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Plenty of hyperbolicity on a class of linear homogeneous jerk differential equations

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Abstract. We consider 3×3 partially hyperbolic linear differential systems over an ergodic flow X^t and derived from the linear homogeneous differential equation $\dddot{x}(t)+\beta(X^t(t))\dot{x}(X^t(t))+\gamma(t)x(t)=0$. Assuming that the partial hyperbolic decomposition $E^s\oplus E^c\oplus E^u$ is proper and displays a zero Lyapunov exponent along the central direction E^c we prove that some C^0 perturbation of the parameters $\beta(t)$ and $\gamma(t)$ can be done in order to obtain non-zero Lyapunov exponents and so a chaotic behaviour of the solution.

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1. Introduction

1.1. Linear homogeneous jerk differential equations

The equations of motion in kinematics relate four quantities; time, space, velocity and acceleration (represented respectively by t, x, v and a) and can be obtained from the assumption that

$$\frac{dx}{dt}=\dot{x}=v \quad \text{and} \quad \frac{dv}{dt}=\dot{v}=a. \tag{1}$$

As we already know from Galileo's inclined plane experiment and his acceleration hypothesis the acceleration a shoud be taken constant. One standard example in kinematics is the parachute problem described by the second order linear homogeneous differential equation

$$\ddot{x}(t)+\alpha\dot{x}(t)+\beta x(t)=0, \tag{2}$$

where the damping term $\alpha\dot{x}(t)$ arises from a first order term related to air resistence and is proportional to the velocity. In the absence of any type of damping (i.e. if $\alpha = 0$) we obtain a conservative system. Indeed, it can be reduced to a linear 2×2 system as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix},$$

where $y = \dot{x}$ becomes traceless when $\alpha = 0$ and, as we will see, traceless systems are conservative. Adding one dimension more we, in a pure theoretical manner, could consider a kinematic approach on the third order linear homogeneous differential equation

$$\dddot{x}(t) + \alpha\ddot{x}(t) + \beta\dot{x}(t) + \gamma x(t) = 0, \tag{3}$$

and the analogous 3×3 linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma & -\beta & -\alpha \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{4}$$

where $y = \dot{x}$ and $z = \dot{y}$. Yet, this seems to be uninteresting due to Galileo's acceleration hypothesis which demands $\dot{z} = 0$. But who says that kinematics with nice applications to real life is solely dedicated to non variable acceleration? Well before Descartes founded the analytic geometry and the algebraic representation along axes back in the first half of the seventeen's century, we humans and some animals already used a cartesian approach in the exquisite labyrinth system which lies inside our ears. Actually, besides the sense of hearing associated with our biological audio system there is another important feature related to the sense of balance and equilibration. This is the core of the so called vestibular system which is endowed with sensors capable of detecting variations in the acceleration. There is a pair of these sensors, called otoliths, in each ear which are capable of detecting linear acceleration. Each pair displays a cartesian-like demeanor in the sense that one otolith detects acceleration in the horizontal direction and another detects acceleration in the vertical direction. Overall, the otoliths react to the first derivative of acceleration commonly called the *jerk*. Individuals with deficient otoliths are likely to have unsatisfactory abilities to sense motions along with a lack of gravity orientation (see [5] for a detailed exposition). Besides this (not obvious) biological appearence of the jerk, considering variable acceleration is considered in several physical situations. With this in mind we denote the jerk by j and assume $j = \dot{a}$ and $\dot{j} = 0$. Overall our 3-dimensional coordinates on (3) will be $(\dot{x}, \dot{y}, \dot{z}) = (v, a, j)$ where in the first two coordinates we take (1) into occount. Jerk equations are ODEs depending on x, v, a and j. Here we will be interested in the linear and homogeneous ones say described by:

$$\dddot{x}(t) + \beta(t)\dot{x}(t) + \gamma(t)x(t) = 0, \tag{5}$$

meaning the damping frictional term $\alpha(t)\ddot{x}(t)$ vanishes and the coefficients $\beta(t)$ and $\gamma(t)$ depend on t. For the same reason described above for (2) we will call the class of linear and homogeneous jerk differential equations *conservative*.

It is well-known that the general solution of differential equations like (2), (3) and (5) when coefficients are constants depends on the roots of the corresponding characteristic polynomial equation. Although the problem hugely increases in difficulty when we allow variation in the parameters we, in some very particular cases, are able to solve these equations by quadratures.

From the theoretical ODE viewpoint we have the initial-value problem for

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma(t) & -\beta(t) & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{6}$$

which consists in finding a solution $x(t)$ of the differential equation (5) that also satisfies the initial conditions $x(t_0) = x_0$, $\dot{x}(t_0) = x_1$ and $\ddot{x}(t_0) = x_2$. This problem has an affirmative solution if $\beta(t)$ and $\gamma(t)$ are continuous on a certain interval. Yet, the tools at our disposal are mainly existential theorems and so a qualitative asymptotic analysis of the behaviour of a solution $x(t)$ must be done since *a priori* we have no idea of the explicit expression of the solution $x(t)$.

2. An abstract framework and statement of the result

2.1. Towards a qualitative approach

In the present work we intend to describe with a certain degree of accuracy the asymptotic behavior of the position, velocity and acceleration of $x(t)$, the solution of (6), for a generic subset of choices of the parameters $\beta(t)$ and $\gamma(t)$. Taking as an example the aforementioned ODE (5) we will be able to describe the limit dynamics of $x(t)$ for arbitrary parameters $\tilde{\beta}(t)$ and $\tilde{\gamma}(t)$ close to $\beta(t)$ and $\gamma(t)$ respectively, where *close* means near in the uniform convergence norm. We follow the steps of the discipline of qualitative theory of differential equations created by Poincaré and Lyapunov which pops up as an alternative to the feeble approach of applying analytic methods to integrate most functions confirmed by Liouville's theory. We rewrite (5) as

$$\dddot{x}(t) + \beta(X^t(w))\dot{x}(t) + \gamma(X^t(w))x(t) = 0 \tag{7}$$

where $X^t \colon M \to M$ for $t \in \mathbb{R}$ and $w \in M$ stands for a given volume-preserving flow, say an $\mathbb{R}$-action. Moreover, we choose X^t to be ergodic with respect to the volume measure ν. This allows us, instead of dealing with a single equation, to consider infinite equations simultaneously each one for each orbit $\cup_{t\in\mathbb{R}} X^t(w)$. Furthermore, differential equations along periodic solutions will be discarded since their volume ν will be zero. The qualitative analysis will be on the Lyapunov exponents of the solution $U(w,t)$ associated with a linear variational equation with infinitesimal generator related to the differential equation (7). Let us now formalize this settings.

2.2. The base dynamics

Let M be a closed Riemannian manifold, ν the volume measure of M, with $\nu(M)=1$ and let $X\colon M\to TM$ be a C^1 vector field. Denote by $X^t\colon M\to M$ the flow associated with X by the relation $\frac{d}{dt}X^t(w)\Big|_{t=0}=X(w)$. Let X be a divergence-free vector field. Then due to the Liouville-Ostrogradski formula (12) X^t *preserves the volume* ν, i.e. if $A\subseteq M$ is a ν-measurable set, then $\nu(X^{-t}(A))=\nu(A)$ for all $t\geq 0$. We say that a vector field X is *ergodic* if X^t preserves ν and given a ν-measurable set $A\subseteq M$, $\nu(A)=0$ or $\nu(A)=1$. For detailed information on conservative flows see [12, § 1.3.6]. We point out that the base dynamics could be given by more general flows since it will be mainly used to codify the orbits.

2.3. The fiber dynamics

Let $M_{3\times 3}$ denote the set of 3×3 matrices with entries in $\mathbb{R}$. Let $C^0(M,M_{3\times 3})$ denote the set of C^0 maps $A\colon M\to M_{3\times 3}$ which we endow with the norm

$$\|A-B\|=\max_{w\in M}\|A(w)-B(w)\|. \tag{8}$$

Let $Id\subset M_{3\times 3}$ denote the identity matrix and let Φ_A^t be the solution of the nonautonomous linear differential equation called the *linear variational equation*

$$\dot{U}(w,t)=A(X^t(w))\cdot U(w,t), \tag{9}$$

satisfying the initial condition $\Phi_A^0(w)=Id$. *Kinetic matrices*, denoted by $\mathcal{K}$, are the subset $\mathcal{K}\subset M_{3\times 3}$ of matrices with the form (4). Let $C^0(M,\mathcal{K})\subset C^0(M,M_{3\times 3})$ stand for the subset of *kinetic infinitesimal generators* which will be defined by

$$A(w)=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -\gamma(w) & -\beta(w) & -\alpha(w)\end{pmatrix}, \tag{10}$$

and are derived from the linear homogeneous jerk differential equation

$$\dddot{x}(t)+\alpha(t)\ddot{x}(t)+\beta(t)\dot{x}(t)+\gamma(t)x(t)=0. \tag{11}$$

Finally, *kinetic traceless matrices*, denoted by $\mathcal{K}_0$, are the subset $\mathcal{K}_0 \subset \mathcal{K}$ of matrices with the form of the 3×3 matrix in (6). From the Liouville-Ostrogradski formula (see e.g. [11]) and since $\Phi^0_A(w) = Id$ we get

$$\det \Phi^t_A(w) = \exp\left(\int_0^t A(X^s(w))\, ds\right), \tag{12}$$

realizing why traceless systems give rise to conservative solutions. Let $C^0(M, \mathcal{K}_0) \subset C^0(M, \mathcal{K})$ stand for the subset of *kinetic traceless infinitesimal generators* which will be defined as in (10) but considering $\alpha = 0$. Hence, $C^0(M, \mathcal{K}_0)$ is derived from the linear homogeneous *conservative* jerk differential equation obtained from (11) by considering $\alpha = 0$. Clearly, $C^0(M, \mathcal{K}_0)$ is a topologically closed set in $C^0(M, M_{3\times 3})$ with respect to the norm (8).

The fiber dynamics will be given by a continuous-time conservative *linear cocycle* over X^t, also called a *linear differential system* (LDS), defined by the map $\Phi^t \colon M \to \mathrm{SL}(3, \mathbb{R})$ on the parameter t where:

(1) the set $\mathrm{SL}(3, \mathbb{R})$ stands for the special linear group of matrices with real entries;
(2) Φ^t is the unique solution of (9), for $A \in C^0(M, \mathcal{K}_0)$, given the initial condition $\Phi^0(w) = Id$ for all $w \in M$. For this reason we denote Φ^t by Φ^t_A;
(3) Φ^t_A is a linear flow, i.e., $\Phi^0(w) = Id$ for all $w \in M$, $\Phi^{t+s}(w) = \Phi^t(X^s(w)) \cdot \Phi^s(w)$, for all $s, t \in \mathbb{R}$ and $w \in M$ and $\Phi^t(w)(au + bv) = a\Phi^t(w)(u) + b\Phi^t(w)(v)$ where $t, a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^3$.

Clearly, when β and γ are periodic coefficients the Floquet theory (see [11]) helps with the analysis and when β and γ are first integrals, i.e. are constant along the orbits of the flow X^t, then (7) can be solved by elementary algorithms present in any book on differential equations (cf. [11, Chapter 6]). The interesting case here is when the parameters vary in time along non-periodic orbits which is actually the case for ν-a.e. orbits as X^t is an ergodic flow.

2.4. Hyperbolic cocycles

From the last section we obtain that fiber dynamics acts on $\mathbb{R}^3_M$. We say that the splitting $E_M = E^1 \oplus E^2 \subset \mathbb{R}^3_M$ is Φ^t_A-invariant if $\Phi^t_A(w)E^i_w = E^i_{X^t(w)}$ for $i = 1, 2$ and $t \in \mathbb{R}$. We say that $E^1 \oplus E^2 \subset \mathbb{R}^3_M$ is a (C, σ)-dominated splitting for Φ^t_A if it is Φ^t_A-invariant and there exist $C > 0$ and $\sigma \in]0, 1[$ such that, for all $w \in M$ and $t \geq 0$ we have:

$$\frac{\|\Phi^t_A(w)|_{E^2_w}\|}{\mathfrak{m}(\Phi^t_A(w)|_{E^1_w})} \leq C\, \sigma^t,$$

where $\mathfrak{m}(\cdot)$ denotes the co-norm of an operator, that is $\mathfrak{m}(A) = \|A^{-1}\|^{-1}$. We say that the subbundle E is *hyperbolic* if either $\|\Phi_A^{-t}(w)\cdot u\| \leq C\,\sigma^t$ (expanding), for all $w \in M$, $t \geq 0$ and any unit vector $u \in E_w$, or $\|\Phi_A^t(w)\cdot u\| \leq C\,\sigma^t$ (contracting), for all $w \in M$, $t \geq 0$ and any unit vector $u \in E_w$.

We say that A is (uniformly) *partially hyperbolic* if there exists a Φ_A^t-invariant dominated splitting $E = E^u \oplus E^c \oplus E^s$ such that E^u is hyperbolic expanding and E^s is hyperbolic contracting; moreover these two subbundles are not simultaneously trivial. We should understand that E^u dominates E^c and E^c dominates E^s. The partial hyperbolic elements in $C^0(M, \mathcal{K}_0)$ are open with respect to the norm (8). Moreover, the maps $A \mapsto E_A^i$ vary continuously for each $i = s, c, u$ when considering (8) in the domain, and the canonic norm on Grassmannians in the co-domain. We refer to [10] for a complete description of this subject.

2.5. Oseledets' theorem

Oseledets' theorem [6,7,10] is valid for both discrete-time and continuous-time *cocycles*. For an LDS $A \in C^0(M, \mathcal{K}_0)$ over X^t Oseledets' theorem asserts that we have for ν-a.e. points $w \in M$ a splitting $\mathbb{R}^3_w = E^1_w \oplus \cdots \oplus E^{k(w)}_w$ (Oseledets splitting) and real numbers $\lambda_1(w) \geq \cdots \geq \lambda_{k(w)}(w)$ (Lyapunov exponents) such that $\Phi^t_w(E^i_w) = E^i_{X^t(w)}$ and

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi_A^t(w)\cdot v^i\| = \lambda_i(A, w)$$

for any $v^i \in E^i_w \backslash \{\vec{0}\}$ and $i = 1, \ldots, k(w)$. In our 3-dimensional and partially hyperbolic case we can have $k(w) = 3$ and in this case all Oseledets subbundles are 1-dimensional, or else we can have $k(w) = 2$ and, in this case, we have a hyperbolic subbundle of dimension 2 (expansive or contractive). Oseledets' theorem allows us to conclude also that:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log|\det(\Phi_A^t(w))| = \sum_{i=1}^{k(w)} \lambda_i(A, w)\dim(E^i_w) \tag{13}$$

which is related to the sub-exponential decrease of the angle between any subspaces of the Oseledets splitting along ν-a.e. orbits (see e.g. [12, § 3.3.5]). We observe that the ergodicity of the flow implies that the Lyapunov exponents and the dimensions of the associated subbundles are ν-a.e. constant and so for simplicity we omit the point by writing $\lambda_i(A)$ instead of $\lambda_i(A, w)$. The hypothesis of being ergodic will also be used for a twofold facilitation of the computation of the Lyapunov exponents because we can switch time averages by space averages and we can perform focus on a single ν-generic orbit. Since in the conservative setting we have $|\det\Phi_A^t(w)| = 1$ for all w and t, by (13), we have $\lambda_1(w) + \lambda_3(w) + \lambda_3(w) = 0$. If M has a partially hyperbolic splitting and displays a zero Lyapunov exponent, then this exponent must define the central fiber $E^c(A)$ and $\lambda^s(A) = -\lambda^u(A)$ meaning the Lyapunov exponent associated to the stable fiber is the symmetric of the Lyapunov exponent associated to the unstable fiber. When there are no zero Lyapunov exponents then $\lambda^s(A) < 0 < \lambda^c(A) < \lambda^u(A)$ or $\lambda^s(A) < \lambda^c(A) < 0 < \lambda^u(A)$.

2.6. Statement of the result

In the present paper we study the asymptotic dynamics of solutions of the linear cocycle Φ_A^t. Our main result is:

Theorem 1. *Let $X^t\colon M \to M$ be an ergodic flow w.r.t. a probability volume measure, $A \in C^0(M, \mathcal{K}_0)$ and assume that the cocycle Φ_A^t has a partially hyperbolic splitting $E^u \oplus E^c \oplus E^s$ over M. Then, either $\lambda^c(A) \neq 0$, or else A may be approximated, in the C^0-topology, by $A_0 \in C^0(M, \mathcal{K}_0)$ for which $\lambda^c(A_0) \neq 0$.*

We observe that if we change in Theorem 1 the set $C^0(M, \mathcal{K}_0)$ for the set of traceless elements in $C^0(M, M_{3\times 3})$ we get essentially a direct consequence of [3]. It is worth noting that as our infinitesimal generators do not evolve in the broader setting of traceless ones our perturbations must be kinetic meaning that our degrees of freedom decrease from 8 (in [3]) to 2 making the problem more difficult. Moreover, traceless infinitesimal generators in $C^0(M, M_{3\times 3})$ model divergence-free C^1 vector fields on 4-dimensional manifolds whereas infinitesimal generators in $C^0(M, \mathcal{K}_0)$ model linear homogeneous conservative differential equations of third order. Despite being somehow related these two settings give rise to two completely nonidentical problems with a different genesis and motivation. We notice that recently in [1] the authors were able to remove the zero Lyapunov exponents (or more generally to remove the trivial Lyapunov spectrum) on kinetic LDSs like (2) under L^p-perturbations on the parameters α, β. Despite being globally different some arguments in [1] resemble the ones in the present paper namely projections on controllable directions and taking advantage of the invariant given by the determinant.

3. Perturbations

3.1. A toy model

We consider the case when the parameters β, γ are constant more precisely such that $\beta(t) = -1$ and $\gamma(t) = 0$. We will get the linear homogeneous conservative jerk differential equation:

$$\dddot{x}(t) - x(t) = 0. \tag{14}$$

We obtain the following linear vectorial differential equation:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}0&1&0\\0&0&1\\0&1&0\end{pmatrix} \cdot \begin{pmatrix}x\\ y\\ z\end{pmatrix} = A \cdot \begin{pmatrix}x\\ y\\ z\end{pmatrix}, \tag{15}$$

such that the infinitesimal generator has eigenvalues $\lambda^u = 1$, $\lambda^c = 0$ and $\lambda^s = -1$ with respective eigendirections $v^u = (1,1,1)$, $v^c = (1,0,0)$ and $v^s = (1,-1,1)$. The system (15) can be writen as

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}1&1&1\\-1&0&1\\1&0&1\end{pmatrix} \cdot \begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix} \cdot \begin{pmatrix}0&-\frac{1}{2}&\frac{1}{2}\\1&0&-1\\0&\frac{1}{2}&\frac{1}{2}\end{pmatrix} \cdot \begin{pmatrix}x\\ y\\ z\end{pmatrix},$$

which allows for an explicit simple solution of the corresponding linear variational equation on a certain eigenvalues basis as

$$\Phi^t_A = \begin{pmatrix}e^{-t}&0&0\\0&1&0\\0&0&e^t\end{pmatrix}. \tag{16}$$

Due to the well known openess and denseness of hyperbolic matrices [8, Proposition 1.11] we have a good chance to obtain a hyperbolic matrix after a perturbation of (15). This follows directly unless our family, which display only two degrees of freedom, is rigid enough to force the presence of a 0 eigenvalue. But this is not the case since, for example,

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}0&1&0\\0&0&1\\\epsilon&1+\epsilon&0\end{pmatrix} \cdot \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \left(\overbrace{\begin{pmatrix}0&1&0\\0&0&1\\0&1&0\end{pmatrix}}^{A} + \overbrace{\begin{pmatrix}0&0&0\\0&0&0\\\epsilon&\epsilon&0\end{pmatrix}}^{P}\right) \cdot \begin{pmatrix}x\\ y\\ z\end{pmatrix} = A_0 \cdot \begin{pmatrix}x\\ y\\ z\end{pmatrix}, \tag{17}$$

has eigenvalues $\hat{\lambda}^u = \frac{1+\sqrt{4\epsilon+1}}{2}$, $\hat{\lambda}^c = \frac{1-\sqrt{4\epsilon+1}}{2}$ and $\hat{\lambda}^s = -1$ with respective eigendirections,

$$\hat{v}^u = \left(-\frac{1-\sqrt{4\epsilon+1}}{\epsilon(1+\sqrt{4\epsilon+1})}, \frac{2}{1+\sqrt{4\epsilon+1}}, 1\right),$$
$$\hat{v}^c = \left(-\frac{1+\sqrt{4\epsilon+1}}{\epsilon(1-\sqrt{4\epsilon+1})}, \frac{2}{1-\sqrt{4\epsilon+1}}, 1\right) \quad \text{and} \quad \hat{v}^s = (1,-1,1).$$

This gives an explicit simple solution of (17) on a certain eigenvalues basis as

$$\Phi^t_{A_0} = \begin{pmatrix}e^{-t}&0&0\\0&e^{\frac{1-\sqrt{4\epsilon+1}}{2}t}&0\\0&0&e^{\frac{1+\sqrt{4\epsilon+1}}{2}t}\end{pmatrix},$$

which no longer has a central direction as (16) had. We carefully choose a perturbation $A_0 = A + P$ that leaves the stable eigenvalue and its eigendirection unchanged. This follows because $P \cdot v^s = \vec{0}$ and so the influence of the vector field P along the direction of v^s is null, or in other words, v^s is an equilibrium point of the flow generated by P. Hence, since the system is conservative we obtain that the amount of increase (or decrease) in the unstable direction passes directly to the central direction resulting in a decrease (or increase) in its eigenvalue $\lambda^c(A_0)$. In other words if $\epsilon < 0$, then $\lambda^c(A_0) > 0$ and $\lambda^u(A_0) < \lambda^u(A)$ and if $\epsilon > 0$, then $\lambda^c(A_0) < 0$ and $\lambda^u(A_0) > \lambda^u(A)$. Be that as it may the linear homogeneous conservative jerk differential equation:

$$\dddot{x}(t) - \epsilon\dot{x}(t) - (1+\epsilon)x(t) = 0, \tag{18}$$

displays nonzero Lyapunov exponents. Furthermore, (18) is also an arbitrarily small perturbation of (14) since $\|A - A_0\| = \epsilon$ can be made as small as we like.

3.2. Capturing the ideia of the toy model

What was the reason for presenting such a trivial reasoning as the one in Sect. 3.1? Indeed, although when we consider non constant coefficients we no longer display a simple linear algebra most of the idea prevails in the general case. Hence, and for future use, we recall now the methodology:

- Perform a local perturbation and keep in mind that the stable direction dynamics should remain equal to the unperturbed system;
- For that choose a perturbation P acting as the null vector field along the stable direction;
- The previous two items send, to the center-unstable space the scheme of a balanced increase/decrease in the Lyapunov exponents;
- Taking into account that the system is conservative the amount we lose (or gain) in the unstable Lyapunov exponent is the amount we gain (or loose) in the central Lyapunov exponent.
- Finally, the ergodicity of the base flow serve the (lazy) purpose of performing just a perturbation that ensures that the process works in a certain orbit and ergodicity demands that the Lyapunov exponent is constant.

3.3. Perturbing Kinetic LDS

The next result produces a local perturbation by an infinitesimal generator vector field acting on $M_{3\times 3}$ and which is 'perpendicular' to a given 1-dimensional direction.

Lemma 3.1. *Let $A \in C^0(M, \mathcal{K}_0)$ and assume that the LDS is partially hyperbolic with splitting $E^u \oplus E^c \oplus E^s$, each of these fibers being 1-dimensional. Let*

$p \in M$ be a non-periodic point. There exists two C^∞ functions $b, g\colon \mathbb{R} \to \mathbb{R}$ such that $b(t), g(t) = 0$ for $t \notin]0,1[$ and for

$$P(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -g(t) & -b(t) & 0 \end{pmatrix} \tag{19}$$

we have that the LDS defined along the orbit of p by $A_0(X^t(p)) = A(X^t(p)) + P(t)$,

(i) *define a ODE of type (5),*
(ii) *is traceless,*
(iii) *has solution $\Phi^t_{A_0} \neq \Phi^t_A$,*
(iv) *is such that $\Phi^1_{A_0}(p) \cdot E^s_p = \Phi^1_A(p) \cdot E^s_p$,*
(v) *$\Phi^1_{A_0}(p) \neq \Phi^1_A(p)$ and*
(vi) *$\|P(t)\| < \epsilon$.*

Proof. Items (i) and (ii) are trivial and (iii) is also trivial if we pick $b \neq 0$ (or $g \neq 0$). Since the splitting is partially hyperbolic into three 1-dimensional directions E^u, E^c and E^s which are Φ^t_A-invariant, we know that there exists $\sigma_t \in \mathbb{R} \setminus 0$ such that

$$\Phi^t_A(0) \cdot E^s_0 = \sigma_t \, E^s_t. \tag{20}$$

Take $\sigma = \sigma_1$. We are left to check that the action of the vector field P does not interfere in the invariant direction E^s. For that we will show that the stable direction E^s is invariant under the action of the vector field $A + P$ if we pick the functions b, g appropriately. For each $t \in \mathbb{R}$ take the smooth map $t \mapsto (x_t, y_t, z_t)$ where (x_t, y_t, z_t) is a unit vector pointing in the direction E^s_t. Taking time derivatives on both sides of (20) we get

$$\dot{\Phi}^t_A(0) \cdot E^s_0 = \dot{\sigma}_t E^s_t + \sigma_t \dot{E}^s_t, \tag{21}$$

and from the linear variational equation (9) we get

$$A(t) \cdot \Phi^t_A(0) \cdot E^s_0 = \dot{\sigma}_t E^s_t + \sigma_t \dot{E}^s_t, \tag{22}$$

and from the invariance in (20) and linearity we obtain

$$\sigma_t A(t) \cdot E^s_t = \dot{\sigma}_t E^s_t + \sigma_t \dot{E}^s_t, \tag{23}$$

that is the ‘instantaneous’ action of the infinitesimal generator at time t is given by

$$A(t) \cdot E^s_t = \frac{\dot{\sigma}_t}{\sigma_t} E^s_t + \dot{E}^s_t. \tag{24}$$

Hence, our goal is to choose P such that

$$[A(t) + P(t)] \cdot E^s_t = \frac{\dot{\sigma}_t}{\sigma_t} E^s_t + \dot{E}^s_t. \tag{25}$$

From (24) and (25) we need to have

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma(t) & -\beta(t) & 0 \end{pmatrix} \cdot \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma(t)-g(t) & -\beta(t)-b(t) & 0 \end{pmatrix} \cdot \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix},$$

that is we pick b, g such that

$$g(t)x_t + b(t)y_t = 0, \tag{26}$$

for all $t \in [0,1]$. To check (iv) we observe that

$$\begin{aligned} A_0(t) \cdot E_t^s \quad &= \quad [A(t)+P(t)] \cdot E_t^s \overset{(25)}{=} \frac{\dot{\sigma}_t}{\sigma_t} E_t^s + \dot{E}_t^s \overset{(24)}{=} A(t) \cdot E_t^s \\ &\overset{(23)+(22)}{=} \sigma_t^{-1} A(t) \cdot \Phi_A^t(0) \cdot E_0^s \overset{(21)}{=} \sigma_t^{-1} \dot{\Phi}_A^t(0) \cdot E_0^s, \end{aligned}$$

and so

$$\Phi_{A+P}^t(0) \cdot E_0^s = \sigma E_1^s.$$

Since (26) means $(g(t), b(t)) \cdot (x_t, y_t) = 0$ we just choose $(g(t), b(t))$ to be an orthogonal vector to (x_t, y_t) with small norm say $(g(t), b(t)) = \epsilon(-y_t, x_t)$. Clearly, when one or both x_t, y_t are 0 we have more choices for $g(t)$ and $b(t)$. Hence we obtain (vi). Finally, g, b can be chosen in a way to avoid the non generic case when $\Phi_{A+P}^1(p) = \Phi_A^1(p)$ and we get (v). $\square$

The next result makes the previous lemma global in the sense that we now define an LDS in $C^0(M, \mathcal{K}_0)$ as a perturbation. Throughout the proofs we make the following assumption: by the conservative flowbox theorem [3] we assume that X^t is trivial and equal to $\frac{\partial}{\partial x} = (1,0,0)$, $p = \vec{0}$ and we still denote the partial hyperbolic splitting by $E^u \oplus E^c \oplus E^s$ because it evolves in the fiber and is not affected by this change of coordinates. Given a sufficiently small $r > 0$ and since p is non-periodic this flowbox can be considered to have length 1 and will be defined by:

$$\mathcal{F}_r(\vec{0}) = \{(t, y, z) \in \mathbb{R}^3 \colon 0 \le t \le 1, \|(0, y, z)\| \le r\}.$$

The points $(0, y, z)$ such that $\|(0, y, z)\| \le r$ form what we call a ball of radius r centered at p and we denote it by $B_r(p)$. With this trivialization in mind we get e.g. that

$$\tilde{\nu}(B_r(p)) = \tilde{\nu}(X^1(B_r(p))), \tag{27}$$

where $\tilde{\nu}$ is the area measure in transversal sections induced by the volume ν.

Lemma 3.2. *Let $A \in C^0(M, \mathcal{K}_0)$, $\epsilon > 0$ and assume that the LDS is partially hyperbolic with splitting $E^u \oplus E^c \oplus E^s$ each of these fibers being 1-dimensional. For any non-periodic point $p \in M$ there is $r_0 = r_0(p) > 0$ such that for each $0 < r \le r_0$ there exists $A_0 \in C^0(M, \mathcal{K}_0)$ satisfying:*

(i) *A_0 is ϵ-C^0-close to A;*

(ii) *$\Phi_{A_0}^1(p) \cdot E_p = \Phi_A^1(p) \cdot E_p$;*

(iii) $\Phi^1_{A_0}(p) \neq \Phi^1_A(p)$;
(iv) $A = A_0$ *outside the flowbox* $X^{[0,1]}(B_r(p))$.

Proof. We will make the perturbation supported in $\mathcal{F}_{r_0}(\vec{0})$ where $r_0 > 0$ is sufficiently small in order for $\mathcal{F}_{r_0}(\vec{0})$ to be a flowbox. Fix a C^∞ bump function ρ satisfying $\rho(s) = 1$, for $|s| \leq \frac{1}{2}$, $\rho(s) = 0$ for $s \geq 1$, and $|\rho| \leq 1$ for all s. For any $0 < r \leq r_0$ we define the map ρ_r by $\rho_r(t) = \rho(\frac{1}{r}t)$. The perturbation $A_0 \in C^0(M, \mathcal{K}_0)$ will be defined by $A_0 = A$ outside $\mathcal{F}_r(\vec{0})$ and for $(t, y, z) \in \mathcal{F}_r(\vec{0})$ will be defined by $A_0 = A + P$ where

$$P(t, y, z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -g(t)\rho_r(\|(0, y, z)\|) & -b(t)\rho_r(\|(0, y, z)\|) & 0 \end{pmatrix}$$

like in Lemma 3.1. Since $\|A - A_0\| = \|P\|$ we can make this norm $\leq \epsilon$ (regardless of the r we take) by just picking g, b sufficiently small and carefully taking into account the invariant (26) to get (i). Conditions (ii) and (iii) are local and so follows from Lemma 3.1. Finally, condition (iv) is trivial. □

3.4. Establishing a parallel with the previous approaches

For the reader familiar with [2–4] we now try to set up the perturbations made in Sect. 3.3 with the ones made in the literature concerning different contexts. The proof of Theorem 1 follows the strategy formulated in [2], and motivated by [9], for the discrete dynamical cocycle (the cocycle given by the derivative) and adapted in [3] for the continuous dynamical cocycle (the cocycle given by the linear Poincaré flow). Indeed, from [3] we get directly that a version of Theorem 1 for linear differential systems evolving in the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$ is true. However, $\dim(\mathfrak{sl}(3, \mathbb{R})) = 8$ and the dimension of the traceless and kinetic LDS is 2 turning the problem much more difficult. The key observation which allows us to obtain the result in our (much more) rigid setting is the subtle note in [2, Theorem 3] saying that perturbations close to the identity and different from the identity map guarantee the growing of the central Lyapunov exponent. Therefore, we do not need a particularly and carefully cooked perturbation (as we did in [3] for convenience) but a much broader type of perturbation. That is the reason why we are still able to obtain nonzero central Lyapunov exponents even when the range of perturbations fall from 8 to 2 degrees of freedom.

Regarding our setting we consider that the subbundles E^s, E^c and E^u are nontrivial, each one is 1-dimensional and, like in the toy model, assume also that, in a neigbourhood of a non-periodic point p, they are given by $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$, respectively. Lemma 3.2 gives

$$\Phi^1_{A_0}(p) \cdot E^s_p = E^s_{X^1(p)}. \tag{28}$$

Let us assume, for the sake of comparing our case with [2], that $\hbar(p) := [\Phi^1_A(p)]^{-1} \cdot \Phi^1_{A_0}(p)$ where $h_*(p)$ is a linear automorphism in $\mathrm{SL}(3,\mathbb{R})$ acting on the fiber $\mathbb{R}^3_p$. Since $[\Phi^0_A(p)]^{-1} \cdot \Phi^0_{A_0}(p) = Id$ we can see the perturbation $\hbar(p)$ as an isotopic deformation of the identity for $t \in [0,1]$ and inside $\mathrm{SL}(3,\mathbb{R})$. When perturbing the original dynamical system f, in [2], it considers the perturbation $g = f \circ h$ where h is a volume-preserving map sufficiently C^1-close to the identity and supported in a small ball centered at p keeping E^s_p invariant and such that its tangent map satisfies

$$Dh(p) \cdot u(p) = y(p)c(p) + z(p)u(p), \tag{29}$$

which is precisely what our $\hbar(p)$ does. We point out again that the choice of h (and $\hbar$) has wide freedom. In our LDS context $\Phi^1_A(p)$, $\hbar(p)$ and $\Phi^1_{A_0}(p)$ will respectively play the part of f, h and g, in [2]. Therefore, $\Phi^1_A(p) \cdot \hbar(p) = \Phi^1_{A_0}(p)$. By (28) and using the fact the A_0 is traceless we get an area invariance in the center-unstable subbundle. Hence, and like [2, Lemma 1.2] when such pertubation can be done in a ball with fixed radius r_0 we obtain

$$I(r_0) := \int_{B_{r_0}(p)} \log\, z(p)\, d\tilde{\nu}(p) < 0. \tag{30}$$

Notice that if we reescale the flowbox under $\varphi_r\colon B_r(p) \to B_{r_0}(p)$ defined by $\varphi_r(0,y,z) = \frac{r_0}{r}(0,y,z)$ where $r \in]0, r_0]$ we get from a change of variables:

$$\begin{aligned} I(r) &:= \int_{B_r(p)} \log\, z(p)\, d\tilde{\nu}(p) = \int_{B_{r_0}(p)} \log\, z(p) \left(\frac{r}{r_0}\right)^2 d\tilde{\nu}(p) \\ &= \left(\frac{r}{r_0}\right)^2 I(r_0) < 0. \end{aligned} \tag{31}$$

4. Proof of Theorem 1

We will be interested in estimating the value of the integral of the logarithm of the unstable action of the LDS perturbation A_0. Of course the unstable direction associated to A_0 is no longer given by the unstable direction associated to the original LDS A. As a first step of the proof of Theorem 1 we change the LDS A_0 for some cocycle having the same Lyapunov exponent for ν-a.e. points.

The two dimensional subspace V_p (where the perturbation P acts) is chosen to be the central-unstable subbundle of $\mathbb{R}^3_p$, where p is an Oseledets' point such that the orbit of p is non-periodic. Now our aim is to show that A_0 as defined in Lemma 3.2 satisfies the conclusion of Theorem 1 and for that it is enough to prove the following result.

Proposition 4.1. *Given A, $\epsilon > 0$ and p and an arbitrarily small $r > 0$ there exists $A_0 \in C^0(M, \mathcal{K}_0)$ satisfying the conclusions of Lemma 3.2 and such that $\lambda^c(A_0) > \lambda^c(A)$.*

4.1. Construction of a useful cocycle Φ

In the fixed setting the space V_p is the space given by the directions y (the unstable direction) and z (the central direction), and W_p is the space generated by x (the stable direction). We recall that $B_r(p)$ denotes the 2-dimensional ball centered at p and of radius r contained in the transversal section (i.e. $B_r(p)$ is the base of the flowbox given by [3]). For $q \in M$ let $u(q)$ denote a unit vector of E^u_q which is Φ^t_A-invariant (but not $\Phi^t_{A_0}$-invariant) and write

$$\Phi^t_A(q) \cdot u(q) = \lambda(t, q)\, u(X^t(q)), \tag{32}$$

where $\lambda(t, q) = \|\Phi^t_A(q) \cdot E^u_q\|$. We are going to consider a perturbation A_0 of the original LDS A and it is crucial that we can estimate its upper Lyapunov exponent. In order to clear the way for this task we will use instead an artificial (and somehow far-fetched) linear 'unstable' cocycle Φ which displays the same upper Lyapunov exponent as A_0 (see Lemma 4.2). The outcome is that computations to estimate the upper Lyapunov exponent on Φ are more direct. Furthermore, Φ coincides with the cocycle associated to A outside a small flowbox $X^{[-1,1]}(B_r(p))$. The cocycle Φ will be produced taking into accunt two main moments:

- A perturbation using Lemma 3.2 in $X^{[0,1]}(B_r(p))$ which, on the one hand, causes the 'unstable' direction to diminish (recall (30)) but, on the other hand, makes another direction v appear which could jeopardize the 'unstable diminishing' and so the asymptotic growth of v must be estimated;
- A correctional impulse $\Psi\colon X^{-1}(B_r(p)) \to \mathbb{R}$ to be defined bellow is introduced in $X^{-1}(B_r(p))$. This multiplicative factor Ψ aims to incorporate in Φ the ignored vector v of the previous point and so adjust the whole process.

We consider the measurable cocycle:

$$\begin{aligned} \Phi^t\colon (M, E^u_M) &\longrightarrow (M, E^u_M) \\ (q, u(q)) &\longmapsto (X^t(q), \lambda\!\!\!{}^{-}(t, q) u(X^t(q)), \end{aligned} \tag{33}$$

where[1] $\lambda\!\!\!{}^{-}(t, q) = \prod_{i=0}^{\lfloor t\rfloor -1} \lambda\!\!\!{}^{-}(1, X^i(q)) \lambda\!\!\!{}^{-}(t - \lfloor t\rfloor, X^{\lfloor t\rfloor}(q))$ if $t > 1$ and, when $t \in [0, 1]$, we use (32) and define $\lambda\!\!\!{}^{-}(t, q)$ by:

(i) $\lambda(t, q)$, if $q \notin X^{[-1-t,1]}(B_r(p))$ or $q \in X^{]-1,-t]}(B_r(p))$,
(ii) $\lambda(t - s, X^s(q))\Psi(X^s(q))\lambda(s, q)$, if $q \in X^{[-1-t,-1]}(B_r(p))$ where $s \geq 0$ is such that $X^s(q) \in X^{-1}(B_r(p))$,

[1] We let $\lfloor t\rfloor$ define the integer part of $t \geq 0$.

(iii) $\lambda(t-s, X^s(q))z(X^s(q))\lambda(s,q)$, if $q \in X^{[-t,0]}(B_r(p))$ where $s \geq 0$ is such that $X^s(q) \in B_r(p)$.
(iv) $\lambda(t+s, X^{-s}(q))[\lambda(s, X^{-s}(q))]^{-1}\, z(X^{-s}(q))$, if $q \in X^{[0,1-t]}(B_r(p))$ where $s \geq 0$ is such that $X^{-s}(q) \in B_r(p)$.
(v) $\lambda(t-s', X^r(q))\lambda(s'+s, X^{-s}(q))[\lambda(s, X^{-s}(q))]^{-1}\, z(X^{-s}(q))$, if $q \in X^{[1-t,1]}(B_r(p))$ where $s, s' \geq 0$ are respectively such that $X^{-s}(q) \in B_r(p)$ and $X^{s'}(q) \in X^1(B_r(p))$.

In order to define the correctional impulse $\Psi(\cdot)$ let us fix $q \in X^{-1}(B_r(p))$ and define $\tau(q)$ as the least positive real number such that there exists $\tilde{q} \in B_r(p)$ with $X^{\tau(q)}(\tilde{q}) = q$. Observe that $X^{\lfloor \tau(q) \rfloor + 1}(\tilde{q}) \in X^{[-1,0[}(B_r(p))$. If there is no recurrence, say if there does not exist such a $\tau(q)$, then we will take $\Psi = 1$. We also have (recall (29))

$$[\Phi_A^1(\tilde{q})]^{-1} \cdot \Phi_{A_0}^1(\tilde{q}) \cdot u(\tilde{q}) = z(\tilde{q})u(\tilde{q}) + v(\tilde{q}), \tag{34}$$

where $v(\tilde{q}) \in E^c$ say $v(\tilde{q}) = y(\tilde{q})c(\tilde{q})$. Therefore, the 'true' perturbation A_0 is such that:

$$\begin{aligned}
\Phi_{A_0}^{\tau(q)}(\tilde{q}) \cdot u(\tilde{q}) &= \Phi_{A_0}^{\tau(q)-1}(X^1(\tilde{q})) \cdot \Phi_{A_0}^1(\tilde{q}) \cdot u(\tilde{q}) \\
&\overset{(34)}{=} \Phi_A^{\tau(q)-1}(X^1(\tilde{q})) \cdot \Phi_A^1(\tilde{q}) \cdot [z(\tilde{q})u(\tilde{q}) + v(\tilde{q})] \\
&= \Phi_A^{\tau(q)}(\tilde{q}) \cdot v(\tilde{q}) + \Phi_A^{\tau(q)}(\tilde{q})\, [z(\tilde{q})u(\tilde{q}))] \\
&= \Phi_A^{\tau(q)}(\tilde{q}) \cdot v(\tilde{q}) + z(\tilde{q})a(\tilde{q}) \left(\prod_{i=0}^{\lfloor \tau(q) \rfloor - 1} \lambda(X^i(\tilde{q})) \right) \cdot u(q),
\end{aligned}$$

where $a(\tilde{q}) = \left\| \Phi_A^{\tau(q) - \lfloor \tau(q) \rfloor}(X^{\lfloor \tau(q) \rfloor}(\tilde{q})) \cdot E^u_{X^{\lfloor \tau(q) \rfloor}(\tilde{q})} \right\|$. In conclusion we have

$$\Phi_{A_0}^{\tau(q)}(\tilde{q}) \cdot u(\tilde{q}) = \Phi_A^{\tau(q)}(\tilde{q}) \cdot v(\tilde{q}) + z(\tilde{q})a(\tilde{q}) \left(\prod_{i=0}^{\lfloor \tau(q) \rfloor - 1} \lambda(X^i(\tilde{q})) \right) \cdot u(q). \tag{35}$$

Since M is compact and the linear differential systems we consider are of class C^0 there is $C_0 > 1$ such that

$$C_0^{-1} < a(q) < C_0 \quad \text{for all} \quad q \in M. \tag{36}$$

Moreover we also have that the 'artificial' cocycle (33) satisfies:

$$\Phi^{\tau(q)}(\tilde{q}, u(\tilde{q})) = \left(\overbrace{X^{\tau(q)}(\tilde{q})}^{\Phi_1}, \overbrace{z(\tilde{q})a(\tilde{q}) \left(\prod_{i=0}^{\lfloor \tau(q) \rfloor - 1} \lambda(X^i(\tilde{q})) \right) \cdot u(q)}^{\Phi_2} \right), \tag{37}$$

according to items (i) and (iii) in the definition of $\lambda\!\!\!^-$. Indeed, taking $s=0$ we get

$$\begin{aligned}\lambda\!\!\!^-(\tau(q),\tilde{q}) &= \lambda(\tau(q),\tilde{q})[\lambda(0,\tilde{q})]^{-1}\,z(\tilde{q}) = \lambda(\tau(q),\tilde{q})\,z(\tilde{q})\\ &= a(\tilde{q})\left(\prod_{i=0}^{\lfloor\tau(q)\rfloor-1}\lambda(X^i(\tilde{q}))\right)z(\tilde{q}).\end{aligned}$$

We notice that Φ_2 the second component of the cocycle (37) does not take into account the 'central' vector $\Phi_A^{\tau(q)}(\tilde{q})\cdot v(\tilde{q})$ that appears in (35). Indeed the abstract linear differential system (37) differs from Φ_{A_0} along the unstable direction by the projection in this direction of the vector $\Phi_A^{\tau(q)}(\tilde{q})\cdot v(\tilde{q})$. This multiplicative effect that must be considered is Ψ. Let π_q^u be the projection from $\mathbb{R}_q^3$ on E_q^u along the new central bundle $E_{A_0}^c$ and $w_u(q)=\pi_q^u(\Phi_A^{\tau(q)}(\tilde{q})\cdot v(\tilde{q}))$ is the correction we have to add to Φ_2. Actually we will take $\Psi(q)$ as the ratio of the norms of these two vectors. So let us define

$$\begin{aligned}\Psi(q) &= \frac{\|\pi_q^u(\Phi_{A_0}^{\tau(q)}(\tilde{q})\cdot u(\tilde{q}))\|}{\|\Phi_2^{\tau(q)}(\tilde{q},u(\tilde{q}))\|}\\ &\overset{(35)+(37)}{=} \frac{\|\pi_q^u[\Phi_A^{\tau(q)}(\tilde{q})\cdot v(\tilde{q})+z(\tilde{q})a(\tilde{q})\left(\prod_{i=0}^{\lfloor\tau(q)\rfloor-1}\lambda(X^i(\tilde{q}))\right)\cdot u(q)]\|}{z(\tilde{q})a(\tilde{q})\left(\prod_{i=0}^{\lfloor\tau(q)\rfloor-1}\lambda(X^i(\tilde{q}))\right)}\\ &= \frac{z(\tilde{q})a(\tilde{q})\left(\prod_{i=0}^{\lfloor\tau(q)\rfloor-1}\lambda(X^i(\tilde{q}))\right)+\|\pi_q^u(\Phi_A^{\tau(q)}(\tilde{q})\cdot v(\tilde{q}))\|}{z(\tilde{q})a(\tilde{q})\left(\prod_{i=0}^{\lfloor\tau(q)\rfloor-1}\lambda(X^i(\tilde{q}))\right)}.\end{aligned}$$

Since $\|u(q)\|=1$ and $\measuredangle(w_u(q),u(q))=0$ we have $\langle w_u(q),u(q)\rangle=\|\pi_q^u(\Phi_A^{\tau(q)}(\tilde{q})\cdot v(\tilde{q}))\|$ and so:

$$\Psi(q)=1+\frac{\langle w_u(q),u(q)\rangle}{z(\tilde{q})a(\tilde{q})\left(\prod_{i=0}^{\lfloor\tau(q)\rfloor-1}\lambda(X^i(\tilde{q}))\right)}. \tag{38}$$

Notice that Ψ is not defined in the whole of M, but for the sake of computing the Lyapunov exponent we have all the elements to get a well-defined Lyapunov exponent. We observe that we constructed a measurable linear differential system over the ν invariant flow X^t, hence by applying Oseledets' Theorem, we conclude that for Lebesgue a.e. points the system has a Lyapunov exponent. From Birkoff's Ergodic Theorem (see e.g. [12]) it follows that this Lyapunov exponent is equal to $\int_M \log(\lambda\!\!\!^-(q))d\nu(q)$.

The next result, which follows the lines of [2, Lemma 1.4], legitimates why we can use the computation of the Lyapunov exponent of Φ to assess the upper Lyapunov exponent of the perturbation A_0.

Lemma 4.2. *For every point q where the Lyapunov exponents are defined we have:*

$$\lim_{t\to\pm\infty}\frac{1}{t}\log\left\|\Phi^t_{A_0}(q)\cdot u(q)\right\| = \lim_{t\to\pm\infty}\frac{1}{t}\log\left\|\Phi^t(q)\right\|. \tag{39}$$

Proof. As X^t is ergodic w.r.t. ν it is enough to show the statement for a ν-generic point $q\in X^{[0,1]}(B_r)$. Pick a return time $T>0$ of $X^{s'}(q)\in X^1(B_r)$ to $X^{[0,1]}(B_r)$ where $s'\in[0,1]$. We have that $\Phi^T(q,u(q))$ is the projection of $\Phi^T_{A_0}(q)\cdot u(q)$ on E^u_A along the center-stable subbundle $E^{cs}_{A_0}$ (recall (35), the definition of Φ and notice that $E^{cs}_{A_0}$ is invariant). Consider the unstable cone field $C^u_\theta(A,p)\subset\mathbb{R}^3_p$ where $p\in M$ and θ is the angle with E^u_A of this cone field in a sense that there exists $\sigma\in]0,1[$ such that

$$\Phi^t_A(p)(C^u_\theta(A,p))\subset C^u_{\sigma^t\theta}(A,X^t(p)). \tag{40}$$

As the perturbation (34) is C^0-close to the identity (see also (29) and the paragraph before) the $s/c/u$-invariant subbundles related to partial hyperbolicity vary continuously when perturbing from Φ_A to Φ_{A_0}. Hence, as E^{cs}_A is transversal to E^u_A we get that the subbundle $E^{cs}_{A_0}$ is also transversal to E^u_A and consequently $\Phi^T_{A_0}(q)\cdot u(q)\subset C^u_\theta(A,X^T(q))$ for some $\theta>0$. Once the iterates of $\Phi^t_{A_0}$ are trapped in a cone field the property (40) guarantees that its exponential growth equals the exponential growth of the projection. □

4.2. End of the proof of Theorem 1

We claim that:

$$\int_M \log\left\|\Phi^1_{A_0}(q)\Big|_{E^u}\right\|\,d\nu(q) = \int_M \log\lambda\!\!\!\!^{-}(1,q)\,d\nu(q). \tag{41}$$

Indeed, on the one hand Birkoff's Ergodic Theorem gives for a ν-generic point q that:

$$\begin{aligned}\lambda^u(A_0) &= \lim_{n\to\pm\infty}\frac{1}{n}\log\left\|\Phi^n_{A_0}(q)\cdot u(q)\right\|\\ &= \lim_{n\to\pm\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\left\|\Phi^1_{A_0}(X^i(q))\cdot u(X^i(q))\right\|\\ &= \int_M \log\left\|\Phi^1_{A_0}(q)\Big|_{E^u}\right\|\,d\nu(q),\end{aligned}$$

and

$$\begin{aligned}\lambda(\Phi) &= \lim_{n\to\pm\infty}\frac{1}{n}\log\|\Phi^n(q)\| = \lim_{n\to\pm\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\left\|\Phi^1(X^i(q))\right\|\\ &= \int_M \log\lambda\!\!\!\!^{-}(1,q)\,d\nu(q),\end{aligned}$$

and on the other hand (39) allow us to conclude (41). Recalling (32) we define

$$\lambda^u(A) = \int_M \log \left\| \Phi_A^1(q)\Big|_{E^u} \right\| \, d\nu(q) = \int_M \log \lambda(1,q)\, d\nu(q)$$

and according to Lemma 4.2

$$\lambda^u(A_0) = \int_M \log \left\| \Phi_{A_0}^1(q)\Big|_{E^u} \right\| \, d\nu(q) = \int_M \log \lambda\!\!\!{}^{-}(1,q)\, d\nu(q).$$

Let us estimate $\lambda^u(A) - \lambda^u(A_0)$. First note that these integrals coincide outside the flowbox $X^{[-2,1]}(B_r(p))$ because time-1 iterates of points in $X^{[-2,1]}$ $(B_r(p))$ are not under perturbations. Now, to simplify our task we use the cocycle property and the definition of Φ obtaining:

$$\begin{aligned}\Phi^3(q) &= \Phi^{1-s}(X^{s+2}(q)) \cdot \Phi^1(X^{s+1}(q)) \cdot \Phi^1(X^s(q)) \cdot \Phi^s(q)\\ &= \|\Phi_A^{1-s}(X^{s+2}(q)) \cdot u(X^{s+2}(q))\| \cdot \Phi^1(X^{s+1}(q))\\ &\quad \cdot \Phi^1(X^s(q)) \cdot \|\Phi_A^s(q) \cdot u(q)\|\end{aligned}$$

for $q \in X^{[-2,-1]}(B_r(p))$ and $s \geq 0$ such that $X^s(q) \in X^{-1}(B_r(p))$. Then we analyse only the difference in $X^{-1}(B_r(p))$ and $B_r(p)$ and consider the measure $\tilde{\nu}$ on transversal sections. So let us begin by computing its difference on $B_r(p)$. Recalling item (iii) in the definition of $\lambda\!\!\!{}^{-}$ with $s = 0$ and $t = 1$ and also (30) we get:

$$\begin{aligned}\int_{B_r(p)} \log \lambda(1,q) - \log(\lambda\!\!\!{}^{-}(1,q))\, d\tilde{\nu}(q) &= \int_{B_r(p)} \log \frac{\lambda(1,q)}{\lambda\!\!\!{}^{-}(1,q)}\, d\tilde{\nu}(q)\\ &= \int_{B_r(p)} \log \frac{\lambda(1,q)}{\lambda(1,q) z(q)}\, d\tilde{\nu}(q)\\ &= -\int_{B_r(p)} \log z(q)\, d\tilde{\nu}(q)\\ &\overset{(31)}{=} -\left(\frac{r}{r_0}\right)^2 I(r_0).\end{aligned}$$

On the other hand by item (ii) in the definition of $\lambda\!\!\!{}^{-}$ with $s = 0$ and $t = 1$ and also (30) we get: $\lambda(t-s, X^s(q))\Psi(X^s(q))\lambda(s,q)$, if $q \in X^{[-1-t,-1]}(B_r(p))$ where $s \geq 0$ is such that $X^s(q) \in X^{-1}(B_r(p))$,

$$\begin{aligned}\int_{X^{-1}(B_r(p))} \log \lambda(1,q) - \log \lambda\!\!\!{}^{-}(1,q)\, d\tilde{\nu}(q) &= \int_{X^{-1}(B_r(p))} \log \frac{\lambda(1,q)}{\lambda\!\!\!{}^{-}(1,q)}\, d\tilde{\nu}(q)\\ &= \int_{X^{-1}(B_r(p))} \log \frac{\lambda(1,q)}{\lambda(1,q)\Psi(q)}\, d\tilde{\nu}(q)\\ &= -\int_{X^{-1}(B_r(p))} \log \Psi(q)\, d\tilde{\nu}(q).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\lambda^u(A)-\lambda^u(A_0) &\geq -\left(\frac{r}{r_0}\right)^2 I(r_0)-\tilde{\nu}(X^{-1}(B_r(p)))\times \max_{w\in X^{-1}(B_r(p))}\log(\Psi(w))\\ &= -r^2\left(\frac{I(r_0)}{r_0^2}+\frac{\tilde{\nu}(X^{-1}(B_r(p)))}{r^2}\max_{w\in X^{-1}(B_r(p))}\log(\Psi(w))\right)\\ &\overset{(27)}{=} -r^2\left(\frac{I(r_0)}{r_0^2}+\pi\max\log\Psi(w)\right).\end{aligned}$$

Since by (30) we have $I(r_0)<0$, to prove that $\lambda^u(A)-\lambda^u(A_0)>0$ it is enough to show that

$$\max_{w\in X^{-1}(B_r(p))}\log\Psi(w)$$

is negligible for small r. For that it suffices to prove that:

Lemma 4.3. *There exist $\sigma\in]0,1[$ and $\tilde{C}>0$ such that for any small r and $w\in X^{-1}(B_r(p))$ one has $|\Psi(w)-1|\leq\tilde{C}\,\sigma^{\tau_r}$, where τ_r is the smallest return time from $B_r(p)$ to $B_r(p)$.*

Proof. As Φ_A is partially hyperbolic we have a dominated splitting between the subbundles E^u_A and E^c_A. Hence, there exist $C_1>0$ and $\sigma\in]0,1[$ such that for any $q\in M$, $t\geq 0$ and unit vectors $u\in E^u_A$ and $c\in E^c_A$ we have

$$\|\Phi^t_A(q,c)\|\leq C_1\,\sigma^t\|\Phi^t_A(q,u)\|. \tag{42}$$

Let $\theta>0$ and consider the center-stable cone $C^{cs}_\theta(A,q)$, a plane $W\subset C^{cs}_\theta(A,q)$ and a vector $v\in C^{cs}_\theta(A,q)\setminus\{\vec{0}\}$. Let also $\pi^u_{q,W}\colon\mathbb{R}^3_q\to E^u_A$ stand for the projection on E^u_A parallel to W. We define a constant C_2 depending on θ by

$$C_2=\max\frac{\|\pi^u_{q,W}(v)\|}{\|v\|}. \tag{43}$$

Now recall that we fixed $q\in X^{-1}(B_r(p))$ and defined $\tau(q)$ as the least positive real number such that there exists $\tilde{q}\in B_r(p)$ with $X^{\tau(q)}(\tilde{q})=q$. We observe that $\tau(q)\geq\tau_r$. Recall also (34) that is

$$[\Phi^1_A(\tilde{q})]^{-1}\cdot\Phi^1_{A_0}(\tilde{q})\cdot u(\tilde{q})=z(\tilde{q})u(\tilde{q})+v(\tilde{q}),$$

and also that $w_u(q)=\pi^u_q(\Phi^{\tau(q)}_A(\tilde{q})\cdot v(\tilde{q}))$ where π^u_q is the projection from $\mathbb{R}^3_q$ on E^u_q along the new central bundle $E^c_{A_0}$. Now, observe that from (43) we get

$$\frac{\|w_u(q)\|}{\|\Phi^{\tau(q)}_A(\tilde{q})\cdot v(\tilde{q})\|}=\frac{\|\pi^u_q(\Phi^{\tau(q)}_A(\tilde{q})\cdot v(\tilde{q}))\|}{\|\Phi^{\tau(q)}_A(\tilde{q})\cdot v(\tilde{q})\|}\leq C_2. \tag{44}$$

Finally, considering $C_3 = \max_{q \in B_1(p)} z(q)^{-1}$, we get:

$$\begin{aligned}
|\Psi(w) - 1| &\overset{(38)}{\leq} \frac{\langle w_u(q), u(q)\rangle}{z(\tilde{q})a(\tilde{q})\prod_{i=0}^{\lfloor\tau(q)\rfloor-1}\lambda(X^i(\tilde{q}))} \overset{(36)}{\leq} \frac{C_0\langle w_u(q), u(q)\rangle}{z(\tilde{q})\prod_{i=0}^{\lfloor\tau(q)\rfloor-1}\lambda(X^i(\tilde{q}))} \\
&\leq \frac{C_0\|w_u(q)\|}{z(\tilde{q})\prod_{i=0}^{\lfloor\tau(q)\rfloor-1}\lambda(X^i(\tilde{q}))} \overset{(44)}{\leq} \frac{C_0\, C_2\|\Phi_A^{\tau(q)}(\tilde{q}) \cdot v(\tilde{q})\|}{z(\tilde{q})\prod_{i=0}^{\lfloor\tau(q)\rfloor-1}\lambda(X^i(\tilde{q}))} \\
&\overset{(42)}{\leq} \frac{C_0\, C_1\, C_2\, \sigma^{\tau(q)}}{z(\tilde{q})} \leq \frac{C_0\, C_1\, C_2\, \sigma^{\tau_r}}{z(\tilde{q})} \leq C_0\, C_1\, C_2\, C_3\, \sigma^{\tau_r},
\end{aligned}$$

and we have the lemma proved by considering $\tilde{C} = C_0\, C_1\, C_2\, C_3$. □

Since τ_r tends to infinity as r goes to zero it follows that if r is small enough, then we get

$$\lambda^u(A) - \lambda^u(A_0) > 0. \tag{45}$$

By the volume preserving assumption and (13) we have,

$$\lambda^u(A) + \lambda^c(A) + \lambda^s(A) = 0 = \lambda^u(A_0) + \lambda^c(A_0) + \lambda^s(A_0). \tag{46}$$

Moreover, as the perturbation A_0 left the stable manifold unchanged the negative Lyapunov exponents remains the same, that is $\lambda^s(A_0) = \lambda^s(A)$, we conclude from (45) and (46) that $\lambda^c(A_0) > \lambda^c(A)$ and Proposition 4.1 is proved.

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