathbf{U} \cap \tilde{M}_f$, such that:

(1) g equals f outside the disjoint union $\cup_{j=0}^{n-1} f^j(\overline{\mathbf{U}})$;
(2) $\mu(\mathbf{K}) > (1 - \kappa)\mu(\mathbf{U})$ and
(3) if $y \in \mathbf{K}$, then $\|Dg_{g^j(y)} - L_j\| < \gamma$ for every $j = 0, \ldots, n-1$.

The Lemma 5.1 bellow is the basic perturbation tool that will be used to construct all our realizable sequences in the two-dimensional context (cf. Lemma 5.2). This low dimensional setting contain all the main differences between the smooth case and the Lipschitz one. Lemma 5.3 bellow (see [9, Lemma 3.4]) is the basic perturbation tool that will be used to construct all our realizable sequences in the higher dimensional context (cf. Lemma 5.4). Lemma 5.4 will be stated without proof.

Let $R_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ denote the rotation of angle $\alpha \in \mathbb{R}$ in $\mathbb{R}^2$ which in canonic coordinates can be written as $R_\alpha(u, v) = (u\cos(\alpha) - v\sin(\alpha), u\sin(\alpha) + v\cos(\alpha))$.

Lemma 5.1 [8, Lemma 3.2] *Let $\epsilon_1 > 0$ and $\kappa \in (0, 1)$. Then there exists $\alpha_0 > 0$ with the following properties. If $|\alpha| \le \alpha_0$ and $r > 0$, then there exists a C^∞ area-preserving diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that:*

(i) $|z| \geq r$ *implies* $h(z) = z$;
(ii) $|z| \leq \sqrt{k}r$ *implies* $h(z) = R_\alpha(z)$;
(iii) $|h(z)| = |z|$ *for all* z;
(iv) $|h(z) - z| \leq \alpha r$ *for all* z;
(v) $\|Dh - I\| \leq \epsilon_1$ *for all* z.

Given $x \in M$ and $\theta \in \mathbb{R}$, we consider the *the rotation of angle* θ *at* x to be the linear map $\hat{R}_\theta = (D\varphi_x)^{-1} R_\theta D\varphi_x : T_x M \to T_x M$.

Lemma 5.2 *Given* $f \in \mathrm{Lipeo}_\mu^\ell(M)$*,* $\epsilon_0 > 0$ *and* $\kappa \in (0, 1)$*, there is an* $\alpha_1 > 0$ *with the following properties. Suppose that* $x \in \tilde{M}_f$ *is not periodic and* $|\theta| \leq \alpha_1$*. Then* $Df_x R_\theta = L_0 \colon T_x M \to T_{f(x)} M$ *is an* (ϵ_0, κ)*-realizable sequence of length* 1 *at* x*.*

Proof Let $\gamma > 0$ be given. Using standard Vitali covering arguments we only need to construct perturbations supported in disks $\mathbf{U} = b(y, r') \subset b(x, r)$. Recall charts (U, φ) and (V, ψ), with $x \in U$ and $f(x) \in V$, and the chart representative of f given by $\hat{f} = \psi \circ f \circ \varphi^{-1} \colon \varphi(U) \to \mathbb{R}^d$. Assume r to be small enough such that $b(x, r) \subset U$. For simplicity, we assume that $\varphi(y) = 0$. Using Lemma 5.1 we find, for each small angle θ, a map $g \in \mathrm{Lipeo}_\mu^\ell(M)$, whose chart representative is $\hat{f} \circ h$, and $\mathbf{K} \subset \mathbf{U} \cap \tilde{M}_f$ satisfying the three properties:

(1) g equals f outside $\overline{\mathbf{U}}$;
(2) $\mu(\mathbf{K}) > (1 - \kappa)\mu(\mathbf{U})$ and
(3) if $z \in \mathbf{K}$, then $\|Dg_z - L_0\| < \gamma$.

Properties (1) and (2) are obvious by construction. As h is C^∞ and f is derivable almost surely we get that Dg_z exists for $z \in \mathbf{K}$. For the last one we have

$$\|Dg_z - L_0\| = \|D\hat{f}_{h(\varphi(z))} Dh_{\varphi(z)} - D\hat{f}_{\varphi(x)} R_\theta\| \leq \|D\hat{f}_{h(\varphi(z))} - D\hat{f}_{\varphi(x)}\| \, \|R_\theta\|$$

which, accordingly Lemma 2.1, can be made smaller than any $\gamma > 0$ once we take $\alpha_1 > 0$ and $r > 0$ sufficiently small. We are left to see that $d_\Lambda(f, g) < \epsilon_0$. Clearly, for small $\alpha_1 > 0$ and $r > 0$ we have $d_0(f, g) < \epsilon_0$. Let $K \subset U$ be a compact set such that $f(K) \cup g(U) \subset V$. Let us take r and ϵ_1 (depending on f and ϵ_0) such that $d_0(f, g) < \epsilon_0$ and

$$(\mathrm{lip}_{\varphi(K)}(\hat{f}) + \mathrm{lip}_{\varphi(K)}(h))\epsilon_1 < \epsilon_0.$$

Now, observing that

$$\mathrm{lip}_{\varphi(K)}(\hat{f}) = \operatorname{ess\,sup}_{u \in \varphi(K)} \|D\hat{f}_u\|,$$

we have

$$\begin{aligned}
\mathrm{lip}_{\varphi,\psi}(f - g) &= \operatorname{ess\,sup}_{u \in \varphi(K)} \|D(\hat{f} - \hat{f} \circ h)_u\| \\
&\leq \operatorname{ess\,sup}_{u \in \varphi(K)} \|D\hat{f}_u - D\hat{f}_u Dh_u\| + \|D\hat{f}_u Dh_u - D\hat{f}_{h(u)} Dh_u\| \\
&\leq \operatorname{ess\,sup}_{u \in \varphi(K)} \|D\hat{f}_u\| \cdot \|D(I - h)_u\| + \|D\hat{f}_u - D\hat{f}_{h(u)}\| \cdot \|Dh_u\| \\
&\leq \mathrm{lip}_{\varphi(K)}(\hat{f})\epsilon_1 + \epsilon_1 \, \mathrm{lip}_{\varphi(K)}(h) \leq \epsilon_0.
\end{aligned}$$

In an analogous way we perform the same computation in order to have $\mathrm{lip}_{\psi,\varphi}(f^{-1} - g^{-1}) < \epsilon_0$, which leads to $d_\Lambda(f, g) < \epsilon_0$. □

Now, let us see what we do when we are in dimension greater than 2. Next result is a key step to obtain Lemma 5.4. It plays the same role of Lemma 5.1 in the proof of Lemma 5.2.

Lemma 5.3 [9, Lemma 3.4] *Given $\epsilon_0 > 0$ and $\sigma \in (0, 1)$, there is $\epsilon > 0$ with the following properties: suppose there is a splitting $\mathbb{R}^n = X \oplus Y$ with $X \perp Y$ and $\dim(Y) = 2$, a right cylinder $\mathcal{A} \oplus \mathcal{B}$ centered at the origin with $\mathcal{A} \subset X$ and $\mathcal{B} \subset Y$, and a linear map $\hat{R}: Y \to Y$ such that $\hat{R}(\mathcal{B}) = \mathcal{B}$ and $\|\hat{R} - I\| < \epsilon$. Then, there exists $\tau > 1$ such that the following holds: let us consider the linear map*

$$\begin{aligned} R: \mathbb{R}^n = X \times Y &\longrightarrow \quad \mathbb{R}^n = X \times Y \\ (u, v) \quad &\longmapsto R(u + v) = u + \hat{R}v \end{aligned}$$

for $a, b > 0$ consider the cylinder $\mathcal{C} = a\mathcal{A} \oplus b\mathcal{B}$. If $a > \tau b$ and $diam(\mathcal{C}) < \epsilon_0$, then there is $h \in Diff^1_\mu(\mathbb{R}^n)$ satisfying:

(i) *$h(z) = z$ for every $z \notin \mathcal{C}$ and $h(z) = R(z)$ for every $z \in \sigma\mathcal{C}$;*
(ii) *$\|Dh_z - I\| < \epsilon_0$ for all z.*

Next result is [9, Lemma 3.3] readapted to our setting. Its proof is basically following [9, Lemma 3.3] and compute the distance d_Λ just like we did in Lemma 5.2. For full details on the quotient spaces along the proof see [9, p. 1441].

Lemma 5.4 *Let $f \in \mathrm{Lipeo}^\ell_\mu(M)$, $\epsilon_0 > 0$, $\kappa > 0$, there exists $\epsilon > 0$ with the following properties: suppose there are a non periodic point $x \in M$, $n \in \mathbb{N}$, and for $j = 0, \ldots, n-1$*

(1) *codimension 2 spaces $X_j \subset T_{f^j(x)}M$ such that $X_j = Df^j_x(X_0)$;*
(2) *ellipsis $\mathcal{B}_j \subset (T_{f^j(x)}M)/X_j$ centered at zero with $\mathcal{B}_j = (Df^j_x/X_0)(\mathcal{B}_0)$;*
(3) *linear maps $\hat{R}_j : (T_{f^j(x)}M)/X_j \to (T_{f^j(x)}M)/X_j$ such that $\hat{R}_j(\mathcal{B}_j) \subset \mathcal{B}_j$ and $\|\hat{R}_j - I\| < \epsilon$.*

Consider the linear maps $R_j : T_{f^j(x)}M \to T_{f^j(x)}M$ such that R_j restricted to X_j is the identity, $R_j(X_j^\perp) = X_j^\perp$ and $R_j/X_j = \hat{R}_j$. Define, for $j \in \{0, 1, ..., n-1\}$

$$L_j = Df_{f^j(x)}R_j : T_{f^j(x)}M \to T_{f^{j+1}(x)}M,$$

Then $\{L_0, \ldots, L_{n-1}\}$ is an (ϵ_0, κ)-realizable sequence of length n at x.

We are now able to state a crucial local perturbation result. Its proof follows from [9, Proposition 3.1].

Proposition 5.5 *Let $f \in \mathrm{Lipeo}^\ell_\mu(M)$, $\epsilon_0 > 0$ and $\kappa \in (0, 1)$. If $m \in \mathbb{N}$ is chosen sufficiently large, then given a non periodic point $y \in \tilde{M}_f$ and a nontrivial splitting $T_yM = E \oplus F$ satisfying*

$$\frac{\|Df^m_y|_{E_y}\|}{\mathfrak{m}(Df^m_y|_{F_y})} \geq \frac{1}{2},$$

there exists an (ϵ_0, κ)-realizable sequence $\{L_0, L_1, \ldots, L_{m-1}\}$ at y of length m and there are nonzero vectors $u \in F$ and $s \in Df^m_y(E)$ such that

$$L_{m-1} \cdots L_0(u) = s.$$

Fix $k \in \{1, \ldots, n-1\}$ and $m \in \mathbb{N}$. The subset of $\tilde{M}_f$ formed by the points $x \in \tilde{M}_f$ such that there exists an m-dominated splitting of index k along the orbit of x is denoted by $\mathcal{D}_k(f, m)$. The set $\Gamma_k(f, m) = M \setminus \overline{\mathcal{D}_k(f, m)}$ is open and each element of it has an iterate where inequality (3) does not hold (for index k). Define $\Gamma^\sharp_k(f, m) = \{x \in \Gamma_k(f, m) \cap \mathcal{O}_f : \lambda_k(f, x) > \lambda_{k+1}(f, x)\}$ and $\Gamma^*_k(f, m) = \{x \in \Gamma^\sharp_k(f, m) : x$ is not periodic$\}$.

Next result give us a way to decrease the Lyapunov exponents by using several times Lemma 5.4 (see [9, Proposition 4.2]). We recall that $\wedge^k(A)$ stands for the kth exterior product of the linear map $A\colon \mathbb{R}^d \to \mathbb{R}^d$. Notice that $\sum_{i=1}^k \lambda_i(f,x)$ is the top Lyapunov exponent associated to the cocycle $\wedge^k Df_x$.

Proposition 5.6 *Let $f \in \mathrm{Lipeo}_\mu^\ell(M)$, $\epsilon_0 > 0$, $\kappa \in (0,1)$, $\delta > 0$ and $k \in \{1,\dots,n-1\}$. For every sufficiently large $m \in \mathbb{N}$ there is a measurable function $N\colon \Gamma_k^*(f,m) \to \mathbb{N}$ with the following properties: for μ-a.e. $x \in \Gamma_k^*(f,m)$ and every $n \geq N(x)$ there exists an (ϵ_0,κ)-realizable sequence $\{\hat{L}_0^{x,n}, \hat{L}_1^{x,n}, \dots, \hat{L}_{m-1}^{x,n}\}$ at x of length n such that*

$$\frac{1}{n}\log\left\|\wedge^k(\hat{L}_{m-1}^{x,n}\cdots\hat{L}_0^{x,n})\right\| \leq \sum_{i=1}^{k-1}\lambda_i(f,x) + \frac{\lambda_k(f,x)+\lambda_{k+1}(f,x)}{2} + \delta.$$

Finally, we put in practice the global strategy like in [9]. Next result is [9, Proposition 4.8] adapted.

Proposition 5.7 *Let $f \in \mathrm{Lipeo}_\mu^\ell(M)$, $\epsilon_0 > 0$, $\delta > 0$ and $k \in \{1,\dots,d-1\}$. For every sufficiently large $m \in \mathbb{N}$ there exists $g \in \mathrm{Lipeo}_\mu^\ell(M)$ such that $d_\Lambda(f,g) < \epsilon_0$, $g = f$ outside the open set $\Gamma_k(f,m)$ and such that*

$$\int_{\Gamma_k(f,m)}\sum_{i=1}^{k}\lambda_i(g,x)\,d\mu(x) \leq \int_{\Gamma_k(f,m)}\sum_{i=1}^{k-1}\lambda_i(f,x) + \frac{\lambda_k(f,x)+\lambda_{k+1}(f,x)}{2}\,d\mu(x) + \delta.$$

Let $\Gamma_k(f,\infty) = \cap_{m\in\mathbb{N}}\Gamma_k(f,m)$. Next result is a consequence of Proposition 5.7.

Proposition 5.8 *Let $f \in \mathrm{Lipeo}_\mu^\ell(M)$, $\epsilon_0 > 0$, $\delta > 0$ and $k \in \{1,\dots,d-1\}$. There exists $g \in \mathrm{Lipeo}_\mu^\ell(M)$ such that*

$$\begin{aligned}\int_M\sum_{i=1}^{k}\lambda_i(g,x)\,d\mu(x) \; &< \int_M\sum_{i=1}^{k}\lambda_i(f,x)\,d\mu(x)\\ &\quad - \int_{\Gamma_k(f,\infty)}\frac{\lambda_k(f,x)-\lambda_{k+1}(f,x)}{2}\,d\mu(x) + \delta.\end{aligned}$$

Proof of Theorem 4 Let $f \in \mathrm{Lipeo}_\mu^\ell(M)$ be a point of continuity of $\mathcal{L}_k(\cdot)$ for all $k = 1,\dots,d-1$. Therefore, for all k we have

$$\int_{\Gamma_k(f,\infty)}\frac{\lambda_k(f,x)-\lambda_{k+1}(f,x)}{2}\,d\mu(x) = 0.$$

And so $\lambda_k(f,x) = \lambda_{k+1}(f,x)$ for μ-a.e. $x \in \Gamma_k(f,\infty)$. Consider $x \in \tilde{M}_f \cap \mathcal{O}_f$. If all the Lyapunov exponents at x are zero, then the proof is over. Otherwise, if for some k we get $\lambda_k(f,x) > \lambda_{k+1}(f,x)$, then $x \notin \Gamma_k(f,\infty)$ and so $x \in \mathcal{D}_k(f,m)$ for some m and the Oseledets splitting is dominated. □

6 Proof of Theorem 3

6.1 Three Technical Lemmata

Next basic lemma is inspired in [18, Lemma 3] which was performed in the hyperbolic setting and re-proved under the dominated splitting assumption in [29, Lemma 3.3]. We will present its proof which follows closely the arguments in [18,29]. But first let us recall some definitions. The formulation of item (ii) of Lemma 6.1 is substantially different from the original one in [18,29] which was based on C^1 assumptions that we were not able to guarantee in the lipeomorphism case. This was the main motivation for the introduction of the ℓ-property defined in Sect. 2.3. The next three technical lemmata are going to be used in the proof of Theorem 3 under a suitable choice of the decomposition given by the Oseledets theorem.

Given a normed space $\mathcal{E}$ and a splitting $\mathcal{E} = E_1 \oplus E_2$ we define $\gamma(E_1, E_2)$ as the maximum of the norms of the projections $\pi_i : \mathcal{E} \to E_i$ $(i = 1, 2)$. A subset $G \subset \mathcal{E}$ is said to be an (E_1, E_2)-*graph* if there exists an open set $U \subset E_2$ and a Lipschitz map $\psi : U \to E_1$ such that $G = \{x + \psi(x) : x \in U\}$. We call $\text{lip}_U(\psi) = \sup_{x \neq y \in U} \frac{\|\psi(x) - \psi(y)\|}{\|x - y\|}$ the *dispersion* of the graph G.

Lemma 6.1 *Let $\alpha, \beta, c, \delta > 0$ be such that*

$$0 < \frac{\frac{1}{2} + \frac{\delta\alpha(1+c)}{c\beta}}{1 - \delta\alpha\frac{1+c}{\beta}} < 1.$$

Let $\mathcal{E}$, $\mathcal{E}'$ be two finite dimensional normed spaces, $\mathcal{E} = E_1 \oplus E_2$ a splitting such that $\gamma(E_1, E_2) \leq \alpha$ and $\mathcal{F} : B(0, r) \subset \mathcal{E} \to \mathcal{E}'$ is a Lipschitz map where $D\mathcal{F}_0$ is defined and satisfying the following properties:

(i) *$D\mathcal{F}_0$ is an isomorphism and $\gamma(D\mathcal{F}_0 \cdot E_1, D\mathcal{F}_0 \cdot E_2) \leq \alpha$;*
(ii) *Denoting by $L = D\mathcal{F}_0|_{E_1}$ and $T = D\mathcal{F}_0|_{E_2}$ and for some small $r > 0$ and $(x, y) \in B(0, r)$ we have*

$$\begin{aligned} \mathcal{F} : E_1 \times E_2 &\longrightarrow \quad D\mathcal{F}_0 \cdot E_1 \times D\mathcal{F}_0 \cdot E_2 \\ (x, y) &\longmapsto (Lx + p(x, y), Ty + q(x, y)) \end{aligned}$$

with remainders $p(x, y)$ and $q(x, y)$ having Lipschitz constants less than δ;
(iii) $\frac{\|D\mathcal{F}_0|_{E_1}\|}{m(D\mathcal{F}_0|_{E_2})} \leq \frac{1}{2}$ *and*
(iv) *$m(D\mathcal{F}_0|_{E_2}) \geq \beta$,*

then, for every (E_1, E_2)-graph $G \subset B(0, r)$ with dispersion $\leq c$, $\mathcal{F}(G)$ is a $(D\mathcal{F}_0 \cdot E_1, D\mathcal{F}_0 \cdot E_2)$-graph with dispersion $\leq c$.

Proof Let $U \subset E_2$ be an open set and $\psi : U \to E_1$ a map whose graph $\{(\psi(v), v)) : v \in U\}$ is G. Then,

$$\mathcal{F}(G) = \{(L\psi(v) + p(\psi(v), v), Tv + q(\psi(v), v)) : v \in U\}.$$

Define $\phi : U \to D\mathcal{F}_0|_{E_2}$ by $\phi(v) = Tv + q(\psi(v), v)$. If $u, w \in U$ then

$$\begin{aligned} \|\phi(v) - \phi(w)\| &\geq \|T(v - w)\| - \|q(\psi(v), v)) - q(\psi(w), w))\| \\ &\geq m(T)\|v - w\| - \delta\alpha(\|\psi(v) - \psi(w)\| + \|v - w\|) \\ &\geq (m(T) - \delta\alpha(1 + c))\|v - w\| \\ &\geq (\beta - \delta\alpha(1 + c))\|v - w\| \end{aligned}$$

and since $\beta - \delta\alpha(1 + c) > 0$ we have that ϕ is a homeomorphism of U onto the open set $\phi(U)$ and we have $\text{lip}(\phi^{-1}) \leq (\beta - \delta\alpha(1 + c))^{-1}$.

Define $\hat{\psi}\colon \phi(U) \to D\mathcal{F}_0|_{E_1}$ by $\hat{\psi}(v) = L\psi(\phi^{-1}(v)) + p(\psi(\phi^{-1}(v)), \phi^{-1}(v)))$. Clearly we have

$$\mathcal{F}(G) = \{(\hat{\psi}(x), x)\colon x \in \phi(U)\},$$

which is a Lipschitz $(D\mathcal{F}_0 \cdot E_1, D\mathcal{F}_0 \cdot E_2)$-graph. Now, to compute the dispersion of $\mathcal{F}(G)$ we write $\hat{\psi} = \tilde{\psi}\phi^{-1}$, where $\tilde{\psi}(w) = L\psi(w) + p(\psi(w), w)$. Then,

$$\begin{aligned}
\|\tilde{\psi}(v) - \tilde{\psi}(w)\| &\leq \|L\psi(v) - L\psi(w)\| + \|p(\psi(v), v)) - p(\psi(w), w))\| \\
&\leq \|L\| \, \|\psi(v) - \psi(w)\| + \delta\alpha(\|\psi(v) - \psi(w)\| + \|v - w\|) \\
&\leq c\|L\| \, \|v - w\| + \delta\alpha(c\|v - w\| + \|v - w\|) \\
&\leq [c\|L\| + \delta\alpha(1 + c)]\|v - w\|
\end{aligned}$$

Now, we obtain that the dispersion of $\mathcal{F}(G)$ is $\leq c$:

$$\begin{aligned}
\sup_{x \neq y \in \phi(U)} \left\{ \frac{\|\hat{\psi}(x) - \hat{\psi}(y)\|}{\|x - y\|} \right\} &= \sup_{x \neq y \in \phi(U)} \left\{ \frac{\|\tilde{\psi}\phi^{-1}(x) - \tilde{\psi}\phi^{-1}(y)\|}{\|x - y\|} \right\} \\
&= \sup_{v \neq w \in U} \left\{ \frac{\|\tilde{\psi}(v) - \tilde{\psi}(w)\|}{\|\phi(v) - \phi(w)\|} \right\} \\
&= \sup_{v \neq w \in U} \left\{ \frac{\|\tilde{\psi}(v) - \tilde{\psi}(w)\|}{\|v - w\|} \frac{\|v - w\|}{\|\phi(v) - \phi(w)\|} \right\} \\
&= \sup_{v \neq w \in U} \left\{ \frac{\|\tilde{\psi}(v) - \tilde{\psi}(w)\|}{\|v - w\|} \right\} \mathrm{lip}(\phi^{-1}) \\
&\leq \frac{c\|L\| + \delta\alpha(1 + c)}{\beta - \delta\alpha(1 + c)} = c\frac{\|L\| + \frac{\delta\alpha(1+c)}{c}}{\beta - \delta\alpha(1 + c)} \leq c\frac{\frac{1}{2}\beta + \frac{\delta\alpha(1+c)}{c}}{\beta - \delta\alpha(1 + c)} \\
&= c\frac{\frac{1}{2} + \frac{\delta\alpha(1+c)}{c\beta}}{1 - \frac{\delta\alpha(1+c)}{\beta}} \leq c\frac{\frac{1}{2} + \frac{\delta\alpha(1+c)}{c\beta}}{1 - \delta\alpha\frac{1+c}{\beta}} < c
\end{aligned}$$

□

Let $g \in \mathrm{Homeo}(M)$, $r > 0$, $x \in M$ and $n \in \mathbb{N}$. We define a *Bowen ball* by:

$$B_n(g, r, x) = \bigcap_{j=0}^{n} g^{-j}(B(g^j(x), r)).$$

In [18] was proved that:

$$h_\mu(g) \geq \sup_{r>0} \int_M h_\mu(g, r, x)\, d\mu(x), \tag{9}$$

where

$$h_\mu(g, r, x) := \limsup_{n \to +\infty} \frac{1}{n} \left(-\log \mu(B_n(g, r, x))\right).$$

From Lemmas 6.1 and 2.1 and a simple induction argument we deduce the following result. For details see [29, Lemma 3.4].

Lemma 6.2 *Let $g \in \mathrm{Lipeo}^{\ell}_{\mu}(M)$ and $\mathcal{D} \subset M$ a g-invariant set. If there is a 1-dominated splitting on $\mathcal{D}$, say $T_{\mathcal{D}}M = E \oplus F$, then for any $c > 0$, there exists $r > 0$ such that for every $x \in \mathcal{D}$ and any (E_x, F_x)-graph G with dispersion $\leq c$ contained in a Bowen ball $B_n(g, x, r)$ $(n \geq 1)$, $g^n(G)$ is a $(Dg^n_x \cdot E_x, Dg^n_x \cdot F_x)$-graph with dispersion $\leq c$.*

Take a map $f \in \mathrm{Lipeo}^{\ell}_{\mu}(M)$, $\epsilon > 0$ and a μ-full measure set $\Sigma \subseteq \tilde{M}_f$ with an $m(x)$-dominated splitting $E \oplus F$ for f of a certain fixed index. Consider $N_\epsilon \in \mathbb{N}$ sufficiently large in order to have $\mu(\Sigma_\epsilon) > 1 - \epsilon$, where $\Sigma_\epsilon = \{x \in \Sigma \colon m(x) \leq N_\epsilon\}$. Observe that $g := f^N$, for $N = N_\epsilon!$, is such that $E \oplus F$ displays a 1-dominated splitting for Dg (cf. [29, p. 1430]).

Lemma 6.3 *Given g, N and ϵ as above, there exists $r > 0$ such that for μ-a.e. $x \in \Sigma_\epsilon$ we have:*

$$h_\mu(g, r, x) \geq N \sum_{i=1}^{\dim(F)} \lambda_i(f, x) - \epsilon. \tag{10}$$

Proof By Lemma 4.1 a dominated splitting can be extended to the closure of Σ_ϵ wherever is defined and moreover vary continuously with the point. So we can fix $a, c > 0$ such that if $x \in \Sigma_\epsilon$ and $y \in B(x, a)$ is a μ-generic point (Rademacher point), then for any linear subspace $G \subseteq T_x M$ which is a $(E(x), F(x))$-graph with dispersion $\leq c$, we have from Lemma 2.1 that

$$\big|\log|\det Dg_y|_G| - \log|\det Dg_x|_F|\big| < \epsilon.$$

and so

$$|\det Dg_y|_G| > |\det Dg_x|_F| e^{-\epsilon}. \tag{11}$$

Feeding Lemma 6.2 with $\mathcal{D} = \overline{\Sigma_\epsilon}$ (mod μ), the 1-dominated splitting $T_{\mathcal{D}}M = E \oplus F$ and the $c > 0$ above, there exists $r > 0$ such that for every $x \in \mathcal{D}$ and any (E_x, F_x)-graph G with dispersion $\leq c$ contained in a Bowen ball $B_n(g, x, r)$, $n \geq 1$, $g^n(G)$ is a $(Dg^n_x \cdot E_x, Dg^n_x \cdot F_x)$-graph with dispersion $\leq c$.

Given $x \in \Sigma_\epsilon$ and $y \in E(x)$ we denote by μ_E the volume-measure in $E(x)$ and by μ_F the volume-measure in $y + F(x)$. There exists $B > 0$ such that the disintegration holds for all $n \geq 1$:

$$\mu(B_n(g, r, x)) = B \int_{E(x)} \mu_F(\mathcal{F}_n(y))\, d\mu_E(y),$$

where $\mathcal{F}_n(y) = (y + F(x)) \cap B_n(g, r, x)$. Hence, Lemma 6.3 is proved once we obtain that for μ-a.e. $x \in \Sigma_\epsilon$ we have:

$$\limsup_{n \to +\infty} \inf_{y \in E(x)} -\frac{1}{n} \log \mu_F(\mathcal{F}_n(y)) \geq N \sum_{i=1}^{\dim(F)} \lambda_i(f, x) - \epsilon. \tag{12}$$

Considering that $\mathcal{F}_n(y) \neq \emptyset$ using Lemma 6.2 we obtain that $g^n(\mathcal{F}_n(y))$ is a $(Dg^n_x \cdot E_x, Dg^n_x \cdot F_x)$-graph with dispersion $\leq c$. Given $r \in (0, a)$ we take

$$D > \sup_{w \in \Sigma_\epsilon} \{\mathrm{Vol}(G) \colon G \subset B(w, r) \text{ is a } (E(x), F(x))\text{-graph with dispersion} \leq c\}.$$

Notice that for $g^n(x) \in \Sigma_\epsilon$ we have $g^n(\mathcal{F}_n(y)) \subseteq g^n(B_n(g, r, x)) \subseteq B(g^n(x), r)$. Thus,

$$D > \mathrm{Vol}(g^n(\mathcal{F}_n(y))) = \int_{\mathcal{F}_n(y)} \left|\det Dg^n_z\Big|_{T_z\mathcal{F}_n(y)}\right| d\mu_E(z). \tag{13}$$

For any $j = 0, 1, \ldots, n$ we have $g^j(\mathcal{F}_n(y)) \subseteq g^j(B_n(g, r, x)) \subseteq B(g^j(x), r) \subseteq B(g^j(x), a)$, and so given any $z \in \mathcal{F}_n(y)$ we have $d(g^j(z), g^j(x)) < a$ for all $j = 0, 1, \ldots, n$. Therefore,

$$\left|\det Dg^n_z\Big|_{T_z\mathcal{F}_n(y)}\right| = \Pi_{j=0}^{n-1}\left|\det Dg_{g^j(z)}\Big|_{T_{g^j(z)}g^j(\mathcal{F}_n(y))}\right| \overset{(11)}{\geq} \Pi_{j=0}^{n-1}\left|\det Dg_{g^j(x)}\Big|_{F(g^j(x))}\right| e^{-\epsilon}$$
$$= \left|\det Dg^n_x\Big|_{F(x)}\right| e^{-n\epsilon}.$$

By (13) and previous inequality we obtain

$$\begin{aligned}\frac{1}{n}\log D &\geq \frac{1}{n}\log\int_{\mathcal{F}_n(y)}\left|\det Dg^n_x\Big|_{F(x)}\right| e^{-n\epsilon}\,d\mu_E(z)\\ &= \frac{1}{n}\log\left[\mu_E(\mathcal{F}_n(y))\left|\det Dg^n_x\Big|_{F(x)}\right| e^{-n\epsilon}\right]\\ &= \frac{1}{n}\log\mu_E(\mathcal{F}_n(y)) + \frac{1}{n}\log\left|\det Dg^n_x\Big|_{F(x)}\right| - \epsilon,\end{aligned}$$

which is equivalent to

$$\inf_{y\in E(x)} -\frac{1}{n}\log\mu_F(\mathcal{F}_n(y)) \geq -\frac{1}{n}\log D + \frac{1}{n}\log\left|\det Dg^n_x\Big|_{F(x)}\right| - \epsilon.$$

Taking limits and recalling that $g = f^N$ we obtain

$$\limsup_{n\to+\infty}\inf_{y\in E(x)} -\frac{1}{n}\log\mu_F(\mathcal{F}_n(y)) \geq \lim_{n\to+\infty}\frac{1}{n}\log\left|\det Dg^n_x\Big|_{F(x)}\right| - \epsilon = N\sum_{i=1}^{\dim(F)}\lambda_i(f, x) - \epsilon,$$

and (12) is proved and thus the lemma. □

6.2 Proof of Theorem 3

We would like to show that for a given $f \in \mathrm{Lipeo}^{\ell}_{\mu}(M)$ if for μ-a.e. $x \in M$ there is an $m(x)$-dominated splitting $E \oplus F$ along the orbit of x (where $m\colon M \to \mathbb{N}$ is an f-invariant measurable function), then

$$h_\mu(f) \geq \int_M \sum_{i=1}^{\dim(F)}\lambda_i(f, x)\,d\mu(x).$$

Notice that it suffices to prove the theorem for a certain fixed dimension of F. Let be given g, N, ϵ and $r > 0$ like in Lemma 6.3. Let $C = \log \mathrm{lip}(f)$, and recall that for μ-a.e. x, $\log\|Df_x\| \leq C$.

We have:

$$
\begin{aligned}
h_\mu(g) &\overset{(9)}{\geq} \sup_{r>0} \int_M h_\mu(g,r,x)\,d\mu(x) \geq \sup_{r>0} \int_{\Sigma_\epsilon} h_\mu(g,r,x)\,d\mu(x) \\
&\overset{(10)}{\geq} \int_{\Sigma_\epsilon} (N \sum_{i=1}^{\dim(F)} \lambda_i(f,x) - \epsilon)\,d\mu(x) \\
&= \int_M N \sum_{i=1}^{\dim(F)} \lambda_i(f,x)\,d\mu(x) - \int_{M\setminus\Sigma_\epsilon} N \sum_{i=1}^{\dim(F)} \lambda_i(f,x)\,d\mu(x) - \epsilon\mu(\Sigma_\epsilon) \\
&\geq \int_M N \sum_{i=1}^{\dim(F)} \lambda_i(f,x)\,d\mu(x) - dNC\mu(M\setminus\Sigma_\epsilon) - \epsilon \\
&\geq \int_M N \sum_{i=1}^{\dim(F)} \lambda_i(f,x)\,d\mu(x) - dNC\epsilon - \epsilon.
\end{aligned}
$$

Therefore,

$$
h_\mu(f) = \frac{1}{N} h_\mu(g) \geq \int_M \sum_{i=1}^{\dim(F)} \lambda_i(f,x)\,d\mu(x) - dC\epsilon - \frac{\epsilon}{N},
$$

and since ϵ is arbitrarily small we get the statement of Theorem 3.

7 Proof of Theorem 1

In this section we are going to prove Theorem 1, which states that Pesin's entropy formula holds in a Λ-generic subset of $\mathrm{Lipeo}^\ell_\mu(M)$. By Theorem 2 we know that there exists a residual set $\mathcal{R} \subset \mathrm{Lipeo}^\ell_\mu(M)$ such that, for each $f \in \mathcal{R}$ and μ-almost every $x \in M$, the the Oseledets splitting of f is either dominated along the orbit of x or else the Lyapunov spectrum of f at x is trivial. Consider $f \in \mathcal{R}$. By Proposition 1 we have for μ-a.e. $x \in M$ that

$$
h_\mu(f) \leq \int_M \sum_{i=1}^{d} \lambda_i^+(f,x)\,d\mu(x),
$$

so we are left to see that

$$
h_\mu(f) \geq \int_M \sum_{i=1}^{d} \lambda_i^+(f,x)\,d\mu(x). \tag{14}
$$

Let $\mathcal{Z} \subseteq M$ stand for the set of points such that the Lyapunov spectrum of f at x is trivial and let $\mathcal{D} \subseteq M$ stand for the set of points such that the Oseledets splitting of f is dominated. Assume that $\mu(\mathcal{Z}), \mu(\mathcal{D}) > 0$ and define for any borelian $A \subseteq M$

$$
\mu_z(A) = \frac{\mu(A\cap\mathcal{Z})}{\mu(\mathcal{Z})} \quad \text{and} \quad \mu_d(A) = \frac{\mu(A\cap\mathcal{D})}{\mu(\mathcal{D})}.
$$

Clearly, $\mu_z(\mathcal{Z}) = 1$, $\mu_d(\mathcal{D}) = 1$ and $\mu = \mu(\mathcal{Z})\mu_z + \mu(\mathcal{D})\mu_d$. Therefore, using the affine property of the metric entropy we get

$$
h_\mu(f) = \mu(\mathcal{Z}) h_{\mu_z}(f) + \mu(\mathcal{D}) h_{\mu_d}(f).
$$

We only have to show that (14) holds for $h_{\mu_z}(f)$ and $h_{\mu_d}(f)$ separately. Since the metric entropy is always non-negative and the Lyapunov exponents of f are all zero in $\mathcal{Z}$ we get:

$$h_{\mu_z}(f) \geq 0 = \int_M \sum_{i=1}^{d} \lambda_i^+(f, x)\, d\mu_z(x).$$

Finally, Theorem 3 gives that:

$$h_{\mu_d}(f) \geq \int_M \sum_{i=1}^{d} \lambda_i^+(f, x)\, d\mu_d(x),$$

and Theorem 1 is proved.

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Compliance with Ethical Standards

Conflicts of interest The authors declare they have no conflicts of interest.

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