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Dynamics of generic 2-dimensional linear differential systems ☆

Mário Bessa

IMPA, Estr. D. Castorina 110, 22460-320 Rio de Janeiro, Brazil

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Abstract

We prove that for a C^0-generic (a dense G_δ) subset of all the 2-dimensional conservative nonautonomous linear differential systems, either Lyapunov exponents are zero or there is a dominated splitting μ almost every point.

Keywords: Ergodic theory; Dominated splitting; Lyapunov exponents

1. Introduction and statement of the results

Linear differential systems are in general morphisms of vector bundles covering a flow. As a standard example we consider a dynamical system given by a C^1 vector field X, with associated flow X^t, and in this case the morphism corresponds to the action of the tangent flow DX^t in the tangent bundle. In this paper we consider the setting of 2-dimensional conservative *linear differential systems*, over continuous μ-invariant flows in compact Hausdorff spaces, where μ is a Borel regular measure. These systems are equipped with a dynamics in the base given by a continuous flow $\varphi^t : X \to X$, and a dynamics in the 2-dimensional tangent bundle, given by a continuous linear cocycle $\Phi^t : X \to GL(2,\mathbb{R})$.

As an example, we consider the *area-preserving* systems, on which $\Phi^t(p) \in SL(2,\mathbb{R})$, and so the infinitesimal generator, given by $A(\varphi^t(p)) = \frac{d}{ds}\Phi^s(p)|_{s=t} \circ [\Phi^t(p)]^{-1}$, is *traceless*. Analogously, other example of conservative systems is the set of *modified area-preserving* systems

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E-mail address: bessa@impa.br.

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$\Phi^t(p) \in GL(2,\mathbb{R})$ verifying for $p \notin \mathrm{Fix}(\varphi^t)$, $\det \Phi^t(p) = \frac{a(p)}{a(\varphi^t(p))}$, where $\mathrm{Fix}(\varphi^t)$ denote the set of fixed points of φ^t and $a : X \to \mathbb{R}$ is a nonnegative subexponential continuous function such that when $p \in \mathrm{Fix}(\varphi^t)$ we have $\det \Phi^t(p) = 1$. This example mimics the volume preserving flows defined on 3-dimensional manifolds, eventually with fixed points, with $a(\cdot) = \|X(\cdot)\|$.

Given a transition matrix $\Phi^t(p)$ we endow the set of associated infinitesimal generators $A : X \to GL(2,\mathbb{R})$ with the C^0-topology, so a residual set is a set which contains a C^0-dense G_δ.

The aim of this work is to describe the asymptotic behavior of $\Phi^t(p)$ for almost all points $p \in X$, namely, their Lyapunov exponents, which are the exponential growth rate of the norm of $\Phi^t(p)|_E$ along the orbits of the flow φ^t in the direction of the 1-dimensional bundle E.

For this purpose we consider a "relaxed" kind of hyperbolicity, called dominated splitting. Recall that hyperbolicity (or exponential dichotomy) guarantees that both 1-dimensional fiber-bundles have exponential behavior, in a way that one bundle contracts when we iterate backward and the other contracts when we iterate forward. In the presence of a dominated splitting we only guarantee that this exponential behavior exists relatively, that is, one of the bundles is *more expanded*, or *less contracted* by Φ^t. As in uniform hyperbolicity, dominated splitting is also uniform, say, the same rates are shared by all points in the set.

We give a general picture for the dynamics of generic conservative 2-dimensional linear differential systems:

Theorem 1. *There is a C^0-residual subset $\mathcal{R}$ of 2-dimensional conservative linear differential systems, such that if $A \in \mathcal{R}$ then for μ-a.e. $p \in X$:*

(a) *$\Phi^t_A(p)$ has a dominated splitting, or*
(b) *the Lyapunov exponents are zero.*

These systems act transitively in the projective space $\mathbb{R}P^1$, and this property is crucial to prove Theorem 1. In general, we may also consider systems with an *accessible condition*, which implies that Φ^t acts transitively in $\mathbb{R}P^1$, so the dichotomy (a) or (b) holds true.

The idea to prove Theorem 1 is the following. We take a conservative linear differential system which is a continuity point of an upper-semicontinuous function and if, for this system, there exists a positive measure set of points with positive Lyapunov exponents ((a) is false) and no dominated splitting ((b) is false) we construct a small C^0-perturbation which allows us to break the continuity obtaining a contradiction. Finally, it is a well-known result (see [15]) that the set of points of continuity of upper-semicontinuous functions is a residual set and Theorem 1 follows. For area-preserving systems and if the measure μ is ergodic, then we obtain uniform hyperbolicity *versus* zero Lyapunov exponents for μ-a.e. point $p \in X$. The same result follows if we consider modified area-preserving systems without fixed points for the flow φ^t and also the ergodicity of the measure μ.

This kind of results first appeared in [16] and in [17] Mañé gave an outline of the proof for conservative diffeomorphisms in surfaces. The complete proof is due to Bochi and may be found in [3]. Next, in a remarkable paper, see [4], Viana and Bochi generalize to multidimensional diffeomorphisms, symplectic diffeomorphisms and discrete cocycles. Also in the setting of discrete cocycles proving abundance of nonzero Lyapunov exponents, we mention the papers of Knill [14] and also Cong [5,6] (see also references therein). In [2] we start the approach of Mañé–Bochi–Viana to the continuous-time case. We point out that some particular examples of genericity of hyperbolicity in C^0-topology on the torus were already explored by Fabbri [8], and Fabbri and Johnson [10]. Several approaches have been proposed for determining the positivity of

Lyapunov exponents for linear differential systems, see Fabbri [9], and Fabbri and Johnson [11]. This last result follows from the paper of Kotani [13].

2. Linear differential systems

2.1. Basic definitions

Let X be a compact Hausdorff space, μ a Borel regular measure and $\varphi^t : X \to X$ a one-parameter family of continuous maps for which μ is φ^t-invariant. A cocycle based on φ^t is defined by a flow $\Phi^t(p)$ differentiable on the time parameter $t \in \mathbb{R}$ and continuous on space parameter $p \in X$, acting on $GL(2,\mathbb{R})$. Together they form the linear skew-product flow:

$$\Psi^t: \quad X \times \mathbb{R}^2 \longrightarrow X \times \mathbb{R}^2, \quad (p, v) \longmapsto \big(\varphi^t(p), \Phi^t(p) \cdot v\big).$$

The flow Φ^t verifies the *cocycle identity*:

$$\Phi^{t+s}(p) = \Phi^s\big(\varphi^t(p)\big) \circ \Phi^t(p),$$

for all $t, s \in \mathbb{R}$ and $p \in X$.

If we define a map $A : X \to GL(2,\mathbb{R})$ in a point $p \in X$ by

$$A(p) = \frac{d}{ds}\Phi^s(p)\big|_{s=0}$$

and along the orbit $\varphi^t(p)$ by

$$A\big(\varphi^t(p)\big) = \frac{d}{ds}\Phi^s(p)\big|_{s=t} \circ \big[\Phi^t(p)\big]^{-1}, \tag{1}$$

then $\Phi^t(p)$ will be the solution of the linear variational equation

$$\frac{d}{ds}u(s)\big|_{s=t} = A\big(\varphi^t(p)\big)u(t), \tag{2}$$

and $\Phi^t(p)$ is also called the fundamental matrix. Given a cocycle Φ^t we can induce the associated A by using (1) and given A we can recover the cocycle by solving the linear variational equation (2), from which we get Φ^t_A. We are interested in two kind of systems, the ones with $\det \Phi^t = 1$ which we call *area-preserving* or *traceless*, denoted by $GL(2,\mathbb{R},\mathrm{Tr}=0)$, and the *modified area-preserving*, denoted by $GL(2,\mathbb{R},\varphi^t)$, by establishing a link to the flow φ^t. To define this setting we need to consider a continuous nonnegative subexponential function $a : X \to \mathbb{R}$ that is non-null outside $\mathrm{Fix}(\varphi^t)$ and we say that A is modified area-preserving if:

$$\det \Phi^t_A(p) = \frac{a(p)}{a(\varphi^t(p))} \quad \text{for all } p \notin \mathrm{Fix}\big(\varphi^t\big) \text{ and } t \in \mathbb{R},$$
$$\det \Phi^t_A(p) = 1 \quad \text{for all } p \in \mathrm{Fix}\big(\varphi^t\big).$$

By Liouville formula we get $e^{\int_0^t \operatorname{Tr} A(\varphi^s(p))\,ds} = \det \Phi^t(p)$, so

$$e^{\int_0^t \operatorname{Tr} A(\varphi^s(p))\,ds} = \frac{a(p)}{a(\varphi^t(p))}.$$

2.2. Topology and conservative perturbations

Consider the set of linear differential systems A which are continuous and denote it by $C^0(X, GL(2,\mathbb{R}))$. We endow $C^0(X, GL(2,\mathbb{R}))$ with the uniform convergence topology defined by $\|A - B\|_0 = \max_{p\in X} \|A(p) - B(p)\|$.

We also define a L^∞-topology, this time on the set of measurable and μ-a.e. bounded maps $L^\infty(X, GL(2,\mathbb{R}))$, such that $\|A - B\|_\infty = \operatorname{ess\,sup} \|A(p) - B(p)\|$.

Therefore we may speak about conservative C^0- (or L^∞)-perturbations of systems $A \in C^0(X, GL(2,\mathbb{R}))$ (or $A \in L^\infty(X, GL(2,\mathbb{R}))$) along the orbit $\varphi^t(p)$ as $A + H$, where $H \in C^0(X, GL(2,\mathbb{R}))$ (or $H \in L^\infty(X, GL(2,\mathbb{R}))$) and $\operatorname{Tr} H(\varphi^t(p)) = 0$. This follows by direct application of Liouville formula, because

$$e^{\int_0^t \operatorname{Tr} A(\varphi^s(p))+\operatorname{Tr} H(\varphi^s(p))\,ds} = e^{\int_0^t \operatorname{Tr} A(\varphi^s(p))\,ds} = \det \Phi^t(p).$$

Given a conservative perturbation of A, say $A + H$, we denote by $\Phi^t_{A+H}(p)$ the solution of the corresponding linear equation (2), i.e. of

$$v'(t) = \big[A(t) + H(t)\big] \cdot v(t).$$

2.3. Oseledets theorem and the entropy function

The Oseledets theorem, see [18], has also an analog version for linear differential systems (see [12] for a simple proof). Moreover, for our particular 2-dimensional conservative linear differential systems we consider in Theorem 2.1 bellow a simplified version of this theorem. Given a system A, let $\mathfrak{O}^+ := \mathfrak{O}^+(A)$ denote the set of points of X with nonzero Lyapunov exponents and let $\mathfrak{O}^0(A)$ denote the set of points with both Lyapunov exponents zero.

Theorem 2.1. *Let Φ^t be as above. For μ-a.e. $p \in X$ there exists the upper-Lyapunov exponent $\lambda^+(p)$ defined by the limit $\lim_{t\to+\infty} \frac{1}{t} \log \|\Phi^t(p)\|$ that is a nonnegative measurable function of p. For μ-a.e. point $p \in \mathfrak{O}^+$ there is a splitting of $\mathbb{R}^2 = N^u_p \oplus N^s_p$ which varies measurably with p such that*:

$$\text{if } \vec{0} \neq v \in N^u_p, \quad \text{then } \lim_{t\to\pm\infty} \frac{1}{t} \log\big\|\Phi^t(p)\cdot v\big\| = \lambda^+(p);$$

$$\text{if } \vec{0} \neq v \in N^s_p, \quad \text{then } \lim_{t\to\pm\infty} \frac{1}{t} \log\big\|\Phi^t(p)\cdot v\big\| = -\lambda^+(p);$$

$$\text{if } \vec{0} \neq v \notin N^u_p, N^s_p, \quad \text{then } \lim_{t\to+\infty} \frac{1}{t} \log\big\|\Phi^t(p)\cdot v\big\| = \lambda^+(p) \quad \text{and}$$

$$\lim_{t\to-\infty} \frac{1}{t} \log\big\|\Phi^t(p)\cdot v\big\| = -\lambda^+(p).$$

Note that the symmetry of the Lyapunov exponents follows from Oseledets theorem because this theorem also gives the equality

$$\lim_{t\to\pm\infty}\frac{1}{t}\log\left|\det\Phi^t(p)\right| = \lambda^+(p)+\lambda^-(p) \tag{3}$$

and therefore in the area-preserving case we have $\det\Phi^t(p)=1$. Consequently, (3) implies that $\lambda^+(p)=-\lambda^-(p)$. For the modified area-preserving case we have the equality, $\det\Phi^t(p)=\frac{a(p)}{a(\varphi^t(p))}$, since $a(\cdot)$ is subexponential and nonzero along nonfixed orbits we get $\lambda^+(p)=-\lambda^-(p)$. For fixed points the former equality follows directly from (3). We define the *entropy function* of the system A, over any measurable, φ^t-invariant set $\Gamma\subseteq X$ by

$$LE(\cdot,\Gamma):\quad GL(2,\mathbb{R})\longrightarrow[0,+\infty),\quad A\longmapsto\int_\Gamma \lambda^+(p)\,d\mu(p),$$

using the subadditivity of the norm we obtain

$$LE(A,\Gamma)=\inf_{n\geqslant 1}\frac{1}{n}\int_\Gamma \log\left\|\Phi^n(p)\right\|\,d\mu(p).$$

Since $LE(\cdot,\Gamma)$ is the infimum of continuous functions it is upper-semicontinuous.

2.4. Hyperbolic structures

Let A be a linear differential system over a flow φ^t, the set $\Lambda\subseteq X$ is said to be *uniformly hyperbolic set* if there exists uniform constants $C>0$ and $\sigma\in(0,1)$ such that for μ-a.e. $p\in\Lambda$ there is a $\Phi_A^t(p)$-invariant decomposition $\mathbb{R}^2=N_p^u\oplus N_p^s$ varying measurably with p and verifying for $t>0$ the following inequalities: $\|\Phi_A^{-t}(p)|_{N_p^u}\|\leqslant C\sigma^t$ and $\|\Phi_A^t(p)|_{N_p^s}\|\leqslant C\sigma^t$. If $\Lambda=X$, then we say that A is *uniformly hyperbolic*. The concept of uniform hyperbolicity is equivalent to the *exponential dichotomy* concept, see [7] for details.

A φ^t-invariant set $\Lambda_m\subseteq X$ has *m-dominated splitting* for A if for μ-a.e. $p\in\Lambda_m$, there is a $\Phi_A^t(p)$-invariant decomposition $\mathbb{R}^2=N_p^u\oplus N_p^s$ varying measurably with p and verifying

$$\frac{\|\Phi_A^m(q)|_{N_q^s}\|}{\|\Phi_A^m(q)|_{N_q^u}\|}\leqslant\frac{1}{2}\quad\text{for any } q=\varphi^t(p).$$

Another definition equivalent to this one is considering constants $C>0$ and $\sigma\in(0,1)$ such that

$$\frac{\|\Phi_A^t(q)|_{N_q^s}\|}{\|\Phi_A^t(q)|_{N_q^u}\|}\leqslant C\sigma^t.$$

In this case we say that the φ^t-invariant set has (C,σ)-dominated splitting.

Let $\Delta(p,m):=\frac{\|\Phi_A^m(p)|_{N_p^s}\|}{\|\Phi_A^m(p)|_{N_p^u}\|}$. We define the following sets:

- $\mathrm{Per}(\varphi^t)=\{p\in X\colon\ p \text{ is a periodic point for the flow } \varphi^t\}$;
- $\Lambda_m(A)=\{p\in X\colon\ \text{the orbit } \varphi^t(p) \text{ has } m\text{-dominated splitting for } \Phi_A^t\}$;

- $\Gamma_m(A) = X - \Lambda_m(A)$;
- $\Gamma_m^+(A) = \Gamma_m(A) \cap \mathfrak{O}^+(A)$;
- $\Gamma_m^*(A) = \{p \in \Gamma_m^+(A)\colon\ p \notin \mathrm{Per}(\varphi^t)\}$;
- $\Delta_m(A) = \{p \in X\colon\ \Delta(p,m) \geqslant \frac{1}{2}\}$.

Before moving on to the proof of Theorem 1 we would like to make some brief comments. First of all note that if $q \in \Gamma_m(A)$, then for some point in the orbit of q, say $\varphi^t(q) = p$, we have $\Delta(p,m) \geqslant 1/2$, and therefore $p \in \Delta_m(A)$. Moreover $\Gamma_m = \bigcup_{t\in\mathbb{R}} \varphi^t(\Delta_m)$. The set Δ_m is of utmost importance because that is where we will apply a perturbation to the original system.

3. Proof of Theorem 1

3.1. Disregarding periodic points in $\mathfrak{O}^+(A)$

Our main objective will be decay $LE(B, \Gamma_m(A))$ for a system B close to the original system A. We say that a flow is *aperiodic* if the measure of periodic points is zero, clearly $\varphi^t : \Gamma_m^*(A) \to \Gamma_m^*(A)$ is aperiodic. We will use Ambrose–Kakutani theorem, see [1], which gives us a special representation of $\varphi^t|_{\Gamma_m^*(A)}$. Next we perturb inside $\Gamma_m^*(A)$ to decrease $LE(B, \Gamma_m^*(A))$. However we have no information about $\Gamma_m(A) - \Gamma_m^*(A)$. For our purposes, points in $\Gamma_m(A)$ with zero Lyapunov exponents will not be a problem, whereas the set $\Gamma_m^+(A) - \Gamma_m^*(A)$ may cause some trouble. Lemma 3.1 says that $\mu(\Gamma_m^+(A) - \Gamma_m^*(A))$ is small (depending on m), therefore we will obtain $LE(B, \Gamma_m^+(A) - \Gamma_m^*(A))$ also small.

We note that for a fixed $m \in \mathbb{N}$ we have $p \in \Gamma_m^+(A) - \Gamma_m^*(A)$ if p is periodic, has positive Lyapunov exponent and belongs to $\Gamma_m(A)$.

Lemma 3.1. *For any $\delta > 0$, there exists $m \in \mathbb{N}$ such that we have $\mu(\Gamma_m^+(A) - \Gamma_m^*(A)) < \delta$.*

Proof. Let P be the measure of all periodic points in $\mathfrak{O}^+(A)$. If $P = 0$ then there is nothing to prove, so consider $P > 0$. Define

$$\mathrm{Per}(n,\lambda) = \bigl\{p \in \mathfrak{O}^+\colon\ \varphi^t(p) = p \text{ for some } t \leqslant n \text{ and } \lambda^+(p) > \lambda\bigr\}.$$

So $\mu(\bigcup_{n\in\mathbb{N}} \mathrm{Per}(n,0)) = P$, therefore for all $\delta > 0$, there exist n_0, λ_0 such that $\mu(\mathrm{Per}(n_0,\lambda_0)) > P - \delta$.

If $p \in \mathrm{Per}(n_0,\lambda_0)$, then p has m' dominated splitting for some m', therefore there exists a large m such that $\mathrm{Per}(n_0,\lambda_0) \subseteq \bigcup_{i=1}^m \Lambda_i$ and we get

$$\Gamma_m^+(A) - \Gamma_m^*(A) \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{Per}(n,0) - \bigcup_{i=1}^m \Lambda_i,$$

so

$$\begin{aligned}\mu\bigl(\Gamma_m^+(A) - \Gamma_m^*(A)\bigr) &< \mu\Biggl(\bigcup_{n\in\mathbb{N}} \mathrm{Per}(n,0) - \bigcup_{i=1}^m \Lambda_i\Biggr) < \mu\Biggl(\bigcup_{n\in\mathbb{N}} \mathrm{Per}(n,0) - \mathrm{Per}(n_0,\lambda_0)\Biggr)\\ &= \mu\Biggl(\bigcup_{n\in\mathbb{N}} \mathrm{Per}(n,0)\Biggr) - \mu\bigl(\mathrm{Per}(n_0,\lambda_0)\bigr) < \delta. \qquad \square\end{aligned}$$

3.2. Perturbations of linear differential systems

We begin by lowering $\|\Phi^t_{A+H}(q)\|$ along a segment of the orbit, this is valid in both settings $GL(2,\mathbb{R},\mathrm{Tr}=0)$ and $GL(2,\mathbb{R},\varphi^t)$. In order to achieve this goal we carry out some perturbations which we explains in the next section.

3.2.1. Small rotations by time-1 perturbation

Lemma 3.2. *Given a conservative system A and $\epsilon>0$, there exists an angle ξ, such that for all $p\in X$ (nonperiodic or with period larger than 1), there exists a system B such that:*

(a) $\|A-B\|<\epsilon$;
(b) *B is supported in $\varphi^t(p)$ for $t\in[0,1]$;*
(c) *B is conservative; and*
(d) *$\Phi^1_B(p)=\Phi^1_A(p)\circ R_\xi$, where R_ξ is a rotation of angle ξ.*

Proof. Let $\eta\in(0,1)$ and $g:\mathbb{R}\to\mathbb{R}$ be the *bump-function* defined by $g(t)=0$ for $t<0$, $g(t)=t$ for $t\in[\eta,1-\eta]$ and $g(t)=1$ for $t\geqslant 1$. Define

$$\Phi^t(p)=\begin{pmatrix} a(t) & b(t)\\ c(t) & d(t)\end{pmatrix}\quad\text{and}\quad R_{\xi g(t)}=\begin{pmatrix}\cos(\xi g(t)) & -\sin(\xi g(t))\\ \sin(\xi g(t)) & \cos(\xi g(t))\end{pmatrix}.$$

We know that $u(t)=\Phi^t(p)$ is a solution of the linear variational equation (2). Take $\Phi^t(p)\times R_{\xi g(t)}$ and compute the time derivative:

$$\begin{aligned}\big(\Phi^t(p)\cdot R_{\xi g(t)}\big)'&=\big(\Phi^t(p)\big)'R_{\xi g(t)}+\Phi^t(p)R'_{\xi g(t)}\\ &=A\big(\varphi^t(p)\big)\Phi^t(p)R_{\xi g(t)}+\Phi^t(p)R'_{\xi g(t)}\\ &=A\big(\varphi^t(p)\big)\Phi^t(p)R_{\xi g(t)}+\Phi^t(p)R'_{\xi g(t)}R_{-\xi g(t)}\big[\Phi^t(p)\big]^{-1}\Phi^t(p)R_{\xi g(t)}\\ &=\big[A\big(\varphi^t(p)\big)+\Phi^t(p)R'_{\xi g(t)}R_{-\xi g(t)}\big[\Phi^t(p)\big]^{-1}\big]\cdot\big(\Phi^t(p)R_{\xi g(t)}\big).\end{aligned}$$

Define $B(\varphi^t(p))=A(\varphi^t(p))+H(\varphi^t(p))$, where

$$H\big(\varphi^t(p)\big)=H(\xi,t)=\Phi^t(p)R'_{\xi g(t)}R_{-\xi g(t)}\big[\Phi^t(p)\big]^{-1}.$$

We conclude that $v(t)=\Phi^t(p)\cdot R_{\xi g(t)}$ is a solution of the linear variational equation

$$\frac{d}{ds}v(s)\Big|_{s=t}=\big[A\big(\varphi^t(p)\big)+H(\xi,t)\big]v(t).\tag{4}$$

Since

$$R'_{\xi g(t)}\cdot R_{-\xi g(t)}=\xi g'(t)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

we easily derive

$$H(\xi,t)=\frac{\xi g'(t)}{\det\Phi^t(p)}\begin{pmatrix} b(t)d(t)+a(t)c(t) & -b(t)^2-a(t)^2\\ d(t)^2+c(t)^2 & -b(t)d(t)-a(t)c(t)\end{pmatrix}.$$

Hence $\operatorname{Tr} H(\xi,t)=0$ and the perturbation is conservative according to our definition, and so (c) follows. Moreover since $g'(t)=0$ for $t\notin\,]0,1[$, its support is $\varphi^t(p)$ for $t\in[0,1]$ and (b) is proved. Since $t\in[0,1]$ and all the terms in the definition of $H(\xi,t)$ are uniformly bounded for all $p\in X$, given any size of perturbation allowed by $\epsilon>0$ we take ξ sufficiently small to guarantee that $\|H\|<\epsilon$ and obtain (a). Finally, for (d), we note that $v(t)=\Phi_A^t(p)\cdot R_{\xi g(t)}$ is solution of (4). So for $t=1$ we obtain $\Phi_B^1(p)=\Phi_A^1(p)\cdot R_\xi$ and the lemma is proved. □

Lemma 3.3. *Given a conservative system A and $\epsilon>0$, there exists an angle ξ, such that for all $p\in X$ (nonperiodic or with period larger than 1), there exists a system B such that*:

(a) $\|A-B\|<\epsilon$;
(b) *B is supported in $\varphi^t(p)$ for $t\in[0,1]$*;
(c) *B is conservative*; *and*
(d) *$\Phi_B^1(p)=\tilde R_\xi\circ\Phi_A^1(p)$, where $\tilde R_\xi$ is an elliptical rotation of angle ξ.*

Proof. We use the same notation of Lemma 3.2. Define the one-parameter elliptical rotation by

$$\tilde R_{\xi g(t)}=\Phi^t(p)\cdot R_{\xi g(t)}\cdot \Phi^{-t}\big(\varphi^t(p)\big). \tag{5}$$

Now we consider $\tilde R_{\xi g(t)}\cdot\Phi^t(p)$ and take time derivatives:

$$\begin{aligned}\big(\tilde R_{\xi g(t)}\cdot\Phi^t(p)\big)'&=\big(\Phi^t(p)\cdot R_{\xi g(t)}\cdot\Phi^{-t}\big(\varphi^t(p)\big)\cdot\Phi^t(p)\big)'=\big(\Phi^t(p)\cdot R_{\xi g(t)}\big)'\\&=\big[A\big(\varphi^t(p)\big)+H\big(\varphi^t(p)\big)\big]\cdot\big(\Phi^t(p)R_{\xi g(t)}\big)\\&=\big[A\big(\varphi^t(p)\big)+H\big(\varphi^t(p)\big)\big]\cdot\big(\tilde R_{\xi g(t)}\cdot\Phi^t(p)\big),\end{aligned}$$

and we reduce to the proof of Lemma 3.2. □

Remark 3.1. We will need Lemma 3.3 to perform some small rotations, and we point out that this lemma gives us an elliptical rotation. So after the change of coordinates the angle may decrease depending on how large the norm of this change of coordinates is. However we can always find $\xi_0<\xi$ depending on $\|\Phi_A^t(p)\|$ (for $t\in[0,1]$) and conclude that the perturbation realizes $\Phi_{A+H}^1(p)=R_{\xi_0}\cdot\Phi_A^1(p)$.

3.2.2. Large rotations by time-m perturbation

Next lemma ensures a control on the norm of $\Phi^t(p)$. In the case of $GL(2,\mathbb{R},\operatorname{Tr}=0)$ we take $\sqrt{a(p)/a(\varphi^t(p))}=1$. Denote by $\measuredangle(N^u_{\varphi^t(p)},N^s_{\varphi^t(p)})$ the angle between the two subspaces of $\mathbb{R}^2$, $N^u_{\varphi^t(p)}$ and $N^s_{\varphi^t(p)}$.

Lemma 3.4. *Let $\xi>0$, $p\in X$ and $d>1$. There exists $E>1$ such that if for all $t\in[0,m]$, $\measuredangle(N^u_{\varphi^t(p)},N^s_{\varphi^t(p)})>\xi$ and $d^{-1}\leqslant\frac{\|\Phi^t(p)|_{N_p^s}\|}{\|\Phi^t(p)|_{N_p^u}\|}\leqslant d$, then $\|\Phi^t(p)\|\leqslant E\sqrt{\frac{a(p)}{a(\varphi^t(p))}}$ for all $t\in[0,m]$.*

Proof. Denote $\measuredangle(N^u_{\varphi^t(p)},N^s_{\varphi^t(p)})=\xi_t>\xi$ for all $t\in[0,m]$. By the volume preserving property we get

$$\big\|\Phi^t(p)\big|_{N_p^s}\big\|a\big(\varphi^t(p)\big)=a(p)\big\|\Phi^t(p)\big|_{N_p^u}\big\|^{-1}\frac{\sin\xi_0}{\sin\xi_t}.$$

Therefore

$$\left\|\Phi^t(p)\big|_{N_p^s}\right\|^2 = \frac{a(p)}{a(\varphi^t(p))}\frac{\|\Phi^t(p)|_{N_p^s}\|}{\|\Phi^t(p)|_{N_p^u}\|}\sin^{-1}\xi_t \leqslant \frac{a(p)d}{a(\varphi^t(p))}\sin^{-1}\xi$$

and so we obtain $\|\Phi^t(p)|_{N_p^s}\| \leqslant \sqrt{\frac{a(p)d}{a(\varphi^t(p))}\sin^{-1}\xi}$. Analogously,

$$\left\|\Phi^t(p)\big|_{N_p^u}\right\|^2 \leqslant \frac{a(p)}{a(\varphi^t(p))}\frac{\|\Phi^t(p)|_{N_p^u}\|}{\|\Phi^t(p)|_{N_p^s}\|}\sin^{-1}\xi_t \leqslant \frac{a(p)d}{a(\varphi^t(p))}\sin^{-1}\xi$$

and we also obtain

$$\left\|\Phi^t(p)\big|_{N_p^u}\right\| \leqslant \sqrt{\frac{a(p)d}{a(\varphi^t(p))}\sin^{-1}\xi}.$$

We conclude that $\|\Phi^t(p)\| \leqslant \sqrt{2\frac{a(p)d}{a(\varphi^t(p))}\sin^{-1}\xi}$, for all $t \in [0,m]$, so the statement hold taking $E = \sqrt{2d/\sin\xi}$. □

In order to perform rotations of large angles we could try, under some particular conditions, to concatenate smoothly several time-1 small rotations until obtain the desired angle. Otherwise, which is easier, we could induce a time-m perturbation to generate the rotations of a given large angle, however this cannot be done in general, because some hyperbolicity in the dynamics obstruct the whole construction.

Under the conditions of Lemma 3.4 it is possible to rotate large angles by time-m keeping the norm of $H(\xi,t)$ for $t \in [0,m]$ small. Since the explicit perturbation is given by

$$B\big(\varphi^s(p)\big) = \frac{d}{dt}\big[\Phi_A^{t+s}(p)\cdot R_{\xi g(s+t)}\cdot \Phi_A^{-s}\big(\varphi^s(p)\big)\big]\big|_{t=0}\Phi_A^s(p)\cdot R_{\xi g(s)}\cdot \Phi_A^{-s}\big(\varphi^s(p)\big),$$

we expect that some control of $\|\Phi_A^t\|$ is needed, so Lemma 3.4 will play an important role.

Lemma 3.5. *Given a conservative system A and $\epsilon, d, \xi > 0$, there exists $m \in \mathbb{N}$, such that if the following conditions are satisfied for $p \in X$ nonperiodic or with period larger than m, namely,*

(1) $\measuredangle(N^u_{\varphi^t(p)}, N^s_{\varphi^t(p)}) > \xi$ *for all* $t \in [0,m]$,
(2) $d^{-1} \leqslant \frac{\|\Phi^t(p)|_{N_p^s}\|}{\|\Phi^t(p)|_{N_p^u}\|} \leqslant d$,

then there exists a system B such that for all $\alpha \in [0,2\pi]$, we have:

(a) $\|A - B\| < \epsilon$;
(b) *B is supported in $\varphi^t(p)$ for $t \in [0,m]$*;
(c) *B is conservative*; *and*
(d) $\Phi_B^1(p) = \Phi_A^1(p) \circ R_\alpha$.

Proof. For any $m \in \mathbb{N}$ we consider $\eta > 0$ close to zero and $g:\mathbb{R} \to \mathbb{R}$ the *bump-function* such that $g(t) = 0$ for $t < 0$, $g(t) = t$ for $t \in [\eta, m-\eta]$ and $g(t) = 1$ for $t \geqslant m$. We use then the same procedure of Lemma 3.2 by defining $R_{\theta g(t)}$. Let $\alpha \in [0, 2\pi]$. Take $\theta < \frac{\epsilon \sin\xi}{4d}$ and $m = \frac{\alpha}{\theta}$. There is no restriction while considering $m \in \mathbb{N}$, by taking a smaller θ. Now fix the function g depending on this m. Clearly we will obtain H such that (b), (c) and (d) are verified. We claim that (a) is also true. By hypothesis we have (1) and (2), so by Lemma 3.4 we conclude that $\|\Phi^t(p)\| < E\sqrt{a(p)/a(\varphi^t(p))}$ and since $E = \sqrt{2d/\sin\xi}$ we get $\theta < \frac{\epsilon}{2E^2}$. Using the same notation of Lemma 3.2, the perturbation is defined, for $t \in [0, m]$, by

$$H(\theta, t) = \frac{\theta g'(t)}{\det \Phi^t(p)} \begin{pmatrix} b(t)d(t) + a(t)c(t) & -b(t)^2 - a(t)^2 \\ d(t)^2 + c(t)^2 & -b(t)d(t) - a(t)c(t) \end{pmatrix}.$$

Consider now the norm given by the maximum and we show that $\|H\| < \epsilon$.

$$\begin{aligned} \|H(\theta,t)\| &\leqslant \frac{\theta g'(t)}{|\det \Phi^t(p)|} \max_{t\in[0,m]} \left\{\pm\left[b(t)d(t) + a(t)c(t)\right], -b(t)^2 - a(t)^2, d(t)^2 + c(t)^2\right\} \\ &\leqslant 2\frac{\theta g'(t)}{|\det \Phi^t(p)|} \left\|\Phi^t(p)\right\|^2. \end{aligned}$$

Now, by Lemma 3.4 we obtain

$$\|H(\theta,t)\| \leqslant 2\frac{\theta g'(t)}{|\det \Phi^t(p)|} \frac{2d}{\sin\xi} \frac{a(p)}{a(\varphi^t(p))} \leqslant 2\frac{\theta g'(t)}{|\det \Phi^t(p)|} E^2 \frac{a(p)}{a(\varphi^t(p))} \leqslant \epsilon,$$

which concludes the proof. □

3.3. Local recurrence argument

Lemma 3.6. *Let A be a continuous conservative system and $\epsilon > 0$. There exists $m \in \mathbb{N}$ such that, given any $p \in \Gamma_m^*(A)$, there exists H satisfying $\|H\| < \epsilon$ and $\Phi^m_{A+H}(N^u_p) = N^s_{\varphi^m(p)}$.*

Proof. Let $\xi > 0$ be given by Lemmas 3.2 and 3.3 in order to guarantee time-1 ϵ-perturbations. Consider the following simple claim, illustrated by Fig. 1, and whose proof may be found in [3].

Claim 1. *Given an angle ξ, there exists $c > 1$, such that if $\Delta(\varphi^t(p), r) > c$, then there is a nonzero vector $v \in N_{\varphi^t(p)}$ such that $\measuredangle(v, N^u_{\varphi^t(p)}) < \xi$ and $\measuredangle(\Phi^r(\varphi^t(p)) \cdot v, N^s_{\varphi^{t+r}(p)}) < \xi$.*

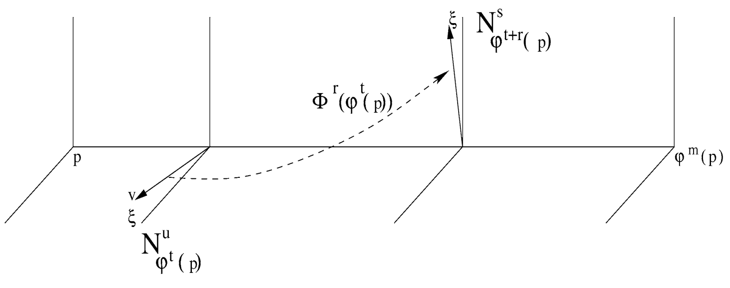

Fig. 1. The action of the dynamics favors our goals.

Let $c > 0$ be given by claim above. Let $E > 1$ be given by Lemma 3.4 depending on ξ and $d = 2c^2$. Let $m \in \mathbb{N}$ be given by Lemma 3.5 and depending on E, hence depending on d and ξ. Consider $p \in \Gamma_m^*(A)$ and $\Phi^t(p)$ for $t \in [0, m]$.

Small angle. If for some $t \in [0, m]$ we have $\measuredangle(N^u_{\varphi^t(p)}, N^s_{\varphi^t(p)}) < \xi$ we use a small rotation by a time-1 perturbation and, if $t + 1 \leqslant m$ we get H such that $\Phi^1_{A+H}(N^u_{\varphi^t(p)}) = N^s_{\varphi^{t+1}(p)}$. If $t + 1 > m$ we get H such that $\Phi^{-1}_{A+H}(N^s_{\varphi^t(p)}) = N^u_{\varphi^{t-1}(p)}$ and in both cases $\|H\| < \epsilon$. Consequently we obtain $\Phi^m_{A+H}(N^u_p) = N^s_{\varphi^m(p)}$.

Now we consider the case when there exists $r, t \in \mathbb{R}$ with $0 \leqslant r + t \leqslant m$ such that $\Delta(\varphi^t(p), r) > c$. We use Claim 1 in order to obtain a vector $v \in N_{\varphi^t(p)}$ such that $\measuredangle(v, N^u_{\varphi^t(p)}) < \xi$ and $\measuredangle(\Phi^r(\varphi^t(p)) \cdot v, N^s_{\varphi^{t+r}(p)}) < \xi$. Now, since ξ is small we make two small rotations at both extremes $\varphi^t(p)$ and $\varphi^{t+r}(p)$. The choice of c sufficient large guarantees disjoint perturbations. Therefore, our first rotation $\Phi^1_{A+H_1}(\varphi^t(p)) = \Phi^1_A(\varphi^t(p)) \cdot R_\xi$, induced by the perturbation H_1, allows us to send $N^u_{\varphi^t(p)}$ into $v\mathbb{R}$, the dynamics of Φ^r_A help us and send this direction into $\Phi^r(\varphi^t(p)) \cdot v$ in time r (see Fig. 1) and another rotation, $\Phi^1_{A+H_2}(\varphi^{t+r-1}(p)) = R_\xi \cdot \Phi^1_A(\varphi^{t+r-1}(p))$, induced by the perturbation H_2, maps $\Phi^r(\varphi^t(p)) \cdot (v\mathbb{R})$ into $N^s_{\varphi^{t+r}(p)}$. Now we concatenate smoothly the five matrix transitions, say,

$$\Phi_A^{m-(t+r)}\big(\varphi^{t+r}(p)\big)\Phi^{r-1}_{A+H_2}\big(\varphi^{t+r-1}(p)\big)\Phi_A^{r-2}\big(\varphi^{t+1}(p)\big)\Phi^1_{A+H_1}\big(\varphi^t(p)\big)\Phi^t_A(p),$$

and we get $\Phi^m_{A+H}(N^u_p) = N^s_{\varphi^m(p)}$. Note that $H(p) = H_1(p) + H_2(p)$ and $\|H_i\| < \epsilon$, for $i = 1, 2$.

Large angle. Finally, we have for all $t \in [0, m]$, $\measuredangle(N^u_{\varphi^t(p)}, N^s_{\varphi^t(p)}) > \xi$ and since for all $r, t \in \mathbb{R}$ with $0 \leqslant r + t \leqslant m$ we have $\Delta(\varphi^t(p), r) < c$. Since $p \in \Delta_m$ we also have $\Delta(p, m) \geqslant 1/2$. Therefore, we conclude that

$$\Delta\big(\varphi^t(p), r\big) = \Delta\big(\varphi^{t+r}(p), m - t - r\big)\Delta(p, m)\Delta(p, t)^{-1} \geqslant \frac{1}{2c^2}.$$

So for $t = 0$ and $r \in [0, m]$ we have $(2c^2)^{-1} \leqslant \Delta(p, r) \leqslant c$ and since $d = 2c^2$ we obtain,

$$d^{-1} \leqslant \frac{\|\Phi^r(p)|_{N^s_p}\|}{\|\Phi^r(p)|_{N^u_p}\|} \leqslant d,$$

for all $r \in [0, m]$. The conditions of Lemma 3.4 are now satisfied and by applying Lemma 3.5 we are able to use rotations by large angles and therefore $\Phi^m_{A+H}(N^u_p) = N^s_{\varphi^m(p)}$ which proves the lemma. □

In the next lemma we only give an outline of the proof and skip technical arguments which may be found in [2,4].

Lemma 3.7. *Let A be a continuous conservative system, $\epsilon > 0$, and $\delta > 0$. There exists $m \in \mathbb{N}$ and a measurable function $T : \Gamma^*_m(A) \to \mathbb{R}$ such that for μ-a.e. $q \in \Gamma^*_m$ and every $t > T(q)$ there exist a traceless $\{H(\varphi^s(q))\}_{s\in\mathbb{R}}$, varying smoothly with s and supported on the segment $\varphi^{[0,t]}(p)$ such that*:

(a) $\|H\| < \epsilon$,
(b) $\frac{1}{t}\log\|\Phi^t_{A+H}(q)\| < \delta$.

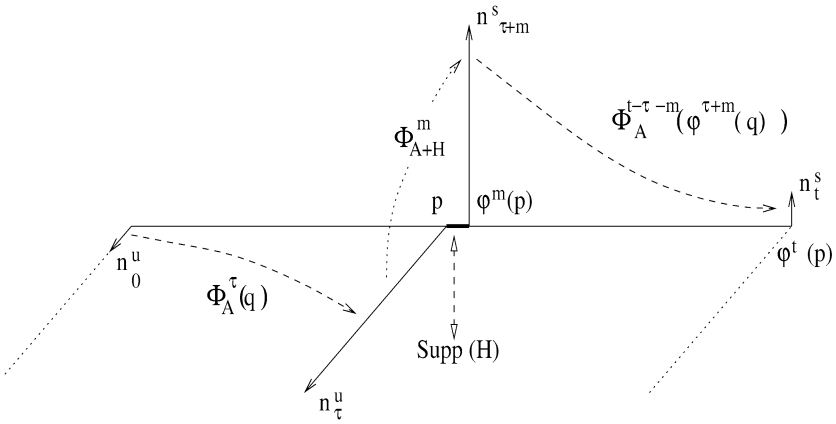

Fig. 2. Behavior of $\|\Phi^t_{A+H}(q)\|$ with H supported in $\bigcup_{r\in[0,m]}\varphi^{r+\tau}(q)$, where $\tau \approx t/2$.

Proof. First, using Lemma 3.6, we choose a sufficiently large m in order to send N^u_p into $N^s_{\varphi^m(p)}$ under ϵ-small C^0-perturbation, for Oseledets regular points $p \in \Delta_m$. So, for our perturbation $A+H$ we obtain $\Phi^m_{A+H}(p)(N^u_p) = N^s_{\varphi^m(p)}$. Given q in the saturated set $\Gamma^*_m(A)$ and using a qualitative recurrence result (see [3, Lemma 3.12]) for all $t > T(q)$ we have to fall into Δ_m approximately in the middle of the journey, say $\varphi^\tau(q) = p$, for $\tau \approx t/2$. Take $t \gg m$. Now we perturb and we get $\Phi^t_{A+H}(q)(N^u_q) = N^s_{\varphi^t(q)}$. The contribution of the exponential growth along the direction N^u_q in the first half, will be annihilated on the other half by an exponential decreasing bundle $N^s_{\varphi^{s+m}(q)}$ implying $\|\Phi^t_{A+H}(q)\| < e^{t\delta}$. That is the reason why we mix the two directions. The idea is shown in Fig. 2. □

3.4. Global recurrence argument

For the global case we construct a special flow by using Ambrose–Kakutani theorem over the aperiodic flow $\varphi^t : \Gamma^*_m \to \Gamma^*_m$, but first we use Lemma 3.1 to increase $m \in \mathbb{N}$ if necessary and obtain

$$\mu\big(\Gamma^+_m(A) - \Gamma^*_m(A)\big) < \delta. \tag{6}$$

Using the measurable function given by Lemma 3.7 we define

$$Z_h = \big\{p \in \Gamma^*_m(A)\colon\ T(p) \leqslant h\big\}.$$

We have that $\lim_{h\to\infty} \mu(\Gamma^*_m(A) - Z_h) = 0$ so holds we take h sufficiently large such that:

$$\mu\big(\Gamma^*_m(A) - Z_h\big) < \delta^2 \mu\big(\Gamma^*_m(A)\big). \tag{7}$$

Let us now increase h and use Oseledets theorem, which is an asymptotic result, to get for points $p \in \mathfrak{O}^0(A)$ the inequality

$$\big\|\Phi^t_A\big\| < e^{t\delta} \quad \text{for all } t \geqslant h. \tag{8}$$

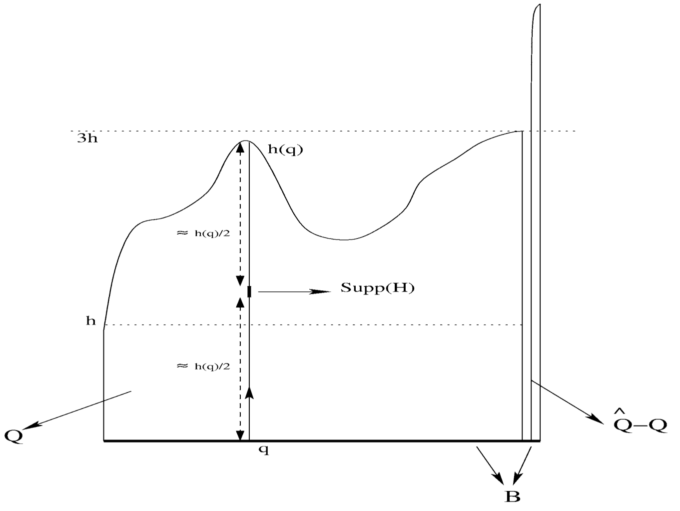

Fig. 3. The Kakutani subcastle Q supports the perturbations.

Suppose that we have a ceiling function over a section $B \subseteq Z_h$ verifying $h(x) \geqslant h$. We denote by $\hat{Q}$ the Kakutani castle with base B. Excluding all towers with height above $3h$ we define a subcastle which we denote by Q (see Fig. 3).

We claim that $\mu(\hat{Q} - Q) < 3\delta^2 \mu(\Gamma_m^*(A))$, as in [3, Lemma 4.2].

Now we will decay the entropy function $LE(\cdot, \Gamma_m(A))$ at A, by a small perturbation $B = A + H$ of the system. We start with a L^∞-perturbation and the idea for the continuous ones comes from noting that $H(\cdot)$ is measurable and therefore, by Lusin's theorem, we have that measurable functions are almost continuous and since we are only interested on almost all points in the base the same result will follow.

For the bounded case we consider the following lemma.

Lemma 3.8. *Let A be a conservative system and $\epsilon, \delta > 0$. Then, there exist $m \in \mathbb{N}$ and a traceless system $H \in L^\infty(X, GL(2, \mathbb{R}))$ such that*:

(a) $\|H\|_\infty < \epsilon$;
(b) $\|H(p)\| = 0$ *for any* $p \notin \Gamma_m(A)$;
(c) $LE(A + H, \Gamma_m(A)) < \delta$.

Proof. Suppose that $\mu(\Gamma_m(A)) > 0$, otherwise, there is nothing to prove.

The equality

$$LE\big(A + H, \Gamma_m(A)\big) = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_{\Gamma_m(A)} \log \big\| \Phi_{A+H}^n(p) \big\| \, d\mu(p)$$

will allow us to prove that $LE(A + H, \Gamma_m(A))$ is small by proving that

$$\frac{1}{t} \int_{\Gamma_m(A)} \log \big\| \Phi_{A+H}^t(p) \big\| \, d\mu(p)$$

is small for a large fixed $t = h\delta^{-1}$.

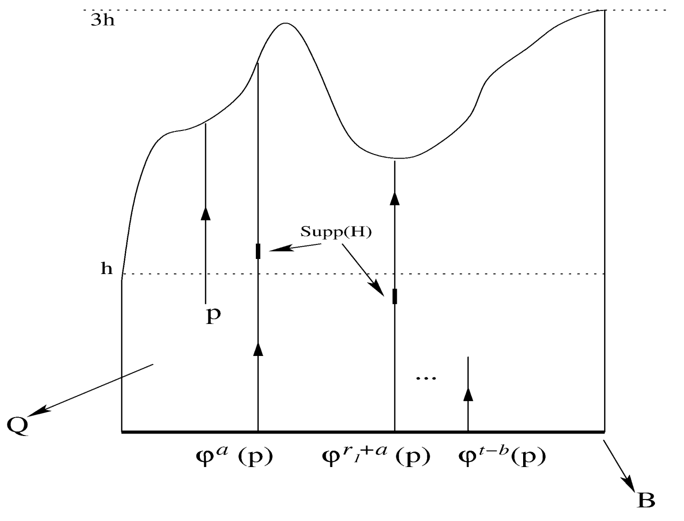

Fig. 4. The global procedure at G.

Note that those points that stay for a long time in Q will necessarily have low contribution for $LE(A+H, \Gamma_m(A))$. So we define

$$G = \left\{ p \in \Gamma_m^*(A)\colon\ \varphi^s(p) \in Q,\ \forall s \in [0, t[\right\}$$

and we claim that

$$\mu\left(\Gamma_m^*(A) - G\right) < 15\delta \tag{9}$$

which is a consequence of [4, Lemma 4.16]. Note that since t is large and the castle Q has height bounded towers and large measure, the orbit leaves Q often, but by (9), it is highly likely to enter Q again. So we split the orbit segment $\varphi^{[0,t]}(p)$ for $p \in G$ by return-times to B, say $t = b + r_n + \cdots + r_2 + r_1 + a$ where all $\varphi^a(p), \varphi^{r_1+a}(p), \varphi^{r_2+r_1+a}(p), \ldots, \varphi^{\sum_{i=1}^n r_i + a}(p)$ are in the base B, and $a, b, r_i \in [0, 3h[$ (see Fig. 4). Given $q \in B$ the height of its tower $h(q)$ verifies $h(q) \geqslant h$, but $B \subseteq Z_h$ so $h(q) \geqslant h \geqslant T(q)$, therefore Lemma 3.7 says that for every $t > T(q)$ there exists a traceless $\{H(\varphi^s(q))\}_{s\in\mathbb{R}}$, varying smoothly with support on the segment $\varphi^{[0,m]}(p)$ such that:

(a) $\|H\| < \epsilon$ and
(b) $\frac{1}{t} \log \|\Phi^t_{A+H}(q)\| < \delta$.

Note that

$$\begin{aligned}
\left\|\Phi^t_{A+H}(p)\right\| &= \left\|\Phi^{b+\sum_{i=1}^n r_i + a}_{A+H}(p)\right\| \\
&\leqslant \left\|\Phi^b_{A+H}\left(\varphi^{\sum_{i=1}^n r_i + a}(p)\right)\right\| \cdot (\ldots) \cdot \left\|\Phi^{r_1}_{A+H}\left(\varphi^a(p)\right)\right\| \cdot \left\|\Phi^a_{A+H}(p)\right\|.
\end{aligned}$$

Take $C = \sup_{p\in M} \|\Phi^1_{A+H}(p)\|$. By (b) and the fact that the towers are smaller than $3h$ we conclude that

$$\left\|\Phi^t_{A+H}(p)\right\| \leqslant C^{3h} \cdot e^{\sum_{i=1}^n r_i\delta} \cdot C^{3h} \leqslant e^{(b+\sum_{i=1}^n r_i\delta+a)\delta} C^{6h} \leqslant e^{t\delta} \cdot C^{6h}$$

and we get $\frac{1}{t}\log\|\Phi^t_{A+H}(p)\| \leqslant \delta(1+6\log C)$. Therefore, since

$$\Gamma_m(A) \supseteq \Gamma_m^+(A) \supseteq \Gamma_m^*(A) \supseteq G,$$

we obtain

$$\begin{aligned} LE\big(A+H,\Gamma_m(A)\big) \leqslant & \int\limits_{\Gamma_m(A)-\Gamma_m^+(A)} \frac{1}{t}\log\left\|\Phi^t_{A+H}(p)\right\| d\mu(p) \\ & + \int\limits_{\Gamma_m^+(A)-\Gamma_m^*(A)} \frac{1}{t}\log\left\|\Phi^t_{A+H}(p)\right\| d\mu(p) \\ & + \int\limits_{\Gamma_m^*(A)-G} \frac{1}{t}\log\left\|\Phi^t_{A+H}(p)\right\| d\mu(p) + \int\limits_{G} \frac{1}{t}\log\left\|\Phi^t_{A+H}(p)\right\| d\mu(p). \end{aligned}$$

Now we use (8), (6), (9) and the fact that $\frac{1}{t}\log\|\Phi^t_{A+H}(p)\| \leqslant \delta(1+6\log C)$ in order to get:

$$LE\big(A+H,\Gamma_m(A)\big) \leqslant \delta + \delta\log C + 15\delta\log C + (1+6\log C)\delta.$$

Substituting δ by $\frac{\delta}{(2+22\log C)}$ along the proof we cause a decay on $LE(\cdot,\Gamma_m(A))$ by a ϵ-small perturbation of the original system. $\square$

In the next lemma we will construct a C^0-perturbation.

Lemma 3.9. *Given a continuous conservative system A and $\epsilon,\delta>0$ there exist $m\in\mathbb{N}$ and a continuous traceless system H_0 such that $B = A + H_0$ verifies*:

(a) $\|A-B\|_\infty < \epsilon$;
(b) $A(p) = B(p)$ *for any* $p \notin \Gamma_m(A)$;
(c) $LE(B,\Gamma_m(A)) < \delta$.

Proof. By Lemma 3.8 we obtain $m\in\mathbb{N}$ and a traceless $H \in L^\infty(X, GL(2,\mathbb{R}))$ such that for $t = h\delta^{-1}$ we have

$$LE\big(A+H,\Gamma_m(A)\big) \leqslant \int\limits_{\Gamma_m(A)} \frac{1}{t}\log\left\|\Phi^t_{A+H}(p)\right\| d\mu(p) < \delta.$$

We now use Lusin's theorem which states that for any measurable function, for instance, H, there is $H_1 \in C^0(X, GL(2,\mathbb{R}))$, such that:

(a) $H_1(p) = H(p)$ for any $p \notin \Gamma_m(A)$;
(b) $\|H_1\|_\infty < \epsilon$;
(c) $\mu(E) = \mu(\{p \in X\colon H_1(p) \neq H(p)\}) < \delta t^{-1}$.

Since for points $p \in E$ we do not necessarily have $\operatorname{Tr} H_1(p) = 0$ we change, say the entry 1-1 of the matrix, obtaining a new matrix H_0 this time with $\operatorname{Tr} H_0(p) = 0$. We define the C^0-perturbation $B = A + H_0$ which verifies $\operatorname{Tr} A = \operatorname{Tr} B$.

Now we define the sets

$$L = \{p \in X\colon H_0(p) = H(p)\} \quad \text{and}$$
$$G_L = \{p \in X\colon \varphi^s(p) \in \Gamma_m(A) \cap L,\ \forall s \in [0,t]\}.$$

Clearly $G_L \subseteq \Gamma_m(A)$ and we have

$$\mu\big(\Gamma_m(A) - G_L\big) \leqslant t\mu(E) \leqslant \delta. \tag{10}$$

Therefore we conclude that

$$\begin{aligned}
LE\big(B, \Gamma_m(A)\big) &= \inf_{n\in\mathbb{N}} \int\limits_{\Gamma_m(A)} \frac{1}{n}\log\big\|\Phi_B^n(p)\big\|\,d\mu(p) \leqslant \int\limits_{\Gamma_m(A)} \frac{1}{t}\log\big\|\Phi_B^t(p)\big\|\,d\mu(p) \\
&= \int\limits_{\Gamma_m(A)-G_L} \frac{1}{t}\log\big\|\Phi_B^t(p)\big\|\,d\mu(p) + \int\limits_{G_L} \frac{1}{t}\log\big\|\Phi_B^t(p)\big\|\,d\mu(p) \\
&= \int\limits_{\Gamma_m(A)-G_L} \frac{1}{t}\log\big\|\Phi_B^t(p)\big\|\,d\mu(p) + \int\limits_{G_L} \frac{1}{t}\log\big\|\Phi_{A+H}^t(p)\big\|\,d\mu(p).
\end{aligned}$$

Let $C = \max_{p\in X} \|\Phi_B^1(p)\|$ and since $G_L \subseteq \Gamma_m(A)$ we get

$$LE\big(B, \Gamma_m(A)\big) \leqslant \mu\big(\Gamma_m(A) - G_L\big)\log C + \int\limits_{\Gamma_m(A)} \frac{1}{t}\log\big\|\Phi_{A+H}^t(p)\big\|\,d\mu(p).$$

Now by (10) and Lemma 3.8 we obtain

$$LE\big(B, \Gamma_m(A)\big) \leqslant \delta\log C + \delta.$$

We then reconstruct the proof replacing δ by $\frac{\delta}{(1+\log C)}$. $\square$

3.5. End of the proof of Theorem 1

Denote by $\Gamma_\infty(A)$ the set $\bigcap_{m\in\mathbb{N}} \Gamma_m(A)$. The following lemma will be useful to prove Theorem 1.

Lemma 3.10. *Given a continuous conservative system A and $\epsilon, \delta > 0$, there exists B, ϵ-close to A such that*

$$LE(B, X) < LE(A, X) - \int\limits_{\Gamma_\infty(A)} \lambda^+(A, p)\, d\mu(p) + \delta.$$

Proof. By Lemma 3.9 there exists $m \in \mathbb{N}$ and a continuous conservative system B such that:

(a) $\|A - B\|_\infty < \epsilon$;
(b) $A(p) = B(p)$ for any $p \notin \Gamma_m(A)$;
(c) $LE(B, \Gamma_m(A)) < \delta$.

So, we have

$$\begin{aligned} LE(B, X) &= LE\big(B, \Gamma_m(A)\big) + LE\big(B, X - \Gamma_m(A)\big) \\ &= LE\big(B, \Gamma_m(A)\big) + LE\big(A, X - \Gamma_m(A)\big) \\ &\leqslant \delta + LE\big(A, X - \Gamma_\infty(A)\big) = \delta + LE(A, X) - LE\big(A, \Gamma_\infty(A)\big). \qquad \square \end{aligned}$$

Theorem 3.11. *Given A in the set of continuous conservative systems, we have that if A is a continuity point of the entropy function $LE(\cdot)$, then for μ-a.e. $p \in X$ the following dichotomy holds*:

(a) *either the Oseledets splitting is dominated, or*
(b) *Lyapunov exponents are zero.*

Proof. Take A a continuous linear differential system (area-preserving or modified area-preserving). Suppose that A is a continuity point for $LE(\cdot)$. Suppose $\mu(X - \bigcup_{m\in\mathbb{N}} \Lambda_m(A)) > 0$, otherwise the statement is proved.

So $\mu(X \cap \bigcap_{m\in\mathbb{N}}(\Gamma_m(A))) = \mu(\bigcap_{m\in\mathbb{N}} \Gamma_m(A)) > 0$, and therefore, $\mu(\Gamma_\infty(A)) > 0$.

Consequently we must have that $LE(A, \Gamma_\infty(A)) = 0$, otherwise by Lemma 3.10 we break the continuity and get a contradiction. So, for any $p \in \mathfrak{O}(A)$ we have zero Lyapunov exponents or if it has positive ones, then $p \notin \Gamma_\infty(A)$ and therefore it has m-dominated splitting for some $m \in \mathbb{N}$. $\square$

The conclusions of Theorem 3.11 are sufficient to guarantee that A is a continuity point of the entropy function. By hypothesis, $X = \mathcal{D} \cup \mathcal{O}$ (mod 0), where $\mathcal{D}$ are points with dominated splitting and $\mathcal{O}$ are points with null exponents. Since $LE(A, \mathcal{O}) = 0$ and LE is upper-semicontinuous we conclude that $LE(B, \mathcal{O})$ is close to $LE(A, \mathcal{O})$, for B close to A. Moreover, $\mathcal{D}$ has also dominated splitting for B with rates of dominated splitting close to the ones belonging to A.

Proof of Theorem 1. Theorem 1 now follows by using the fact that the set of points of continuity of upper-semicontinuous functions is a residual set, see [15]. $\square$

4. Some consequences of Theorem 1

4.1. Ergodic flows

Corollary 4.1. *If μ is ergodic, then there is a residual subset $\mathcal{R}$ of area-preserving systems such that for every $A \in \mathcal{R}$ we have Φ_A^t uniformly hyperbolic or Lyapunov exponents are zero for μ-a.e. point $p \in X$.*

Proof. Take the residual $\mathcal{R}$ given by Theorem 1 and $A \in \mathcal{R}$. If $\mu(\Lambda_m) = 0$ for all m, then the corollary follows, otherwise if $\mu(\Lambda_m) > 0$ for some m, we have a full measure set Λ_m with m-dominated splitting, because Λ_m is φ^t-invariant and μ is ergodic. □

Conservativeness yields $\det \Phi^t(p) = 1$ for all $t \in \mathbb{R}$ and $p \in X$. Given a μ-generic point $p \in M$ we have

$$\sin\big(\measuredangle\big(N_p^u, N_p^s\big)\big) = \big\|\Phi^t(p)\big|_{N_p^u}\big\| \cdot \big\|\Phi^t(p)\big|_{N_p^s}\big\| \sin\big(\measuredangle\big(N_{\varphi^t(p)}^u, N_{\varphi^t(p)}^s\big)\big). \tag{11}$$

Claim 2. *If Λ_m has m-dominated splitting, then for all $p \in \Lambda_m$ we have $\measuredangle(N_p^u, N_p^s) \geqslant \alpha > 0$.*

Since $\Phi^m(p)$ is a linear isomorphism its co-norm

$$m\big(\Phi^m(p)\big) = \inf_{\|v\|=1} \big\|\Phi^m(p) \cdot v\big\|$$

is given by $\|[\Phi^m(p)]^{-1}\|^{-1}$. Let $u \in N_p^u$ and $s \in N_p^s$ be unitary vectors. Since $\sin(\frac{\measuredangle(N_p^u, N_p^s)}{2}) = \frac{\|u-s\|}{2}$, we prove that $\|u - s\|$ is bounded away from zero. By dominated splitting we have $2\|\Phi^m(p) \cdot s\| \leqslant \|\Phi^m(p) \cdot u\|$, so

$$2\big\|\Phi^m(p) \cdot s\big\| \leqslant \big\|\Phi^m(p) \cdot (u - s + s)\big\| \leqslant \big\|\Phi^m(p) \cdot (u - s)\big\| + \big\|\Phi^m(p) \cdot s\big\|,$$

therefore

$$\big\|\Phi^m(p) \cdot s\big\| \leqslant \big\|\Phi^m(p)\big\| \big\|(u - s)\big\|.$$

Since $\|[\Phi^m(p)]^{-1}\|^{-1} \leqslant \|\Phi^m(p) \cdot s\|$ we obtain

$$\big\|\big[\Phi^m(p)\big]^{-1}\big\|^{-1} \leqslant \big\|\Phi^m(p)\big\| \big\|(u - s)\big\|$$

and therefore

$$\big\|\big[\Phi^m(p)\big]^{-1}\big\|^{-1} \big\|\Phi^m(p)\big\|^{-1} \leqslant \|u - s\|.$$

Now we just note that $\Phi^m(p)$ is continuous and X is compact and the claim follows.

Claim 2 and (11) implies that

$$\big\|\Phi^t(p)\big|_{N_p^u}\big\| . \big\|\Phi^t(p)\big|_{N_p^s}\big\| \geqslant \sin^{-1} \alpha.$$

Again by dominated splitting, there exist constants $C > 0$ and $\sigma \in (0, 1)$ such that

$$C\sigma^t \geqslant \frac{\|\Phi^t(p)|_{N_p^s}\|}{\|\Phi^t(p)|_{N_p^u}\|} \geqslant \sin^{-1}\alpha \left\|\Phi^{-t}(p)\big|_{N_p^u}\right\|^2,$$

and consequently

$$\left\|\Phi^{-t}(p)\big|_{N_p^u}\right\| \leqslant \sqrt{C\sin^{-1}\alpha\big(\sigma^{t/2}\big)}.$$

Now it suffices to take the constants of uniform hyperbolicity $C' = \sqrt{C\sin^{-1}\alpha}$ and $\sigma' = \sqrt{\sigma}$ and to proceed analogously for N^s in order to prove the corollary.

Corollary 4.2. *If μ is ergodic and $\mathrm{Fix}(\varphi^t) = \emptyset$, then there is a residual subset $\mathcal{R}$ of modified area-preserving systems such that for every $A \in \mathcal{R}$ either Φ_A^t is uniformly hyperbolic or $LE(A) = 0$.*

Proof. Since $a(\cdot)$ is a non-null continuous function on a compact set X, the quotient $\frac{a(\cdot)}{a(\varphi^t(\cdot))}$ has an upper bound K and a lower bound K^{-1}. Conservativeness, dominated splitting and the nonexistence of fixed points for the flow φ^t guarantees that

$$C\sigma^t \geqslant \frac{\|\Phi^t(p)|_{N_p^s}\|}{\|\Phi^t(p)|_{N_p^u}\|} \geqslant \sin^{-1}\alpha K^2 \left\|\Phi^{-t}(p)\big|_{N_p^u}\right\|^2,$$

and we proceed as in Corollary 4.1. □

4.2. Further statements in the conservative setting

Given a matrix M, denote by M^{T} the matrix transpose. Let

$$O(2,\mathbb{R}) = \left\{M \in GL(2,\mathbb{R}):\ M^{\mathrm{T}}M = Id\right\}$$

be the orthogonal group. The special orthogonal group is defined by

$$SO(2,\mathbb{R}) = O(2,\mathbb{R}) \cap SL(2,\mathbb{R}).$$

Take $S \in SO(2,\mathbb{R})$. We consider the systems A such that their fundamental matrix verifies:

$$\left(\Phi_A^t\right)^{\mathrm{T}} S\Phi_A^t = S. \tag{12}$$

In this case for the system A the following equality

$$A(t)^{\mathrm{T}}S + SA(t) = 0 \tag{13}$$

holds. Let us denote by $\mathfrak{S}$ the set of systems where the matrix transition Φ_A^t evolves on elements of a special orthogonal group, i.e. (12) is verified. An example of this kind of systems is when $S = J$, where J is the standard 2×2 symplectic matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These are the symplectic systems, that in our 2-dimensional setting, are reduced to the traceless systems already considered.

The following lemma says that Theorem 1 is also true for $A \in \mathfrak{S}$.

Lemma 4.3. *If $A \in \mathfrak{S}$, then $A + H \in \mathfrak{S}$, where H is the perturbation defined by Lemmas* 3.2*,* 3.3 *and* 3.5.

Proof. Given $A \in \mathfrak{S}$, by (13) we have $A(t)^{\mathrm{T}}S + SA(t) = 0$ for $S \in SO(2,\mathbb{R})$. We want to prove that $[A(t)+H(t)]^{\mathrm{T}}S + S[A(t)+H(t)] = 0$ and it is sufficient to show that $H(t)^{\mathrm{T}}S + SH(t) = 0$.

Since $H(t) = \Phi^t(p)\xi g'(t)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)[\Phi^t(p)]^{-1}$ we have

$$\begin{aligned} H(t)^{\mathrm{T}}S + SH(t) &= \left[\Phi^t(p)\xi g'(t)J\left[\Phi^t(p)\right]^{-1}\right]^{\mathrm{T}}S + S\Phi^t(p)\xi g'(t)J\left[\Phi^t(p)\right]^{-1} \\ &= \left(\left[\Phi^t(p)\right]^{-1}\right)^{\mathrm{T}}\xi g'(t)J\left(\Phi^t(p)\right)^{\mathrm{T}}S + S\Phi^t(p)\xi g'(t)J\left[\Phi^t(p)\right]^{-1}. \end{aligned}$$

Using (12) in the last expression we obtain

$$\begin{aligned} H(t)^{\mathrm{T}}S + SH(t) &= \left(\left[\Phi^t(p)\right]^{-1}\right)^{\mathrm{T}}\xi g'(t)JS\left[\Phi^t(p)\right]^{-1} + \left[\left(\Phi^t(p)\right)^{\mathrm{T}}\right]^{-1}S\xi g'(t)J\left[\Phi^t(p)\right]^{-1} \\ &= \xi g'(t)\left(\left[\Phi^t(p)\right]^{-1}\right)^{\mathrm{T}}[JS + SJ]\left[\Phi^t(p)\right]^{-1}. \end{aligned}$$

Now $JS + SJ = 0$ because $S \in SO(2,\mathbb{R})$ and the lemma is proved. □

5. Generalizations to other systems

5.1. Beyond the conservative setting

In the proof of Theorem 1 we considered a system A and we built up a small C^0-perturbation B with small norm. One of the steps to decrease the norm was the mixing of two directions in $\mathbb{R}P^1$: the Oseledets 1-dimensional subspaces N^u and N^s. For this purpose it was crucial that:

(I) The transition matrix Φ_B^t had a transitively action over $\mathbb{R}P^1$.
(II) The perturbation B remained inside our original set of systems.

Once we guarantee these two properties we are able to obtain Lemma 3.6, therefore Theorem 1 follows by direct application of the local and the global recurrence arguments.

So Theorem 1 is also true for systems where (I) and (II) are valid. For this reason we use a definition (cf. [4, Definition 1.2]) of a more general class of systems on which Theorem 1 must be true.

Definition 5.1. We say that a set of systems S have the *accessible property* if for all $C > 0$ and $\epsilon > 0$, there exist $\tau \in \mathbb{R}$ and $\xi > 0$ verifying the following:

Given $u, v \in \mathbb{R}P^1$ with $\measuredangle(u,v) < \xi$ and $A \in S$ with $\|A\| < C$, there exists $B \in S$ such that:

(i) $\|A - B\| < \epsilon$;
(ii) $\Phi_B^\tau \cdot u = \Phi_A^\tau \cdot v$.

The techniques and constructions in this paper explicitly prove that the area-preserving systems, the modified area-preserving systems and the systems considered in Section 4.2 actually satisfy the accessibility property. Furthermore, the systems with fundamental matrix on $GL(2,\mathbb{R})$ are also accessible. Since Definition 5.1 is somewhat abstract similar techniques must be developed for other systems if one aims to prove identical results.

5.2. Multidimensional case

For linear differential systems of nonautonomous differential equations with dimension greater or equal than three the proof relies on the study of the exterior product of order $n = \dim N^u$, where N^u is the subspace associated to the positive Lyapunov exponents ($\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A)$). By a more elaborate and careful technique we are able to mix directions with different exponential behavior by a ϵ-C^0-perturbation. This procedure guarantees that $\|\bigwedge^n \Phi^t_{A+H}(p)\|$ ($\bigwedge^n$ denotes the nth wedge product) decays abruptly when compared with $\|\bigwedge^n \Phi^t_A(p)\|$, and once again we produce a discontinuity in the *entropy function* and as in the proof of Theorem 1 we obtain the two possible cases for μ-a.e. $p \in X$, dominated splitting or zero Lyapunov exponents. The proof follows the strategy in the discrete case, see [4], and will appear in a forthcoming paper.

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