Advanced Nonlinear Studies **8** (2008), 191–211

Dynamics of Generic Multidimensional Linear Differential Systems

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Received in revised form 10 July 2007
Communicated by Russell Johnson

Abstract

We prove that there exists a residual subset $\mathcal{R}$ (with respect to the C^0 topology) of d-dimensional linear differential systems based in a μ-invariant flow and with transition matrix evolving in $GL(d,\mathbb{R})$ such that if $A \in \mathcal{R}$, then, for μ-a.e. point, the Oseledets splitting along the orbit is dominated (uniform projective hyperbolicity) or else the Lyapunov spectrum is trivial. Moreover, in the conservative setting, we obtain the dichotomy: dominated splitting versus zero Lyapunov exponents.

2000 Mathematics Subject Classification. Primary: 37D25, 37D30; Secondary: 34D08..
Keywords. Linear Differential Systems, Dominated splitting, Lyapunov exponents, Multiplicative ergodic theorem.

1 Introduction

Let $\varphi^t\colon X \to X$ be a continuous flow defined in a compact Hausdorff space X and $A\colon X \to \mathfrak{sl}(d,\mathbb{R})$ be a continuous map, where $\mathfrak{sl}(d,\mathbb{R})$ is the Lie algebra of all $d \times d$ matrices with trace equal to zero. Given any $p \in X$, the solution $\Phi_A^t(p)$ of the non-autonomous linear differential equation $u(t)' = A(\varphi^t(\cdot)) \cdot u(t)$, with initial condition $\Phi_A^0(p) = \mathrm{Id}$, is a linear flow which lies in the special linear group $SL(d,\mathbb{R})$. A typical example is the linear Poincaré flow (see [10] B.3) of a divergence-free (zero divergence) vector field F defined in a $d+1$-dimensional manifold and such that $\|F(p)\| = 1$ for all regular points p. However, the linear Poincaré flow of any divergence-free vector field F in general does not evolve in $SL(d,\mathbb{R})$. Nevertheless, it still has a restriction which is invariant, as we will s ee in Section 2.1 when we define the linear differential systems and which mimics the volume-preserving flows.

Moreover, beyond the conservative setting, we will consider general linear differential systems with solutions in the linear group $GL(d,\mathbb{R})$, which can be induced by a flow in manifolds with dimension $d+1$.

Given a linear differential system A, the Lyapunov exponents measure the asymptotic exponential growth rate of $\|\Phi^t_A(p)\cdot v\|$ for $v\in\mathbb{R}^d_p$. These real numbers play a central role in ergodic theory and the absence of zero exponents yields a valuable description of the dynamics of A. Therefore, it is very important to detect zero Lyapunov exponents. If $A(t)=A$ is constant (e.g. the flow is over a fixed point), then the Lyapunov exponents are exactly the real parts of the eigenvalues of A. In general, the eigenvalues of the matrix $A(t)$ are meaningless if one aims to study the asymptotic solutions. If we assume that φ^t leaves invariant a Borel regular probability measure μ, then, due to the multiplicative ergodic theorem (see for example [17]), we have that the Lyapunov exponents are well defined for almost every orbit.

Throughout this work we use a weak form of hyperbolicity, called dominated splitting which, broadly speaking, means that we have an invariant splitting along the orbit into two subspaces such that one is most expanding (or less contracted) than the other, by uniform rates.

The main target of this paper is to understand in detail the Lyapunov exponents for typical continuous-time general families of linear differential systems in any dimension. By using a much more careful and elaborate technique, we obtain, in particular, the higher dimensional generalization of the theorems in [4]. Let us start with our main result.

Theorem 1.1 *There exists a C^0-residual subset $\mathcal{R}$ of d-dimensional linear differential systems with fundamental matrix evolving in $GL(d,\mathbb{R})$ and over a μ-invariant flow $\varphi^t\colon X\to X$ where μ is a Borel regular probability measure, such that, if $A\in\mathcal{R}$, then for μ-a.e. point $x\in X$ we have dominated splitting or else the Lyapunov exponents are all equal.*

Since we are going to make conservative perturbations and, in conservative setting, having equal Lyapunov exponents is tantamount to all exponents being zero, we derive the following Corollary.

Corollary 1.1 *There exists a C^0-residual subset $\mathcal{R}$ of d-dimensional conservative linear differential systems over a μ-invariant flow $\varphi^t\colon X\to X$ where μ is a Borel regular probability measure, such that if $A\in\mathcal{R}$, then for μ-a.e. point $x\in X$ we have dominated splitting or else the Lyapunov exponents are all zero.*

We point out that the first example of these kind of dichotomies appeared in the groundbreaking approach of Mañé (see [21]). In [22] Mañé gave an outline of what could be the proof for area-preserving diffeomorphisms on surfaces. Then, in [7], Bochi combined the arguments outlined in [22] with new ideas to complete the proof. Next, in a seminal paper of Bochi and Viana (see [8]) the dichotomy was generalized to any dimension, to symplectomorphisms and to discrete cocycles. Later we used the approach of Mañé-Bochi-Viana, in the context of the continuous-time systems, and we proved the 2-dimensional linear differential systems version (see [4]) and also the 3-dimensional volume-preserving flows case (see [5]).

Let us mention now some results in the opposite direction to ours: in [6] we proved that, in the setting of dynamical linear differential systems, ergodicity and dominated splitting assure that zero Lyapunov can be removed by small C^1-perturbations, at least when the central direction is 1-dimensional. We also mention the early work of Millionshchikov (see [23] and [24]) where abundance (dense and open set with respect to C^0-topology) of a simple spectrum (all Lyapunov exponents are different) is proved for a class of linear differential systems. Fabbri, in [13], proved the C^0-genericity (open and dense) of hyperbolicity on the torus for 2-dimensional linear differential systems (see also Fabbri-Johnson [15]). For determining the positivity of Lyapunov exponents we mention that Knill (see [19]) proved that for a C^0-dense set of 2-dimensional bounded and measurable conservative discrete cocycles we have positive exponents. We remark that Fabbri's result is the continuous-time counterpart of Knill's theorem on tori. Later, in [3], Arnold and Cong used a different strategy and generalized Knill's result to $GL(d,\mathbb{R})$ valued discrete cocycles. A very interesting and recent result of Cong (see [11]) says that a generic bounded cocycle has simple spectrum, moreover the Oseledets splitting is dominated. As a consequence of this result, in the conservative 2-dimensional discrete and bounded case we have abundance of uniform hyperbolicity. Going back to linear differential systems we also recall the results of Fabbri [14], Fabbri-Johnson [16] and the early paper of Kotani [18]. Furthermore, Nerurkar (see [25]) proved the positivity of Lyapunov exponents for a dense set in a class of conservative linear differential systems.

It is possible to prove the dichotomy of Theorem 1.1 for systems with solutions evolving in more general subgroups of $GL(d,\mathbb{R})$. Actually, in order to obtain similar results these systems must satisfy the accessibility condition (see [4] Definition 5.1) which guarantees that we can mix directions and thus it can be shown that the strategy of the proof still works. Note that in [25] Nerurkar also used a definition of accessibility. The approach in this work explicitly proves that conservative systems and systems evolving in the general linear group actually are accessible. Parallel perturbations must be developed for other kind of systems, like for example, linear differential systems with transition matrix evolving in the symplectic group.

2 Preliminaries

2.1 Linear differential systems

We consider a non-atomic probability space (X,μ) where X is a compact and Hausdorff space and μ is a Borel regular probability measure. Let $\varphi^t\colon X\to X$ be a flow continuous in the space parameter and C^1 in the time parameter and assume that μ is φ^t-invariant. For $d\in\mathbb{N}$ let $A\colon X\to GL(d,\mathbb{R})$ be a continuous map. For each $p\in X$ we consider the non-autonomous linear differential equation:

$$\frac{d}{dt}u(s)|_{s=t}=A(\varphi^t(p))\cdot u(t), \tag{2.1}$$

called the linear variational equation. We say that the solution of (2.1) is the fundamental matrix related to the system A. The solution of (2.1) is a linear flow $\Phi_A^t(p)\colon \mathbb{R}^d_p \to \mathbb{R}^d_{\varphi^t(p)}$ which may be seen as the skew-product flow,

$$\begin{array}{rccc} \Phi^t\colon & X \times \mathbb{R}^d_p & \longrightarrow & X \times \mathbb{R}^d_{\varphi^t(p)} \\ & (p, v) & \longrightarrow & (\varphi^t(p), \Phi^t_A(p) \cdot v). \end{array}$$

Since for all $p \in X$ we have $A(p) = \frac{d}{dt}\Phi^t_A(p)|_{t=0}$, Φ^t_A is also called the infinitesimal generator of A. We recall a basic relation which says that for all $p \in X$ and $t \in \mathbb{R}$ we have $\Phi_A^{t+s}(p) = \Phi^s_A(\varphi^t(p)) \circ \Phi^t_A(p)$.

We will be interested in two of the most common systems; the traceless ones, where $\text{Tr}(A) = 0$ and the systems where $\Phi^t_A \in GL(d, \mathbb{R})$, that is where Φ^t_A evolves in the linear group of matrices with non-zero determinant. We denote the last-mentioned systems by $\mathcal{G}$ and the traceless systems by $\mathcal{T}$. Another kind of systems which are of special interest are the modified volume-preserving systems which simulate the volume-preserving vector fields in manifolds with dimension $d + 1$. To define formally these systems we consider a continuous function $a\colon X \to \mathbb{R}$ which is nonnegative and has subexponential growth, that is for $p \in X$ we have

$$\lim_{t\to\pm\infty} \frac{1}{t} \log(a(\varphi^t(p))) = 0. \tag{2.2}$$

Moreover, if $\text{Fix}(\varphi^t)$ denotes the set of fixed points of φ^t, then $a(X \setminus \text{Fix}(\varphi^t)) \neq 0$. We say that A is modified volume-preserving, denoting by $\mathcal{T}_a$ if:

$$\det\Phi^t_A(p) = \begin{cases} 1, & \text{if } p \in \text{Fix}(\varphi^t) \\ \frac{a(p)}{a(\varphi^t(p))}, & \text{if } p \notin \text{Fix}(\varphi^t). \end{cases}$$

The linear Poincaré flow of a C^1 volume-preserving flow φ^t is the most typical example of these kind of systems. In this case we have $a(\cdot) = \|\frac{d}{dt}\varphi^t(\cdot)|_{t=0}\|$.

Let $A \in \mathcal{T}$, then any conservative perturbation of A, say $A + H$, must satisfy $\Phi^t_{A+H} \in SL(d, \mathbb{R})$. It is immediate to see that if $H \in \mathcal{T}$, then $A + H \in \mathcal{T}$. Using the Liouville formula,

$$\det\Phi^t_A(p) = \exp\left(\int_0^t \text{Tr}A(\varphi^s(p))\right) ds,$$

it is straightforward to see that if $A \in \mathcal{T}_a$, respectively $A \in \mathcal{G}$, and $H \in \mathcal{T}$, then $A + H \in \mathcal{T}_a$, respectively $A + H \in \mathcal{G}$.

To compute the distance between linear differential systems we will basically work with two norms; the uniform convergence norm (or the C^0-norm) which is given by

$$\|A - B\| = \max_{p\in X} \|A(p) - B(p)\|$$

and the L^∞-norm, which is given by

$$\|A - B\|_\infty = \text{esssup}\|A(p) - B(p)\|.$$

2.2 Multiplicative ergodic theorem

In our setting the Oseledets Theorem [26] guarantees that for μ-a.e. point $p \in X$, there exists a Φ_A^t-invariant splitting called Oseledets' splitting of the fiber $\mathbb{R}_p^d = E_p^1 \oplus ...E_p^{k(p)}$ and real numbers called Lyapunov exponents $\hat{\lambda}_1(p) > ... > \hat{\lambda}_{k(p)}(p)$, with $k(p) \leq d$, such that:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi_A^t(p) \cdot v^i\| = \hat{\lambda}_i(p), \tag{2.3}$$

for any $v^i \in E_p^i \setminus \{\vec{0}\}$ and $i = 1, ..., k(p)$. If we do not count the multiplicities, then we have $\lambda_1(p) \geq \lambda_2(p) \geq ... \geq \lambda_d(p)$. Moreover, given any of these subspaces E^i and E^j, the angle between them along the orbit has subexponential growth, that is

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \sin(\measuredangle(E^i_{\varphi^t(p)}, E^j_{\varphi^t(p)})) = 0. \tag{2.4}$$

If the flow φ^t is ergodic, then the Lyapunov exponents and the dimensions of the associated subbundles are μ-a.e. constant. For a simplified proof of this theorem for linear differential systems see [17]. We denote by $\mathcal{O}(A)$ the μ-generic points given by the multiplicative ergodic theorem.

2.3 Multilinear operators algebra

Let $\mathcal{H}$ be a Hilbert space and $n \in \mathbb{N}$. The n^{th} exterior product of $\mathcal{H}$, denoted by $\wedge^n(\mathcal{H})$, is also a vector space. If $\dim(\mathcal{H}) = d$, then $\dim(\wedge^n(\mathcal{H})) = \binom{d}{n}$. Given an orthonormal basis of $\mathcal{H}$, $\{e_j\}_{j\in J}$, then the family of exterior products $e_{j_1} \wedge e_{j_2} \wedge ... \wedge e_{j_n}$ for $j_1 < ... < j_n$ with $j_\alpha \in J$ constitutes an orthonormal basis of $\wedge^n(\mathcal{H})$. Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and a linear operator $L\colon \mathcal{H}_1 \to \mathcal{H}_1$ we define the operator $\wedge^n(L)$ by

$$\begin{array}{cccc} \wedge^n(L)\colon & \wedge^n(\mathcal{H}_1) & \longrightarrow & \wedge^n(\mathcal{H}_2) \\ & \psi_1 \wedge ... \wedge \psi_n & \longrightarrow & L(\psi_1) \wedge ... \wedge L(\psi_n). \end{array}$$

Note that given a linear differential system A over φ^t, and since for the operators $L_t\colon \mathcal{H}_p \to \mathcal{H}_{\varphi^t(p)}$ and $L_s\colon \mathcal{H}_{\varphi^t(p)} \to \mathcal{H}_{\varphi^{t+s}(p)}$ we have $\wedge^n(L_sL_t) = \wedge^n(L_s) \wedge^n(L_t)$, we obtain that $\wedge^n(\Phi_A^t)$ is also a linear differential system over φ^t which we also denote by $\wedge^n(A)$. For details on multilinear algebra of operators in Hilbert spaces see [27] chapter V.

This operator, in the particular case of $\dim(\mathcal{H}) = d$, will be very useful to prove our results since we can recover the spectrum and splitting information of the dynamics of $\wedge^n(\Phi_A^t)$ from the one obtained by applying Oseledets' Theorem to Φ_A^t. This information will be for the same full measure set and with this approach we deduce our results. Next we present the multiplicative ergodic theorem for exterior power (for a proof see [2] Theorem 5.3.1).

Lemma 2.1 *The Lyapunov exponents $\lambda_i^{\wedge n}(p)$ for $i \in \{1, ..., \binom{d}{n}\}$ (repeated with multiplicity) of the n^{th} exterior product operator $\wedge^n(A)$ at p are the numbers:*

$$\sum_{j=1}^{n} \lambda_{i_j}(p), \;\; \textit{where } 1 \leq i_1 < ... < i_n \leq d.$$

This nondecreasing sequence starts with $\lambda_1^{\wedge n}(p) = \lambda_1(p) + \lambda_2(p) + ... + \lambda_n(p)$ *and ends with* $\lambda_{q(n)}^{\wedge n}(p) = \lambda_{d+1-n}(p) + \lambda_{d+2-n}(p) + ... + \lambda_d(p)$*. Moreover the splitting of* $\wedge^n(\mathbb{R}_p^d(i))$ *for* $0 \leq i \leq q(n)$ *(of* $\wedge^n(A)$*) associated to* $\lambda_i^{\wedge n}(p)$ *can be obtained from the splitting* $\mathbb{R}_p^d(i)$ *(of* A*) as follows; take an Oseledets basis* $\{e_1(p), ..., e_d(p)\}$ *of* $\mathbb{R}_p^d$ *such that* $e_i(p) \in E_p^k$ *for* $\dim(E_p^1) + ... + \dim(E_p^{k-1}) < i \leq \dim(E_p^1) + ... + \dim(E_p^k)$*. Then the Oseledets space is generated by the* n*-vectors:*

$$e_{i_1} \wedge ... \wedge e_{i_n} \text{ such that } 1 \leq i_1 < ... < i_n \leq d \text{ and } \sum_{j=1}^{n} \lambda_{i_j}(p) = \lambda_i^{\wedge n}(p).$$

2.4 Dominated splitting

Let $\varphi^t \colon X \to X$ be a flow and $\Lambda \subseteq X$ a φ^t-invariant set. Let A be a linear differential system. Given any linear map L we denote by $\mathfrak{m}(L)$ the conorm which is defined by $\|L^{-1}\|^{-1} = \inf_{v \neq \vec{0}} \|L \cdot v\|$. We say that $\mathbb{R}_\Lambda^d = U_\Lambda \oplus S_\Lambda$ is an m-dominated splitting for A over Λ if $\Phi_A^t(p) \cdot U_p = U_{\varphi^t(p)}$ and $\Phi_A^t(p) \cdot S_p = S_{\varphi^t(p)}$ for $p \in \Lambda$ and $t \in \mathbb{R}$, the dimensions of U_p and S_p are constant on Λ and for every $p \in \Lambda$ the following inequality holds:

$$\frac{\|\Phi_A^m(p)|_{S_p}\|}{\mathfrak{m}(\Phi_A^m(p)|_{U_p})} \leq \frac{1}{2}. \tag{2.5}$$

We note that dominated splitting is a weak form of uniform hyperbolicity (or exponential dichotomy, see [12]). The dimension of U is called the index of the splitting. Every m-dominated splitting over Λ can be extended to an m-dominated splitting over the closure of Λ. Also the angles between the subbundles of a dominated splitting are uniformly bounded away from zero. Moreover, for a fixed index the dominated splitting is unique. For the detailed proofs of these properties about dominated structures see [10] Section B.1. Notice that an m-dominated splitting at $p \in X$ means that one has m-dominated splitting over the closure of the orbit of p.

Given A as above, $n \in \{1, ..., d-1\}$ and $m \in \mathbb{N}$ we denote by $\Lambda_n(A, m) \subseteq X$ the set of points $p \in X$ such that there exists an m-dominated splitting of index n along the orbit of x. Clearly the set $\Gamma_n(A, m) = X \setminus \Lambda_n(A, m)$ is open.

In the sequel we will be interested in dominated structures over the orbit of points $p \in \mathcal{O}(A)$ and related to the natural flow invariant decomposition given by the Oseledets Theorem; $U_p = E_p^1 \oplus ... \oplus E_p^n$ and $S_p = E_p^{n+1} \oplus ... \oplus E_p^d$ where the subspaces E_p^i for $i = \{1, ..., d\}$ are the ones obtained in Section 2.2 and $n \in \{1, ..., d-1\}$ is a fixed index. Therefore, we define the set $\Gamma_n^\sharp(A, m)$ of points $p \in \Gamma_n(A, m) \cap \mathcal{O}(A)$ such that $\lambda_n(A, p) > \lambda_{n+1}(A, p)$. Let $\Gamma_n^*(A, m)$ denote the set of nonperiodic points of $\Gamma_n^\sharp(A, m)$. Finally, let

$$\Gamma_n(A, \infty) = \bigcap_{m \in \mathbb{N}} \Gamma_n(A, m) \text{ and } \Gamma_n^\sharp(A, \infty) = \bigcap_{m \in \mathbb{N}} \Gamma_n^\sharp(A, m).$$

In Lemma 4.1 of [8] it is proved that for every system A and $n \in \{1, ..., d-1\}$, the set $\Gamma_n^\sharp(A, \infty)$ contains no periodic points. Actually,

$$\forall \delta > 0, \exists m_0 \in \mathbb{N} \colon \forall m \geq m_0, \;\; \mu(\Gamma_n^\sharp(A, m) \setminus \Gamma_n^*(A, m)) < \delta. \tag{2.6}$$

The statement (2.6) will be very useful because our perturbations must be supported along large orbit segments and the existence of periodic points complicates our task.

Given a system A, $n \in \{0, 1, ..., d-1\}$ and $m \in \mathbb{N}$ we consider the measurable function

$$\begin{array}{rccc} \rho_{A,m}\colon & \Gamma_n^*(A,m) & \longrightarrow & \mathbb{R} \\ & p & \longrightarrow & \frac{\|\Phi_A^m(p)|_{S_p}\|}{\mathfrak{m}(\Phi_A^m(p)|_{U_p})}. \end{array}$$

Now we define the set

$$\Delta_n^*(A,m) = \left\{ p \in \Gamma_n^*(A,m)\colon \rho_{A,m}(p) > \frac{1}{2} \right\}.$$

Notice that

$$\bigcup_{t \in \mathbb{R}} \varphi^t(\Delta_n^*(A,m)) = \Gamma_n^*(A,m). \tag{2.7}$$

The next simple lemma will be useful in Section 4.

Lemma 2.2 *Given $\Delta_n^*(A,m)$ and $\Gamma_n^*(A,m)$ as in (2.7) if $\mu(\Gamma_n^*(A,m)) > 0$, then $\mu(\Delta_n^*(A,m)) > 0$.*

Proof. We claim that

$$\Gamma_1 = \bigcup_{t \in \mathbb{Q}} \varphi^t(\Delta_n^*(A,m)) = \Gamma_n^*(A,m).$$

Notice that $\Gamma_1 \subseteq \Gamma_n^*(A,m)$ is trivial so we just have to prove that $\Gamma_n^*(A,m) \subseteq \Gamma_1$. Take $z \in \Gamma_n^*(A,m)$, then there exist $t \in \mathbb{R}$ and $p \in \Delta_n^*(A,m)$ such that $\varphi^t(p) = z$. If $t \in \mathbb{Q}$, then $z \in \Gamma_1$ and we are done. Otherwise, using the continuity of the function $t \to \rho_{A,m}(\varphi^t(q))$ (for q fixed), there exist a small enough $s > 0$ and $r \in \mathbb{Q}$ such that $\varphi^s(p) \in \Delta_n^*(A,m)$ and $\varphi^r(\varphi^s(p)) = z$. Hence $z \in \Gamma_1$ and $\Gamma_1 = \Gamma_n^*(A,m)$.

Finally, if $\mu(\Delta_n^*(A,m)) = 0$, then we have $\mu(\Gamma_1) = 0$ and the lemma is proved.

2.5 Entropy functions

Let us consider the following function where $\mathcal{L}$ is one of the sets $\mathcal{T}$, $\mathcal{T}_a$ or $\mathcal{G}$:

$$\begin{array}{rccc} \mathcal{E}_n\colon & \mathcal{L} & \longrightarrow & [0,+\infty) \\ & A & \longmapsto & \int_X \lambda_1(\wedge^n(A),p)d\mu(p). \end{array} \tag{2.8}$$

With this function we compute the integrated largest Lyapunov exponent of the n^{th} exterior power operator. We consider also the function $\mathcal{E}_n(A,\Gamma)$, where $\Gamma \subseteq X$ is a φ^t-invariant set, defined by:

$$\mathcal{E}_n(A,\Gamma) = \int_\Gamma \lambda_1(\wedge^n(A),p)d\mu(p).$$

Let us denote $\Sigma_n(A,p) = \lambda_1(A,p) + ... + \lambda_n(A,p)$. By using Lemma 2.1 we conclude that for $n = 1, ..., d-1$ we have $\Sigma_n(A,p) = \lambda_1(\wedge^n(A), p)$ and therefore we obtain $\mathcal{E}_n(A,\Gamma) = \mathcal{E}_1(\wedge^n(A),\Gamma)$. By using Proposition 2.2 of [8] we get that:

$$\mathcal{E}_n(A,\Gamma) = \inf_{k\in\mathbb{N}} \frac{1}{k}\int_\Gamma \log \| \wedge^n (\Phi_A^k(p))\| d\mu(p), \tag{2.9}$$

and so the entropy function (2.8) is upper semi-continuous for all $n \in \{1, ..., d-1\}$.

Let us consider now the function:

$$\begin{array}{rccc} \mathcal{E}\colon & \mathcal{L} & \longrightarrow & \mathbb{R}^{d-1} \\ & A & \longmapsto & (\mathcal{E}_1(A), \mathcal{E}_2(A), ..., \mathcal{E}_{d-1}(A)). \end{array}$$

In Section 5 we derive Theorem 1.1 from the following proposition. This proposition follows from Lemma 4.4 which will be proved in Section 4.

Proposition 2.1 *If A is a continuity point of $\mathcal{E}$, then there exist two disjoint measurable sets E and D with $\mu(E \cup D) = 1$ verifying the following: for the linear system A we have, for any $p \in E$, that the spectrum is trivial and, for any $p \in D$, we have that the Oseledets splitting is dominated along the orbit of p.*

3 Perturbations of linear differential systems

We begin by proving a basic perturbation lemma which will be the main tool for proving our results.

Lemma 3.1 *Given $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$) and $\epsilon > 0$, there exists $\xi_0 > 0$ (depending on A and ϵ), such that given any $\xi \in (0, \xi_0)$, any $p \in X$ (non-periodic or with period larger than 1) and any 2-dimensional vector space $V_p \subset \mathbb{R}^d_p$, there exists a measurable traceless system H (depending on ξ and p) such that:*

1. *$\|H\| < \epsilon$;*
2. *H is supported in $\varphi^t(p)$ for $t \in [0,1]$;*
3. *If W_p is the orthogonal complement of V_p in $\mathbb{R}^d_p$, then $\Phi^t_{A+H}(p) = \Phi^t_A(p)$ on W_p;*
4. *$\Phi^1_{A+H}(p) \cdot v = \Phi^1_A(p) \circ R_\xi \cdot v$, $\forall v \in V_p$, where R_ξ is the rotation of angle ξ on V_p;*

Proof. Take $K = \max_{p\in X} \|\Phi_A^{\pm t}(p)\|$ for $t \in [0,1]$. We claim that it is sufficient to take $\xi_0 > 0$ such that:

$$\xi_0 < \frac{\epsilon}{2K^2}.$$

Let $\eta\colon \mathbb{R} \to [0,1]$ be any C^∞ function such that $\eta(t) = 0$ for $t \le 0$, $\eta(t) = 1$ for $t \ge 1$, and $0 \le \eta'(t) \le 2$, for all t. We define the 1-parameter family of linear maps $\Psi^t(p)\colon \mathbb{R}^d_p \to \mathbb{R}^d_p$ for $t \in [0,1]$ as follows; we fix two orthonormal basis $\{u_1, u_2\}$ of V_p and $\{u_3, u_4, ..., u_d\}$

of W_p. For $\theta \in [0, 2\pi]$, we consider the rotation of angle θ whose matrix relative to the basis $\{u_1, u_2\}$ is

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Since $V_p \oplus W_p = \mathbb{R}^d_p$, given any $u \in \mathbb{R}^d_p$ we decompose $u = u_V + u_W$, where $u_V \in V_p$ and $u_W \in W_p$. For $t \in \mathbb{R}$ and $\xi \in (0, \xi_0)$ we define

$$\mathcal{R}^t \cdot u = R_{\eta(t)\xi}(u_V) + u_W.$$

Now we consider the 1-parameter family of linear maps $\Psi^t(p)\colon \mathbb{R}^d_p \to \mathbb{R}^d_{\varphi^t(p)}$ where $\Psi^t(p) = \Phi^t_A(p) \circ \mathcal{R}^t$. We take time derivatives and we obtain:

$$\begin{aligned} (\Psi^t(p))' &= (\Phi^t_A(p))'\mathcal{R}^t + \Phi^t_A(p)(\mathcal{R}^t)' = \\ &= A(\varphi^t(p))\Phi^t_A(p)\mathcal{R}^t + \Phi^t_A(p)(\mathcal{R}^t)' = \\ &= A(\varphi^t(p))\Psi^t(p) + \Phi^t_A(p)(\mathcal{R}^t)'(\Psi^t(p))^{-1}\Psi^t(p) = \\ &= \big[A(\varphi^t(p)) + H(\varphi^t(p))\big] \cdot \Psi^t(p). \end{aligned}$$

Hence we define the perturbation by,

$$H(\varphi^t(p)) = \Phi^t_A(p)(\mathfrak{R}^t)'(\mathfrak{R}^t)^{-1}(\Phi^t_A(p))^{-1},$$

where $(\mathfrak{R}^t)'$ and $(\mathfrak{R}^t)^{-1}$ are respectively $(\mathcal{R}^t)'$ and $(\mathcal{R}^t)^{-1}$ but written in the canonical base of $\mathbb{R}^d_p$ instead. Since

$$(\mathcal{R}^t)' \cdot u = \eta'(t)\xi \begin{pmatrix} -\sin(\eta(t)\xi) & -\cos(\eta(t)\xi) \\ \cos(\eta(t)\xi) & -\sin(\eta(t)\xi) \end{pmatrix} \cdot u_V$$

and also

$$(\mathcal{R}^t)^{-1} \cdot u_V = R_{-\eta(t)\xi}(u_V) = \begin{pmatrix} \cos(\eta(t)\xi) & \sin(\eta(t)\xi) \\ -\sin(\eta(t)\xi) & \cos(\eta(t)\xi) \end{pmatrix} \cdot u_V,$$

we obtain that if $u_V = (\psi_1, \psi_2, \overbrace{0, 0, ..., 0}^{(d-2)\times})$ (in the coordinate system $\{u_1, ..., u_d\}$) then,

$$(\mathcal{R}^t)'(\mathcal{R}^t)^{-1} \cdot u = \xi\eta'(t)(-\psi_2, \psi_1, \overbrace{0, 0, ..., 0}^{(d-2)\times}).$$

Clearly we have $\mathrm{Tr}((\mathcal{R}^t)'(\mathcal{R}^t)^{-1}) = 0$ and since the trace is invariant by any change of coordinates we obtain $\mathrm{Tr}((\mathfrak{R}^t)'(\mathfrak{R}^t)^{-1}) = 0$ and consequently

$$\mathrm{Tr}(\Phi^t_A(p)(\mathfrak{R}^t)'(\mathfrak{R}^t)^{-1}(\Phi^t_A(p))^{-1}) = 0.$$

Therefore we define the linear differential system $B = A + H$ which is measurable and clearly conservative. In fact, as we mention previously, if $A \in \mathcal{T}_a$ (respectively $A \in \mathcal{G}$) and

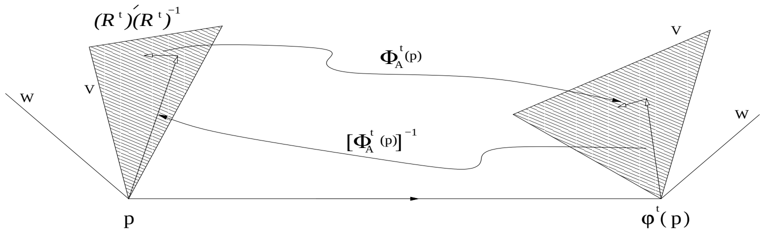

Figure 1: The action of the perturbation $H(\varphi^t(p))$, $t \in [0,1]$.

$Tr(H) = 0$, then $A + H \in \mathcal{T}_a$ (respectively $A \in \mathcal{G}$). In Figure 1 we give the geometric idea of how H acts. Now to prove 1. we compute the norm of H:

$$\begin{aligned} \|H(\varphi^t(p))\| &= \|\Phi_A^t(p)(\mathfrak{R}^t)'(\mathfrak{R}^t)^{-1}(\Phi_A^t(p))^{-1}\| \leq \\ &\leq K^2\|(\mathcal{R}^t)'(\mathcal{R}^t)^{-1}\| \leq 2K^2\xi \leq 2K^2\xi_0 < \epsilon. \end{aligned}$$

Moreover, by choice of η, we have that Supp(H) is $\varphi^t(p)$ for $t \in [0,1]$ and 2. is proved. Note that the perturbed system B generates the linear flow $\Phi_{A+H}^t(p)$ which is the same as Ψ^t, hence given $u \in W_p$ we have:

$$\Phi_B^t(p) \cdot u = \Psi^t(p) \cdot u = \Phi_A^t(p)[R_{\eta(t)\xi}(u_V) + u_W] = \Phi_A^t(p) \cdot u_W = \Phi_A^t(p) \cdot u,$$

and 3. follows. Finally to prove 4. taking $u \in V_p$ we obtain,

$$\begin{aligned} \Phi_B^1(p) \cdot u &= \Psi^1(p) \cdot u = \Phi_A^1(p) \circ \mathcal{R}^1 \cdot u = \Phi_A^1(p)[R_{\eta(1)\xi}(u_V) + u_W] = \\ &= \Phi_A^1(p) R_\xi(u_V) = \Phi_A^1(p) \circ R_\xi \cdot u, \end{aligned}$$

and Lemma 3.1 is proved.

Lemma 3.2 *Given $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$) and $\epsilon > 0$, there exists $\xi_0 > 0$ (depending on A and ϵ), such that given any $\xi \in (0, \xi_0)$, any $p \in X$ (non-periodic or with period larger than 1) and any 2-dimensional vector space $V_{\varphi^1(p)} \subset \mathbb{R}^d_{\varphi^1(p)}$, there exists a measurable traceless system H (depending on ξ and p) such that:*

1. *$\|H\| < \epsilon$;*
2. *H is supported in $\varphi^t(p)$ for $t \in [0,1]$;*
3. *If W_p is the orthogonal complement of V_p in $\mathbb{R}^d_p$, then $\Phi_{A+H}^t(p) = \Phi_A^t(p)$ on W_p;*
4. *$\Phi_{A+H}^1(p) \cdot v = \tilde{R}_\xi \circ \Phi_A^1(p) \cdot v$, $\forall v \in V_p$, where $\tilde{R}_\xi$ is the elliptical rotation of angle ξ on $V_{\varphi^1(p)}$;*

Proof. We keep the same notation of Lemma 3.1 and we define the following 1-parameter linear map acting on $\mathbb{R}^d_{\varphi^t(p)}$:

$$\tilde{\mathcal{R}}^t = \Phi^t_A(p) \cdot \mathcal{R}^t \cdot [\Phi^t_A(\varphi^t(p))]^{-1}.$$

We denote $\Psi^t = \tilde{\mathcal{R}}^t \cdot \Phi^t_A$ and taking time derivatives we obtain,

$$\begin{aligned}
(\Psi^t)' &= (\tilde{\mathcal{R}}^t \cdot \Phi^t_A)' = (\Phi^t_A \cdot \mathcal{R}^t)' = \\
&= [A(\varphi^t(p)) + H(\varphi^t(p))] \cdot (\Phi^t_A \cdot \mathcal{R}^t) = \\
&= [A(\varphi^t(p)) + H(\varphi^t(p))] \cdot (\Phi^t_A \cdot \mathcal{R}^t \cdot (\Phi^t_A)^{-1} \cdot \Phi^t_A) = \\
&= [A(\varphi^t(p)) + H(\varphi^t(p))] \cdot (\tilde{\mathcal{R}}^t \cdot \Phi^t_A) = \\
&= [A(\varphi^t(p)) + H(\varphi^t(p))] \cdot \Psi^t,
\end{aligned}$$

now it is analogous to the proof of Lemma 3.1. The lemma is proved.

In the next lemma, and under certain hypotheses, we produce a small norm perturbation along a large orbit segment which allows us to perform a rotation by a large angle. This will enable us, in Lemma 3.4, to mix the expansion rates of a given splitting.

First we introduce some notation that will be used in Lemma 3.3. Given a system A, $m \in \mathbb{N}$, $p \in X$ and a nontrivial splitting of index $n \in \{1, ..., d-1\}$ at $p \in X$, $U_p \oplus S_p = \mathbb{R}^d_p$, we define a codimension 2 subspace of $\mathbb{R}^d_p$, denoted by $\mathcal{W}_p$, in the following way:
Definition of $\mathcal{W}_p$: Denote by $U_t \oplus S_t$ the image of the splitting $U_p \oplus S_p$ by $\Phi^t_A(p)$, that is, $U_t = U_{\varphi^t(p)} = \Phi^t_A(p)(U_p)$ and $S_t = S_{\varphi^t(p)} = \Phi^t_A(p)(S_p)$. Now we consider three unit vectors $\nu \in U_p$, $\eta \in S_p$ and $\eta' \in S_m$ such that:

$$\|\Phi^m_A(p) \cdot \nu\| = \mathfrak{m}(\Phi^m_A(p)|_{U_p}) \ , \ \|\Phi^m_A(p) \cdot \eta\| = \|\Phi^m_A(p)|_{S_p}\| \text{ and } \eta' = \frac{\Phi^m_A(p) \cdot \eta}{\|\Phi^m_A(p) \cdot \eta\|}.$$

Define $G_p = U_p \cap \nu^\perp$, $H_m = S_m \cap \eta'^\perp$ and $H_p = \Phi^{-m}_A(\varphi^m(p))(H_m) \subset S_p$. Consider unit vectors $v_p = \nu$ and $w_p \in S_p \cap H_p^\perp$. Finally, define $\mathcal{W}_p = G_p \oplus H_p$.

Lemma 3.3 *Given $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$) and $\epsilon, c, \xi > 0$. There exists $m_0(\epsilon, c, \xi) = m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ we have the following; let $p \in X$ be any non-periodic point and $U_t \oplus S_t$ be a nontrivial splitting of index $n \in \{1, ..., d-1\}$ over the orbit of p. Suppose that for all $t, r \in [0, m]$ with $0 \leq t + r \leq m$ we have:*

(i) $\measuredangle(S_t, U_t) > \xi$;

(ii) $\frac{\|\Phi^r_A(\varphi^t(p))|_{S_t}\|}{\mathfrak{m}(\Phi^r_A(\varphi^t(p))|_{U_t})} \leq c$;

(iii) $\frac{\|\Phi^m_A(p)|_{S_p}\|}{\mathfrak{m}(\Phi^m_A(p)|_{U_p})} \geq \frac{1}{2}$.

Then, for $\mathcal{W}_p \subset \mathbb{R}^d_p$, $v_p \in U_p$ and $w_p \in S_p$ as defined above, if $\alpha = \measuredangle(v_p + \mathcal{W}_p, w_p + \mathcal{W}_p)$ then, there exists a measurable traceless system H, such that:

1. $\|H\| < \epsilon$;

2. *H is supported in $\varphi^t(p)$ for $t \in [0,m]$;*

3. $\Phi^t_{A+H}(p) = \Phi^t_A(p)$ *on* $\mathcal{W}_p$;

4. $\Phi^m_{A+H}(p) = \Phi^m_A(p) \circ R_\alpha$*, where R_α is a rotation of angle α on $\mathbb{R}^d_p/\mathcal{W}_p$.*

Proof. Once again we use the ideas of the proof of Lemma 3.1. We consider $\theta_0 > 0$ and $m_0 \in \mathbb{N}$ such that:

$$\theta_0 < \frac{\epsilon \sin^6(\xi)}{16c} \text{ and } m_0 \geq \frac{2\pi}{\theta_0}.$$

Therefore, for any $m \geq m_0$, we can choose $\theta \in (0,\theta_0)$ such that $m\theta = \alpha$. Now we take a C^∞ function $\eta\colon \mathbb{R} \to [0,m]$ such that $\eta(t) = 0$ for $t \leq 0$, $\eta(t) = m$ for $t \geq m$, and $0 \leq \eta'(t) \leq 2$, for all t. Finally, we define $R_{\eta(t)\theta}\colon \mathbb{R}^d_p/\mathcal{W}_p \to \mathbb{R}^d_p/\mathcal{W}_p$ like in Lemma 3.1, in order to obtain a family of linear maps $\mathcal{R}_t\colon \mathbb{R}^d_p \to \mathbb{R}^d_p$ such that; $\mathcal{R}_t|_{\mathcal{W}_p} = Id$, $\mathcal{R}_t(\mathcal{W}_p^\perp) = \mathcal{W}_p^\perp$ and $\mathcal{R}_t/\mathcal{W}_p = R_{\eta(t)\theta}$. Like in Lemma 3.1 we obtain a perturbation

$$H(\varphi^t(p)) = \Phi^t_A(p)(\mathcal{R}^t)'(\mathcal{R}^t)^{-1}(\Phi^t_A(p))^{-1} \text{ for } t \in [0,m].$$

Let us estimate $\|H(\cdot)\|$. Take $\Phi^t_A(p)/\mathcal{W}_p\colon \mathbb{R}^d_p/\mathcal{W}_p \to \mathbb{R}^d_{\varphi^t(p)}/\Phi^t_A(p)(\mathcal{W}_p)$ the induced linear map from the quotient space $\mathbb{R}^d_p/\mathcal{W}_p$ into the quotient space $\mathbb{R}^d_{\varphi^t(p)}/\Phi^t_A(p)(\mathcal{W}_p)$. It follows directly from Lemma 3.8 of [8] that (i), (ii) and (iii) implies the following inequality for all $t \in [0,m]$,

$$\frac{\|\Phi^t_A(p)/\mathcal{W}_p\|}{\mathfrak{m}(\Phi^t_A(p)/\mathcal{W}_p)} \leq \frac{8c}{\sin^6(\xi)}.$$

Since for $v \in \mathcal{W}_p$ and $t \in \mathbb{R}$ we have $H(\varphi^t(p)) \cdot v = \vec{0}$ we obtain that,

$$\begin{aligned}
\|H(\varphi^t(p))\| &= \|(\Phi^t_A(p)/\mathcal{W}_p)(\mathcal{R}^t)'(\mathcal{R}^t)^{-1}(\Phi^t_A(p)/\mathcal{W}_p)^{-1}\| \leq \\
&\leq 2\theta\|\Phi^t_A(p)/\mathcal{W}_p\|\|(\Phi^t_A(p)/\mathcal{W}_p)^{-1}\| = \\
&= 2\theta\frac{\|\Phi^t_A(p)/\mathcal{W}_p\|}{\mathfrak{m}(\Phi^t_A(p)/\mathcal{W}_p)} \leq \\
&\leq 2\theta\frac{8c}{\sin^6(\xi)} < 2\theta_0\frac{8c}{\sin^6(\xi)} < \epsilon.
\end{aligned}$$

Therefore 1. follows. The conclusions 2. and 3. are immediate. Finally, to prove 4., we note that in time-m we rotate $\eta(m)\theta = m\theta = \alpha$ and the lemma is proved.

The following lemma will be crucial in the sequel.

Lemma 3.4 *Given a system $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$) and $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ we have the following property: for all non-periodic point p with a splitting $\mathbb{R}^d_p = U_p \oplus S_p$ satisfying*

$$\frac{\|\Phi^m_A(p)|_{S_p}\|}{\mathfrak{m}(\Phi^m_A(p)|_{U_p})} \geq \frac{1}{2}, \tag{3.10}$$

there exists a measurable traceless system H supported in $\varphi^{[0,m]}(p)$, with $\|H\| < \epsilon$ and such that there exist vectors $\mathfrak{u} \in U_p \setminus \{\vec{0}\}$ and $\mathfrak{s} \in \Phi^m_A(S_p) \setminus \{\vec{0}\}$ satisfying $\Phi^m_{A+H}(p)(\mathfrak{u}) = \mathfrak{s}$.

Proof. Let $\xi_0 > 0$ be the smaller one given by both Lemma 3.1 and Lemma 3.2 (depending on $\epsilon > 0$ and A). Let also $c > 0$ be such that:

$$c > (\sin \xi_0)^{-2} \text{ and } c \geq \max_{p \in X} \left\{ \frac{\|\Phi_A^1(p)\|}{\mathfrak{m}(\Phi_A^1(p))} \right\}. \tag{3.11}$$

Take $m_0 \in \mathbb{N}$ given by Lemma 3.3 which, we recall, has to be at least $\frac{32c\pi}{\epsilon \sin^6(\xi_0)}$.
Small angle: Recall the notation $S_t = \Phi_A^t(p)(S_p)$ and $U_t = \Phi_A^t(p)(U_p)$ for $t \in [0, m]$. First we assume that

$$\exists t \in [0, m] \text{ such that } \measuredangle(S_t, U_t) \leq \xi_0. \tag{3.12}$$

Then we take unit vectors $s_t \in S_t$ and $u_t \in U_t$ with $\measuredangle(s_t, u_t) < \xi_0$. If $t \in [0, m-1]$, then we use Lemma 3.1 with $V_{\varphi^t(p)} = \langle s_t, u_t \rangle$ (where $\langle e_1, e_2 \rangle$ denotes the vector space spanned by e_1 and e_2) and we define $H(\varphi^{t+r}(p))$ for $r \in [0, 1]$ and zero otherwise. On the other hand, if $t \in (m-1, m]$, then we use Lemma 3.2 and define $H(\varphi^r(p))$ for $r \in [t-1, t]$ and zero otherwise. In both cases we obtain vectors $\mathfrak{u} \in U_p \setminus \{\vec{0}\}$ and $\mathfrak{s} \in \Phi^m(S_p) \setminus \{\vec{0}\}$ such that $\Phi_{A+H}^m(p)(\mathfrak{u}) = \mathfrak{s}$.

Now we assume that there exist $r, t \in \mathbb{R}$ with $0 \leq r + t \leq m$ such that:

$$\frac{\|\Phi_A^r(\varphi^t(p))|_{S_t}\|}{\mathfrak{m}(\Phi_A^r(\varphi^t(p))|_{U_t})} \geq c. \tag{3.13}$$

We choose unit vectors $s_t \in S_t$ and $u_t \in U_t$ which realizes both norms, that is $\|\Phi_A^r(\varphi^t(p)) \cdot s_t\| = \|\Phi_A^r(\varphi^t(p))|_{S_t}\|$ and $\|\Phi_A^r(\varphi^t(p)) \cdot u_t\| = \mathfrak{m}(\Phi_A^r(\varphi^t(p))|_{U_t})$. We define also the unit vectors,

$$u_{t+r} = \frac{\Phi_A^r(\varphi^t(p)) \cdot u_t}{\|\Phi_A^r(\varphi^t(p)) \cdot u_t\|} \in U_{t+r} \text{ and } s_{t+r} = \frac{\Phi_A^r(\varphi^t(p)) \cdot s_t}{\|\Phi_A^r(\varphi^t(p)) \cdot s_t\|} \in S_{t+r}.$$

The vector $\hat{u}_t = u_t + \sin(\xi_0) s_t$ satisfy $\measuredangle(\hat{u}_t, u_t) < \xi_0$ so an ϵ-small perturbation B_1 given by Lemma 3.1 with $V_{\varphi^t(p)} = \langle s_t, u_t \rangle$ will send u_t into $\mathbb{R}\Phi_A^1(\varphi^t(p)) \cdot (\hat{u}_t)$.

Let $\gamma = \|\Phi_A^r(\varphi^t(p)) \cdot u_t\| (\sin \xi_0 \|\Phi_A^r(\varphi^t(p)) \cdot s_t\|)^{-1}$ we define a vector in $\mathbb{R}_{\varphi^{t+r}(p)}$ by $\hat{s}_{t+r} = \gamma u_{t+r} + s_{t+r}$. We have that,

$$\begin{aligned}
\Phi_A^r(\varphi^t(p)) \cdot \hat{u}_t &= \Phi_A^r(\varphi^t(p)) \cdot u_t + \sin(\xi_0) \Phi_A^r(\varphi^t(p)) \cdot s_t = \\
&= \Phi_A^r(\varphi^t(p)) \cdot u_t + \frac{\|\Phi_A^r(\varphi^t(p)) \cdot u_t\|}{\gamma \|\Phi_A^r(\varphi^t(p)) \cdot s_t\|} \Phi_A^r(\varphi^t(p)) \cdot s_t = \\
&= \gamma^{-1} \|\Phi_A^r(\varphi^t(p)) \cdot u_t\| . (\gamma u_{t+r} + s_{t+r}) = \\
&= \gamma^{-1} \|\Phi_A^r(\varphi^t(p)) \cdot u_t\| . \hat{s}_{t+r}.
\end{aligned}$$

Hence the vectors $\Phi_A^r(\varphi^t(p)) \cdot \hat{u}_t$ and $\hat{s}_{t+r}$ are colinear. Moreover, by (3.11), (3.13) and definition of γ, u_t and s_t, we have

$$\gamma = \frac{\mathfrak{m}(\Phi_A^r(\varphi^t(p))|_{U_t})}{\|\Phi_A^r(\varphi^t(p))|_{S_t}\|} (\sin \xi_0)^{-1} \leq (c \sin \xi_0)^{-1} < \sin \xi_0.$$

Therefore we obtain that $\measuredangle(s_{t+r}, \hat{s}_{t+r}) < \xi_0$ and using Lemma 3.2 we are able to produce a time-1 ϵ-perturbation B_2 based at $\varphi^{t+r-1}(p)$ such that $\Phi_{B_2}^1(\varphi^{t+r-1}(p)) = R_{\xi_0} \circ$

$\Phi^1_A(\varphi^{t+r-1}(p))$, where R_{ξ_0} acts in $V_{\varphi^{t+r}(p)} = \langle s_{t+r}, \hat{s}_{t+r}\rangle$ and sends $\hat{s}_{t+r}$ into s_{t+r}. We note that choosing $c > 0$ sufficiently large, see (3.11), guarantees disjoint perturbations. Now we concatenate as follows:

$$\mathbb{R}u_0 \overset{\Phi^t_A(p)}{\longrightarrow} \mathbb{R}u_t \overset{\Phi^1_{B_1}(\varphi^t(p))}{\longrightarrow} \mathbb{R}[\Phi^1_A(\varphi^t(p))\cdot \hat{u}_t] \overset{\Phi^{r-1}_A(\varphi^{t+1}(p))}{\longrightarrow} \mathbb{R}[\Phi^r_A(\varphi^t(p))\cdot \hat{u}_t].$$

We go back by time-1 and then we perform our second perturbation B_2:

$$\mathbb{R}[\Phi^{-1}_A(\Phi^r_A(\varphi^t(p))\cdot \hat{u}_t)] \overset{\Phi^1_{B_2}(\varphi^{t+r-1}(p))}{\longrightarrow} \mathbb{R}s_{t+r} \overset{\Phi^{m-t-r}_A(\varphi^{t+r}(p))}{\longrightarrow} \mathbb{R}s_m.$$

Large angle: Using the same notation of Lemma 3.3, we begin by defining $G_m = \Phi^m_A(p)(G_p)$ and $\mathcal{W}_m = G_m \oplus H_m$. Notice that $\Phi^m_A(p)(\mathcal{W}_p) = \mathcal{W}_m$. Now, we treat the case when we do not have (3.12) and also (3.13). In this case, the fact that the angles are bounded away from ξ_0, the condition

$$\forall r,t\in\mathbb{R}\colon 0\le t+r\le m \text{ we have } \frac{\|\Phi^r_A(\varphi^t(p))|_{S_t}\|}{\mathfrak{m}(\Phi^r_A(\varphi^t(p))|_{U_t})}\le c,$$

and the hypothesis (3.10) will allow us to use Lemma 3.3 and obtain a measurable traceless system H, such that $\|H\| < \epsilon$, H is supported in $\varphi^t(p)$ for $t \in [0,m]$, $\Phi^t_{A+H}(p) = \Phi^t_A(p)$ on $\mathcal{W}_p$ and $\Phi^m_{A+H}(p) = \Phi^m_A(p)\circ R_\alpha$, where R_α is a rotation of angle $\alpha = \measuredangle(v_p+\mathcal{W}_p, w_p+\mathcal{W}_p)$ on $\mathbb{R}^d_p/\mathcal{W}_p$. Then we obtain,

$$\Phi^m_{A+H}(p)(v_p+\mathcal{W}_p) = \Phi^m_A(p)\circ R_\alpha(v_p+\mathcal{W}_p) = \Phi^m_A(p)\cdot w_p + \mathcal{W}_m.$$

Finally, let us see that there exists nonzero vectors $\mathfrak{u}\in U_p$ and $\mathfrak{s}\in S_m$ such that $\Phi^m_{A+H}(p)(\mathfrak{u}) = \mathfrak{s}$. Since $\mathcal{W}_m = G_m\oplus H_m$ there exist $g_m\in G_m$ and $h_m \in H_m$ such that,

$$\Phi^m_{A+H}(p)\cdot v_p = \Phi^m_A(p)\cdot w_p + g_m + h_m. \tag{3.14}$$

Let $g_p = \Phi^{-m}_A(\varphi^m(p))\cdot g_m$. Note that $\Phi^m_{A+H}(p)\cdot g_p = g_m$, because $g_p\in G_p$, $G_p\subset \mathcal{W}_p\cap U_p$ and $\Phi^t_{A+H}(p) = \Phi^t_A(p)$ on $\mathcal{W}_p$.

Consider the vector $\mathfrak{u}\in U_p$ defined by $\mathfrak{u} = v_p - g_p$. Since

$$\Phi^m_{A+H}(p)\cdot\mathfrak{u} = \Phi^m_{A+H}(p)\cdot v_p - \Phi^m_{A+H}(p)\cdot g_p = \Phi^m_{A+H}(p)\cdot v_p - g_m,$$

using (3.14) we obtain $\Phi^m_{A+H}(p)\cdot\mathfrak{u} = \Phi^m_A(p)\cdot w_p + h_m \in S_m$ and Lemma 3.4 is proved.

4 On the decay of the entropy function

The next lemma is the flow version of Lemma 3.12 of [7] and is easily obtained from it.

Lemma 4.1 *Let $\varphi^t\colon X\to X$ be a measurable μ-invariant flow, $\gamma>0$, $\Delta\subseteq X$ such that $\mu(\Delta)>0$ and $\Gamma = \bigcup_{t\in\mathbb{R}}\varphi^t(\Delta)$. There exists a measurable function $T\colon\Gamma\to\mathbb{N}$ such that for μ-a.e. $p\in\Gamma$ and for all $t\ge T(p)$ there exists some $s\in[0,t]$ satisfying $|\frac{s}{t}-\frac{1}{2}|<\gamma$ and $\varphi^s(p)\in\Delta$.*

Proof. Let $\Delta_1 = \underset{t\in[0,1]}{\cup} \varphi^t(\Delta)$ and $f = \varphi^1$. Notice that $\Gamma = \underset{n\in\mathbb{Z}}{\cup} f^n(\Delta_1)$. Now we apply Lemma 3.12 of [7] to f, Γ and Δ_1. Therefore, there exists a measurable function $N_0\colon \Gamma \to \mathbb{N}$ (depending on γ) such that for any $p \in \Gamma$ and for any $n \geq N_0(p)$, there exists $\ell \in \{0, 1, ..., n\}$ such that $|\frac{\ell}{n} - \frac{1}{2}| < \frac{\gamma}{2}$ and $f^\ell(p) \in \Delta_1$.

Since $f^\ell(p) \in \Delta_1$ there exists $s \in [\ell - 1, \ell]$ such that $\varphi^s(p) \in \Delta$. Take $T_0 \in \mathbb{R}$ such that for all $t \geq T_0$ we have $\left|\frac{[t]}{t} - 1\right| < \gamma$ and $\frac{1}{[t]} < \frac{\gamma}{2}$.

For $p \in \Gamma$ we define $T(p) = \max\{N_0(p), T_0\}$. Hence, for all $t \geq T(p)$ and s as above (depending on $[t]$) we have that

$$\frac{s}{t} - \frac{1}{2} \geq \frac{\ell - 1}{[t]} - \frac{1}{2} = \frac{\ell}{[t]} - \frac{1}{2} - \frac{1}{[t]} > -\frac{\gamma}{2} - \frac{1}{[t]} > -\gamma,$$

and also

$$\frac{s}{t} - \frac{1}{2} \leq \frac{\ell}{t} - \frac{1}{2} = \left(\frac{\ell}{[t]} - \frac{1}{2}\right)\frac{[t]}{t} + \left(\frac{[t]}{t} - 1\right)\frac{1}{2} < \gamma.$$

Therefore, we obtain $|\frac{s}{t} - \frac{1}{2}| < \gamma$ finishing the proof of the lemma.

The next lemma gives us a local strategy to use the absence of dominated splitting and the different Lyapunov exponents in order to cause a decay, by a small perturbation, of the largest Lyapunov exponent of the n^{th} exterior power system. We follow Proposition 4.2 of [8] adapting it to the flow setting. We only give the main steps of the proof, for all the details see [8].

Lemma 4.2 *Let $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$) be such that there exists a positive measure subset of $\mathcal{O}(A)$ satisfying the following hypotheses;*

(H1) not all the Lyapunov exponents of the cocycle Φ_A^t are equal;

(H2) for all $m \in \mathbb{N}$ the cocycle Φ_A^t does not admit an m-dominated Oseledets' splitting.

Then, given any $\epsilon, \delta > 0$, there exist $n \in \{1, ..., d-1\}$ and $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ the following is true: there exists a measurable function $\tilde{T}\colon \Gamma_n^*(A, m) \to \mathbb{R}$ such that for μ-a.e. point $q \in \Gamma_n^*(A, m)$ and every $t > \tilde{T}(q)$ there exists $H \in \mathcal{T}$ supported on the segment $\varphi^s(q)$ for $s \in [0, m]$ such that

1. *$\|H\| < \epsilon$;*
2. *$\frac{1}{t}\log\| \wedge^n (\Phi_{A+H}^t(q))\| < \frac{1}{2}(\Sigma_{n-1}(A, q) + \Sigma_{n+1}(A, q)) + \delta$.*

Proof. Using hypothesis (H1) we obtain that there exists $n \in \{1, ..., d-1\}$ such that $\lambda_n(q) > \lambda_{n+1}(q)$ for any q in a positive measure subset of $\mathcal{O}(A)$. Then, hypothesis (H2), implies that $\mu(\Gamma_n^\sharp(A, m)) > 0$ for all $m \in \mathbb{N}$.

Choice of m_0: By (2.6) there exists $m_1 \in \mathbb{N}$ such that for all $m \geq m_1$ we have $\mu(\Gamma_n^*(A, m)) > 0$. For $m \geq m_1$ let $\Delta_n^*(A, m) \subset \Gamma_n^*(A, m)$ be like in (2.7). By Lemma 2.2 we conclude that $\mu(\Delta_n^*(A, m)) > 0$ for any $m \geq m_1$. Now let $m_0 \in \mathbb{N}$ be any integer greater than m_1 as well as large enough so that conclusion of Lemma 3.4 apply.

Notice that we have,

$$\frac{1}{2}[\Sigma_{n-1}(A,q)+\Sigma_{n+1}(A,q)]=\lambda_1(q)+...+\lambda_{n-1}(q)+\frac{\lambda_n(q)+\lambda_{n+1}(q)}{2}. \quad (4.15)$$

Let $\mathbb{R}^d_{\Gamma^*_n(A,m)}=U\oplus S$, where U corresponds to the vector space spanned by the Lyapunov exponents $\lambda_1(q)$,...,$\lambda_n(q)$ and S corresponds to the vector space spanned by $\lambda_{n+1}(q)$,...,$\lambda_d(q)$. We recall that, by definition of $\Gamma^*_n(A,m)$, we have $\lambda_n(q)>\lambda_{n+1}(q)$.

Choice of $\tilde{T}$: By Lemma 4.1 for any μ-generic point $q\in\Gamma^*_n(A,m)$, there exists $T(q)$ such that for all $t\geq T(q)$ and $s\approx\frac{t}{2}$ we have $p=\varphi^s(q)\in\Delta^*_n(A,m)$. To define $\tilde{T}(q)$ we increase, if necessary, $T(q)$ (depending on A, q, m and δ) exactly as in the definition of $N(\cdot)$ in Proposition 4.2 of [8] in order to obtain, not only useful estimates given by Oseledets' Theorem but also $\tilde{T}(q)>>m$. These estimates will be used later when we compute the size of the perturbation.

Let E_q be the vector space associated to $\lambda_1^{\wedge n}(q)$ (the largest Lyapunov exponent of the n^{th} exterior power system) and F_q be the vector space associated to the other Lyapunov exponents. We obtain a splitting $\wedge^n(\mathbb{R}^d)=E\oplus F$. Since both $\lambda_i(q)'s$ and $\lambda_j^{\wedge n}(q)'s$ are written in nonincreasing order it follows that,

$$\lambda_1^{\wedge n}(q)=\sum_{i=1}^{n}\lambda_i(q) \text{ and } \lambda_2^{\wedge n}(q)=\sum_{i=1}^{n-1}\lambda_i(q)+\lambda_{n+1}(q). \quad (4.16)$$

Since $\lambda_n(q)>\lambda_{n+1}(q)$ we get $\lambda_1^{\wedge n}(q)>\lambda_2^{\wedge n}(q)$ and also that $\dim(E_q)=1$. By using Lemma 4.4 of [8] and Lemma 3.4 we get that, $\wedge^n(\Phi^m_{A+H}(p))\colon\ \wedge^n(\mathbb{R}^d_p)\to\wedge^n(\mathbb{R}^d_{\varphi^m(p)})$ satisfies the property,

$$\wedge^n(\Phi^m_{A+H}(p))(E_p)\subset F_{\varphi^m(p)}. \quad (4.17)$$

We decompose the action of the map $\wedge^n(\Phi^m_{A+H}(p))$ in three steps (see Figure 2); the first (between q and p) and the third (between $\varphi^m(p)$ and $\varphi^t(q)$), with matrix in the basis induced by the Oseledets directions and with respect to the splitting $E\oplus F$ which we denote respectively by:

$$A_1=\begin{pmatrix}A_1^{uu} & 0\\ 0 & A_1^{ss}\end{pmatrix} \text{ and } A_2=\begin{pmatrix}A_2^{uu} & 0\\ 0 & A_2^{ss}\end{pmatrix}.$$

The second step (between p and $\varphi^m(p)$), with matrix in the basis induced by the Oseledets directions and with respect to the splitting $E\oplus F$ which we denote by:

$$B=\begin{pmatrix}B^{uu} & B^{us}\\ B^{su} & B^{ss}\end{pmatrix}.$$

The outcome of the inclusion (4.17) is that $B^{uu}=0$. Therefore we obtain:

$$\wedge^n(\Phi^t_{A+H}(q))=\begin{pmatrix}0 & A_2^{uu}B^{us}A_1^{ss}\\ A_2^{ss}B^{su}A_1^{uu} & A_2^{ss}B^{ss}A_1^{ss}\end{pmatrix}. \quad (4.18)$$

By choice of $\tilde{T}$ we have $t>>m$ so the logarithm of the norm of the entries of B, that only depend on m, when divided by t are less than a small fraction of δ. Moreover, as $p=\varphi^s(q)$ with $s\approx t/2$, we obtain that

- $\|A_1^{ss}\| < \exp\left(s\left(\lambda_2^{\wedge n}(q)+\delta\right)\right)$;
- $\|A_1^{uu}\| < \exp\left(s\left(\lambda_1^{\wedge n}(q)+\delta\right)\right)$;
- $\|A_2^{ss}\| < \exp\left((t-s-m)\left(\lambda_2^{\wedge n}(q)+\delta\right)\right)$ and
- $\|A_2^{uu}\| < \exp\left((t-s-m)\left(\lambda_1^{\wedge n}(q)+\delta\right)\right)$.

These estimates follow by choice of $\tilde{T}(p)$ $(< t)$ and are obtained using the asymptotic properties (2.3) and (2.4) given by Oseledets' Theorem applied to the dynamics of $\wedge^n(\Phi_A^t)$.

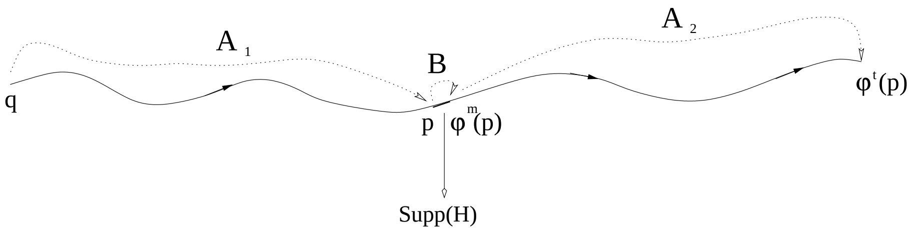

Figure 2: The $A_i's$ for $i = 1, 2$ goes approximately $\frac{t}{2}$ where $t >> m$.

We obtain estimates for the norm of the entries in (4.18) and it is straightforward to see that

$$\log \| \wedge^n (\Phi_{A+H}^t(q))\| < t\left(\frac{\lambda_1^{\wedge n}(q)+\lambda_2^{\wedge n}(q)}{2}+k\delta\right),$$

for some $k > 0$. We switch δ by δ/k along the proof and using (4.15) and (4.16) we obtain,

$$\begin{aligned}
\frac{1}{t}\log \| \wedge^n (\Phi_{A+H}^t(q))\| &< \frac{1}{2}\left(\sum_{i=1}^{n}\lambda_i(q)+\sum_{i=1}^{n-1}\lambda_i(q)+\lambda_{n+1}(q)\right)+\delta = \\
&= \lambda_1(q)+...+\lambda_{n-1}(q)+\frac{\lambda_n(q)+\lambda_{n+1}(q)}{2}+\delta = \\
&= \frac{1}{2}(\Sigma_{n-1}(A,q)+\Sigma_{n+1}(A,q))+\delta,
\end{aligned}$$

and the lemma is proved.

In the next lemma we make the previous lemma global. We only present a brief overview of the proof. For the complete details see [8] and [5].

Lemma 4.3 *Let $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$) be such that there exists a positive measure subset of $\mathcal{O}(A)$ satisfying the hypotheses (H1) and (H2) of Lemma 4.2. Take any $\epsilon, \delta > 0$. Then, there exist $n \in \{1, ..., d-1\}$ and $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, there exists a continuous system $B = A + H$ (with $Tr(A) = Tr(B)$) such that,*

1. *$H(\cdot) = [0]$ outside the open set $\Gamma_n(A, m)$;*

2. $\|H\|_\infty < \epsilon$;

3. $\int_{\Gamma_n(A,m)} \Sigma_n(B,q)d\mu(q) < \delta + \int_{\Gamma_n(A,m)} \frac{1}{2}(\Sigma_{n-1}(A,q) + \Sigma_{n+1}(A,q))d\mu(q)$.

To prove this lemma we recall that the Ambrose-Kakutani Theorem (see [1]) says that any aperiodic flow φ^t (i.e. the set of periodic points of φ^t has zero measure) flow is isomorphic to some special flow. Now, by Lemma 4.2 and by Ambrose-Kakutani's Theorem, we can develop a tower argument over the aperiodic flow $\varphi^t\colon \Gamma_n^*(A,m) \to \Gamma_n^*(A,m)$. This strategy, which is completely described in Proposition 4.8, Proposition 7.3 and Lemma 7.4 of [8] (see also [5] for the ingredients in the flow framework), allows us to construct a measurable system $\tilde{B}$ such that $\|A - \tilde{B}\|_\infty < \epsilon/2$. Once the global measurable system $\tilde{B}$ is constructed and, since Luzin's theorem asserts that measurable functions are almost continuous, we produce the continuous system. The highlights of the proof may be seen in [9] Section 2.4.

Now we define the discontinuity "jump" of the function $\mathcal{E}_n$ defined in Section 2.5 by:

$$J_n(A) = \int_{\Gamma_n(A,\infty)} \lambda_n(A,p) - \lambda_{n+1}(A,p)d\mu(p).$$

In the next lemma we follow [8] (Proposition 4.17):

Lemma 4.4 *Let $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$) be such that there exists a positive measure subset of $\mathcal{O}(A)$ satisfying the hypotheses (H1) and (H2) of Lemma 4.2. Take any $\epsilon, \delta > 0$. There exists $n \in \{1, ..., d-1\}$ with $J_n(A) > 0$ and there exists $B \in \mathcal{T}$ (respectively in $\mathcal{T}_a$ or $\mathcal{G}$) ϵ-close to A such that*

$$\int_X \Sigma_n(B,\cdot)d\mu < \int_X \Sigma_n(A,\cdot)d\mu - 2J_n(A) + \delta.$$

Proof. By Lemma 4.3 we obtain B such that $A = B$ outside $\Gamma_n(A,m)$ such that,

$$\int_{\Gamma_n(A,m)} \Sigma_n(B,\cdot)d\mu < \delta + \int_{\Gamma_n(A,m)} \frac{\Sigma_{n-1}(A,\cdot) + \Sigma_{n+1}(A,\cdot)}{2} d\mu.$$

Clearly $X = \Gamma_n(A,m) \sqcup (X \setminus \Gamma_n(A,m))$ so we split the integral:

$$\int_X \Sigma_n(B,\cdot)d\mu < \delta + \int_{\Gamma_n(A,m)} \frac{\Sigma_{n-1}(A,\cdot) + \Sigma_{n+1}(A,\cdot)}{2} d\mu + \int_{X\setminus\Gamma_n(A,m)} \Sigma_n(A,\cdot)d\mu.$$

Now since $\Sigma_n(A,\cdot) = \lambda_1(A,\cdot) + ... + \lambda_n(A,\cdot)$ we note that,

$$2J_n(A) = \int_{\Gamma_n(A,\infty)} \left(\Sigma_n(A,\cdot) - \frac{\Sigma_{n-1}(A,\cdot) + \Sigma_{n+1}(A,\cdot)}{2}\right) d\mu.$$

Moreover, since $\Gamma_n(A,m) \supset \Gamma_n(A,\infty)$, we obtain,

$$-\int_{\Gamma_n(A,m)} \left(\Sigma_n(A,\cdot) - \frac{\Sigma_{n-1}(A,\cdot) + \Sigma_{n+1}(A,\cdot)}{2}\right) d\mu \leq -2J_n(A).$$

Or equivalently

$$\int_{\Gamma_n(A,m)} \left(\frac{\Sigma_{n-1}(A,\cdot) + \Sigma_{n+1}(A,\cdot)}{2} \right) d\mu \leq -2J_n(A) + \int_{\Gamma_n(A,m)} \Sigma_n(A,\cdot) d\mu,$$

and the lemma is proved.

5 End of the proof of the results

Proposition 2.1 is a direct consequence of Lemma 4.4. To prove Theorem 1.1, we note that the continuity points of an upper semi-continuous function is a residual set (see [20]). Hence by Proposition 2.1 and the fact that $\mathcal{E}$ is upper semi-continuous we obtain the conclusion of Theorem 1.1.

The proof of Corollary 1.1 now follows easily, because all the perturbations we did are traceless. Actually, take $A \in \mathcal{R}$ (the residual given by Theorem 1.1) and $p \in \mathcal{O}(A)$ with a trivial spectrum. It is a consequence of the Oseledets Theorem that:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log |\det(\Phi_A^t(p))| = \sum_{i=1}^{k(p)} \lambda_i(p).dim(E_p^i). \tag{5.19}$$

So if $A \in \mathcal{T}$, then $\det(\Phi_A^t(p)) = 1$, and we obtain $\sum_{i=1}^{k(p)} \lambda_i(p).dim(E_p^i) = 0$. Since all Lyapunov exponents are equal, they must all be zero. If $A \in \mathcal{T}_a$, then

$$\det(\Phi_A^t(p)) = \frac{a(p)}{a(\varphi^t(p))}.$$

By (5.19) and (2.2) we obtain

$$\begin{aligned}
\lim_{t\to\pm\infty} \frac{1}{t} \log |\det(\Phi_A^t(p))| &= \lim_{t\to\pm\infty} \frac{1}{t} \log \left(\frac{a(p)}{a(\varphi^t(p))} \right) = \\
&= \lim_{t\to\pm\infty} \frac{1}{t} \log a(p) - \lim_{t\to\pm\infty} \frac{1}{t} \log a(\varphi^t(p)) = \\
&= -\lim_{t\to\pm\infty} \frac{1}{t} \log a(\varphi^t(p)) = \\
&= 0.
\end{aligned}$$

The last equality follows by the fact that $a(\cdot)$ is subexponential. Therefore all Lyapunov exponents are zero. Corollary 1.1 is now proved.

Remark 5.1 Given $A \in \mathcal{T}$ (or $\mathcal{T}_a$), then if for μ-a.e. point $p \in X$, the Oseledets splitting of A is dominated or trivial at p, then A is a continuity point of $\mathcal{E}$. This follows from semi-continuity and also from the fact that if we perturb the system A a little bit we still have a dominated splitting.

Acknowledgments

This work was supported by FCT(Fundação para a Ciência e a Tecnologia)/ FSE(Fundo Social Europeu) SFRH/BPD/20890/2004.

I would like to thank the referee for useful suggestions and corrections given.

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