Bull Braz Math Soc, New Series (2018) 49:73–87
https://doi.org/10.1007/s00574-017-0048-6

Positivity of the Top Lyapunov Exponent for Cocycles on Semisimple Lie Groups over Hyperbolic Bases

Mário Bessa[1] · Jairo Bochi[2] · Michel Cambrainha[3] · Carlos Matheus[4] · Paulo Varandas[5] · Disheng Xu[6]

Received: 16 January 2017 / Accepted: 28 June 2017 / Published online: 4 July 2017

Abstract A theorem of Viana says that almost all cocycles over any hyperbolic system have nonvanishing Lyapunov exponents. In this note we extend this result to cocycles on any noncompact classical semisimple Lie group.

Bessa is partially supported by FCT—'Fundação para a Ciência e a Tecnologia', through Centro de Matemática e Aplicações (CMA-UBI), Universidade da Beira Interior, project UID/MAT/00212/2013. Bochi is partially supported by project Fondecyt 1140202 (Chile). Varandas is partially supported by CNPq-Brazil.

✉ Paulo Varandas
paulo.varandas@ufba.br
http://www.pgmat.ufba.br/varandas

Mário Bessa
bessa@ubi.pt
http://cmup.fc.up.pt/cmup/bessa/

Jairo Bochi
jairo.bochi@mat.uc.cl
http://www.mat.uc.cl/~jairo.bochi/

Michel Cambrainha
michel.cambrainha@uniriotec.br

Carlos Matheus
matheus.cmss@gmail.com

Disheng Xu
disheng.xu@imj-prg.fr
https://webusers.imj-prg.fr/disheng.xu

1 Departamento de Matemática, Universidade da Beira Interior, Rua Marquês d'Ávila e Bolama, 6201-001 Covilhã, Portugal

2 Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna, 4860 Santiago, Chile

Keywords Lyapunov exponents · Linear cocycles · Semisimple Lie groups

1 Introduction

Lyapunov exponents are ubiquitous in differentiable dynamics (Barreira and Pesin 2002), control theory (Colonius and Kliemann 2000), random walks on Lie groups (Furman 2002), one-dimensional Schrödinger operators (Bourgain 2005), among other fields. Let us recall the basic definitions. Consider (M, μ) a probability space, $f: M \to M$ a measure-preserving discrete-time dynamical system and $A: M \to \mathbb{R}^{d\times d}$ a (at least) measurable matrix-valued map. The pair (A, f) is called a *linear cocycle*. We form the products:

$$A^{(n)}(x) := A(f^{n-1}(x)) \cdots A(f(x))A(x). \tag{1.1}$$

Let $\|\cdot\|$ be any matrix norm, and assume that $\log^+ \|A\|$ is μ-integrable. The *(top) Lyapunov exponent* of the cocycle is

$$\lambda_1(A, f, x) := \lim_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x)\|, \tag{1.2}$$

which by the subadditive ergodic theorem is well-defined (possibly $-\infty$) for μ-almost every x, and is independent of the choice of norm. If μ is ergodic, then the Lyapunov exponent is almost everywhere equal to a constant, which we denote by $\lambda_1(A, f, \mu)$.

The Lyapunov exponent is a very subtle object of study. Let us explain the type of question we are interested in. Consider maps A taking values in the group $\mathrm{SL}(d, \mathbb{R})$. In that case, the Lyapunov exponent is nonnegative, and it is reasonable to expect that it should be positive except in some degenerate or fragile situations. As a result in this direction, Knill (1992) proved that for any base dynamics (f, μ) where the measure μ is ergodic and non-atomic, $\lambda_1(A, f, \mu) > 0$ for all maps A in a dense subset of the space $L^\infty(M, \mathrm{SL}(2, \mathbb{R}))$. Still in $d = 2$, this result was extended to virtually any regularity class (continuous, Hölder, smooth, analytic) by Avila (2011). The case $d > 2$ remains unsolved, though similar results have been obtained by Xu (2017) for some other matrix groups as the symplectic groups. In general, the sets of maps where the Lyapunov exponents are positive are believed to be not only dense, but also "large" in a probabilistic sense (see Avila 2011).

[3] Departamento de Matemática, Universidade Federal do Estado do Rio de Janeiro, Av. Pasteur 458, Rio de Janeiro 22290-240, Brazil

[4] Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), 93439 Villetaneuse, France

[5] Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, Salvador 40170-110, Brazil

[6] Université Paris Diderot, Sorbonne Paris Cité, Institut de Mathématiques de Jussieu-Paris Rive gauche, UMR 7586, CNRS, Sorbonne Universités, UPMC Université Paris 06, 75013 Paris, France

However, in low regularity as L^∞ or C^0, these sets can be "small" in a topological sense; indeed they can be locally meager (Bochi 2002; Bochi and Viana 2005).

Historically, the first case to be studied was random products of i.i.d. matrices, which fits in the general setting of linear cocycles by taking $f : \mathrm{SL}(d, \mathbb{R})^{\mathbb{Z}} \to \mathrm{SL}(d, \mathbb{R})^{\mathbb{Z}}$ the shift map $f((B_n)_{n\in\mathbb{Z}}) = (B_{n+1})_{n\in\mathbb{Z}}$, μ a probability measure on $\mathrm{SL}(d, \mathbb{R})^{\mathbb{Z}}$ of the form $\mu = \nu^{\mathbb{Z}}$ where ν is a probability measure on $\mathrm{SL}(d, \mathbb{R})$ with finite support, and $A : \mathrm{SL}(d, \mathbb{R})^{\mathbb{Z}} \to \mathrm{SL}(d, \mathbb{R})$ is the matrix-valued map $A((B_n)_{n\in\mathbb{Z}}) = B_0$. Furstenberg showed that that the Lyapunov exponent is positive under explicit mild conditions (Theorem 8.6 in Furstenberg 1963). Finer results were later obtained (still in the i.i.d. case) by Guivarc'h and Raugi (1989) and Gol'dsheid and Margulis (1989), among others.

As first shown in Ledrappier's seminal paper (Ledrappier 1986), and later vigorously expressed in the work of Viana and collaborators (Bonatti et al. 2003; Bonatti and Viana 2004; Avila and Viana 2007; Viana 2008; Avila and Viana 2010; Avila et al. 2013), the philosophy of random i.i.d. products of matrices can be adapted to other contexts, where Bernoulli shifts are replaced by more general classes of dynamical systems with hyperbolic behavior, at least under certain conditions on the maps A. A landmark result, proved by Viana (2008), can be stated informally as follows: if the dynamics (f, μ) is nonuniformly hyperbolic (and satisfies an additional technical but natural hypothesis) then, in spaces of sufficiently regular (at least Hölder) maps $A : M \to \mathrm{SL}(d, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, positivity of the Lyapunov exponent occurs on a set which is large in both a topological and in a probabilistic sense.

This note provides a extension of that theorem. More precisely, Viana posed the following problem (Viana 2008, Problem 4): *Characterize the class of groups $G \subset \mathrm{GL}(d, \mathbb{K})$ for which the theorem is valid. Does it include the symplectic group* $\mathrm{Sp}(d, \mathbb{K})$? Here we show that the class of groups for which Viana's theorem is valid includes all noncompact classical semisimple groups of matrices, in particular the symplectic groups, and also pseudo-unitary groups, quaternionic groups, etc.

The proof of our result has two new key ingredients. The first main ingredient is that the matrix functions consisting of evaluations at periodic points are submersions as functions of the cocycle (cf. Proposition 3.4). The second main ingredient is that a pair of matrices in a noncompact classical semisimple group whose projective actions preserve a common probability measure is contained in a semi-algebraic set of positive codimension (cf. Corollary 3.13). By combining these ingredients, we deduce our main result by proving that every cocycle with some zero Lyapunov exponent has a neighborhood (in the appropriate space) whose cocycles with zero exponents are contained in a Whitney stratified set of arbitrarily large codimension.

Let us mention that when f is quasiperiodic (and so lies in a region antipodal to hyperbolicity in the dynamical universe), the study of Lyapunov exponents forms another huge area of research: see for instance Avila and Krikorian (2015) and Duarte and Klein (2014) and references therein. Finally, we note that for *derivative cocycles* (i.e., where $A = Df$) very few general results are known, except in low topologies (Bochi 2002; Bochi and Viana 2005; Avila et al. 2016).

2 Precise Setting

In this section we recall some basic notions about multiplicative ergodic theory, and then state our results. The reader is referred to Barreira and Pesin (2002) and Viana (2008) for more details and references.

2.1 Lyapunov Exponents

The top Lyapunov exponent of a cocycle (A, f) was defined in (1.2). In general, we define Lyapunov exponents $\lambda_1(A, f, x) \geq \lambda_2(A, f, x) \geq \cdots \geq \lambda_d(A, f, x)$ by

$$\lambda_i(A, f, x) := \lim_{n\to\infty} \frac{1}{n} \log \sigma_i(A^{(n)}(x)), \tag{2.1}$$

where $\sigma_i(\cdot)$ denotes the i-th singular value.

2.2 Hyperbolic Measures and Local Product Structure

Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism[1] of a compact manifold M, and let μ be a invariant Borel probability measure. Suppose that μ is *hyperbolic*, that is, the Lyapunov exponents of the derivative cocycle Df are all different from zero at μ-almost every point x. So, by Oseledets theorem, we can split the tangent bundle $T_x M$ as the sum of the subspaces E^{u}_x and E^{s}_x corresponding to positive and negative Lyapunov exponents of the derivative cocycle $A = Df$, respectively, defined by (2.1).

Since f is a $C^{1+\alpha}$ diffeomorphism, given a hyperbolic probability measure μ, Pesin's stable manifold theorem (see e.g. Barreira and Pesin 2002) says that, for μ-almost every x, there exists a C^1-embedded disk $W^{\mathrm{s}}_{\mathrm{loc}}(x)$ (local stable manifold at x) such that $T_y W^{\mathrm{s}}_{\mathrm{loc}}(x) = E^{\mathrm{s}}_y$, it is forward invariant, i.e. $f(W^{\mathrm{s}}_{\mathrm{loc}}(x)) \subset W^{\mathrm{s}}_{\mathrm{loc}}(f(x))$, and the following holds: given $0 < \tau_x < |\lambda_{1+\dim E^{\mathrm{u}}_x}(Df, f, x)|$, there exists $K_x > 0$ such that $d(f^n(y), f^n(z)) \leq K_x e^{-n\tau_x} d(y, z)$ for every $y, z \in W^{\mathrm{s}}_{\mathrm{loc}}(x)$. Local unstable manifolds $W^{\mathrm{u}}_{\mathrm{loc}}(x)$ are defined analogously using E^{u}_x and f^{-1}.

Moreover, since local invariant manifolds and the constants above vary measurably with the point x one can select *hyperbolic blocks* $\mathcal{H}(K, \tau)$ in such a way that $K_x \leq K$ and $\tau_x \geq \tau$ for all $x \in \mathcal{H}(K, \tau)$, the local manifolds $W^{\mathrm{s}}_{\mathrm{loc}}(x)$ and $W^{\mathrm{u}}_{\mathrm{loc}}(x)$ vary continuously with $x \in \mathcal{H}(K, \tau)$; moreover, $\mu(\mathcal{H}(K, \tau)) \to 1$ as $K \to \infty$ and $\tau \to 0$. In particular, if $x \in \mathcal{H}(K, \tau)$ and $\delta > 0$ is small enough, then for every y, $z \in B(x, \delta) \cap \mathcal{H}(K, \tau)$, the intersection $W^{\mathrm{u}}_{\mathrm{loc}}(y) \cap W^{\mathrm{s}}_{\mathrm{loc}}(z)$ is transverse and consists of a unique point, denoted $[y, z]$.

[1] We need $C^{1+\alpha}$ regularity in order to apply the so-called *Pesin's theory* (providing a measurable family of invariant manifolds with many good properties). In fact, it is known (see Bonatti et al. 2013 for instance) that Pesin's theory may fail in C^1 regularity.

For each $x \in \mathcal{H}(K, \tau)$, define sets:

$$\mathcal{N}_x^{\mathrm{u}}(\delta) := \{[x, y] \in W_{\mathrm{loc}}^{\mathrm{u}}(x) : y \in \mathcal{H}(K, \tau) \cap B(x, \delta)\},$$
$$\mathcal{N}_x^{\mathrm{s}}(\delta) := \{[y, x] \in W_{\mathrm{loc}}^{\mathrm{s}}(x) : y \in \mathcal{H}(K, \tau) \cap B(x, \delta)\}.$$

Let $\mathcal{N}_x(\delta)$ be the image of $\mathcal{N}_x^{\mathrm{u}}(\delta) \times \mathcal{N}_x^{\mathrm{s}}(\delta)$ under the map $[\cdot, \cdot]$. This is a small "box" neighborhood of x in the block $\mathcal{H}(K, \tau)$, and (reducing δ if necessary) the following map is a homeomorphism:

$$\Upsilon_x : \mathcal{N}_x(\delta) \to \mathcal{N}_x^{\mathrm{u}}(\delta) \times \mathcal{N}_x^{\mathrm{s}}(\delta)$$
$$y \mapsto ([x, y], [y, x])$$

Definition 2.1 (Viana 2008, p. 646) The hyperbolic measure μ has *local product structure* if for every (K, τ), every small $\delta > 0$ as before, and every $x \in \mathcal{H}(K, \tau)$, the measure $\mu|_{\mathcal{N}_x(\delta)}$ is equivalent to the product measure $\mu_x^{\mathrm{u}} \times \mu_x^{\mathrm{s}}$, where μ_x^i denotes the conditional measure of $(\Upsilon_x)_*(\mu|_{\mathcal{N}_x(\delta)})$ on $\mathcal{N}_x^i(\delta)$, for $i \in \{\mathrm{u}, \mathrm{s}\}$.

2.3 Space of Cocycles

The relevant functional spaces of linear cocycles for our subsequent discussion are defined as follows. Let G be a Lie subgroup of $\mathrm{GL}(d, \mathbb{C})$, let M be a Riemannian compact manifold M, and let $(r, \nu) \in \mathbb{N} \times [0, 1] - \{(0, 0)\}$. Let $C^{r,\nu}(M, G)$ denote the set of maps $A : M \to G$ of class C^r such that $D^r A$ is ν-Hölder continuous if $\nu > 0$. We equip this set with the topology induced by the distance:

$$d_{r,\nu}(A, B) := \sup_{0 \le j \le r} \|D^j(A - B)(x)\| + \sup_{x \ne y} \frac{\|D^r(A - B)(x) - D^r(A - B)(y)\|}{d(x, y)^\nu},$$

where the last term is omitted if $\nu = 0$. Then $C^{r,\nu}(M, G)$ is a Banach manifold (see e.g. Palais 1968).

2.4 Statement of the Results

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let $d \ge 2$. Let $\mathbf{G}$ be a $\mathbb{K}$-algebraic subgroup of $\mathrm{SL}(d, \mathbb{C})$, i.e., a group of complex $d \times d$ matrices of determinant 1, defined by polynomial equations with coefficients in $\mathbb{K}$. Denote $G := \mathbf{G} \cap \mathrm{SL}(d, \mathbb{K})$. Henceforth we will assume the following properties:

(1) $\mathbf{G}$ is *connected* (or equivalently, $\mathbf{G}$ is irreducible as an algebraic set);
(2) G is *semisimple* (or equivalently, $\mathbf{G}$ is semisimple);
(3) G is *noncompact*;
(4) G *acts irreducibly* on $\mathbb{K}^d$, that is, the only subspaces $V \subset \mathbb{K}^d$ invariant under the whole action of G are the trivial subspaces $V = \{0\}$ and $V = \mathbb{K}^d$.

Our assumptions are satisfied by all noncompact classical groups G, that is, the special linear groups $\mathrm{SL}(d, \mathbb{K})$ for $d \geq 2$ and $\mathrm{SL}(n, \mathbb{H}) \simeq \mathrm{SU}^*(2n)$ for $n \geq 1$ (where $\mathbb{H}$ is the field of quaternions), the symplectic groups $\mathrm{Sp}(n, \mathbb{K})$, $\mathrm{Sp}(n, m)$ for $n, m \geq 1$, the special indefinite orthogonal groups $\mathrm{SO}(m, n)$ for $m, n \geq 1$, $m + n \geq 3$, the special unitary groups $\mathrm{SU}(m, n)$ for $m, n \geq 1$, and the quaternionic orthogonal groups $\mathrm{SO}^*(2n)$ for $n \geq 2$. We refer the reader to Knapp (2002, pp. 110–118) for definitions and basic properties of classical semisimple Lie groups.

The following result is exactly Theorem A in Viana (2008) when $G = \mathrm{SL}(d, \mathbb{K})$:

Theorem A *Let G be a group of matrices satisfying the hypotheses above. Let f be a $C^{1+\alpha}$-diffeomorphism of a compact manifold M. Let μ be a f-invariant ergodic* hyperbolic *non-atomic probability measure with* local product structure. *Let $(r, \nu) \in \mathbb{N} \times [0, 1] - \{(0, 0)\}$. Then there exists an open and dense subset $\mathcal{G}$ of $C^{r,\nu}(M, G)$ such that for any $A \in \mathcal{G}$, the cocycle (A, f) has at least one positive Lyapunov exponent at μ-a.e. point. Moreover, the complement of $\mathcal{G}$ in $C^{r,\nu}(M, G)$ has infinite codimension.*

The last statement means that the complement of $\mathcal{G}$ is locally contained in Whitney stratified sets (see Gibson 1976) of arbitrarily large codimension. In particular, $\mathcal{G}$ is large in a very strong probabilistic sense. Arguing exactly as in Viana (2008, p. 676), we obtain the following consequence in the non-ergodic case:

Corollary 2.2 *Let G be a group of matrices satisfying the hypotheses above. Let f be a $C^{1+\alpha}$-diffeomorphism of a compact manifold M. Let μ be a f-invariant ergodic* hyperbolic *non-atomic probability measure with* local product structure. *Let $(r, \nu) \in \mathbb{N} \times [0, 1] - \{(0, 0)\}$. Then there exists a residual subset $\mathcal{R}$ of $C^{r,\nu}(M, G)$ such that for any $A \in \mathcal{G}$, the cocycle (A, f) has at least one positive Lyapunov exponent at μ-a.e. point.*

Following Viana (2008), we require hyperbolicity and local product structure in the statements above. The point is to ensure the existence of the so-called *stable and unstable holonomies*, which allow one to compare the linear actions of the cocycle at different points of the phase space. As it turns out, it is possible to derive the existence of such holonomies in other settings including certain linear cocycles over volume preserving and accessible partially hyperbolic dynamics: in particular, even though we have not checked all details, it seems plausible that the techniques introduced in Avila et al. (2013) together with the arguments in this paper might lead to variants of our results for typical Hölder cocycles over volume preserving, center-bunched and accessible partially hyperbolic C^2-diffeomorphisms.

Finally, let us comment on the properties (1)–(4) of G above. First, hypothesis (3) is clearly a necessary condition for the existence of positive Lyapunov exponents: otherwise, all Lyapunov exponents would vanish. Second, in the context of Theorem A, the set of cocycles with zero Lyapunov exponents exhibit a rigidity phenomenon: the projective action of pairs of matrices at some suitable periodic points preserve a common invariant measure (cf. Proposition 3.5 and (3.11)). On the other hand, our hypotheses (1), (2), (3) and (4) ensure that the pairs of matrices in G whose projective actions preserve a common measure belong to a proper semi-algebraic set $Z \subset G \times G$

(cf. Corollary 3.13). These results permit to deduce Theorem A from the fact that certain evaluation maps are submersions (cf. Proposition 3.4).

3 Proofs

Here we review some intermediate results from Viana (2008) in Sects. 3.1 and 3.2, then we recall some algebraic facts in Sect. 3.4, and finally we prove Theorem A in Sect. 3.5.

3.1 Holonomies

In this and in the next subsection, we assume that f is a $C^{1+\alpha}$-diffeomorphism of a Riemannian compact manifold M preserving a non-atomic hyperbolic measure μ with local product structure, and that $A \in C^{r,\nu}(M, \mathrm{SL}(d, \mathbb{K}))$ for some $(r, \nu) \in \mathbb{N} \times [0, 1] - \{(0, 0)\}$ (and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

A key insight from Viana (2008) is that the vanishing of Lyapunov exponents of the cocycle (A, f, μ) implies the existence of a dynamical structure called *stable and unstable holonomies*.

More concretely, let f be a $C^{1+\alpha}$-diffeomorphism of a Riemannian compact manifold M preserving a non-atomic hyperbolic measure with local product structure. Let $A \in C^0(M, \mathrm{SL}(d, \mathbb{K}))$ be a continuous linear cocycle.

Definition 3.1 Given $N \geq 1$ and $\theta > 0$, let $\mathscr{D}_A(N, \theta)$ denote the set of points $x \in M$ satisfying:

$$\prod_{j=0}^{k-1} \|A^{(N)}(f^{jN}(x))\| \, \|A^{(N)}(f^{jN}(x))^{-1}\| \leq e^{kN\theta} \quad \text{for all } \ k \in \mathbb{N}.$$

We say that $\mathcal{O}$ is a *holonomy block for* A if it is a compact subset of $\mathcal{H}(K, \tau) \cap \mathscr{D}_A(N, \theta)$ for some constants K, τ, N, θ satisfying $3\theta < \tau$.

By Viana (2008, Corollary 2.4), if all Lyapunov exponents of (A, f) vanish at μ-almost every point then there exist holonomy blocks of measure arbitrarily close to 1.

By Viana (2008, Proposition 2.5), the limits

$$H^{\mathrm{s}}_{A,x,y} = H^{\mathrm{s}}_{x,y} := \lim_{n \to +\infty} A^{(n)}(y)^{-1} A^{(n)}(x) \quad \text{and}$$
$$H^{\mathrm{u}}_{A,x,z} = H^{\mathrm{u}}_{x,z} := \lim_{n \to -\infty} A^{(n)}(z)^{-1} A^{(n)}(x),$$

called *stable and unstable holonomies*, exist whenever x belongs to a holonomy block $\mathcal{O}$, $y \in W^{\mathrm{s}}_{\mathrm{loc}}(x)$ and $z \in W^{\mathrm{u}}_{\mathrm{loc}}(x)$. These holonomy maps depend differentiably on the cocycle:

Proposition 3.2 (Viana 2008, Lemma 2.9) *Given a periodic point p in a holonomy block and points $y \in W^{s}_{loc}(p)$ and $z \in W^{u}_{loc}(p)$, the maps $B \mapsto H^{s}_{B,p,y}$ and $B \mapsto H^{u}_{B,z,p}$ from a small neighborhood $\mathcal{U}$ of A to* $\mathrm{SL}(d,\mathbb{C})$ *are C^1, with derivatives*:

$$
\begin{aligned}
\partial_B H^{s}_{B,p,y} : \dot{B} \mapsto & \sum_{i=0}^{\infty} B^{(i)}(y)^{-1}[H^{s}_{B,f^i(p),f^i(y)} B(f^i(p))^{-1} \cdot \dot{B}(f^i(p)) \\
& - B(f^i(y))^{-1} \dot{B}(f^i(y)) H^{s}_{B,f^i(p),f^i(y)}] \cdot B^{(i)}(p) \qquad (3.1) \\
\partial_B H^{u}_{B,z,p} : \dot{B} \mapsto & \sum_{i=1}^{\infty} B^{(-i)}(p)^{-1}[H^{u}_{B,f^{-i}(z),f^{-i}(p)} B(f^{-i}(z)) \dot{B}(f^{-i}(z)) \\
& - B(f^{-i}(p)) \dot{B}(f^{-i}(p)) H^{u}_{B,f^{-i}(z),f^{-i}(p)}] \cdot B^{(-i)}(z) \qquad (3.2)
\end{aligned}
$$

Remark 3.3 In this statement, it is implicit the fact ensured by Viana (2008, Corollary 2.11) that the same holonomy block works for all $B \in \mathcal{U}$.

Suppose $\mathcal{O}_i$, where $i = 1, \dots, l$, are holonomy blocks of (f, A) containing horseshoes H_i associated to periodic points $p_i \in \mathcal{O}_i$ of minimal periods κ_i and some homoclinic points of p_i, say $z_i \in W^{u}_{loc}(p_i)$ and $f^{q_i}(z_i) \in W^{s}_{loc}(p_i)$, $q_i > 0$, such that $p_i, z_i \in \mathrm{supp}(\mu|\mathcal{O}_i \cap f^{-\kappa_i}(\mathcal{O}_i))$. By the remark above we know there is a neighborhood $\mathcal{U}$ of A such that all the same holonomy blocks $\mathcal{O}_i$ (hence p_i, z_i, q_i) still work for any $B \in \mathcal{U}$. Then we have the following important property:

Proposition 3.4 *Let* $\mathbf{G}$ *be a $\mathbb{K}$-algebraic subgroup of* $\mathrm{SL}(d,\mathbb{C})$ *and* $G := \mathbf{G} \cap \mathrm{SL}(d,\mathbb{K})$. *Assume that for each $1 \le i \le l$ the set $\mathcal{O}_i \subset M$ denotes a holonomy block with respect to all cocycles in an open set $\mathcal{U}$ of Hölder continuous cocycles taking values on G. Given periodic points $p_i \in \mathcal{O}_i$ of minimal periods κ_i and homoclinic points z_i such that $z_i \in W^{u}_{loc}(p_i)$ and $f^{q_i}(z_i) \in W^{s}_{loc}(p_i)$, $q_i > 0$, the map*

$$
\begin{aligned}
\Phi : \mathcal{U} &\to G^{2l} \\
B &\mapsto (g_{1,1}(B), \dots, g_{l,1}(B), g_{1,2}(B), \dots, g_{l,2}(B))
\end{aligned}
$$

is a submersion at every $B \in \mathcal{U}$, where

$$
g_{i,1}(B) := B^{(\kappa_i)}(p_i) \quad \text{and} \quad g_{i,2}(B) := H^{s}_{B,f^{q_i}(z_i),p_i} \circ B^{(q_i)}(z_i) \circ H^{u}_{B,p_i,z_i} \qquad (3.3)
$$

Proof Fix V_{z_i}, V_{p_i} some neighborhoods of p_i and z_i. Without loss of generality we could assume $V_{z_i}, V_{p_i}, i = 1, \dots, l$ are small enough such that

$$
\begin{aligned}
f^n(p_i) \cap V_{z_j} &= \emptyset, \quad \forall i, j, n \\
f^n(p_i) \cap V_{p_j} &= \emptyset \quad \text{except if } i = j \text{ and } \kappa_i | n \qquad (3.4) \\
f^n(z_i) \cap V_{z_j} &= \emptyset \quad \text{except if } i = j \text{ and } n = 0
\end{aligned}
$$

We claim that the derivative of map Φ is surjective at every point of $\mathcal{U}$, even when restricted to the subspace of tangent vectors $\dot{B}$ supported on $\bigcup_i V_{p_i} \cup \bigcup_i V_{z_i}$. In fact for every $B \in \mathcal{U}$, for tangent vectors $\dot{B}$ supported on $\bigcup_i V_{p_i} \cup \bigcup_i V_{z_i}$ we will prove that the derivative of Φ has the following lower triangular form:

$$\begin{aligned}\partial_B \Phi^T(\dot{B}) &= (\partial_B g_{1,1}(\dot{B}), \ldots, \partial_B g_{l,1}(\dot{B}),\\ &\partial_B g_{1,2}(\dot{B}), \ldots, \partial_B g_{l,2}(\dot{B}))^T = \begin{pmatrix} \partial\Phi_{1,1} & 0 \\ * & \partial\Phi_{2,2} \end{pmatrix} \cdot \begin{pmatrix} \dot{B}_p \\ \dot{B}_z \end{pmatrix}\end{aligned} \tag{3.5}$$

where $\dot{B}_p = (\dot{B}(p_1), \ldots, \dot{B}(p_l))^T$, $\dot{B}_z = (\dot{B}(z_1), \ldots, \dot{B}(z_l))^T$ and $\partial\Phi_{1,1}$, $\partial\Phi_{2,2}$ are two diagonal surjective linear maps.

By (3.4), we easily get

$$\begin{aligned}&\partial_B(g_{i,1}(\dot{B}))\\ &= \begin{cases} g_{i,1}(B) \cdot B(p_i)^{-1} \cdot \dot{B}(p_i), & \text{if } \operatorname{supp}(\dot{B}) \subset V_{p_i}, \\ 0, & \text{if } \operatorname{supp}(\dot{B}) \subset V_{p_j}, \quad j \neq i \ \text{ or } \ V_{z_j}, 1 \leq j \leq l. \end{cases}\end{aligned} \tag{3.6}$$

By $g_{i,2}$'s definition,

$$\begin{aligned}\partial_B(g_{i,2}(\dot{B})) = &\ \partial_B H^{\mathrm{s}}_{B, f^{q_i}(z_i), p_i}(\dot{B}) \cdot B^{(q_i)}(z_i) \cdot H^{\mathrm{u}}_{B, p_i, z_i}\\ &+ H^{\mathrm{s}}_{B, f^{q_i}(z_i), p_i} \cdot \partial_B B^{(q_i)}(z_i)(\dot{B}) \cdot H^{\mathrm{u}}_{B, p_i, z_i}\\ &+ H^{\mathrm{s}}_{B, f^{q_i}(z_i), p_i} \cdot B^{(q_i)}(z_i) \cdot \partial_B H^{\mathrm{u}}_{B, p_i, z_i}(\dot{B})\end{aligned} \tag{3.7}$$

By (3.1), (3.2) and (3.4), for any j,

$$\partial_B H^{\mathrm{s}}_{B, f^{q_i}(z_i), p_i}(\dot{B}) = \partial_B H^{\mathrm{u}}_{B, p_i, z_i}(\dot{B}) = 0 \text{ if } \operatorname{supp}(\dot{B}) \subset V_{z_j} \tag{3.8}$$

and

$$\partial_B B^{(q_i)}(z_i)(\dot{B}) = \begin{cases} B^{(q_i)}(z_i) \cdot B(z_i)^{-1} \cdot \dot{B}(z_i) & \text{if } \operatorname{supp}(\dot{B}) \subset V_{z_i}, \\ 0, & \text{if } \operatorname{supp}(\dot{B}) \subset V_{z_j}, \quad j \neq i. \end{cases} \tag{3.9}$$

Combining (3.7), (3.8) and (3.9) we get

$$\begin{aligned}&\partial_B(g_{i,2}(\dot{B}))\\ &= \begin{cases} H^{\mathrm{s}}_{B, f^{q_i}(z_i), p_i} \cdot B^{(q_i)}(z_i) \cdot B(z_i)^{-1} \cdot \dot{B}(z_i) \cdot H^{\mathrm{u}}_{B, p_i, z_i} & \text{if } \operatorname{supp}(\dot{B}) \subset V_{z_i}, \\ 0, & \text{if } \operatorname{supp}(\dot{B}) \subset V_{z_j}, j \neq i. \end{cases}\end{aligned} \tag{3.10}$$

Then by (3.10), (3.6) and invertibility of $H^{\mathrm{s,u}}$ and B, we get (3.5). As explained before, Proposition 3.4 follows. □

3.2 Disintegrations

Let f_A denote the induced *projectivized cocycle*, that is, the skew-product map on $M \times \mathbb{P}^{d-1}(\mathbb{K})$ defined by $(x, [v]) \mapsto (f(x), [A(x)v])$.

By compactness of the projective space, the projectivized cocycle f_A always has invariant probability measures m on $M \times \mathbb{P}^{d-1}(\mathbb{K})$ projecting down to μ on M. Any such measure m can be disintegrated (in an essentially unique way) into a family of measures m_z on $\{z\} \times \mathbb{P}^{d-1}(\mathbb{K})$, $z \in M$, in the sense that $m(C) = \int m_z(C \cap (\{z\} \times \mathbb{P}^{d-1}(\mathbb{K})))\, d\mu(z)$ for all measurable subsets $C \subset M \times \mathbb{P}^{d-1}(\mathbb{K})$: see Bogachev (2007, Section 10.6).

As explained in Sect. 2.2, (f, μ) has hyperbolic blocks $\mathcal{H}(K, \tau)$ of almost full μ-measure. Given a holonomy block $\mathcal{O}$ of positive μ-measure inside a hyperbolic block $\mathcal{H}(K, \tau)$, $\delta > 0$ sufficiently small (depending on K and τ) and a point $x \in \operatorname{supp}(\mu|\mathcal{O})$, we denote by $\mathcal{N}_x(\mathcal{O}, \delta)$, $\mathcal{N}_x^{\mathrm{u}}(\mathcal{O}, \delta)$ and $\mathcal{N}_x^{\mathrm{s}}(\mathcal{O}, \delta)$ the subsets of $\mathcal{N}_x(\delta)$, $\mathcal{N}_x^{\mathrm{u}}(\delta)$ and $\mathcal{N}_x^{\mathrm{s}}(\delta)$ obtained by replacing $\mathcal{H}(K, \tau)$ by $\mathcal{O}$ in the definitions.

The next result extracted from Viana (2008, Proposition 3.5) says that the disintegration behaves in a rigid way when all Lyapunov exponents of the cocycle vanish. For simplicity, the action of a linear map L on the projective space is also denoted by L.

Proposition 3.5 *Suppose that all Lyapunov exponents of (A, f) vanish at μ-almost every point.*

If $\mathcal{O}$ is a holonomy block of positive μ-measure, $\delta > 0$ is sufficiently small and $x \in \operatorname{supp}(\mu|\mathcal{O})$, then every f_A-invariant probability measure m on $M \times \mathbb{P}^{d-1}(\mathbb{K})$ projecting down to μ on M admits a disintegration $\{m_z : z \in M\}$ such that the function $\operatorname{supp}(\mu|\mathcal{N}_x(\mathcal{O}, \delta)) \ni z \mapsto m_z$ is continuous in the weak topology and, moreover,*

$$(H^{\mathrm{s}}_{y,z})_* m_y = m_z = (H^{\mathrm{u}}_{w,z})_* m_w$$

for all $y, z, w \in \operatorname{supp}(\mu|N_x(\mathcal{O}, \delta))$ with $y \in W^{\mathrm{s}}_{\mathrm{loc}}(z)$ and $w \in W^{\mathrm{u}}_{\mathrm{loc}}(z)$.

We shall exploit the rigidity condition in the previous proposition through the following result extracted from Viana (2008, Proposition 4.5) ensuring the existence of holonomy blocks containing periodic points and some of its homoclinic points when all Lyapunov exponents vanish in a set of positive measure.

Proposition 3.6 *Suppose that all Lyapunov exponents of (A, f) vanish at μ-almost every point. Then for any $l > 0$, there exists l holonomy blocks $\mathcal{O}_i$, $1 \leq i \leq l$, containing horseshoes H_i associated to periodic points (with different orbits) $p_i \in \mathcal{O}_i$ of period $\pi(p_i)$ and some homoclinic points z_i of p_i such that $z_i \in \operatorname{supp}(\mu|\mathcal{O}_i \cap f^{-\pi(p_i)}(\mathcal{O}_i))$.*

3.3 Some Facts About Semi-algebraic Sets

Recall that a subset of $\mathbb{R}^n$ is called *semi-algebraic* if it is defined by finitely many polynomial inequalities[2] (see Gibson 1976, p. 17), and the dimension of a semi-algebraic set is the maximal local dimension near regular points (see Gibson 1976, p. 18). By the Tarski–Seidenberg theorem, the image $Y = f(X)$ of a semi-algebraic subset X of $\mathbb{R}^n$ under any polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^p$ is also a semi-algebraic subset with dimension $\dim(Y) \leq \dim(X)$: cf. Gibson (1976, pp. 18 and 28).

Semi-algebraic sets are not necessarily smooth; however, their singular points are relatively "tame". In order to state this tameness properly, we need some classical concepts from differential topology. A *stratification* of a set $W \subset \mathbb{R}^n$ is a locally finite partition of W into smooth submanifolds, called *strata*. A stratification is called *Whitney* if it satisfies the following regularity condition: Suppose that (x_i) is a sequence of points in a stratum X converging to a point $x \in X$, (y_i) is a sequence in a stratum Y converging to the same point $x \in X$; suppose also that $x_i \neq y_i$ for each i and that the lines containing $x_i - y_i$ converge to a one-dimensional subspace $L \subset \mathbb{R}^n$; finally suppose that the tangent spaces $T_{y_i}Y$ converge to a space $T \subset \mathbb{R}^n$; then we have $L \subset T$. Using local coordinates, the ambient space $\mathbb{R}^n$ in the definitions above can be replaced by any smooth Banach manifold. See Gibson (1976, pp. 9–11) for more details.

For later use, we list in the proposition below some basic properties of Whitney stratified sets:

Proposition 3.7 *Any semi-algebraic set is Whitney stratified. The product of Whitney stratified sets is Whitney stratified (and their codimensions add). The pre-image of a Whitney stratified set under a submersion is Whitney stratified (and the codimension is preserved).*

Proof (*Indication of proof*) It is shown in Gibson (1976, p. 20) that semi-algebraic sets are Whitney stratified. The fact of products of Whitney stratified sets are Whitney stratified is stated in Gibson (1976, p. 16). Finally, the fact that the pre-image of a Whitney stratified set under a submersion is Whitney stratified is a particular case of the statement (1.4) in Gibson (1976, p. 14). □

3.4 Some Facts About Linear Algebraic Groups

Recall that G is an algebraic group of matrices satisfying the hypotheses listed at Sect. 2.4. In this subsection we collect some algebraic facts that we will use.

The following property, shown by Breuillard (2008, Lemma 6.8) (see also Aoun 2013, Lemma 7.7) only needs the fact that G is algebraic and semisimple (hypothesis (2)):

Proposition 3.8 *There exists a proper algebraic subvariety $V \subset G \times G$ such that any pair of elements $(g_1, g_2) \in (G \times G) - V$ generates a Zariski dense subgroup of G.*

[2] I.e., a semi-algebraic set is an element of the smallest Boolean ring of subsets of $\mathbb{R}^n$ containing all subsets of the form $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : P(x_1, \ldots, x_n) > 0\}$ with $P \in \mathbb{R}[X_1, \ldots, X_n]$.

The following is Proposition 3.2.15 in Zimmer (1984), and uses our hypotheses (1) and (4):

Proposition 3.9 *Let m be a probability measure on the projective space $\mathbb{P}^{d-1}(\mathbb{K})$, and let G_m be the set of elements of G whose projective actions preserve m. Then*:

(i) *G_m is compact, or*
(ii) *G_m is contained in a proper algebraic subgroup of G.*

Remark 3.10 Actually G_m is an amenable subgroup, by Theorem 2.7 in Moore (1979), but we will not need this fact.

Remark 3.11 If $\mathbb{K} = \mathbb{R}$ then property (i) in Proposition 3.9 actually implies (under our hypothesis (3)) property (ii). Indeed, by a well-known fact (Knapp 2002, Proposition 4.6), every compact subgroup of $\mathrm{SL}(d, \mathbb{R})$ preserves a positive definite quadratic form, and in particular is $\mathbb{R}$-algebraic.[3]

Lemma 3.12 *Suppose $\mathbb{K} = \mathbb{C}$. Then there is a semi-algebraic set $W \subset G \times G$ of positive codimension such that for any pair of elements $(g_1, g_2) \in (G \times G) - W$, the group they generate is not contained in any compact subgroup of G.*

Proof Let K be a maximal compact subgroup of G. We think of G as a complexification of K: in particular, the Lie algebra of G is the tensor product over $\mathbb{R}$ of $\mathbb{C}$ and the Lie algebra of K, and, *a fortiori*, $\dim_{\mathbb{R}}(G) = 2 \cdot \dim_{\mathbb{R}}(K)$. Consider a maximal Abelian subgroup A of G and the corresponding decomposition $G = KAK$ coming from the diffeomorphism

$$K \times \exp(\mathfrak{p}) \to G \quad \text{where} \quad \mathfrak{p} = \bigcup_{k \in K} \mathrm{Ad}(k) \cdot \mathfrak{a},$$

Ad(.) denotes the adjoint action, and $\mathfrak{a}$ is the Lie algebra of A. Note that $\dim_{\mathbb{R}}(G) > \dim_{\mathbb{R}}(K) + \dim_{\mathbb{R}}(A)$ (as one can infer, for instance, from the Killing–Cartan classification of simple complex Lie groups via Dynkin diagrams: see, e.g., Knapp 2002 for more details).

Define $\Phi : K \times A \times K \times K \to G \times G$ by $\Phi(k, a, u, v) = (kaua^{-1}k^{-1}, kava^{-1}k^{-1})$. Note that Φ is a polynomial map between semi-algebraic sets. Hence, its image $W := \Phi(K \times A \times K \times K)$ is a semi-algebraic set of dimension

$$\dim_{\mathbb{R}}(W) \le \dim_{\mathbb{R}}(K \times A \times K \times K) = (\dim_{\mathbb{R}}(K) + \dim_{\mathbb{R}}(A)) + 2 \cdot \dim_{\mathbb{R}}(K).$$

Since $\dim_{\mathbb{R}}(G) > \dim_{\mathbb{R}}(K) + \dim_{\mathbb{R}}(A)$ and $\dim_{\mathbb{R}}(G \times G) = 2 \cdot \dim_{\mathbb{R}}(G) = \dim_{\mathbb{R}}(G) + 2 \cdot \dim_{\mathbb{R}}(K)$, it follows that

$$\dim_{\mathbb{R}}(W) \le (\dim_{\mathbb{R}}(K) + \dim_{\mathbb{R}}(A)) + 2 \cdot \dim_{\mathbb{R}}(K) < \dim_{\mathbb{R}}(G \times G).$$

[3] These implications fail in the complex case; for example the compact group $\mathrm{SU}(d)$ is Zariski-dense in $\mathrm{SL}(d, \mathbb{C})$.

In summary, W is a semi-algebraic subset of $G \times G$ of positive codimension. Therefore, the proof of the lemma will be complete once we show that if $(g_1, g_2) \in G \times G$ generates a group contained in a compact subgroup of G, then $(g_1, g_2) \in W$. In this direction, we observe that K is a maximal compact subgroup of G, so that if the closure of the subgroup generated by g_1 and g_2 is compact, then there exists $g \in G$ such that $g_1, g_2 \in gKg^{-1}$, say $g_1 = gxg^{-1}$ and $g_2 = gyg^{-1}$ with $x, y \in K$. On the other hand, the decomposition $G = KAK$ allows us to write $g = kak'$ for some $k, k' \in K$ and $a \in A$. It follows that

$$\begin{aligned}(g_1, g_2) &= (gxg^{-1}, gyg^{-1}) = (ka(k'xk'^{-1})a^{-1}k^{-1}, ka(k'yk'^{-1})a^{-1}k^{-1})\\ &= \Phi(k, a, u, v)\end{aligned}$$

with $k \in K, a \in A, u = k'xk'^{-1} \in K$ and $v = k'yk'^{-1} \in K$, i.e., $(g_1, g_2) \in W$. This completes the proof. □

By combining the previous results, we deduce the following:

Corollary 3.13 *There is a semi-algebraic set $Z \subset G \times G$ of positive codimension such that no pair of elements $(g_1, g_2) \in (G \times G) - Z$ admits a common invariant measure on projective space $\mathbb{P}^{d-1}(\mathbb{K})$.*

Proof If $\mathbb{K} = \mathbb{R}$ then we let $Z = V$ be the proper algebraic subvariety of $G \times G$ described in Proposition 3.8 above. Otherwise, if $\mathbb{K} = \mathbb{C}$ then we let $Z = V \cup W$ where W is given by Lemma 3.12.

Now consider a pair of elements $(g_1, g_2) \in G \times G$ that admit a common invariant measure m on $\mathbb{P}^{d-1}(\mathbb{K})$. If $\mathbb{K} = \mathbb{C}$ then (g_1, g_2) belongs to either W or V, according to which property (i) or (ii) holds in Proposition 3.9. If $\mathbb{K} = \mathbb{R}$ then by Remark 3.11 we know that property (ii) holds, so $(g_1, g_2) \in V$. □

3.5 Proof of Theorem A

Let G be an algebraic group of matrices satisfying the hypotheses listed at Sect. 2.4. Let f be a $C^{1+\alpha}$-diffeomorphism of a compact manifold M. Let μ be a f-invariant ergodic hyperbolic non-atomic probability measure with local product structure.

Let $A \in C^{r,\nu}(M, G)$ be a cocycle whose Lyapunov exponents vanish at μ-almost every point. To prove Theorem A, we only need to prove that for any $l > 0$ there exists a neighborhood $\mathcal{U} \subset C^{r,\nu}(M, G)$ of A such that the cocycles in $\mathcal{U}$ with vanishing Lyapunov exponents are contained in a Whitney stratified set with codimension $\geq l$.

By Proposition 3.6, we can find l holonomy blocks $\mathcal{O}_i$ of positive μ-measure containing horseshoes H_i associated to distinct periodic points $p_i \in \mathcal{O}_i$, $1 \leq i \leq l$ of minimal periods κ_i, and some homoclinic points $z_i \in \mathcal{O}_i$ of p_i, $z_i \in W^u_{loc}(p_i)$, $f^{q_i}(z_i) \in W^s_{loc}(p_i)$ such that $p_i, z_i \in \operatorname{supp}(\mu|\mathcal{O} \cap f^{-\kappa_i}(\mathcal{O}_i))$ and $q_i > 0$. Moreover the same $\mathcal{O}_i, p_i, z_i, q_i$ work for any B in a small neighborhood $\mathcal{U}$ of A.

Then by Proposition 3.5, for any $A' \in \mathcal{U}$ with vanishing Lyapunov exponents, for any $1 \leq i \leq l$ the projective actions of the matrices

$$g_{i,1}(A') := A'^{(\kappa_i)}(p_i) \quad \text{and} \quad g_{i,2}(A') := H^{s}_{A', f^{q_i}(z_i), p_i} \circ A'^{(q_i)}(z_i) \circ H^{u}_{A', p_i, z_i} \tag{3.11}$$

preserve a common probability measure $m_{p_i}(A')$ on $\mathbb{P}^{d-1}(\mathbb{K})$. Thus, for any i, the pair $(g_{i,1}(A'), g_{i,2}(A'))$ belongs to the semi-algebraic set Z of positive codimension in $G \times G$ given by Corollary 3.13. Recall from Proposition 3.7 that:

- semi-algebraic sets are Whitney stratified;
- products of Whitney stratified sets is Whitney stratified and codimensions add;
- pre-images of Whitney stratified sets under submersions are Whitney stratified, and codimension is preserved.

Therefore by Proposition 3.4 we conclude that all such A' lie in a Whitney stratified subset of codimension $\geq l$. This completes the proof. □

Acknowledgements The authors are grateful to the referee for a careful reading of the manuscript and for useful suggestions that helped to improve the presentation of the paper.

References

Aoun, R.: Transience of algebraic varieties in linear groups—applications to generic Zariski density. Ann. Inst. Fourier (Grenoble) **63**, 2049–2080 (2013)

Avila, A.: Density of positive Lyapunov exponents for SL(2, ℝ)-cocycles. J. Am. Math. Soc. **24**(4), 999–1014 (2011)

Avila, A., Crovisier, S., Wilkinson, A.: Diffeomorphisms with positive metric entropy. Publ. Math. Inst. Hautes Études Sci. **124**, 319–347 (2016)

Avila, A., Krikorian, R.: Monotonic cocycles. Invent. Math. **202**(1), 271–331 (2015)

Avila, A., Viana, M.: Simplicity of Lyapunov spectra: a sufficient criterion. Port. Math. **64**, 311–376 (2007)

Avila, A., Viana, M.: Extremal Lyapunov exponents: an invariance principle and applications. Invent. Math. **181**(1), 115–189 (2010)

Avila, A., Santamaria, J., Viana, M.: Holonomy invariance: rough regularity and applications to Lyapunov exponents. Astérisque **358**, 13–74 (2013)

Barreira, L., Pesin, Y.: Exponents and Smooth Ergodic Theory. University Lecture Series, 23. American Mathematical Society, Providence, RI, xii+151 (2002)

Bochi, J.: Genericity of zero Lyapunov exponents. Ergod. Theory Dyn. Syst. **22**, 1667–1696 (2002)

Bochi, J., Viana, M.: The Lyapunov exponents of generic volume-preserving and symplectic maps. Ann. Math. **161**, 1423–1485 (2005)

Bogachev, V.I.: Measure Theory, vol. 2. Springer, Berlin (2007)

Bonatti, C., Crovisier, S., Shinohara, K.: The $C^{1+\alpha}$ hypothesis in Pesin theory revisited. J. Mod. Dyn. **7**(4), 605–618 (2013)

Bonatti, C., Gómez-Mont, X., Viana, M.: Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices. Ann. Inst. H. Poincaré Anal. Non Linéaire **20**, 579–624 (2003)

Bonatti, C., Viana, M.: Lyapunov exponents with multiplicity 1 for deterministic products of matrices. Ergod. Theory Dyn. Syst. **24**, 1295–1330 (2004)

Bourgain, J.: Green's Function Estimates for Lattice Schrödinger Operators and Applications. Annals of Mathematics Studies, vol. 158. Princeton University Press, Princeton (2005)

Breuillard, E.: A strong Tits alternative. arXiv:0804.1395 (2008)

Colonius, F., Kliemann, W.: The Dynamics of Control. With an appendix by Lars Grüne. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston (2000)

Duarte, P., Klein, S.: Positive Lyapunov exponents for higher dimensional quasiperiodic cocycles. Commun. Math. Phys. **332**(1), 189–219 (2014)

Furman, A.: Random walks on groups and random transformations. In: Hasselblatt. B., Katok, A. (eds.) Handbook of Dynamical Systems, vol. 1A, pp. 931–1014. North-Holland, Amsterdam (2002)

Furstenberg, H.: Noncommuting random products. Trans. Am. Math. Soc. **108**, 377–428 (1963)

Gibson, C.G., Wirthmüller, K., du Plessis, A.A., Looijenga, E.J.N.: Topological Stability of Smooth Mappings. Lecture Notes in Mathematics, vol. 552. Springer, Berlin (1976)

Gol'dsheid, Y.I., Margulis, G.A.: Lyapunov exponents of a product of random matrices. Russ. Math. Surv. **44**(5), 11–71 (1989)

Guivarc'h, Y., Raugi, A.: Propriétés de contraction d'un semi-groupe de matrices inversibles. Coefficients de Liapunoff d'un produit de matrices aléatoires indépendantes. Isr. J. Math. **65**(2), 165–196 (1989)

Knapp, A.: Lie Groups Beyond an Introduction. Progress in Mathematics, vol. 140, 2nd edn. Birkhäuser Boston, Inc., Boston (2002)

Knill, O.: Positive Lyapunov exponents for a dense set of bounded measurable $SL(2, \mathbb{R})$-cocycles. Ergod. Theory Dyn. Syst. **12**(2), 319–331 (1992)

Ledrappier, F.: Positivity of the exponent for stationary sequences of matrices. In: Arnold, L., Wihstutz, V. (eds.) Lyapunov Exponents (Bremen, 1984). Lecture Notes in Mathematics, vol. 1886, pp. 56–73, Springer, New York (1986)

Moore, C.: Amenable groups of semi-simple groups and proximal flows. Isr. J. Math. **34**, 121–138 (1979)

Palais, R.: Foundations of Global Non-linear Analysis. Benjamin, New York (1968)

Viana, M.: Almost all cocycles over any hyperbolic system have nonvanishing Lyapunov exponents. Ann. Math. **167**(2), 643–680 (2008)

Xu, D.: Density of positive Lyapunov exponents for symplectic cocycles. J. Eur. Math. Soc. (JEMS) (2017) **(to appear)**

Zimmer, R.J.: Ergodic Theory and Semisimple Groups. Monographs in Mathematics, vol. 81. Birkhäuser Verlag, Basel (1984)