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Flat vacua of maximal supergravity

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INTRODUCTION

String theory is currently the most promising candidate for describing gravity at high energies, where quantum effects become relevant, thus opening the way towards a unified description of all fundamental interactions. It appeared for the first time in the late 1960's as an attempt to describe the observed spectrum of hadrons and their interactions (before being replaced by the more successful quantum chromodynamics) and its innovation lies in the fact that the fundamental degrees of freedom are not point-like particles but extended objects: one-dimensional strings and, in later developments, higher dimensional branes. The spectrum of this theory comes up with a massless spin-2 particle, which was not compatible with the intention of using string theory as a theory of strong interactions. Instead, in 1974, Scherk and Schwarz suggested [1] that this particle could be interpreted as a graviton, thus offering a quantum description of the gravitational interaction; they also proved how to recover Einstein's theory of general relativity in a proper low energy limit.

In order to be a consistent theory of quantum gravity, thus solving the problem of non-renormalizability, string theory requires two ingredients: the introduction of extra dimensions and supersymmetry between bosonic and fermionic degrees of freedom. Then, a central question is how to recover from string theory the 4-dimensional spacetime we live in.

The issue can be addressed by adopting an effective field theory approach: realistic field theories in 4 dimensions should be interpreted as a low-energy limit of the more fundamental string theory. The 4 spacetime dimensions can be obtained as the result of a compactification procedure, tracing back to the work by Kaluza and Klein, which is based on the assumption that extra dimensions parametrize a compact space small enough to remain hidden to any present experiment.

A crucial point that makes string theory a good candidate for unification of all interactions and matter is that it is so constrained to have basically no free parameters. On the other hand, when we try to get back physics in 4 dimensions through compactification, a lot of freedom arises in the choice of the internal space and its metric. In other words, the compactification procedure completely breaks the uniqueness of string theory: each possible low energy effective theory is constructed about a different vacuum of string theory and the number of such vacua is so huge to result in an apparent loss of any predictivity.

At low energies and after compactification of the extra dimensions, string theory is described by gauged supergravity theories. Supergravity, in a lower-dimensional perspective, is the result of turning global supersymmetry into a local gauge symmetry and it allows to combine the features of supersymmetry with a description of the gravitational interaction. Once we interpret supergravity theories as a low-energy version of superstring theory, their particle spectrum, their masses, couplings and symmetries are essentially determined by the geometry of the space where compactification is performed, together with the possible fluxes one can turn on.

The problem of vacuum selection in string theory is then intimately related with the study of the possible vacua of supergravity theories. The subject can be also approached from a modern, bottom-up perspective, in the framework of the *swampland program*. The swampland can be defined as the set of apparently consistent quantum field theories in 4 dimensions that do not admit a completion into a quantum gravity theory in the ultraviolet regime.

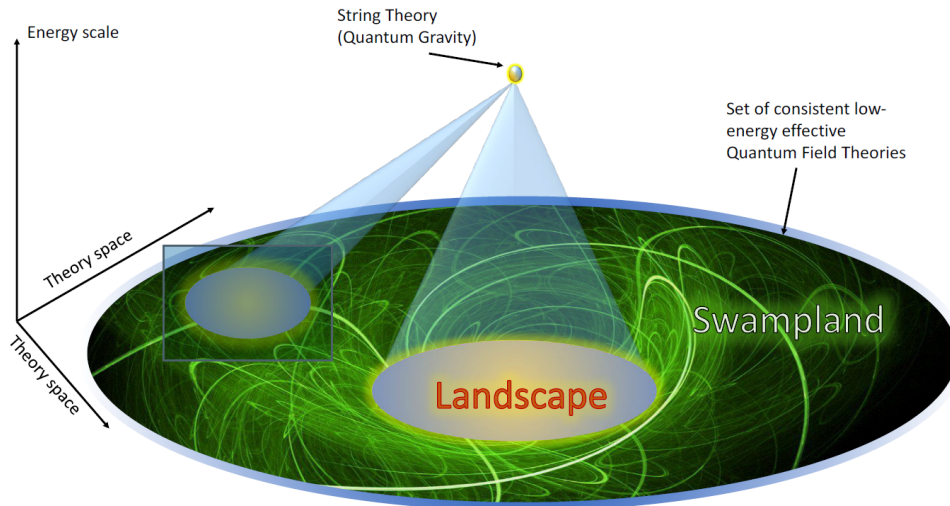


Figure 1: Schematic illustration of the string landscape and the swampland within the space of quantum field theories in 4 dimensions (from [2]).

The definition was introduced in [3], in order to stress and possibly quantify how, despite the vastness of the string landscape, it can be viewed as relatively small if compared to the swampland. The original idea (also found in [4]) was to present some finiteness properties (regarding, for instance, the volume of the scalar field space, or the number of matter fields) that could allow to identify the boundaries of the landscape.

In general, the swampland program aims at specifying some criteria, formulated exclusively in terms of properties of the low-energy effective theories, to select field theories which admit a quantum gravity UV completion, isolating them from the swampland of all other consistent-looking theories. In the definition of swampland, it is not by chance that we used a generic notion of quantum gravity rather than string

theory. Indeed, in principle, swampland criteria should regard universal properties of the lower-dimensional theories that are valid no matter whether the correct quantum gravity theory is actually a string theory or not. However, being string theory the only UV complete theory of gravity known to us, almost all of the swampland program is unavoidably developed in the framework of string theory.

The criteria are usually stated in the form of conjectures because they cannot be rigorously proved in a mathematical sense, but rather they are motivated by some physical evidence. Therefore, the various conjectures are not all at the same footing, in the sense that the evidence supporting them can have varying levels of rigour. The most common approach [2] to the development of swampland conjectures is to use known string vacua and the corresponding low-energy theories as a type of experimental data. If a given conjecture is satisfied by the known vacua, we can say that this conjecture is supported by string theory data. A review of the various conjectures can be found in [2], [5].

In this framework, the study of 4-dimensional theories and their vacuum structure can lead to interesting results if we select a well-defined set of theories and try to obtain, at least for these theories, complete results. This thesis focuses on the maximal supergravity theory in 4 dimensions, with $\mathcal{N} = 8$ supersymmetry generators, whose peculiarities are that the matter content is completely constrained by supersymmetry and its higher dimensional origin is well known since its discovery. In particular, we will look for its Minkowski vacua in the case of $\mathcal{N} = 6, 5, 4$ residual supersymmetries, analysing in which cases vacua can actually be found.

The work is organized as follows.

- In chapter 2 we describe the general features of gauged supergravity theories: starting from a discussion of the global symmetries of a generic supergravity theory, it is shown that any gaugeable symmetry must be a subgroup of the U-duality group, emerging as a generalization of electric-magnetic duality, and how the gauging procedure leads to the introduction of a non-trivial scalar potential. The embedding tensor formalism, which will be a fundamental tool in chapters 4 and 5, is also introduced. In the second part of the chapter, we deal with flux compactifications and their direct correspondence with the presence of a gauging in the low-energy effective theory. Non-trivial fluxes over the internal manifold for some higher-order form field strength and compactifications on twisted tori (geometric fluxes) are considered, deriving at any step the resulting gauge algebra.
- The discussion of flux compactifications continues in chapter 3, where the focus is shifted to non-geometric fluxes. Their introduction is needed in order to match the most general gauge algebra one expects to find from a lower-dimensional analysis with the algebra that can be obtained from the compactification procedure.

Particular attention is devoted to the discussion of duality twists, the role of T-duality transformations and the equivalence between duality twists and orbifolds. Eventually, we present the possibility of introducing an asymmetry between left-moving and right-moving string coordinates, which is not unnatural in the context of string theory but completely spoils a geometric interpretation of the internal compact space.

- The maximal supergravity theory in 4 dimensions, with $\mathcal{N} = 8$ supersymmetry generators, is the topic of chapter 4. The scalar sector of the theory can be characterized as a coset manifold, $E_{7(7)}/SU(8)$, where the isometry group $E_{7(7)}$ is also the U-duality group of the theory. The gauge group of the theory and its embedding into $E_{7(7)}$ can be encoded in the embedding tensor, so that the lagrangian, and in particular the scalar potential, can be written in a formally $E_{7(7)}$ -covariant way without explicitly fixing the gauge group.
- In chapter 5, the analysis of Minkowski vacua for the maximal supergravity theory is performed. The technique adopted, combining the embedding tensor formalism with the transitive action of the duality group on the scalar manifold, allows to recover information on the whole scalar manifold just by computing the scalar potential and its derivatives at the origin of the scalar manifold, so that the computational procedure is considerably simplified. The results obtained are summarized in 6.

1.1 VACUA OF SUPERGRAVITY THEORIES

The notion of supergravity vacua will be crucial throughout the thesis. Having a supergravity theory, we call *vacua* the field configurations of this theory in which the metric is maximally symmetric. In 4 spacetime dimensions, this means that the metric is Minkowski, de Sitter or Anti-de Sitter; to be compatible with Lorentz invariance, all other fields with spin different from zero must vanish, while scalar fields must take constant values. Then, in presence of a non-trivial scalar potential, vacuum configurations correspond to critical points of the potential. The value of the potential at the critical point gives the vacuum energy density and acts as a cosmological constant, thus establishing if the vacuum solution corresponds to a Minkowski, de Sitter or Anti-de Sitter spacetime.

Some remarks are worthwhile [6]:

- the most general maximally symmetric field configurations could involve, apart from constant scalar fields, also non-trivial Lorentz invariant fermionic condensates, such as gaugino condensates $\langle \bar{\lambda}\lambda \rangle \neq 0$. However, at energies far below the condensation scale, they would be described in an effective field theory as additional scalar fields;

- what we call supergravity vacua are a subcase of the vacuum solutions to Einstein equations, i.e. those solutions where the only contribution to the energy momentum tensor comes from a vacuum energy density. The two concepts do not coincide because vacuum solutions in general relativity allow for a spacetime metric which is not necessarily maximally symmetric.

In this thesis we have chosen to focus on Minkowski vacua, where the scalar potential takes a vanishing expectation value, and their higher-dimensional origin from string compactification. However, there are plenty of reasons to investigate also de Sitter and Anti-de Sitter vacuum configurations. The existence of de Sitter vacua, in particular, can be very interesting also from a cosmological perspective, because, due to the positive effective cosmological constant at the vacua, they could provide theoretical models for inflation. On the other hand, Anti-de Sitter vacua can describe, via the holographic correspondence, conformal fixed points of some dual (three-dimensional) field theory, then they could take a role in the analysis of the renormalization group flow in the dual framework.

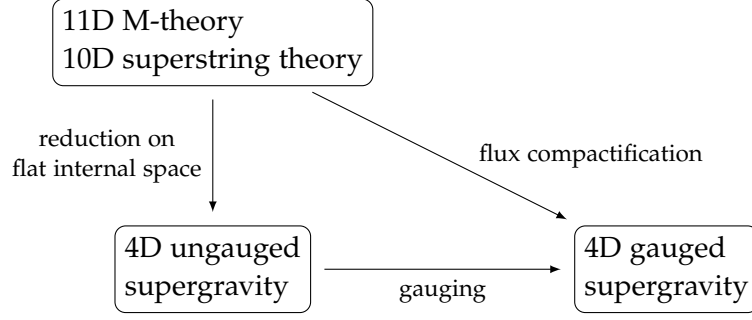
FLUX COMPACTIFICATIONS AND GAUGED SUPERGRAVITY

Gauged supergravity arises starting from a theory containing n_V abelian vector fields, under which none of the matter fields are charged, namely an *ungauged supergravity*. A gauged supergravity theory can be considered as a deformation of the corresponding ungauged theory, obtained if some global symmetry group is promoted to a local symmetry, coupling it to the formerly abelian vector fields of the theory and introducing minimal couplings with the matter fields. In the context of gauged supergravity, however, the most general gaugeable group is not a subgroup of the global symmetries of the lagrangian, but rather of a global symmetry of the equations of motion, the so-called *U-duality group* G_U .

Once the duality group of the ungauged theory is fixed, all possible gaugings can be described through the *embedding tensor*, which parametrizes the way the gauge group is embedded into G_U . This formalism was originally developed for three-dimensional supergravity theories in [7], [8] and then extended to higher dimensions. Writing the action in terms of the embedding tensor allows to keep "implicit" the choice of the gauging and then to formally restore the global symmetries of the ungauged model, which are in general broken by the gauging procedure, in particular symplectic covariance.

On the other hand, supergravity theories in four dimensions can be seen as low-energy effective theories deriving from compactifications of ten-dimensional superstring theories or eleven-dimensional M-theory. In this framework, the geometry of the compact internal manifold and the way compactification is performed affect the field content, but also the symmetries of the lower-dimensional theory. Compactification on a Ricci-flat internal manifold in absence of fluxes gives an ungauged supergravity theory, but if the compactification is made in presence of fluxes (either fluxes of higher-order form field strengths, geometric or non-geometric fluxes) then gauged supergravities come up.

In this chapter, the main features of gauged supergravity theories will be presented, starting from the description of gaugeable global symmetry groups in the ungauged theory, and then the concept of flux compactification will be introduced, with particular attention to the way of deriving the gauge algebra of the reduced theory.



2.1 GLOBAL SYMMETRIES AND GAUGING PROCEDURE

In theories of $\mathcal{N} = 1$ supergravity, the scalar potential is the result of two different contributions: F-terms, coming from an holomorphic function, the so-called superpotential, and D-terms, which are related to the gauging ([9], [10]). For $\mathcal{N} \geq 2$, F-terms are no longer allowed and then the scalar potential is completely determined by the gauging. This means that for ungauged theories of extended supergravity no scalar potential can arise; as a consequence, all scalar fields of the theory should remain massless. Having a non-trivial potential is also crucial for describing scenarios of spontaneous supersymmetry breaking, since vacua of supersymmetric theories coincide with its critical points: in absence of the scalar potential, a fully supersymmetric Minkowski spacetime is the only possible maximally symmetric vacuum configuration, while vacua of gauged supergravity can show different supersymmetry breaking patterns.

Since the gauging procedure starts by promoting some of the global symmetries of the theory to local symmetries, it is important to highlight which kind of global symmetries our theories could show.

2.1.1 Electric-magnetic duality

For any field theory containing vector fields, there exist a natural generalization, first described by Gaillard and Zumino [11], of the electric-magnetic duality showed by Maxwell's equations, which can be expressed as the symmetry of Bianchi identities and equations of motion for the gauge potential A under the exchange of the field strength $F = dA$ and its Hodge dual \tilde{F} .

Having a theory with n_V abelian vector fields A^I , coupled non-minimally with other fields ζ^i , the lagrangian

$$e^{-1}\mathcal{L} = \frac{1}{4}\mathcal{I}_{IJ}F^I{}_{\mu\nu}F^{J\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I{}_{\mu\nu}\tilde{F}^{J\mu\nu} + \frac{1}{2}\mathcal{O}_I{}^{\mu\nu}F^I{}_{\mu\nu} + e^{-1}\mathcal{L}_{\text{rest}} \quad (2.1)$$

(where \mathcal{I}_{IJ} , \mathcal{R}_{IJ} are scalar field-dependent symmetric matrices and $\mathcal{O}_I{}^{\mu\nu}$ is a tensor with generic dependence on the fields ζ^i and their first order derivatives, while $\mathcal{L}_{\text{rest}}$

includes all terms which are not function of the A^I) is invariant under local abelian $U(1)^{n_V}$ transformations, under which the vector fields transform as $A^I_\mu \rightarrow A^I_\mu + \partial_\mu \Lambda^I$. The Bianchi identities for the vector fields $dF^I = 0$ are still valid, but equations of motion are modified by non minimal couplings. However, if we define

$$\tilde{G}_{J\mu\nu} = 2 \frac{\partial \mathcal{L}(F^I, \zeta^i, \partial \zeta^i)}{\partial F^{J\mu\nu}} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_J^{\rho\sigma}, \quad (2.2)$$

the new equations of motion for vector fields can be written again as Bianchi identities for the dual field strengths G_I

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial F^{I\mu\nu}} = 0 \quad \Leftrightarrow \quad dG_I = 0, \quad (2.3)$$

and (2.3) implies that one can define, at least locally, n_V dual 1-forms A_I such that $dA_I = G_I$. The system of equations $\{dF^I = 0, dG_I = 0\}$ is in principle invariant under general linear transformations of $GL(2n_V, \mathbb{R})$ mixing the field strengths F^I and the G_I , but the symmetry has to be constrained if we want to maintain the definition of the G_I in terms of the F^I . In particular, we can consider infinitesimal transformations of the form $S = \mathbb{I} + \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are $n_V \times n_V$ real matrices, whose action on the field strengths is

$$\begin{aligned} \delta F^I &= A^I_J F^J + B^{IJ} G_J, \\ \delta G_I &= C_{IJ} F^J + D_I^J G_J, \end{aligned} \quad (2.4)$$

and analogously on the dual field strengths $\{\tilde{F}^I, \tilde{G}_I\}$. Varying both the LHS and the RHS of eq.(2.2) and using the explicit expression (2.1) of the lagrangian to write the G_I, \tilde{G}_I in terms of the F^I, \tilde{F}^I , one can obtain the variations for $\mathcal{I}_{IJ}, \mathcal{R}_{IJ}$ and then consistency requires that $C = C^T, B = B^T, D = -A^T$, i. e. that the symmetry group is $Sp(2n_V, \mathbb{R})$. Still, symplectic invariance of the system of Bianchi identities and equations of motion for the vector fields does not imply automatically invariance of the remaining equations of motion, nor invariance of the lagrangian, which transforms as

$$\delta \mathcal{L} = \frac{1}{4} C_{IJ} F^I_{\mu\nu} \tilde{F}^{J\mu\nu} + \frac{1}{4} B^{IJ} G_{I\mu\nu} \tilde{G}_J^{\mu\nu} + \delta \mathcal{L}_{\text{rest}}. \quad (2.5)$$

Neglecting for the moment the transformation rules of $\mathcal{L}_{\text{rest}}$, duality rotations which do not affect the lagrangian are those with $B = C = 0$ (but the term $F^I C_{IJ} \tilde{F}^I$ just corresponds to a total derivative).

2.1.2 Global symmetries and symplectic frames

Since we are considering theories containing not only the vector fields, but also different types of bosons and fermions, we should ask now how the electric-magnetic duality

group acts on the other fields and which group of transformations G_U leaves all their equations of motion invariant. It turns out that this invariance can be obtained if we impose that the duality transformations do not change the lagrangian sector which we called $\mathcal{L}_{\text{rest}}$ [6].

For the scalar sector of the lagrangian, the global symmetry group coincides with the isometry group G of the scalar manifold. If the action of this group on the scalar fields comes together with a linear action on the vector field strengths and their duals, which in particular should define a $2n_v$ -dimensional symplectic representation \mathcal{R}_V of G , then the group is a global symmetry group of the field equations. This is always the case in theories of extended supergravity, because supersymmetry connects vector fields with (at least some of) the scalar fields¹, so that transformations of the ones and of the others necessarily come together and then we have $G_U = G$. In the most general case in which the theory includes fields not having a direct coupling to vector fields, if G_{inert} is the global symmetry of such fields the largest global symmetry group of the equations of motion is $G_{\text{global}} = G_U \times G_{\text{inert}}$.

As already mentioned, G_{global} is not a symmetry of the lagrangian, which could be modified by generic transformations of $G_U = G \subset \text{Sp}(2n_V, \mathbb{R})$. Different lagrangians correspond to different *symplectic frames*, i.e. to different choices of which elements, in the basis of the $2n_V$ -dimensional representation $\mathcal{R}_V(G)$, are taken to be the vector fields appearing in the lagrangian, namely the electric vector fields A^I , and which are their "magnetic" duals A_I . Elements of the double quotient space

$$\text{GL}(n_V, \mathbb{R}) \setminus \text{Sp}(2n_V, \mathbb{R}) / G \quad (2.6)$$

are in 1-to-1 correspondence with lagrangians that cannot be obtained one from the other by local redefinitions of the scalar and vector fields ([12], [6]) (in particular, $\text{GL}(n_V, \mathbb{R})$ transformations give a local redefinition of the electric vector fields in the lagrangian, while G acts on both scalars, as a group of isometries on $\mathcal{M}_{\text{scalar}}$, and vectors, with its representation $\mathcal{R}_V(G)$). Each of them will have a different global symmetry group, defined by the subgroup $G_{\text{electric}} \subset G$ sending electric field strengths into themselves. In the context of ungauged supergravities, these inequivalent lagrangians all share the same equations of motion and then give the same physics, but the choice of the symplectic frame becomes relevant when the gauging is introduced and vectors become minimally coupled to other fields.

2.1.3 General gauging procedure

From now on, we will always assume that the most general group of global symmetry of the equations of motion is simply $G_{\text{global}} = G = \text{Iso}(\mathcal{M}_{\text{scalar}})$. This is always guaranteed

¹ Scalar and vector fields always sit in the same multiplets in the case of $\mathcal{N} \geq 3$ theories, while for $\mathcal{N} = 2$ scalar fields are in both vector multiplets and hypermultiplets, then isometries of the manifold described by the latter correspond to trivial transformations of the field strengths [12].

for supersymmetric theories with $\mathcal{N} \geq 4$, since vectors and scalars necessarily are in the same multiplets. The action of isometries (seen as global symmetries) on the scalar fields is

$$\delta\varphi^i = \alpha^I \bar{\zeta}_I^i, \quad (2.7)$$

where $\bar{\zeta}_I$ are the Killing vectors of the scalar manifold, verifying $\nabla_{(i}\bar{\zeta}_{Ij)} = 0$. Gauging such transformations means, as usual, that we require symmetry under transformations with spacetime-dependent parameters $\alpha^I(x)$. In supersymmetric theories, since the field content is arranged in multiplets of fixed length, the addition of new vector fields is not permitted, then the gauge fields of the theory are taken from the n_V independent vector fields of the ungauged theory. As a consequence, we can derive a constraint on the dimension of the gauge group, which is

$$\dim(G_{\text{gauge}}) \leq n_V. \quad (2.8)$$

Moreover, since the field strengths should transform in the adjoint representation of the gauge group, but we already know that they transform in a $2n_V$ -dimensional representation of G , which is usually the fundamental one, we get

$$\text{adj}(G_{\text{gauge}}) \subset \text{fund}(G).$$

Once some of the global isometries are turned into local transformations, the introduction of covariant derivatives for the scalar fields

$$\partial_\mu \varphi^i \rightarrow \widehat{\partial}_\mu \varphi^i = \partial_\mu \varphi^i + g A_\mu^I \bar{\zeta}_I^i \quad (2.9)$$

is needed in order to restore gauge symmetry of the scalar kinetic term. The formerly abelian gauge transformations of vector fields are modified to include terms depending on the structure constants of the gauge group

$$\delta A_\mu^I = \partial_\mu \alpha^I(x) + g f_{JK}^I A_\mu^J \alpha^K \quad (2.10)$$

and also the field strengths are replaced by non-abelian ones

$$\mathcal{F}_{\mu\nu}^I = 2\partial_{[\mu} A_{\nu]}^I + g f_{JK}^I A_\mu^J A_\nu^K. \quad (2.11)$$

Here we do not reabsorb the coupling constant g into the vector fields in order to make clear the order in g of each modification to the lagrangian.

The introduction of covariant derivatives and non-abelian gauge transformations, while ensuring gauge invariance of the theory, on the other hand breaks supersymmetry. In order to restore it, mass-like terms for fermionic fields have to be introduced in the lagrangian (see [13]):

$$e^{-1} \mathcal{L}_{\text{mass}} = g \left(A_{ij} \bar{\psi}_\mu^i \gamma^{\mu\nu} \psi_\nu^j + B_{Ai} \bar{\chi}^A \gamma^\mu \psi_\mu^i + C_{AB} \bar{\chi}^A \chi^B \right) + h.c., \quad (2.12)$$

where ψ_μ^i and χ^A generically denote gravitini and spin-1/2 fermions contained in the theory (indices i and A label some representations of the R-symmetry group), while tensors A_{ij} , B_{Ai} , C_{AB} depend on the scalar fields. These mass terms cancel all supersymmetry violating terms linear in g if they come together with modifications of supersymmetry transformation rules for fermions

$$\delta\psi_\mu^i = \delta_0\psi_\mu^i - gA^{ij}\gamma_\mu\epsilon_j \quad , \quad \delta\chi^A = \delta_0\chi^A - gB^{Ai}\epsilon_i \quad , \quad (2.13)$$

where δ_0 denotes the supersymmetry variations before gauging and ϵ_i are the infinitesimal parameters of supersymmetry transformations. Finally, in order to restore supersymmetry also at higher order in g , the addition of a term of order g^2 is required:

$$e^{-1}\mathcal{L}_{\text{pot}} = g^2(B^{Ai}B_{Ai} - A^{ij}A_{ij}) \quad , \quad (2.14)$$

which gives the gauging-dependent potential for the scalar fields.

2.2 THE EMBEDDING TENSOR FORMALISM

As anticipated, a very efficient way to describe gauged supergravities and also to perform a systematic analysis of all possible gaugings of a given supergravity theory is through the embedding tensor formalism. Once the embedding tensor is defined, it completely parametrizes not only the gauge group, but also the way it is embedded into G and eventually into $\text{Sp}(2n_V, \mathbb{R})$.

We will discuss this formalism under the assumption that the scalar manifold is a homogeneous space, i.e. that the isometry group G has a transitive action on the manifold (the group action allows to go from any point to any other point over the manifold). This means that the manifold can be seen as a *coset manifold* G/H , where H is the isotropy subgroup of G leaving a chosen point invariant. Each point ϕ of such a manifold can be described through its coset representative, which is an element of the group G denoted by $L(\phi)$. The assumption, though simplifying the formal treatment of gauging procedure, is not restrictive in the context of extended supersymmetry. Indeed, in all theories with $\mathcal{N} \geq 3$ and in part of the theories with $\mathcal{N} = 2$ the scalar manifold is constrained by supersymmetry to have the form of a coset manifold (see [6], [12] for the explicit coset structure of all $\mathcal{N} \geq 3$ scalar manifolds).

If $t_\alpha \in \mathfrak{g}$ are the generators of the isometry group G , we can specify the generators of G_{gauge} as linear combinations of the t_α

$$T_A = m_A^\alpha t_\alpha \quad \text{with } A = 1, \dots, \dim(G_{\text{gauge}}) \quad , \quad (2.15)$$

where m_A^α is a constant matrix, completely generic but for the fact that the resulting generators T_A must have closed commutators in order to correctly define an algebra². In a similar way, the vector fields chosen to define the gauge connection are

$$A_\mu^A = A_\mu^M n_M^A, \quad (2.16)$$

where $A^M = \{A^I, A_I\}$. These informations can be encoded in a single object if we renounce keeping indices A over the adjoint representation of G_{gauge} and only use representations of G . In this way we can define the embedding tensor

$$\Theta_M^\alpha = n_M^A m_A^\alpha. \quad (2.17)$$

It transforms under G in the product of its fundamental and adjoint representations, hence allowing to write the action and the equations of motion in a G -covariant way (G -covariance is broken only when we assign a specific value to Θ_M^α , thus fixing the gauging). In this formalism, we can define gauge generators as

$$X_M = \Theta_M^\alpha t_\alpha \quad (2.18)$$

even if, being $2n_V$, they are not linearly independent. The vector and scalar fields before the gauging transform under the global symmetry group G (using the formalism of coset representatives for scalar fields) as

$$\delta L = \Lambda^\alpha (t_\alpha L + L w_\alpha), \quad (2.19a)$$

$$\delta A_\mu^M = -\Lambda^\alpha (t_\alpha)_N^M A_\mu^N, \quad (2.19b)$$

where the Λ^α are constant parameters, w_α is the so-called H -compensator, which is an element in the Lie algebra of H (see (A.9), (A.10), (A.11)), and $(t_\alpha)_N^M$ are G generators in their $2n_V$ -dimensional representation. Once the gauging is introduced and covariant derivatives are defined, the theory is invariant under the following transformations:

$$\delta L = g \Lambda^M (X_M L + L \Theta_M^\alpha w_\alpha), \quad (2.20a)$$

$$\delta A_\mu^M = \partial_\mu \Lambda^M - g A_\mu^N X_{NP}^M \Lambda^P \equiv \widehat{\partial}_\mu \Lambda^M, \quad (2.20b)$$

where $X_{NP}^M = \Theta_N^\alpha (t_\alpha)_P^M$ and Λ^M is now a local parameter $\Lambda^M(x)$.

2.2.1 Constraints on the embedding tensor

In order for the embedding tensor to define a consistent Lagrangian with local gauge symmetry, it has to satisfy two types of constraints [13]. The first set of constraints is

² Actually, the case in which the generators T_A close in a Lie algebra is not the most general one. When the theory contains antisymmetric tensor fields, in presence of non-abelian gauge couplings the gauged algebra can take the structure of a *free differential algebra* (FDA) (see [14], [15] and section 2.4.1).

quadratic and comes from requiring that the tensor Θ is invariant under the action of the generators X_M , i.e. that it does not transform under the gauge group. It can be expressed as

$$\delta_M \Theta_N^\alpha = \Theta_M^\beta \delta_\beta \Theta_N^\alpha = \Theta_N^\beta (t_\beta)_M^P \Theta_P^\alpha + \Theta_M^\beta (t_\beta)_\gamma^\alpha \Theta_N^\gamma = 0, \quad (2.21)$$

where the generators in the adjoint representation $(t_\beta)_\gamma^\alpha$ are nothing but the structure constants of \mathfrak{g} algebra. If the LHS of (2.21) is contracted with a generator t_α , the quadratic constraint on Θ allows to derive the closure of the gauge algebra

$$[X_M, X_N] = -X_{MN}^P X_P. \quad (2.22)$$

A further quadratic constraint is needed since we have defined $2n_V$ generators X_M , but at most n_V mutually local vector fields can enter the gauging procedure, consistently with (2.8). The condition

$$\Theta_M^\alpha \Theta_N^\beta \Omega^{MN} = 0, \quad (2.23)$$

where Ω^{MN} is the $2n_V \times 2n_V$ symplectic matrix, guarantees that electric and magnetic charges are mutually local, so that a symplectic frame can be always chosen in which the gauging is purely electric [16]. This locality requirement is automatically verified if the other constraints are imposed whenever scalar and vector fields are always in the same multiplets, as it is the case for $\mathcal{N} \geq 3$ theories.

The linear constraints on Θ are imposed by supersymmetry: the embedding tensor transforms under the product of the fundamental and adjoint representations of the duality group, which is reducible, but supersymmetry requires that Θ only transforms in a subset of the representations obtained by decomposition. This can be expressed as a linear projection $\mathbb{P}\Theta = 0$, where \mathbb{P} projects on a subset of the representations contained in the product of fundamental and adjoint ones. The explicit expression of this constraint depends on the spacetime dimension and on the number of supercharges, but in 4 dimensions it can be expressed as

$$X_{(MNP)} = X_{(MN}^Q \Omega_{P)Q} = 0. \quad (2.24)$$

2.3 FLUX COMPACTIFICATIONS AND GAUGING

Gauged supergravity theories have acquired a special interest in the context of flux compactifications of higher-dimensional theories, because they provide an example of deriving, from superstring theory or M-theory, models with phenomenologically desirable properties, such as mechanisms of supersymmetry breaking, or the introduction of mass terms.

In particular, standard dimensional reduction usually generates massless scalar fields (associated with the geometry of the internal manifold), which is problematic in a "realistic" 4-dimensional field theory. Indeed, such fields would generate long range

interactions that are unacceptable in our observed universe, i.e. are not compatible with the limits imposed by fifth force experiments. The introduction of fluxes, for instance by giving a non-trivial expectation value to some tensor field, causes instead, in the lower-dimensional theory obtained after reduction, the introduction of minimal couplings between vector and scalar fields and the emergence of a scalar potential giving masses to the moduli fields, all features typical of the gauging procedure.

2.3.1 Kaluza–Klein reduction

The simplest way to perform a dimensional reduction is through spontaneous compactification, which generalizes Kaluza–Klein procedure first followed in an attempt to unify gravity and electromagnetism in 4 dimensions starting from a 5-dimensional theory of pure gravity ([17], [18], [19]). The original idea has been widely used and developed because it provides a way to connect string theories, which are only consistent in a precise spacetime dimension ($D = 26$ for bosonic string theories and $D = 10$ for superstring theories), with 4-dimensional field theories [20]. Physics in 4 dimensions, in particular supersymmetric theories and extended supergravities, can be recovered as low energy approximation of an higher-dimensional theory.

Having a theory in D dimensions, we look for solutions where the D -dimensional spacetime geometry can be split as

$$M_D \longrightarrow M_4 \times M_{\text{int}} , \quad (2.25)$$

where M_{int} is a $(D - 4)$ -dimensional compact manifold. In particular, having a Ricci-flat internal manifold (as it is the case for a torus or a Calabi–Yau space) allows to solve the D -dimensional equations of motion for the metric $R_{ij}^D = 0$ in such a way that the manifold is split into

$$M_D = \text{Mink}_4 \times M_{\text{int}} . \quad (2.26)$$

If the D -dimensional fields are expanded in normal modes of M_{int} , the coefficients of the expansion can be interpreted as fields satisfying their own equations of motion in 4 dimensions. Under the assumption of a sufficiently small size of the internal manifold, the lower-dimensional effective theory is obtained by retaining, among these fields, only the massless ones.

Such a procedure allows to obtain ungauged supergravity theories from compactification of superstring or M-theory. As an example of how the field content of a 4-dimensional theory arises from the spectrum of the original theory, we mention that $\mathcal{N} = 8$ supergravity, which we will discuss in the following chapters, can be derived from 11-dimensional supergravity, by performing Kaluza–Klein reduction over a 7-dimensional torus \mathbb{T}^7 . The 11D theory, first described in [21], contains a

graviton, a gravitino and a 3-form field \mathcal{A} . After dimensional reduction, the 4D fields (for simplicity we just focus on the bosonic sector) are

$$g_{ij} \longrightarrow \begin{cases} g_{\mu\nu} & 1 \text{ graviton} \\ g_{\mu M} & 7 \text{ vector fields} \\ g_{MN} & 28 \text{ scalar fields} \end{cases} \quad (2.27)$$

$$\mathcal{A}_{ijk} \longrightarrow \begin{cases} \mathcal{A}_{\mu\nu\rho} & \text{no physical degrees of freedom in 4D} \\ \mathcal{A}_{\mu\nu M} & 7 \text{ tensors} \\ \mathcal{A}_{\mu MN} & 21 \text{ vector fields} \\ \mathcal{A}_{MNP} & 35 \text{ scalar fields} \end{cases} \quad (2.28)$$

where the notation employed splits the coordinates of the original 11D manifold into compact and non-compact ones

$$x^i = (x^\mu, y^M) \quad \text{with } i = 0, \dots, 10; \mu = 0, \dots, 3; M = 4, \dots, 10.$$

It is immediate to see that, after dualizing the 7 massless tensors to massless scalar fields, we get back the bosonic spectrum of 1 graviton, 28 vectors and 70 scalars that characterizes maximal supergravity in 4 dimensions.

The abelian $U(1)^{28}$ gauge symmetry of the theory originates in part from the gauge symmetry of the 3-form \mathcal{A} , i.e. invariance of the action under transformations of type $\mathcal{A} \rightarrow \mathcal{A} + d\Sigma$ where Σ is a 2-form, and in part from the invariance under diffeomorphisms. This can be observed by writing explicitly the reduced metric

$$ds_D^2 = \underbrace{g_{\mu\nu} dx^\mu dx^\nu}_{ds_4^2} + g_{MN} (G^M_\mu dx^\mu + dy^M) (G^N_\nu dx^\nu + dy^N), \quad (2.29)$$

where, to match the previous definition, $g_{\mu M} = G^N_\mu g_{NM}$. In a Kaluza–Klein reduction, all the fields only depend on the non-compact coordinates $x^{\mu 3}$

$$g_{\mu\nu} = g_{\mu\nu}(x) \quad , \quad g_{MN} = g_{MN}(x) \quad , \quad G^M_\mu = G^M_\mu(x) \quad ,$$

and the original invariance under diffeomorphisms of the D-dimensional theory is broken. Since the dy^M transform under residual diffeomorphisms as $dy^M \rightarrow dy^M + d\omega^M(x)$, to keep invariance of the vielbein $G^M_\mu dx^\mu + dy^M$ it is necessary that the so-called Kaluza–Klein vector fields G^M_μ transform as $G^M_\mu \rightarrow G^M_\mu - \partial_\mu \omega^M$, which is precisely a gauge transformation in 4 dimensions.

³ In other words, the vector fields ∂_M generate isometries for the theory [22].

2.3.2 Fluxes and the origins of gauging

A generalization of spontaneous compactifications is obtained if we admit the presence of non-vanishing fluxes. If we still take as our model the 11-dimensional supergravity theory (discussed in [23]), the 3-form \mathcal{A} is associated to a "field strength", i.e. a 4-form H such that, at least locally, $H = d\mathcal{A}$; introducing a constant flux for H means giving it a non-trivial expectation value over the internal manifold, of type $\langle H_{MNPQ} \rangle = h_{MNPQ} \neq 0$. In absence of fluxes, the reduced 3-form could be written as

$$\begin{aligned} \mathcal{A}(x, y) = & \frac{1}{3!} \mathcal{A}_{\mu\nu\rho}(x) dx^\mu \wedge dx^\nu \wedge dx^\rho + \frac{1}{2} \mathcal{A}_{\mu\nu M}(x) dx^\mu \wedge dx^\nu \wedge dy^M + \\ & + \frac{1}{2} \mathcal{A}_{\mu MN}(x) dx^\mu \wedge dy^M \wedge dy^N + \frac{1}{3!} \mathcal{A}_{MNP}(x) dy^M \wedge dy^N \wedge dy^P, \end{aligned} \quad (2.30)$$

where all the 4-dimensional fields only depend on the non-compact coordinates. The constant flux is the result of adding to the expression (2.30) a 3-form $\sigma = \sigma(y)$ (so that $\mathcal{A} \rightarrow \mathcal{A} + \sigma$) with explicit dependence on the compact coordinates y^M , whose exterior derivative is by definition

$$d\sigma = h = \frac{1}{4!} h_{MNPQ} dy^M \wedge dy^N \wedge dy^P \wedge dy^Q. \quad (2.31)$$

How the introduction of a flux corresponds, at the level of the 4-dimensional effective theory, to a gauging can be easily showed by just considering the kinetic term for the 3-form in the 11-dimensional lagrangian $-\frac{1}{24} H_{ijkl} H^{ijkl}$. After dimensional reduction, it gives rise to the kinetic terms for scalar and vector fields derived from the 3-form

$$\partial_\mu \mathcal{A}_{MNP} \partial^\mu \mathcal{A}_{QRS} g^{MQ} g^{NR} g^{PS}, \quad \partial_{[\mu} \mathcal{A}_{\nu]MN} \partial^{[\mu} \mathcal{A}^{\nu]}_{PQ} g^{MP} g^{NQ}, \quad (2.32)$$

where we can observe that scalar fields define a σ model, with a metric on the scalar manifold which depends on the scalar fields (in this case the ones coming from dimensional reduction of the 11D metric), and also non-minimal couplings between vectors and scalars appear in the gauge kinetic term.

In presence of fluxes, new terms appear in the 4-dimensional lagrangian: first of all, a contribution playing the role of potential for the scalar fields

$$h_{IJKL} h_{MNPQ} g^{IM} g^{JN} g^{KP} g^{LQ} \equiv V(g_{MN}). \quad (2.33)$$

On the other hand, kinetic terms for scalars and vectors are modified respectively by couplings of the form

$$\partial_\mu \mathcal{A}_{MNP} h_{IJKL} g^{\mu I} g^{MJ} g^{NK} g^{PL}, \quad \partial_{[\mu} \mathcal{A}_{\nu]MN} h_{IJKL} g^{\mu I} g^{\nu J} g^{MK} g^{NL}, \quad (2.34)$$

which allow to recover both covariant derivatives for scalar fields and non-abelian field strengths

$$\widehat{\partial}_\mu \mathcal{A}_{MNP} = \partial_\mu \mathcal{A}_{MNP} + h_{MNPQ} G_\mu^Q, \quad H_{\mu\nu MN} = 2\partial_{[\mu} \mathcal{A}_{\nu]MN} + h_{MNPQ} G_\mu^P G_\nu^Q. \quad (2.35)$$

These results show again how the constants h_{MNPQ} , which we interpret as fluxes over the internal manifold of the higher-dimensional theory, give direct information about the gauging of the corresponding 4D supergravity theory and also about the embedding of the gauge group.

From the expression of non-abelian field strengths in eq.(2.35), it is immediate to notice that the constant fluxes h_{MNPQ} have the role of structure constants of the non-abelian gauge group. More precisely, denoting with $A^a = \{G^M, \mathcal{A}_{NP}\}$ the gauge vector fields, we can introduce the associated generators spanning the Lie algebra of the gauge group

$$T_a = \{Z_M, W^{NP}\} \quad (2.36)$$

and then eq.(2.35) translates into the following non-vanishing commutator:

$$[Z_M, Z_N] = h_{MNPQ} W^{PQ} . \quad (2.37)$$

In other words, this relation tells us that the commutator of two infinitesimal transformations under diffeomorphisms of the internal manifold results in a gauge transformation. If we do not introduce other types of fluxes, all the remaining commutators of the algebra will vanish

$$[Z_M, W^{NP}] = 0 \quad , \quad [W^{MN}, W^{PQ}] = 0 ; \quad (2.38)$$

but there exist other possible ways to deform the theory and introduce mass parameters, then switching on additional commutators.

2.4 REDUCTION ON TWISTED TORI

In the previous paragraph, fluxes have been introduced arising from a dependence on internal coordinates of some components of the 3-form \mathcal{A} . Another possibility is to include fluxes coming from internal coordinate dependence of the metric components; such a dependence can be completely general, provided that the internal compact manifold has an isometry group with a transitive action on it and with the same dimension as M_{int} .

This procedure amounts to introducing deformations of the geometry of the internal manifold; they can be described, if we consider the expression (2.29) for the reduced metric in the case of compactifications on a flat space, by replacing the coordinate basis dy^M with a basis of (nowhere-vanishing) 1-forms η^M depending on the internal coordinates

$$\eta^M = N_N^M(y) dy^N . \quad (2.39)$$

The matrix N_N^M defines a "twisting" of the vielbeins with respect to the coordinate basis dy^M and dimensional reductions performed in this context are usually referred

to as compactifications on *twisted tori*⁴. At the level of the effective 4D theory, this procedure is equivalent to the mechanism, first described by Scherk, Schwarz [24] and Cremmer [25], for introducing mass couplings for the moduli fields in theories coming from string compactifications. The interpretation of (2.39) is that we could perform compactification still on a flat compact manifold, but changing some "boundary conditions" on it, i.e. imposing that the theory is invariant under some additional (usually discrete) group of transformations [26]. As a consequence, the coordinate basis dy^M may be no more well defined globally on the internal manifold, but (2.39) allows to get vielbeins which are globally defined.

The 2-forms obtained from exterior derivatives of the vielbeins η^M can be expanded as

$$d\eta^M = \frac{1}{2}\tau^M_{NP}\eta^N \wedge \eta^P . \quad (2.40)$$

Following [22], we can define a basis of vector fields Z_M on the internal manifold which are dual to the 1-forms: $Z_M\eta^N = \delta_M^N$. They are the Killing vectors generating the group of isometries of the internal space; this group, as already mentioned, necessarily has the same dimension of M_{int} . The Z_M vectors generalize the homonymous generators defined in (2.36), which in the case of dimensional reduction over a flat internal manifold just corresponded to translations of the compact coordinates.

The constant parameters τ^M_{NP} , in order for the gauge algebra to be well-defined, must satisfy

$$\tau^M_{R[N}\tau^R_{PQ]} = 0 , \quad (2.41)$$

which can be easily obtained from (2.40) exploiting the property of exterior derivatives $d^2 = 0$, and

$$\tau^M_{MN} = 0 , \quad (2.42)$$

which comes from requiring invariance under isometries of the internal volume element $V_{\text{int}} = \frac{1}{(D-4)!}\varepsilon_{M_1\dots M_{D-4}}\eta^{M_1} \wedge \dots \wedge \eta^{M_{D-4}}$, i.e. $\mathcal{L}_{Z_M}V_{\text{int}} = 0$.

2.4.1 Gauge algebra

In generic dimensional reductions, one has to find, starting from gauge transformations and internal space diffeomorphisms acting on the fields of the D -dimensional theory, the structure of gauge transformations for the 4-dimensional fields of the effective theory. Once this analysis is performed, gauge transformations of vector fields can be used to determine the gauge algebra, i.e. to find its structure constants. Indeed, all the commutators between generators of the algebra can be obtained by evaluating products of infinitesimal gauge transformations over the vector fields, which for sure provide a faithful representation of the gauge group. This method is described in detail in

⁴ Despite the name, the formalism developed can be also applied for more general reductions, where the internal manifold has not necessarily the geometry of an n -torus, as explained in [22].

[22], where dimensional reduction is performed starting from 10-dimensional heterotic string theory.

Calling T_a the algebra generators and ω^a the corresponding infinitesimal parameters, taking two gauge transformations $g = \exp(i\omega_1^a T_a)$, $h = \exp(i\omega_2^a T_a)$ and exploiting Baker–Hausdorff formula, we find

$$h^{-1} \cdot g^{-1} \cdot h \cdot g = 1 + \omega_1^a \omega_2^b [T_a, T_b] + O(\omega^3) = 1 + \omega_1^a \omega_2^b f_{ab}{}^c T_c + O(\omega^3). \quad (2.43)$$

Hence, replacing g, h with the explicit expression of gauge transformations for vector fields, one can derive the structure constants $f_{ab}{}^c$.

The application of this procedure to the compactification of M-theory on twisted \mathbb{T}^7 in presence of fluxes can be found in [23]. In this case, taking the algebra generators (2.36) and infinitesimal parameters $\omega^a = \{\omega^M, \Sigma_{NP}\}$, infinitesimal gauge transformations for the vector fields coming from reduction of the metric and the 3-form are

$$\begin{aligned} \delta G_{\mu}^M &= \partial_{\mu} \omega^M - \tau_{NP}^M \omega^N G_{\mu}^P, \\ \delta \mathcal{A}_{\mu MN} &= \partial_{\mu} \Sigma_{MN} + 2G_{\mu}^P \tau_{P[M}^S \Sigma_{N]S} - 2\omega^Q \tau_{Q[M}^S \mathcal{A}_{\mu N]S} + \\ &\quad - \omega^R h_{RMNQ} G_{\mu}^Q - \tau_{MN}^S (\Sigma_{\mu S} + \Sigma_{SR} G_{\mu}^R), \end{aligned}$$

and the non-vanishing commutators of the resulting gauge algebra are

$$\begin{aligned} [Z_M, Z_N] &= h_{MNPQ} W^{PQ} + \tau_{MN}^P Z_P, \\ [Z_M, W^{NP}] &= 2\tau_{MQ}^{[N} W^{P]Q}. \end{aligned} \quad (2.44)$$

The full algebra does not verify, in general, vanishing of Jacobi identities, because it cannot be described as an ordinary Lie algebra: it has a more complicated structure, which can be formalized employing Free Differential Algebras (see [14] for a general definition). These algebras are characterized by generalized Maurer–Cartan equations, which do not involve only 1-forms, but also higher-rank forms, in this case the 2-forms $\mathcal{A}_{\mu MN}$. Integrability conditions of the FDA constrain the algebra structure constants [15]: in addition to (2.41), one also obtains

$$\tau_{[IJ}^N h_{KLM]N} = 0, \quad (2.45)$$

which has the 11-dimensional interpretation of a Bianchi identity for the 4-form field strength h . The Jacobi identities for vector fields Z_M take the form

$$[Z_M, [Z_N, Z_P]] + [Z_N, [Z_P, Z_M]] + [Z_P, [Z_M, Z_N]] = W^{QR} \tau_{QR}^S h_{MNPS} \quad (2.46)$$

and then the Lie algebra structure can be recovered if $\tau_{QR}^S h_{MNPS} = 0$.

From a purely 4-dimensional perspective, whether the gauge algebra is a Lie algebra or not is not an intrinsic property of the gauge group, but rather depends on

the particular symplectic frame, i.e. on the way G_{gauge} is embedded into G . Then, even in the cases when the gauge algebra of the theory obtained from compactification is just a free differential algebra, the Lie algebra structure can be recovered through a duality rotation that makes the gauging purely electric, as it is guaranteed by the locality constraint on the embedding tensor. In such a frame, the magnetic charges disappear and the tensor gauge fields are dualized to scalar fields (see also section [4.2.3](#)).

NON-GEOMETRIC FLUXES, ASYMMETRIC ORBIFOLDS AND
FLAT VACUA

In the last part of chapter 2, we have discussed the gauge algebra obtained from compactification in presence of the so-called *geometric fluxes*. The algebra structure (2.44), however, is not the most general one can obtain. If we denote by Z_M again the gauge generators arising from dimensional reduction of the metric and by \mathcal{X}^M the generators associated with the vector fields obtained from reduction of a 2-form B_{ij} ¹, in principle we could have [27]

$$[Z_M, Z_N] = h_{MNP} \mathcal{X}^P + \tau_{MN}^P Z_P, \quad (3.1a)$$

$$[Z_M, \mathcal{X}^N] = \tilde{\tau}_{MP}^N \mathcal{X}^P + Q_M^{NP} Z_P, \quad (3.1b)$$

$$[\mathcal{X}^M, \mathcal{X}^N] = \tilde{Q}_P^{MN} \mathcal{X}^P + R^{MNP} Z_P. \quad (3.1c)$$

Such supergravity gaugings always exist from the lower dimensional point of view and, actually, they can be also consistently realized in the full string theory [28]. On the other hand, a general gauging containing the so-called *Q-fluxes* and *R-fluxes* cannot be obtained by a geometric compactification of a higher dimensional supergravity theory.

Then, in this chapter we focus on the string origin of supergravity theories in presence of Q- and R-fluxes. In general, when there is no supergravity compactification that gives rise to a certain gauged supergravity theory in lower dimensions, we say that the lifting is *non-geometric* [27]. They can result from intrinsically stringy constructions such as reductions with duality twists, T-fold reductions or asymmetric orbifolds. From the point of view of string theory as a two-dimensional conformal field theory, the need of such constructions is somewhat obvious, in the sense that there is no reason a priori why the target space should have a conventional geometric interpretation.

Another way of seeing the difference between geometric and non-geometric string compactifications is the following [29]: a string solution is geometric when the background fields constitute a spacetime manifold whose transition functions between overlapping coordinate patches only involve standard diffeomorphisms and possibly

¹ The change in notation is motivated by the fact that we now consider mainly compactification of 10-dimensional theories, while in the previous chapter we took as a prototype compactification of 11-dimensional supergravity, where the field content includes a 3-form field.

gauge transformations. Conversely, if the transition functions are allowed to involve some duality transformation, the corresponding backgrounds preserve only locally the structure of a Riemannian manifold, but not globally. When the duality transformation is a T-duality, the corresponding background is called T-fold (the name was proposed in [30]) and, from the point of view of (3.1), corresponds to turning on Q-fluxes.

There exist another class of backgrounds that do not look like conventional spaces even locally: they are connected with the presence of R-fluxes. It can be shown that in some cases such backgrounds have a point in moduli space that minimises a scalar potential and at which the theory can be constructed by a special type of asymmetric orbifold, so that they can be thought of as giving deformations of asymmetric orbifolds.

In this chapter, in order to show how non-geometric fluxes can appear in the gauge algebra, we first describe the general features of compactification in presence of duality twists. Later, we introduce the orbifold formalism, focusing on the equivalence between compactification with duality twists and orbifolds, the role of orbifold points in allowing for T-duality transformations and the possibility of decoupling left-moving and right moving coordinates in the case of asymmetric orbifolds. Finally, we present some results, obtained from string theory computations, about compactifications in presence of asymmetric orbifolds and their consequences in terms of partial supersymmetry breaking in the reduced 4-dimensional theory.

3.1 DUALITY TWISTS AND NON-GEOMETRIC FLUXES

We describe compactifications with duality twists, which can be interpreted as a generalization of the Scherk–Schwarz reductions described in 2.4. Starting from a supergravity theory with a global symmetry group G , we can introduce twists in the compact directions by an element of the group G .

We will introduce the general formalism considering, for simplicity, the case of twisted reductions of a theory in $D + 1$ dimensions on a circle of radius r , parametrized by a periodic coordinate $y \sim y + 2\pi r$. The fields should sit in some representation of G , i.e. they transform under the action of the group according to $\psi \rightarrow g[\psi]$, for any $g \in G$. In the twisted reduction, the fields are chosen to have a dependence on the circle coordinate of type

$$\psi(x^\mu, y) = g(y) [\psi(x^\mu)] , \quad \mu = 0, \dots, D - 1 . \quad (3.2)$$

Fixing the choice of $g(y)$ means taking a section of a principal fiber bundle over S^1 with structure group G (whose fibers for any point of S^1 are homeomorphic to the group G itself).

In order to have a consistent reduction, we have to require the reduced theory to be independent of y , which means that

$$g(y) = \exp \left(\frac{My}{2\pi r} \right) , \quad (3.3)$$

where M is an element in the Lie algebra of G . Going around the circle once, the fields are not periodic, but acquire a non-trivial *monodromy* given by

$$\mathcal{M}(g) = g(2\pi r) g(0)^{-1} = \exp M. \quad (3.4)$$

The *twist group* of the bundle is defined to be the discrete abelian subgroup of G generated by the monodromy \mathcal{M} [31]. Its elements can be viewed as the twists obtained if we go around the circle more than once. If the twist group is a finite group of order n , then the principal bundle has a trivial n -fold covering; in other words, if the twist group is \mathbb{Z}_n we can take a larger circle of radius nr as a covering of the original circle of radius r and the principal bundle built on this base space will have a trivial monodromy, since $\mathcal{M}^n = \mathbb{1}$.

While the low-energy effective action of our theory is invariant under G , if we consider the full quantum theory the symmetry is broken to a discrete subgroup

$$G(\mathbb{Z}) = G \cap \text{Sp}(2k, \mathbb{Z}), \quad (3.5)$$

where k is the number of independent vector fields of the theory. The subgroup $G(\mathbb{Z})$ has the property to preserve the self-dual lattice of electric and magnetic charges [32]. Therefore, while in classical supergravity any element of G can be used as monodromy, a consistent twisted reduction that can be lifted to string theory requires that the monodromy is in $G(\mathbb{Z})$ [33].

After the twisted reduction, the theory in D dimensions will be a gauged supergravity, where the gauge symmetry group is the 1-dimensional subgroup of G generated by the Lie algebra element M and the gauge field is the vector field arising from Kaluza–Klein reduction of the metric. The reduced theory does not depend, actually, on the choice of $g(y)$, but only on its conjugacy class in G . Indeed, if we change the twist from $g(y)$ to $hg(y)h^{-1}$, where h is a constant element of G , the resulting theory is equivalent to the original one up to a field redefinition $\psi \rightarrow h[\psi]$ [33]. Then, eventually the reductions are classified by conjugacy classes of M in the Lie algebra.

M plays the role of a mass matrix for the reduced theory; the fermion mass terms and the modifications of the supersymmetry transformation rules that characterize the gauged supergravity (as discussed in chapter 2) depend linearly on M , while the scalar potential, which is a generalization of the Scherk–Schwarz potential [24], is a quadratic function of the mass matrix [31].

3.1.1 *T-duality twists and Q-fluxes*

We want now to give some details about the gauge algebra resulting from compactification with duality twists, i.e. to show how the Q- and R-fluxes come out. To this purpose, we consider, following [27], [28], theories obtained by compactifications from $D + d + 1$ dimensions on d -dimensional twisted tori (in presence of a 3-form field

strength). The reduced theory in $D + 1$ dimensions is covariant under the action of an $O(d, d)$ group, as it was shown by Kaloper and Myers [22]. In the case we start from an heterotic string theory, the $O(d, d)$ symmetry is contained in a larger $O(d, d + 16)$ symmetry group, while for type II strings it is a subgroup of the U-duality group.

Then, we want to further reduce on a circle introducing a twist. In the string theory, the monodromy associated to any twisted reduction has to be in the discrete T-duality group $O(d, d; \mathbb{Z})$

$$\psi(x^\mu, y) = \exp\left(\frac{My}{2\pi r}\right) [\psi(x^\mu)] \quad \text{with} \quad \mathcal{M} = \exp M \in O(d, d; \mathbb{Z}) . \quad (3.6)$$

In order to understand the resulting gauge algebra structure, we introduce a notation that allows to distinguish between the gauge generators coming from the S^1 reduction, labeled by the circle coordinate y , and the remaining ones, which result from the previous compactification over the d -torus

$$T_a = \{Z_y, \mathcal{X}^y, T_\alpha\} , \quad T_\alpha = \{Z_I, \mathcal{X}^I\} . \quad (3.7)$$

The non-vanishing commutators of the gauge algebra for our reduced D -dimensional theory are

$$[Z_y, T_\alpha] = M_\alpha{}^\beta T_\beta , \quad (3.8)$$

where $M_\alpha{}^\beta$ is a proper $2d$ -dimensional matrix representation of the Lie algebra element M . It can be decomposed, in the basis $T_\alpha = \{Z_I, \mathcal{X}^I\}$, as

$$M_\alpha{}^\beta = \begin{pmatrix} W_I{}^J & U_{IJ} \\ V^{IJ} & (W^T)^I{}_J \end{pmatrix} , \quad (3.9)$$

where U, V, W are $d \times d$ matrices and U, V are antisymmetric, as it is required by the fact that M should belong to the Lie algebra of $O(d, d)$. Then, the gauge algebra can be decomposed accordingly as

$$[Z_y, Z_I] = W_I{}^J Z_J + U_{IJ} \mathcal{X}^J , \quad (3.10a)$$

$$[Z_y, \mathcal{X}^I] = -W_J{}^I \mathcal{X}^J + V^{IJ} Z_J , \quad (3.10b)$$

$$[Z_I, Z_J] = [\mathcal{X}^I, \mathcal{X}^J] = [\mathcal{X}^y, Z_y] = [\mathcal{X}^y, Z_I] = [\mathcal{X}^y, \mathcal{X}^I] = 0 . \quad (3.10c)$$

We can easily compare the above commutators with the most general gauge algebra (3.1), obtaining the following non-vanishing fluxes:

$$\tau_{yI}^J = \tilde{\tau}_{yI}^J = W_I{}^J , \quad h_{yIJ} = U_{IJ} , \quad Q_y{}^{IJ} = V^{IJ} . \quad (3.11)$$

Then, whenever we choose $V^{IJ} \neq 0$, we obtain that T-duality twists can give rise to a non-geometric compactification. Still, we have not described a way to turn on non-trivial $\tilde{Q}_M{}^{NP}$ or R^{MNP} fluxes.

3.2 ORBIFOLDS

Orbifolds can be defined as topological spaces obtained by taking the quotient of a manifold by the action of a discrete group. Strictly speaking, this requirement does not hold globally: an orbifold just needs to be locally homeomorphic to the quotient of an euclidean space by a discrete group. If the group action is free, i.e. without fixed points, the resulting mathematical object is still a manifold, while when fixed points exist they result in singularities at the orbifold level.

If we consider a theory defined on such spaces, we have to take into account also the action of the discrete group on the states of the theory. The Hilbert space of the orbifolded theory includes the states (among all the states of the original theory before taking the quotient by a discrete group) that are invariant under the group action, plus the *twisted sectors* of states that are close up to non-trivial group transformations.

The fact that strings can consistently propagate on orbifolds was already pointed out in [34], [35]. Putting together this result with the fact that, in string theory, left-moving and right-moving excitations are allowed to live on different spaces (as it happens for heterotic strings), in [36] it was proposed for the first time to consider *asymmetric orbifolds*, where the left-moving coordinates of a string theory live on one orbifold and the right-moving ones live on another orbifold.

The purpose of this section is to show how further generalizations of the gauge algebra (3.10) are naturally realized in the context of orbifolds.

3.2.1 Again T-duality

In the previous section we have seen how to generate non-geometric Q-fluxes from compactifications on a T-fold with monodromy in $O(d, d; \mathbb{Z})$. This means that we have reduced our theory from $D + d + 1$ to D dimensions on an internal space that can be locally described as $T^d \times S^1$, allowing for twists in the T-duality group of the d -torus. Starting from this class of reductions, we can ask what is the action of T-duality on the resulting theories. In particular, we can consider two different kinds of duality transformations: those acting on the T^d fiber and those acting on the base space S^1 .

The first type of T-duality can be parametrized by an element of $O(d, d; \mathbb{Z})$ that modifies the mass matrix M specifying the twist in the following way:

$$M \rightarrow M' = \mathcal{O}M\mathcal{O}^{-1}, \quad \mathcal{O} \in O(d, d; \mathbb{Z}). \quad (3.12)$$

The metric and the 2-form field, which keep a dependence on the S^1 coordinate y , are dualized with a linear fractional transformation depending on the matrix \mathcal{O} and acting on their sum $\mathcal{E}(y) = G(y) + B(y)$ (according to the Buscher rules [37]).

It can be easily shown [28] that the above described procedure allows to start from a block-diagonal twist-matrix

$$M = \begin{pmatrix} W & 0 \\ 0 & -W^T \end{pmatrix}, \quad (3.13)$$

corresponding to a compactification in presence of geometric fluxes $\tau, \tilde{\tau}$ and end up with both geometric and non geometric fluxes $(h, \tau, \tilde{\tau}, Q)$. Then, in this case Q-fluxes are simply obtained by T-dualizing some geometric flux; we can distinguish them from "truly non-geometric" Q-fluxes, which correspond to those cases when the matrix M generating the monodromy is not in the same conjugacy class of "geometric" matrices of the form (3.13).

3.2.2 Orbifolds and twisted reductions

The situation changes completely when we try to perform a T-duality along the S^1 direction. The reason is that T-duality in its usual formulation requires translations in the direction in which we dualize to be isometries, but the ansatz (3.6) for the twisted reduction introduces an explicit dependence of the fields on the y coordinate. However, there can be special points in the moduli space of our theory where translation along the y direction actually becomes an isometry and then T-duality is allowed. These are the *orbifold points*, because at such points the twisted reduction becomes equivalent to an orbifold by a discrete symmetry of the d-torus followed by a shift along S^1 .

A detailed analysis of the connection between orbifolds and duality twist can be found in [31], where the case of strings compactified on $T^2 \times S^1$ is taken as a prototype. First, we focus on duality twists of elliptic type, i.e. those obtained if the generator M in the Lie algebra of $O(d, d)$ is compact. This in turn happens if M is (or is conjugate to) an element of the block-diagonal subgroup $O(d) \times O(d) \subset O(d, d)$. It can be shown that, in the elliptic case, the monodromy \mathcal{M} always generates a twist group of finite order, which means that $\mathcal{M}^n = \mathbb{1}$ for some finite $n \in \mathbb{N}$ and then the twist group is isomorphic to \mathbb{Z}_n .

In general, the twist group acts on the fields, and in particular on the points of moduli space, in a non-trivial way, but in the case there are points invariant under the action of \mathbb{Z}_n , i.e. points at which \mathbb{Z}_n becomes a symmetry for the theory, then a description in terms of orbifolds is possible. Remarkably, as clarified in [31], such points correspond to stable minima with zero energy of the scalar potential arising from compactification. It is immediate to see how we can construct an orbifold by the \mathbb{Z}_n action. If the duality twist is defined along a circle of radius r , then we can take its n -fold cover, a circle of radius $R = nr$ parametrized by a coordinate Y with periodicity $Y \sim Y + 2\pi R$, so that all fields are periodic around this larger circle. In the orbifolded

theory, the \mathbb{Z}_n group acts with rotations generated by the monodromy \mathcal{M} together with order n shifts along Y

$$Y \rightarrow Y + \frac{2\pi R}{n} = Y + 2\pi r .$$

In the case of twisted reductions over $T^2 \times S^1$, the allowed orbifolds are associated to the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$, which are the only possible discrete symmetries of the lattice associated with a 2-torus. They are in 1-to-1 correspondence with the 4 conjugacy classes (apart from the trivial one) of elliptic monodromies $\mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_6$.

Once we have sketched this description in terms of orbifolds at the stable points of the scalar potential, we can perform a T-duality transformation along the S^1 direction. The effect of T-duality is to exchange the circle of radius R with a circle of radius $\tilde{R} = 1/R$ and coordinate $\tilde{Y} \sim \tilde{Y} + 2\pi\tilde{R}$; the action of the \mathbb{Z}_n orbifold after dualization involves a shift of order n along the dual coordinate \tilde{Y} .

Then, T-duality can be carried out exactly at the orbifold point, but as soon as we go away from this point the fields will acquire a non-trivial dependence on the circle coordinate. A new background can be still constructed consistently by exploiting deformations around the orbifold point after dualization [27].

Under T-duality along the circle direction, the gauge generator Z_y is conjugated to \mathcal{X}^y , so that a gauge algebra of type (3.8) becomes

$$[\mathcal{X}^y, T_\alpha] = M_\alpha{}^\beta T_\beta . \quad (3.14)$$

In this context, it is possible to obtain non-trivial R -fluxes. Indeed, if we start from a twist specified by a generator M with non-vanishing U, V, W , then after performing T-duality we get a compactification with $\tilde{\tau}, Q, \tilde{Q}$ and R fluxes. In details, the correspondence is

$$h \rightarrow \tilde{\tau}, \quad \tau \rightarrow Q, \quad \tilde{\tau} \rightarrow \tilde{Q}, \quad Q \rightarrow R, \quad (3.15)$$

as it is reported in [28].

3.2.3 Asymmetric orbifolds and R -fluxes

Up to now, we have not specified a distinction between symmetric and asymmetric orbifolds. Since we are considering twisted reductions over spaces of the form $T^d \times S^1$, the orbifold action involves a duality twist and a shift, so that we will have different cases depending on whether the asymmetry concerns the T^d fiber or the base space S^1 .

Given that an orbifold is specified by its monodromy matrix $\mathcal{M} \in O(d, d)$, we can always interpret it as a rotation acting on the doubled torus T^{2d} parametrized by the standard "geometric" coordinates y^I , with $I = 1, \dots, d$, and their duals \tilde{y}^I , or, equivalently, by the left-moving and right-moving coordinates y_L^I, y_R^I . We can choose to

specify the orbifold action in terms of the y^l, \tilde{y}^l , or y_L^l, y_R^l with a simple change of basis for the monodromy matrix \mathcal{M} .

Once the basis has been fixed, we can see explicitly whether the orbifold is symmetric or asymmetric in the torus coordinates. In order to give some very simple examples, we will consider for a moment that $d = 2$, so that we have a 2-torus specified by one complex coordinate $z = y^1 + iy^2$. The symmetric action of a \mathbb{Z}_k orbifold can be written as

$$\mathcal{M}_k \quad : \quad \begin{cases} z_L \rightarrow e^{2\pi i/k} z_L \\ z_R \rightarrow e^{2\pi i/k} z_R \end{cases} . \quad (3.16)$$

Starting from a symmetric orbifold, if we act with a T-duality transformation, as in (3.12), the orbifold is modified to

$$\mathcal{M}_k \quad : \quad \begin{cases} z_L \rightarrow e^{2\pi i/k} z_L \\ z_R \rightarrow e^{-2\pi i/k} z_R \end{cases} . \quad (3.17)$$

This is an asymmetric orbifold, because the action on left-moving and right-moving coordinates is different, but it cannot give rise to truly non-geometric fluxes because the compactification with geometric fluxes can be recovered via a duality transformation.

A truly asymmetric orbifold can be obtained, instead, if we consider \mathbb{Z}_k rotations acting only on the left-moving coordinates

$$\mathcal{M}_k \quad : \quad \begin{cases} z_L \rightarrow e^{2\pi i/k} z_L \\ z_R \rightarrow z_R \end{cases} . \quad (3.18)$$

In this case, the monodromy matrix cannot be dualized to a symmetric one, and indeed such an orbifold would generate non-geometric fluxes. Some examples of symmetric, asymmetric and truly asymmetric orbifolds for higher-dimensional tori can be found in [28].

As a further generalization of the above discussion, we can consider orbifolds where asymmetry lies not only in the twist acting on the torus coordinates, but also in the shifts along the circle direction, in the sense that the orbifold action involves asymmetric shifts for the coordinates y_L, y_R . Such an orbifold is specified by a group of type $\mathbb{Z}_n \times \mathbb{Z}_m$, associated with two elements M, \tilde{M} in the $\mathfrak{o}(d, d)$ algebra, where M generates an order n rotation, while \tilde{M} generates an order m rotation, with the requirement that

$$[M, \tilde{M}] = 0 . \quad (3.19)$$

The gauge algebra in this case is given by

$$\begin{aligned} [Z_y, T_\alpha] &= M_\alpha^\beta T_\beta , \\ [\mathcal{X}^y, T_\alpha] &= \tilde{M}_\alpha^\beta T_\beta . \end{aligned} \quad (3.20)$$

As proposed in [27], this suggests that, away from the orbifold point, the fields acquire a dependence on the S^1 coordinates of type

$$\psi(x^\mu, y, \tilde{y}) = \exp\left(\frac{My}{2\pi r}\right) \exp\left(\frac{\tilde{M}\tilde{y}}{2\pi\tilde{r}}\right) [\psi(x^\mu)] . \quad (3.21)$$

The dependence on both y and its dual coordinate \tilde{y} is a signature of a non-geometric background that is not geometric even locally, associated with the presence of R -fluxes that cannot be eliminated with a duality transformation, as we are going to see.

The matrix \tilde{M} can be parametrized, in analogy with M , as

$$\tilde{M}_\alpha{}^\beta = \begin{pmatrix} \tilde{W}_I{}^J & \tilde{U}_{IJ} \\ \tilde{V}^{IJ} & (\tilde{W}^T)_J{}^I \end{pmatrix} , \quad (3.22)$$

so that we obtain

$$[Z_y, Z_I] = W_I{}^J Z_J + U_{IJ} \mathcal{X}^J , \quad (3.23a)$$

$$[Z_y, \mathcal{X}^I] = -W_J{}^I \mathcal{X}^J + V^{IJ} Z_J , \quad (3.23b)$$

$$[\mathcal{X}^y, Z_I] = \tilde{W}_I{}^J Z_J + \tilde{U}_{IJ} \mathcal{X}^J , \quad (3.23c)$$

$$[\mathcal{X}^y, \mathcal{X}^I] = -\tilde{W}_J{}^I \mathcal{X}^J + \tilde{V}^{IJ} Z_J . \quad (3.23d)$$

If we compare this result with the gauge algebra (3.1), we can now identify all the fluxes as

$$\begin{aligned} \tau_{yI}^J &= \tilde{\tau}_{yI}^J = W_I{}^J , & h_{yIJ} &= U_{IJ} , & Q_y^{IJ} &= V^{IJ} , \\ \tilde{\tau}_{IJ}^y &= -\tilde{U}_{IJ} , & Q_I{}^{yJ} &= -\tilde{Q}_I{}^{yJ} = -\tilde{W}_I{}^J , & R^{yIJ} &= \tilde{V}^{IJ} . \end{aligned} \quad (3.24)$$

3.3 DIMENSIONAL REDUCTIONS

As already outlined in chapter 2, compactification of supersymmetric theories gives rise to gauged supergravity theories in lower dimensions, whose features and symmetries mainly depend on the way compactification is performed and the fluxes that have been introduced.

We are mainly interested in extended supergravity theories in 4 dimensions. The introduction of non-geometric backgrounds and in particular of genuine left-right asymmetry (by means of asymmetric orbifolds) allows to obtain extended supergravities

with $\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R$ supersymmetry generators starting from vacuum configurations of type II superstrings. The possible configurations are described in detail in [38]: they are

$$\begin{aligned}
\mathcal{N}_8 &\leftrightarrow \mathcal{N}_L = 4, \mathcal{N}_R = 4, \\
\mathcal{N}_6 &\leftrightarrow \mathcal{N}_L = 2, \mathcal{N}_R = 4, \\
\mathcal{N}_5 &\leftrightarrow \mathcal{N}_L = 1, \mathcal{N}_R = 4, \\
\mathcal{N}_4 &\leftrightarrow \mathcal{N}_L = 2, \mathcal{N}_R = 2 \quad \text{or} \quad \mathcal{N}_L = 0, \mathcal{N}_R = 4, \\
\mathcal{N}_3 &\leftrightarrow \mathcal{N}_L = 1, \mathcal{N}_R = 2, \\
\mathcal{N}_2 &\leftrightarrow \mathcal{N}_L = 1, \mathcal{N}_R = 1 \quad \text{or} \quad \mathcal{N}_L = 0, \mathcal{N}_R = 2, \\
\mathcal{N}_1 &\leftrightarrow \mathcal{N}_L = 0, \mathcal{N}_R = 1,
\end{aligned}$$

up to the exchange $\mathcal{N}_L \leftrightarrow \mathcal{N}_R$. From the above scheme it is clear that only the cases of \mathcal{N} even, different from 6, admit a left-right symmetric description, while for $\mathcal{N} = 6, 5, 3, 1$ an asymmetric background is required.

In particular, models with $\mathcal{N} = 3$ can be constructed by performing a geometric \mathbb{Z}_2 orbifold, whose action is given by a twist (resulting in a theory with residual $\mathcal{N} = 4 = 2_L + 2_R$) and a shift that avoids massless states in the twisted sectors. This is followed by a non-geometric projection (acting only on left or only on right coordinates) that allows for a further supersymmetry breaking $\mathcal{N} = 4 \rightarrow \mathcal{N} = 3$.

The results of [38] are obtained from a string theory approach, but it could be interesting to investigate their 4-dimensional counterpart. Namely, we should be able to reproduce the same orbifolds as gaugings of some supergravity theory. Then, we could take the known stringy results as a guide to investigate gauged supergravities where the gauging produces specific supersymmetry breaking patterns at the vacuum.

 $\mathcal{N} = 8$ SUPERGRAVITY IN 4 DIMENSIONS

In 4 spacetime dimensions, the maximal number of allowed supersymmetry generators is $\mathcal{N} = 8$. The field content of the resulting supergravity theory is completely constrained by supersymmetry: the unique (CPT-self-conjugate) supermultiplet contains

- a graviton (spin 2), which can be described by the vielbein $e^a{}_\mu$,
- 8 gravitini ψ_μ^i , $i = 1, \dots, 8$,
- $\binom{\mathcal{N}}{2} = 28$ vector fields A_μ^I , where in agreement with the notation introduced in [2.1](#) we denote with the upper index I only the electric vector fields representing the independent degrees of freedom of the theory, while the 28 dual "magnetic" vectors are $A_{\mu I}$,
- $\binom{\mathcal{N}}{3} = 56$ spin 1/2 fermions $\chi^{ijk} = \chi^{[ijk]}$,
- $\binom{\mathcal{N}}{4} = 70$ real scalar fields $\varphi^{ijkl} = \varphi^{[ijkl]}$.

The R-symmetry group of the theory is $SU(8)$ ¹; gravitinos, spin 1/2 fermions, scalar fields transform respectively in its **8**, **56**, **70** representation and the indices i, j, k, l used to label the fields denote precisely these transformation properties.

The $\mathcal{N} = 8$ theory was constructed by Cremmer and Julia in [\[39\]](#), [\[40\]](#) exploiting dimensional reduction from the 11-dimensional supergravity, then in [\[41\]](#) it was derived entirely within a 4-dimensional context.

We mentioned in sections [2.1](#), [2.2](#) how the structure and the symmetries of the scalar sector play an important role in the description of extended supergravity theories and their gaugings. Then, a starting point in the analysis of the maximal theory is to characterize its scalar manifold: it can be described as a homogeneous space

$$\mathcal{M}_{\text{scalar}} = E_{7(7)}/SU(8) , \tag{4.1}$$

¹ The largest R-symmetry group allowed for $\mathcal{N} = 8$ supergravity would be in principle $U(8)$, but fields can transform non-trivially only under its $SU(8)$ subgroup because the supermultiplet is CPT-self-conjugate [\[6\]](#).

as explained in [40]. In this chapter we will first give some details on the $E_{7(7)}$ group and the structure of the scalar manifold, then describe the lagrangian realization of the maximal theory. Subsequently, we will focus on the gaugings of the theory, adopting the embedding tensor technique described in chapter 2. We will discuss some implications of the constraints imposed on the embedding tensor and the main effects of the gauging procedure on the different fields of the theory will be analyzed. Finally, we will show how to describe the gauging via an object called T -tensor, obtained from the embedding tensor and coset representatives; the decomposition of this tensor into irreducible representations of $SU(8)$ will reveal particularly useful in the following chapter.

4.1 THE SCALAR MANIFOLD

4.1.1 The exceptional group $E_{7(7)}$

To describe $E_{7(7)}$, we can start from its generators, considering the infinitesimal group action in the fundamental (56) representation. Following [42], we assume that the 56-dimensional representation space is spanned by two antisymmetric rank-2 tensors $x^M = (x_{AB}, y^{CD})$, where indices take values $A, B, C, D \dots = 1, \dots, 8$. The infinitesimal transformation laws are

$$\begin{aligned}\delta x_{AB} &= \Lambda_A^C x_{CB} + \Lambda_B^C x_{AC} + \Sigma_{ABCD} y^{CD}, \\ \delta y^{AB} &= \Lambda'^A_C y^{CB} + \Lambda'^B_C y^{AC} + \frac{1}{24} \varepsilon^{ABCDEFGH} \Sigma_{EFGH} x_{CD},\end{aligned}\tag{4.2}$$

where $\Lambda_A^B = -\Lambda'^B_A$, $\Lambda_A^A = 0$, while Σ^{ABCD} is a totally antisymmetric tensor. If we define $\Lambda_{AB}^{CD} = 2 \Lambda_{[A}^{[C} \delta_{B]}^{D]}$, the \mathfrak{e}_7 algebra generators in the fundamental representation can be expressed as

$$(t_\alpha)_M^N = \begin{pmatrix} \Lambda_{AB}^{EF} & \Sigma_{ABGH} \\ \star \Sigma^{CDEF} & \Lambda'^{CD}_{GH} \end{pmatrix},\tag{4.3}$$

where $\star \Sigma$ denotes the Hodge dual of the tensor Σ . It is immediate to see that the 133 independent generators correspond to the 63 degrees of freedom of the traceless Λ_A^B plus the 70 degrees of freedom of the antisymmetric Σ^{ABCD} .

The generators with $\Sigma = 0$ select an $SL(8, \mathbb{R})$ subgroup, which is the maximal subgroup of $E_{7(7)}$. Indeed, tensors Λ_A^B , Λ'^A_B can be interpreted as generators of $SL(8, \mathbb{R})$ respectively in the **8** and **8'** representations; analogously, Λ_{AB}^{CD} and Λ'^{AB}_{CD} correspond to the **28** and **28'** representations of $SL(8, \mathbb{R})$ resulting from the splitting of the $E_{7(7)}$ **56** representation ($\mathbf{56} \rightarrow \mathbf{28} \oplus \mathbf{28}'$). This can be easily observed from the block diagonal structure assumed by (4.3) once $\Sigma = 0$ has been imposed.

In order to see how the $SU(8)$ subgroup of $E_{7(7)}$ comes out, it is useful to separate the tensor $\Lambda_A{}^B$ in the symmetric and the antisymmetric parts, while the tensor Σ can be split into self-dual and anti-self-dual components:

$$\Lambda_A{}^B \longrightarrow \begin{cases} (\Lambda^s)_A{}^B = (\Lambda^s)_B{}^A & (35 \text{ d.o.f.}) \\ (\Lambda^a)_A{}^B = -(\Lambda^a)_B{}^A & (28 \text{ d.o.f.}) \end{cases} \quad (4.4)$$

$$\Sigma_{ABCD} \longrightarrow \begin{cases} \star(\Sigma^d)^{ABCD} = (\Sigma^d)_{ABCD} & (35 \text{ d.o.f.}) \\ \star(\Sigma^a)^{ABCD} = -(\Sigma^a)_{ABCD} & (35 \text{ d.o.f.}) \end{cases} \quad (4.5)$$

Among $\mathfrak{e}_{7(7)}$ generators, those verifying $\Lambda = \Lambda^a, \Sigma = \Sigma^a$ span the 63-dimensional Lie algebra of $SU(8)$; they are all compact generators, indeed $SU(8)$ is the maximal compact subgroup of $E_{7(7)}$ ². The signature of the Killing metric on the \mathfrak{e}_7 algebra, which comes from the difference between the number of non-compact and compact generators, is $70 - 63 = 7$, that's why the real form of the exceptional group E_7 appearing as duality group in the maximal supergravity theory is denoted with $E_{7(7)}$.

4.1.2 A complex basis

Once the structure of E_7 generators has been described, it is useful to change the basis (in the previous section indices denoted with capital letters $A, B \dots$ correspond to vector representations of $SL(8, \mathbb{R})$) in order to make easier the identification of $SU(8)$ generators and eventually the definition of coset representatives for the coset manifold $E_{7(7)}/SU(8)$. The appropriate basis is a complex one, of type $(z, \bar{z}) = (x + iy, x - iy)$, in which the infinitesimal $SU(8)$ transformations take the form

$$\delta_{SU(8)}(x_{AB} \pm iy^{AB}) = \left((\Lambda^a)_{AB}{}^{CD} \pm i \star(\Sigma^a)^{ABCD} \right) (x_{AB} \pm iy^{AB}), \quad (4.6)$$

where notation becomes consistent from the point of view of $SL(8, \mathbb{R})$ covariance only if we take into account relations (4.4), (4.5). If we consider the remaining generators, orthogonal to $SU(8)$ ones with respect to the Killing metric, their action exchanges complex coordinates with their conjugates

$$\delta_{\perp}(x_{AB} \pm iy^{AB}) = \left((\Lambda^s)_{AB}{}^{CD} \pm i \star(\Sigma^d)^{ABCD} \right) (x_{AB} \mp iy^{AB}). \quad (4.7)$$

This change of basis can be correctly reproduced through chiral Γ matrices, obtained from a Clifford algebra $\text{Cliff}(8)$ in 8 dimensions ([40], [43]). Such an algebra is defined by matrices $\Gamma^A, A = 1, \dots, 8$ satisfying the anticommutation relations

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}\mathbb{1}, \quad (4.8)$$

² It is not exactly $SU(8)$ to be contained in $E_{7(7)}$, but $SU(8)/\mathbb{Z}_2$, which still has dimension 63; the same is true for the $SL(8, \mathbb{R})$ subgroup, which is actually $SL(8, \mathbb{R})/\mathbb{Z}_2$ [40].

where η^{AB} could be in principle a diagonal matrix with arbitrary number of +1 and -1 along the diagonal, but for our purposes we can restrict to the case in which it is an euclidean metric $\eta^{AB} = \pm\delta^{AB}$. Antisymmetrized products of the Γ^A matrices are defined as $\Gamma^{AB\dots D} = \Gamma^{[A}\Gamma^B\dots\Gamma^D]$ and, as in any even dimension $d = 2n$, it is also possible to define a chirality matrix $\Gamma^{d+1} = i^n\Gamma^1\dots\Gamma^d$ anticommuting with all the Γ^A , which is useful in order to split the 2^n -dimensional representation of $\text{Cliff}(8)$ into two chiral representation of $\text{SO}(d)$. For the algebra $\text{Cliff}(d)$, matrices with two antisymmetrized indices Γ^{AB} satisfy the following properties:

$$[\Gamma^{AB}, \Gamma^{CD}] = -8\eta^{[A[C}\Gamma^{B]D]}, \quad (4.9a)$$

$$\Gamma^{AB}_{ij}\Gamma^{ij}_{CD} = 16\delta^{AB}_{CD}, \quad (4.9b)$$

where (4.9a) allows to identify the set of Γ^{AB} matrices as a complete basis of the Lie algebra generating $\text{SO}(8)$. They play a role similar to the $\gamma^{\mu\nu}$ matrices built out of the Dirac matrices γ^μ in 4 dimensions, which convert the Lorentz group generators from vector to spinor representation, allowing to define covariant derivatives for spinors in curved spacetime. Indeed, matrices $(\Gamma^{AB})^{ij}$ make it possible to interpolate between indices $A, B\dots$, referring to $\text{SL}(8, \mathbb{R})$ representation, and indices $i, j\dots$ that label $\text{SU}(8)$ representations, thanks to a property of the $\text{SO}(8)$ subgroup, which is common to $\text{SL}(8, \mathbb{R})$ and $\text{SU}(8)$, called *triality* [44]. This property is essentially the local equivalence between vector and spinor representations of the $\mathfrak{su}(8)$ algebra; it allows to interchange the two types of indices, so that the $(\Gamma^{AB})^{ij}$ can be also interpreted as matrices $(\Gamma^{ij})^{AB}$ coming from a $\text{Cliff}(8)$ algebra of matrices Γ^i .

Thus, complex coordinates can be defined as

$$z_{ij} = \frac{1}{4\sqrt{2}}\left(\Gamma^{AB}_{ij}x_{AB} + i\Gamma_{ABij}y^{AB}\right), \quad \bar{z}^{ij} = \frac{1}{4\sqrt{2}}\left(\Gamma^{ABij}x_{AB} - i\Gamma_{AB}^{ij}y^{AB}\right). \quad (4.10)$$

We can also apply the change of basis to $E_{7(7)}$ generators, obtaining

$$\begin{aligned} \lambda_{ij}{}^{kl} &= \frac{1}{32}\Gamma^{AB}_{ij}\left((\Lambda^a)_{AB}{}^{CD} + i\star(\Sigma^a)^{ABCD}\right)\Gamma^{CDkl}, \\ \sigma_{ijkl} &= \frac{1}{32}\Gamma^{AB}_{ij}\left((\Lambda^s)_{AB}{}^{CD} + i\star(\Sigma^d)^{ABCD}\right)\Gamma^{CD}{}_{kl}, \end{aligned} \quad (4.11)$$

where the tensor $\lambda_{ij}{}^{kl}$ encodes all $\text{SU}(8)$ generators, while the totally antisymmetric σ_{ijkl} correspond to orthogonal generators. The change of basis can be described with a more compact notation if one defines a 56×56 matrix constructed from the Γ^{AB}

$$S_{\underline{M}}{}^{\underline{N}} = \frac{1}{4\sqrt{2}}\begin{pmatrix} \Gamma_{ij}{}^{AB} & i\Gamma_{ijCD} \\ \Gamma^{klAB} & -i\Gamma^{kl}{}_{CD} \end{pmatrix}, \quad (4.12)$$

where from now on we denote with underlined capital letters $\underline{M}, \underline{N} \dots = 1, \dots, 56$ indices referring to the complex basis. Applying the change-of-basis matrix (4.12) (and its inverse) allows to obtain the generic form of $E_{7(7)}$ generators in the complex basis

$$(t_\alpha)_{\underline{M}}^{\underline{N}} = S_{\underline{M}}^{\underline{P}} (t_\alpha)_P^Q S^+_{\underline{Q}}{}^{\underline{N}} = \begin{pmatrix} \lambda_{ij}{}^{mn} & \sigma_{ijpq} \\ \bar{\sigma}^{klmn} & \bar{\lambda}{}^{kl}{}_{pq} \end{pmatrix}. \quad (4.13)$$

4.1.3 The coset manifold

Using only the generators orthogonal to $SU(8)$ and in particular establishing a correspondence between the antisymmetric form σ_{ijkl} and the 70 scalar fields φ_{ijkl} , we can easily define coset representatives of $E_{7(7)}/SU(8)$ as³

$$L(\varphi)_{\underline{M}}^{\underline{N}} = \exp \begin{pmatrix} 0 & \varphi_{ijpq} \\ \varphi^{klmn} & 0 \end{pmatrix} = \begin{pmatrix} u_{ij}{}^{mn} & -v_{ijpq} \\ -v^{klmn} & u^{kl}{}_{pq} \end{pmatrix}. \quad (4.14)$$

Here and in the following, even if not written explicitly, it is understood that raising or lowering all indices of type $i, j \dots$ corresponds to complex conjugation, so that, for instance, $\varphi_{ijkl} = \bar{\varphi}^{ijkl}$, or $u_{ij}{}^{kl} = \bar{u}{}^{ij}{}_{kl}$ (the upper or lower position of indices $i, j \dots$ refers to the representations $\mathbf{8}$ or $\bar{\mathbf{8}}$ of $SU(8)$).

Another possibility, which will reveal useful in the following, is to write coset representatives in a mixed basis, as

$$L(\varphi)_{\underline{M}}^{\underline{N}} = S^+_{\underline{M}}{}^{\underline{P}} L(\varphi)_{\underline{P}}^{\underline{N}}, \quad (4.15)$$

so that they can be used to interpolate between $SL(8, \mathbb{R})$ -covariant and $SU(8)$ -covariant objects. This can be the case when couplings between scalar and vector fields appear.

In section 2.1.2, we discussed how the isometry group of the scalar manifold (here $E_{7(7)}$) coincides with the duality group of global symmetries of the equations of motion, acting linearly on the vector fields contained in the theory, but it is not a symmetry group for the lagrangian. If we stick to ungauged supergravity, the same equations of motion can be obtained from a set of inequivalent lagrangians, corresponding in the case of maximal supergravity to the double quotient space

$$GL(28, \mathbb{R}) \setminus Sp(56, \mathbb{R})/E_{7(7)}, \quad (4.16)$$

(from the general expression (2.6)). Elements of (4.16) can be explicitly represented through matrices $E \in Sp(56, \mathbb{R})$ that define the embedding of $E_{7(7)}$ into $Sp(56, \mathbb{R})$, or equivalently the embedding of the 28 electric vector fields into the 56-dimensional

³ Here we exploit an Iwasawa decomposition of $E_{7(7)}$, which is a non-compact form of E_7 , with respect to its maximal compact subgroup $SU(8)$ [39].

representation space of $E_{7(7)}$ [45]. As a consequence, the action of the duality group $G_U = E_{7(7)}$ on vector field strengths and on scalar fields is not the same, but transformations of the ones and the others are related by a constant E matrix:

$$\begin{aligned}\delta F^M &= F^N (t_\alpha)_N{}^M, \\ \delta L(\phi)_{\hat{M}}{}^N &= (E^{-1})_{\hat{M}}{}^P (t_\alpha)_P{}^Q E_Q{}^{\hat{R}} L(\phi)_{\hat{R}}{}^N,\end{aligned}\quad (4.17)$$

where $t_\alpha \in \mathfrak{e}_{7(7)}$. We use for the moment a notation distinguishing between indices $M, N \dots$ and $\hat{M}, \hat{N} \dots$, which both refer to some 56-dimensional representation of $E_{7(7)}$, but denote different basis: the former correspond to the basis where the 28 electric vector fields and their magnetic duals are $A^M = (A^I, A_I)$, while the latter are associated to the basis in which we have first defined the infinitesimal $E_{7(7)}$ action (equation (4.2)), with basis vectors $x^{\hat{M}} = (x_{AB}, y^{CD})$. We will not keep this distinction in any other part of this thesis: following [45], we will indeed always reabsorb transformations $E_M{}^{\hat{N}}$ into the definition of coset representatives

$$L(\phi)_{\hat{M}}{}^N \rightarrow L(\phi)_M{}^N = E_M{}^{\hat{P}} L(\phi)_{\hat{P}}{}^N = (L_M{}^{ij}, L_{Mmn}) = \begin{pmatrix} L_I{}^{ij} & L_{I mn} \\ L^J{}_{ij} & L^J{}_{mn} \end{pmatrix}. \quad (4.18)$$

The drawback is that the object $L(\phi)_M{}^N$ used as coset representative is no more an element of $E_{7(7)}$ (it only belongs to $\text{Sp}(56, \mathbb{R})$), unless $E = \mathbb{1}$.

Once we have fixed coset representatives, we can define the Maurer–Cartan form, taking values in $\mathfrak{e}_{7(7)}$ (see section A.3),

$$\Omega(\phi) = L^{-1}(\phi) dL(\phi) = \mathcal{V}^a(\phi) t_a + \omega^i(\phi) t_i, \quad (4.19)$$

where t_i and t_a denote respectively generators of the $\mathfrak{su}(8)$ subalgebra and the remaining generators of $\mathfrak{e}_{7(7)}$. It is important to remark that the Maurer–Cartan form is not affected by changes of basis such as (4.15), nor it depends on transformations (4.18), because they are all realized through constant matrices. This means that the choice of the symplectic frame, encoded in E, has no consequence on the coset manifold geometry, even though it affects the explicit expression of the lagrangian and also the embedding tensor of the gauged theory.

The vielbein \mathcal{V} and the $\text{SU}(8)$ -connection ω can be found by exploiting the block-diagonal structure (4.13) of generators in the complex basis

$$(L^{-1}(\phi) dL(\phi))_{\underline{M}}{}^{\underline{N}} = \begin{pmatrix} \omega_{ij}{}^{mn} & \mathcal{V}_{ijpq} \\ \mathcal{V}^{klmn} & \omega^{kl}{}_{pq} \end{pmatrix}. \quad (4.20)$$

By definition of symplectic matrices, the coset representatives always satisfy $L^T \Omega L = \Omega$, which in the mixed basis becomes

$$(L^T)_{\underline{N}}{}^{\underline{M}} \Omega^{\underline{NP}} L_{\underline{P}}{}^{\underline{Q}} = \Omega^{\underline{MQ}}. \quad (4.21)$$

The symplectic form Ω in the real basis takes the form $\Omega^{MN} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$, while in the complex basis is $\Omega^{\underline{MN}} = \begin{pmatrix} 0 & i\mathbb{1} \\ -i\mathbb{1} & 0 \end{pmatrix}$. The invariance relation (4.21) allows to easily obtain the inverse of coset representatives as

$$(L^{-1})_{\underline{M}}^{\underline{N}} = (\Omega^{-1})_{\underline{MP}} (L^T)^P_Q \Omega^{QN} = \begin{pmatrix} -iL^I_{ij} & iL_{Jij} \\ iL^{Ikl} & -iL_{Jkl} \end{pmatrix}. \quad (4.22)$$

Then, the following expressions can be derived:

$$\begin{aligned} \mathcal{V}_x{}^{ijkl} &= i L_{Mij} \Omega^{MN} \partial_x L_{Nkl} = -i L^I_{ij} \partial_x L_{Ikl} + i L_{Iij} \partial_x L^I_{kl}, \\ \omega_x{}^{ijkl} &= i L_{Mij} \Omega^{MN} \partial_x L_N{}^{kl} = -i L^I_{ij} \partial_x L_I{}^{kl} + i L_{Iij} \partial_x L^{Ikl}, \end{aligned} \quad (4.23)$$

where indices $x, y, \dots = 1, \dots, 70$ refer to curved coordinates ϕ^x on the scalar manifold. To ensure compatibility with the Lie algebra $\mathfrak{e}_{7(7)}$ [45], $\mathcal{V}_x{}^{ijkl}$ is a self-dual SU(8) tensor

$$\mathcal{V}_x{}^{ijkl} = \frac{1}{24} \varepsilon^{ijklmnpq} \mathcal{V}_x{}_{mnpq}. \quad (4.24)$$

For the same reason, the SU(8)-connection can be reduced to a traceless rank-2 tensor

$$\omega_x{}^{ijkl} = \delta_{[i}^{[k} \omega_x{}^{l]} \leftrightarrow \omega_x{}^j{}_i = \frac{2}{3} \omega_x{}^{kj}{}_i. \quad (4.25)$$

The connection ω can be used to define an SU(8)-covariant derivative, as in any coset manifold

$$\partial_x L \rightarrow \mathcal{D}_x L = \partial_x L - \omega^i{}_x L t_i. \quad (4.26)$$

The action of covariant derivatives on coset representatives (A.23) can be written, exploiting the index structure, as

$$\mathcal{D}_x L_M{}^{\underline{N}} = \partial_x L_M{}^{\underline{N}} - L_M{}^{\underline{P}} \omega_x{}^{\underline{N}}_{\underline{P}}, \quad (4.27)$$

where the connection, according to (4.20), is $\omega_{\underline{M}}{}^{\underline{N}} = \begin{pmatrix} \omega_{ij}{}^{mn} & 0 \\ 0 & \omega^{kl}{}_{pq} \end{pmatrix}$. The 1-form ω_x behaves as a gauge field under SU(8), as in (A.16b); it is a *composite* connection, in the sense that it is not associated with propagating vector degrees of freedom, but instead is defined from coset representatives in order to get rid of the redundancy in parametrizing $E_{7(7)}/\text{SU}(8)$. It allows to build covariant derivatives also for fermionic fields, which transform linearly under SU(8) (derivatives appearing in the ungauged lagrangian will be both spacetime and SU(8) covariant).

The vielbein is related to the covariant derivative of coset representatives by (A.24), which in coordinates becomes

$$\mathcal{D}_x L_M{}^{ij} = L_{Mkl} \mathcal{V}_x{}^{klj}. \quad (4.28)$$

This identity allows to write the scalar kinetic term of the ungauged lagrangian in terms of the vielbein, as in [46]:

$$e^{-1} \mathcal{L}_{\text{kin}} = -\frac{1}{12} \mathcal{V}_\mu^{ijkl} \mathcal{V}^\mu_{ijkl}, \quad (4.29)$$

where we use a pull-back to switch from curved coordinates on the coset manifold to spacetime coordinates: $\mathcal{V}_\mu = \partial_\mu \varphi^x \mathcal{V}_x$.

4.2 GAUGING THE THEORY

The gauging procedure consists in promoting a subgroup of the U-duality group $G_U = E_{7(7)}$, which is a symmetry group of the equations of motion and not necessarily of the lagrangian, as we stressed in chapter 2, to a (local) gauge symmetry group. In order to construct the gauged theory and to obtain a lagrangian which is gauge-invariant but at the same time preserves supersymmetry, the fundamental steps to follow are the ones described in 2.1.3, which eventually lead to the introduction of fermionic mass terms and of a scalar potential.

The maximal supergravity theory was first studied with an $SO(8)$ gauging, since it was already known that in extended supergravity with \mathcal{N} supersymmetry generators the $\frac{\mathcal{N}(\mathcal{N}-1)}{2}$ vector fields appearing in the graviton multiplet can be used to build the connection of an $SO(\mathcal{N})$ gauge group. Of course, this is not the only possible choice, but it is not obvious in principle whether a given group can be gauged or not. The best way to select all the admissible gaugings for the $\mathcal{N} = 8$ theory is to adopt the embedding tensor formalism described in 2.2 (see [45], [46]), which allows to study the gauged theory without fixing a priori the gauge group.

4.2.1 The embedding tensor and its constraints

The embedding tensor Θ_M^α parametrizes the gauge group G_{gauge} as a subgroup embedded into $E_{7(7)}$; this happens through the definition of gauge generators, as in (2.18),

$$X_M \equiv \Theta_M^\alpha t_\alpha \quad \text{with } M = 1, \dots, 56, \quad \alpha = 1, \dots, 133, \quad (4.30)$$

where t_α are the generators of $\mathfrak{e}_{7(7)}$. As it is shown by the index structure, the embedding tensor transforms in the tensor product of the **56** (fundamental) and **133** (adjoint) representations of $E_{7(7)}$. Its rank, i.e. the number of independent generators among the X_M , gives the dimension of the gauge group, which cannot exceed the number of electric vector fields: $\dim(G_{\text{gauge}}) \leq 28$.

In order to correspond to a consistent gauging, the embedding tensor has to satisfy the set of linear and quadratic constraints discussed in 2.2.1. The linear constraint, also called *representation constraint*, follows from requiring the cancellation of the terms associated to supersymmetry variation of the gauged lagrangian at order linear in the

coupling constant (and then linear in Θ). In maximal supergravity, the representation space in which the embedding tensor is defined can be decomposed according to

$$56 \otimes 133 = 56 \oplus 912 \oplus 6480 , \quad (4.31)$$

but supersymmetry restricts Θ_M^α to be in the **912** representation. This condition, given the decomposition (4.31), implies the equations

$$(t_\alpha)_M^N \Theta_N^\alpha = 0 \quad , \quad (t_\gamma t_\beta \gamma^{\alpha\beta})_M^N \Theta_N^\gamma = -\frac{1}{2} \Theta_M^\alpha , \quad (4.32)$$

where $\gamma^{\alpha\beta}$ is the inverse of the Cartan–Killing metric defined on the Lie algebra [46].

The quadratic constraint is (2.21), coming from the requirement of gauge invariance of the theory; it implies (2.22), which is interpreted as the closure of the gauge algebra. However, from the relation $[X_M, X_N] = -X_{MN}^P X_P$ it does not automatically follow that the quantities $-X_{MN}^P = -\Theta_M^\alpha (t_\alpha)_N^P$ are the structure constants of the algebra; it only guarantees that the two quantities coincide after contraction with the gauge generators, i.e.

$$-X_{MN}^P X_P = f_{MN}^P X_P , \quad (4.33)$$

where f_{MN}^P are meant to be the correct structure constants. In the generic case, we can split X_{MN}^P into an antisymmetric and a symmetric component (with respect to the first two indices

$$X_{MN}^P = X_{[MN]}^P + Z^P_{MN} , \quad (4.34)$$

where $Z^P_{MN} = Z^P_{(MN)}$ and consistency with (2.22) requires that $Z^P_{MN} X_P = 0$.

The presence of non-vanishing Z^P_{MN} has an important consequence⁴: if we assume that the structure constants coincide with the antisymmetrized $X_{[MN]}^P$, they fail to satisfy ordinary Jacobi identities [13]

$$X_{[MN]}^P X_{[QP]}^R + X_{[NQ]}^P X_{[MP]}^R + X_{[QM]}^P X_{[NP]}^R = -Z^R_{P[Q} X_{MN]}^P , \quad (4.35)$$

where the RHS vanishes only after contraction with the X_R . These identities are actually the analogue of (2.46), which were obtained from dimensional reduction in presence of fluxes. As already mentioned, the resulting gauge algebra is not a Lie algebra in general, but rather it can be described consistently as a free differential algebra [16], provided we introduce some tensor gauge fields $B_{\mu\nu}$. The explicit 4-dimensional construction will be described in section 4.2.3.

Once the linear representation constraint is imposed, the quadratic one (2.21) is equivalent to the locality constraint (2.23), which can be also expressed as

$$\Theta^{I[\alpha} \Theta_I^{\beta]} = 0 , \quad (4.36)$$

⁴ For the theory of maximal supergravity in 4 dimensions, Z^P_{MN} are never vanishing because it can be shown [46] that requiring $X_{MN}^P = X_{[MN]}^P$ would violate (4.32).

where we again adopt the convention that splits vector indices in the electric and the magnetic part, then $\Theta_M^\alpha = \begin{pmatrix} \Theta_I^\alpha \\ \Theta^{I\alpha} \end{pmatrix}$. This condition implies that, for any gauge group, it is possible to choose a symplectic frame, i.e. an embedding of G_{gauge} into $\text{Sp}(56, \mathbb{R})$, encoded by a matrix E in the double quotient (4.16), such that

$$\Theta^{I\alpha} = 0. \quad (4.37)$$

As a consequence, when we construct the gauge connection, which is in general

$$A_\mu = A_\mu^M X_M = (A_\mu^I \Theta_I^\alpha + A_{\mu I} \Theta^{I\alpha}) t_\alpha, \quad (4.38)$$

only electric vector fields A_μ^I are involved: equation (4.37) defines a purely electric gauging. However, in order to keep $E_{7(7)}$ covariance, in the following we will not assume to be necessarily in the electric frame and then we will keep also magnetic vector fields.

In general, different values of the embedding tensor correspond to either different gauge groups or different embeddings into $E_{7(7)}$ of the same gauge group. However, once the symplectic frame has been fixed, there is not a one-to-one correspondence between allowed embedding tensors and inequivalent gauged supergravity models, because many distinct values of the embedding tensors can correspond to theories related between each other by U-duality transformations. A classification of the inequivalent theories, relying on the analysis of some duality-invariant tensorial quantities constructed from the embedding tensor, can be found in [47].

4.2.2 The scalar sector

Once we have introduced the embedding tensor that describes the gauge group, the first step in the construction of a gauge invariant lagrangian is the definition of a covariant derivative from the connection (4.38)

$$\partial_\mu \rightarrow \widehat{\partial}_\mu = \partial_\mu - g A_\mu^M X_M = \partial_\mu - g A_\mu^M \Theta_M^\alpha t_\alpha, \quad (4.39)$$

where g is the gauge coupling constant and the symbol $\widehat{}$ denotes, here and in the following, gauge-covariantized quantities. We can observe from this definition that the embedding tensor Θ_M^α plays the role of a charge matrix.

We want now to analyze how the gauging modifies the structure of the scalar sector, again exploiting the formalism of coset manifolds. To begin with, we can define a gauge-covariant Maurer–Cartan form (to compare with (4.19))

$$\widehat{\Omega} = L^{-1} \widehat{d}L \equiv L^{-1} (d - g A^M X_M) L = \widehat{\mathcal{V}}^a t_a + \widehat{\omega}^i t_i, \quad (4.40)$$

where we have introduced the gauged versions of vielbein and SU(8)-connection. Making explicit the above expression, they are

$$\widehat{\mathcal{V}}^a = \mathcal{V}^a - gA^M \Theta_M^\alpha \underbrace{(L^{-1}t_\alpha L)^a}_{\xi_\alpha^x \mathcal{V}_x^a}, \quad (4.41a)$$

$$\widehat{\omega}^i = \omega^i - gA^M \Theta_M^\alpha \underbrace{(L^{-1}t_\alpha L)^i}_{\mathcal{P}_\alpha^i}, \quad (4.41b)$$

where, comparing with the definitions given in section A.3, the gauged vielbein $\widehat{\mathcal{V}}$ and SU(8)-connection $\widehat{\omega}$ are written in terms of the Killing vectors ξ_α^x and prepotentials \mathcal{P}_α^i respectively. In components,

$$\widehat{\mathcal{V}}_{\mu ijkl} = \mathcal{V}_{\mu ijkl} - igA_\mu^M L_{Nij} \Omega^{NP} X_{MP}^Q L_{Qkl}, \quad (4.42a)$$

$$\widehat{\omega}_{\mu ij}{}^{kl} = \omega_{\mu ij}{}^{kl} - igA_\mu^M L_{Nij} \Omega^{NP} X_{MP}^Q L_Q{}^{kl}, \quad (4.42b)$$

where $\mathcal{V}_{\mu ijkl}$ and $\omega_{\mu ij}{}^{kl}$ can be taken from (4.23), once we have replaced the derivatives ∂_x with the pull-back $\partial_\mu = (\partial_\mu \varphi^x) \partial_x$. We can introduce now a covariant derivative for coset representatives that is both gauge and SU(8)-covariant

$$\widehat{\mathcal{D}}_\mu L = \partial_\mu L - L \widehat{\omega}_\mu - gA_\mu^M X_M, \quad (4.43)$$

which in components becomes

$$\widehat{\mathcal{D}}_\mu L_M{}^{ij} = \partial_\mu L_M{}^{ij} - L_M{}^{kl} \widehat{\omega}_{\mu kl}{}^{ij} - gA_\mu^N X_{NM}{}^P L_P{}^{ij}. \quad (4.44)$$

The gauged version of the identity (A.24) means that the vielbein can be expressed as

$$\widehat{\mathcal{V}}_{\mu ijkl} = iL_{Mij} \Omega^{MN} \widehat{\mathcal{D}}_\mu L_{Nkl}, \quad (4.45)$$

and this quantity enters the definition of the gauge-covariant kinetic term for the scalar fields, in analogy with the ungauged case.

4.2.3 The vector sector

The set of electric and magnetic vector fields should transform under the gauge group as in (2.20b). Then, the natural choice for the non-abelian field strengths of the theory is

$$\mathcal{F}^M{}_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}^M + gX_{[NP]}^M A_\mu^N A_\nu^P, \quad (4.46)$$

which follows from the Ricci identity $[\widehat{\partial}_\mu, \widehat{\partial}_\nu] = -g\mathcal{F}^M{}_{\mu\nu} X_M$, and we would expect it to be enough to guarantee gauge invariance of the vector kinetic term. Actually, this is not the case, because field strengths defined as in (4.46) transform in the following way:

$$\delta \mathcal{F}^M{}_{\mu\nu} = -g\Lambda^N X_{NP}{}^M \mathcal{F}^P{}_{\mu\nu} + 2gZ^M{}_{NP} (\Lambda^N \mathcal{F}^P{}_{\mu\nu} - A_{[\mu}^N A_{\nu]}^P), \quad (4.47)$$

where only the first term corresponds to an ordinary covariant transformation, while the second one would require the addition of new terms in the lagrangian in order to be compensated.

The standard solution ([16], [48]) is to introduce tensor fields, in particular 2-forms, coupled to vectors through the Z^P_{MN} (it is a Stückelberg-like coupling, similar to those appearing in massive deformations of supergravity [13]) and then define modified field strengths

$$H^M_{\mu\nu} \equiv \mathcal{F}^M_{\mu\nu} + Z^M_{NP} B^{NP}_{\mu\nu}. \quad (4.48)$$

The new gauge transformation rules for vector and tensor fields are

$$\begin{aligned} \delta A_\mu^M &= \widehat{\partial}_\mu \Lambda^M - Z^M_{NP} \Sigma_\mu^{NP}, \\ \delta B^{MN}_{\mu\nu} &= 2\widehat{\partial}_{[\mu} \Sigma_{\nu]}^{MN} - 2\Lambda^{[M} H^{N]}_{\mu\nu} + 2A_{[\mu}^{(M} \delta A_{\nu]}^N, \end{aligned} \quad (4.49)$$

where Σ^{MN} is the gauge parameter associated to the tensor field B^{MN} . These transformation rules can be interpreted as the relations defining the free differential algebra associated to the gauging. As a consequence of (4.49), the modified field strength transforms covariantly:

$$\delta H^M_{\mu\nu} = -X_{PN}^M \Lambda^P H^N_{\mu\nu}. \quad (4.50)$$

In order to guarantee that the full lagrangian is invariant under the vector gauge transformations, we need to add new terms to the action [46]:

$$\begin{aligned} e^{-1} \mathcal{L}_{\text{top}} &= \frac{i}{8} g \varepsilon^{\mu\nu\rho\sigma} \Theta^{I\alpha} B_{\mu\nu\alpha} \left(2\partial_\rho A_{\sigma I} + g X_{MNI} A_\rho^M A_\sigma^N - \frac{1}{4} \Theta_I^\beta B_{\rho\sigma\beta} \right) + \\ &+ \frac{i}{3} g \varepsilon^{\mu\nu\rho\sigma} X_{MNI} A_\mu^M A_\nu^N \left(\partial_\rho A_\sigma^I + \frac{1}{4} g X_{PQ}^I A_\rho^P A_\sigma^Q \right) + \\ &+ \frac{i}{6} g \varepsilon^{\mu\nu\rho\sigma} X_{MN}^I A_\mu^M A_\nu^N \left(\partial_\rho A_{\sigma I} + \frac{1}{4} g X_{PQI} A_\rho^P A_\sigma^Q \right), \end{aligned} \quad (4.51)$$

where we identify $B_\alpha = -(t_\alpha)_{NP} B^{NP}$. Here, in the first line we have a topological coupling between the antisymmetric tensor B^{MN} and the magnetic vectors A_I , while the remaining terms are a generalization of the Chern–Simons-like couplings that are needed in order to ensure gauge invariance even in the case of electric gauging.

Clearly the tensor fields B^{MN} cannot add new degrees of freedom to the theory; they should instead be dual to some of the scalar fields of the ungauged lagrangian. Such a duality relation can be obtained explicitly from the equations of motion for the vector fields, so that substituting the solution to these equations in the action eliminates the tensor fields, which is equivalent to perform a duality rotation to the electric frame. However, the presence of tensors B^{MN} in the action if we don't fix the symplectic frame is not problematic: indeed, it is in agreement with the possible higher-dimensional origin of supergravity theories, since tensor fields come up naturally from flux compactification, as we discussed in 2.4.

4.3 THE T -TENSOR

After describing how the gauging procedure and the requirement of gauge invariance modify the scalar coset manifold, as well as the effects on the vector sector of the lagrangian, we have to consider, according to the general procedure outlined in 2.1.3, which modifications are needed in order to restore supersymmetry invariance of the lagrangian after the gauging. This step, which introduces terms linear and quadratic in the gauge coupling constant g , can be conveniently formalized through the T -tensor, which was first introduced in [41]. It can be defined as the embedding tensor "dressed" with coset representatives (in order to couple to fermions), then it depends on the embedding tensor, but also on scalar fields:

$$T_{\underline{M}}{}^{\alpha}(\Theta, \varphi) t_{\alpha} = (L^{-1})_{\underline{M}}{}^M \Theta_M{}^{\beta} (L^{-1} t_{\beta} L). \quad (4.52)$$

Equivalently, the T -tensor can be expressed, in the complex basis of the **56** representation, as

$$T_{\underline{MN}}{}^{\underline{P}}(\Theta, \varphi) = (L^{-1})_{\underline{M}}{}^M (L^{-1})_{\underline{N}}{}^N X_{MN}{}^P L_P{}^{\underline{P}}. \quad (4.53)$$

From these definitions, it is clear that T is an $SU(8)$ -covariant tensor, i.e. its indices transform under local $SU(8)$ transformations.

The constraints T obeys are a direct consequence of the constraints on the embedding tensor. In particular, T inherits the representation constraint on Θ

$$\mathbb{F}T(\Theta, \varphi) = 0 \quad \forall \varphi, \quad (4.54)$$

which forces T , just like Θ , to be in the **912** representation of $E_{7(7)}$. If this representation is decomposed according to the $SU(8)$ subgroup of $E_{7(7)}$, we get

$$\mathbf{912} \rightarrow \mathbf{36} \oplus \overline{\mathbf{36}} \oplus \mathbf{420} \oplus \overline{\mathbf{420}}. \quad (4.55)$$

In other words, the components of the T -tensor can be parametrized in terms of simpler tensors

$$T_{\underline{MN}}{}^{\underline{P}} \rightarrow (A_1^{ij}, A_{1ij}, A_{2i}{}^{jkl}, A_{2i}{}^i{}_{jkl}), \quad (4.56)$$

where A_1 is symmetric, while A_2 is antisymmetric in the last three indices and traceless

$$A_1^{ij} = A_1^{ji}, \quad A_{2i}{}^{jkl} = A_{2i}{}^{[jkl]}, \quad A_{2i}{}^i{}_{jk} = 0. \quad (4.57)$$

As usual with complex indices, we have $A_{1ij} = \overline{A_1^{ij}}$, $A_{2i}{}^i{}_{jkl} = \overline{A_{2i}{}^{jkl}}$.

If the tensor $T_{\underline{MN}}{}^{\underline{P}} = ((T)_{ij}{}^{\underline{P}}, (T)^{ij}{}_{\underline{N}}{}^{\underline{P}})$ is decomposed in blocks according to

$$(T)^{ij}{}_{\underline{N}}{}^{\underline{P}} = \begin{pmatrix} \frac{2}{3} \delta_{[k}^{[p} T_{l]}{}^{q]ij} & T_{mnpq}{}^{ij} \\ \frac{1}{24} \epsilon^{klrstuvw} T_{tuvw}{}^{ij} & -\frac{2}{3} \delta_{[r}^{[m} T_{s]}{}^{n]ij} \end{pmatrix}, \quad (4.58)$$

the final expression of T components in terms of A_1 and A_2 is [46]

$$\begin{aligned} T_k{}^{lij} &= -\frac{3}{4}A_{2k}{}^{lij} - \frac{3}{2}A_1{}^{l[i}\delta^j]{}_k, \\ T_{klmn}{}^{ij} &= -\frac{4}{3}\delta^{[i}T^{j]}{}_{[klmn]}. \end{aligned} \quad (4.59)$$

The gauging procedure always requires to modify the supersymmetry transformation rules for fermions, as already shown in (2.13), and in the case of the $\mathcal{N} = 8$ theory the tensors giving the shifts are precisely A_1 and A_2 :

$$\delta\psi_\mu{}^i = \delta_0\psi_\mu{}^i + \sqrt{2}gA_1{}^{ij}\gamma_\mu\epsilon_j, \quad \delta\chi^{ijk} = \delta_0\chi^{ijk} - 2gA_2{}^l{}_{ijk}\epsilon_l. \quad (4.60)$$

At linear order in g , this modification comes together with the introduction in the lagrangian of Yukawa terms for fermions, also referred to as mass-like terms, as in (2.12),

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{Yuk}} &= g\left(\frac{1}{\sqrt{2}}A_{1ij}\bar{\psi}_\mu{}^i\gamma^{\mu\nu}\bar{\psi}_\nu{}^j + \frac{1}{6}A_{2i}{}^{jkl}\bar{\psi}_\mu{}^i\gamma^\mu\chi_{jkl} + \right. \\ &\quad \left. + \frac{\sqrt{2}}{144}\varepsilon^{ijklmnpq}\bar{\chi}_{ijk}A_{2r}{}^{lmn}\chi_{pqr}\right) + h.c.. \end{aligned} \quad (4.61)$$

To conclude, at order g^2 supersymmetry requires the addition of a scalar-dependent term that, as already discussed, plays the role of the scalar potential

$$V = g^2\left(\frac{1}{24}A_{2n}{}^{jkl}A_{2n}{}^{jkl} - \frac{3}{4}A_{1kl}A_1{}^{kl}\right). \quad (4.62)$$

4.4 THE GAUGED LAGRANGIAN

We report here, for completeness, the whole expression of the gauged lagrangian, giving some comments on the terms that have not been discussed in the rest of the chapter.

$$\begin{aligned} e^{-1}\mathcal{L} &= \frac{1}{2}R - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\bar{\psi}_\mu{}^i\gamma_\nu\widehat{\mathcal{D}}_\rho\psi_{\sigma i} - \bar{\psi}_\mu{}^i\overleftarrow{\widehat{\mathcal{D}}}_\rho\gamma_\nu\psi_{\sigma i}\right) + \\ &\quad - \frac{1}{12}\left(\bar{\chi}^{ijk}\gamma^\mu\widehat{\mathcal{D}}_\mu\chi_{ijk} - \bar{\chi}^{ijk}\overleftarrow{\widehat{\mathcal{D}}}_\mu\gamma^\mu\chi_{ijk}\right) - \frac{1}{12}|\widehat{\mathcal{V}}_\mu{}^{ijkl}|^2 + \\ &\quad - \frac{\sqrt{2}}{6}\left(\bar{\chi}_{ijk}\gamma^\nu\gamma^\mu\psi_{\nu l}\widehat{\mathcal{V}}_\mu{}^{ijkl} + h.c.\right) + \left(H^{I+\mu\nu}\mathcal{O}_{I+\mu\nu} + h.c.\right) + \\ &\quad - \frac{i}{4}\left(\mathcal{N}_{IJ}H^{I+\mu\nu}H^{J+\mu\nu} - \overline{\mathcal{N}}_{IJ}H^{I-\mu\nu}H^{J-\mu\nu}\right) + \\ &\quad + e^{-1}\mathcal{L}_{\text{top}} + e^{-1}\mathcal{L}_{\text{Yuk}} - V. \end{aligned} \quad (4.63)$$

The first terms in the lagrangian are the Einstein–Hilbert and Rarita–Schwinger kinetic terms respectively for the graviton and the gravitini, followed by the Dirac lagrangian for fermions, the scalar kinetic term, expressed as a function of the vielbein, and the Noether couplings between scalars and fermions, required by supersymmetry invariance.

The next terms correspond to the vector kinetic terms and the couplings of the vector fields with the fermions and the tensor fields. Here we have replaced the usual non-abelian field strengths \mathcal{F}^M with the modified field strengths H^M defined in (4.48); following [16], we split them into self-dual and anti-self-dual components H^{M+} , H^{M-} normalized in such a way that

$$H_{\mu\nu}^M = H_{\mu\nu}^{M+} + H_{\mu\nu}^{M-} . \quad (4.64)$$

The tensor $\mathcal{O}^{M+}_{\mu\nu}$ is defined according to

$$\mathcal{O}^{+ij}_{\mu\nu} = \frac{\sqrt{2}}{2} \bar{\psi}_\rho^i \gamma^{[\rho} \gamma_{\mu\nu} \gamma^{\sigma]} \psi_\sigma^j - \frac{1}{2} \bar{\psi}_{\rho k} \gamma_{\mu\nu} \gamma^\rho \chi^{ijk} - \frac{\sqrt{2}}{144} \varepsilon^{ijklmnpq} \bar{\chi}_{klm} \gamma_{\mu\nu} \chi_{npq} , \quad (4.65)$$

and $\frac{i}{4} \mathcal{O}^{+ij}_{\mu\nu} = L^{Iij} \mathcal{O}_I^+_{\mu\nu}$ [46].

FLAT SUPERSYMMETRIC VACUA OF MAXIMAL SUPERGRAVITY

In the previous chapter we have outlined the main features of the theory of maximal supergravity in 4 dimensions, with particular attention to the origin and the expression of the scalar potential; now, we focus on the study of its Minkowski vacua. The technique we use to identify the possible vacua has been first employed in [49], [43]. It exploits the embedding tensor formalism, together with the transitive action of the duality group on the scalar manifold. Indeed, the scalar potential can be expressed as a quadratic function of the embedding tensor and a non-linear function of the scalar fields, but different points of the scalar manifold can be related one to another by a duality transformation, as well as different values of the embedding tensor. As a consequence, we can recover information on the whole scalar manifold just by computing the scalar potential and its derivatives at the origin of the scalar manifold. In particular, the extremization condition and the requirement of vanishing potential, which are needed to select a Minkowski vacuum, can be expressed as quadratic equations for the embedding tensor, to be solved simultaneously with the quadratic and linear constraints on the embedding tensor that guarantee to have a consistent gauging.

Although this procedure considerably simplifies the scan of allowed vacuum configurations, it is still not possible to solve analytically the problem in full generality. We will concentrate, instead, on verifying the presence of flat vacua in the case of specific supersymmetry breaking patterns. While partial supersymmetry breaking to an even number of supersymmetries is kinematically possible, and indeed some vacua are found, partial supersymmetry breaking to an odd number of residual supersymmetries is extremely constrained.

5.1 THE TECHNIQUE

Preliminarily, we make an important remark: the scalar potential can be expressed in terms of a real, symmetric, field-dependent matrix, \mathcal{M}_{MN} , defined from coset representatives as

$$\mathcal{M}_{MN} = L_M^{ij} L_{Nij} + L_N^{ij} L_{Mij}. \quad (5.1)$$

It is positive definite and its inverse is

$$\mathcal{M}^{MN} = \Omega^{MP} \Omega^{NQ} \mathcal{M}_{PQ} . \quad (5.2)$$

The scalar potential, written in terms of \mathcal{M}_{MN} , takes the form

$$V = \frac{1}{672} g^2 \left(X_{MN}{}^R X_{PQ}{}^S \mathcal{M}^{MP} \mathcal{M}^{NQ} \mathcal{M}_{RS} + 7 X_{MN}{}^Q X_{PQ}{}^N \mathcal{M}^{MP} \right) , \quad (5.3)$$

as derived in [46]. This equation makes clear the quadratic dependence of the potential on the embedding tensor. At the same time, being \mathcal{M}_{MN} manifestly $SU(8)$ invariant, we can see how $SU(8)$ transformations do not affect V .

In order to find the possible Minkowski vacua of our theory, in principle we should select a viable gauging (i.e. an embedding tensor verifying the quadratic and linear constraints) and, once the choice of the gauging has fixed the expression of the scalar potential $V = V(\varphi)$, impose the extremization conditions

$$\frac{\partial V(\varphi)}{\partial \varphi^{ijkl}} = 0 ,$$

together with the equation $V(\varphi) = 0$ (since we are interested in flat vacua). This system of equations is not easy to solve in full generality due to the dependence of $V(\varphi)$ on the 70 scalar fields: even if we employ an Iwasawa decomposition, the potential will have an exponential dependence on the fields associated with the generators of the Cartan subalgebra and a polynomial dependence on those corresponding to the nilpotent generators [6], [43].

Our approach to the study of vacua is based instead on the structure of the scalar manifold as a homogeneous space: any point can be connected to any other by an $E_{7(7)}$ transformation. On the other hand, elements of $E_{7(7)}$ act linearly on the embedding tensor. The key point in our analysis is that the scalar potential, which is in general a function of the embedding tensor and of coset representatives $V = V(L, \Theta)$, has a dependence of type

$$V = V(L^{-1}\Theta) , \quad (5.4)$$

where $L^{-1}\Theta$ denotes the tensor combination where the embedding tensor indices are fully contracted. Exploiting homogeneity of $\mathcal{M}_{\text{scalar}}$, we can map any (critical) point to a fixed point φ' , at the price of changing accordingly the form of the embedding tensor. Denoting with $U \in E_{7(7)}$ the group element mapping a generic point φ into the chosen φ' , we have

$$\begin{aligned} UL(\varphi) &= L(\varphi')h(\varphi, \varphi') , \quad h \in SU(8) , \\ U\Theta &= \Theta' , \end{aligned} \quad (5.5)$$

and the simultaneous application of the above transformations leaves the scalar potential invariant

$$V(L, \Theta) = V(UL, U\Theta) , \quad U \in E_{7(7)} . \quad (5.6)$$

In particular, we will perform all calculations at the origin of the scalar manifold, i.e. $\varphi' = 0$. The choice is motivated by the fact that this point is invariant under the action of the isotropy group $SU(8)$. As a consequence, we can consider only transformations U corresponding to the non-compact directions of $E_{7(7)}$, which parametrize nothing but the coset manifold $E_{7(7)}/SU(8)$. Then, there is a one-to-one correspondence between points of the scalar manifold and transformations of the embedding tensor.

Concretely, the scalar potential is defined from the irreducible components of the T -tensor (4.53). Starting from a given value of the tensor $T(\varphi, \Theta)$, if we consider the transformation U such that $UL(\varphi) = L(0)h(\varphi)$, the same transformation acts on the gauge algebra generators as

$$X_{MN}{}^P = \Theta_M{}^\alpha (t_\alpha)_N{}^P \quad \rightarrow \quad X'_{MN}{}^P = U_M{}^Q U_N{}^R X_{QR}{}^S U^{-1}{}_S{}^P. \quad (5.7)$$

If we take the tensor T' computed from modified generators $X'_{MN}{}^P$ at the point $\varphi' = 0$, it is immediate to see that the equality

$$T'_{\underline{MN}}{}^{\underline{P}}(0) = L^{-1}(0)_{\underline{M}}{}^M L^{-1}(0)_{\underline{N}}{}^N X'_{MN}{}^P L(0)_P{}^{\underline{P}} = T_{\underline{MN}}{}^{\underline{P}}(\varphi) \quad (5.8)$$

holds up to the action of $h(\varphi)$, which, as already stressed, has no impact at the level of the scalar potential.

In other words, we have shown that, as far as the analysis of the scalar potential is concerned, moving on the scalar manifold via a transformation $U \in E_{7(7)}$ is the same as modifying the embedding of the gauge group into $E_{7(7)}$ with the inverse transformation U^{-1} . Thus, it is not restrictive to look for vacua of the theory just by sitting at the origin of the scalar manifold. In this way, we can scan over all allowed embedding tensor values at once and the advantage is that the system to solve in order to select a vacuum configuration reduces to a set of quadratic equations for the embedding tensor.

5.2 MINKOWSKI VACUA

The quadratic constraints (2.21) (or equivalently (4.36)) to impose on the embedding tensor in order to have a consistent gauging are equivalent to the following quadratic identities for the tensors A_1 and A_2 :

$$0 = A_2^k{}_{lij} A_2 n^{mij} - A_2^i{}_{kij} A_2^m{}_{nij} - 4A_2^{(k}{}_{lini} A_1^{m)i} - 4A_2^{(n}{}_{mki} A_1 l)_i + \\ - 2\delta_l{}^m A_1 n_i A_1^{ki} + 2\delta_n{}^k A_1 l_i A_1^{mi}, \quad (5.9)$$

$$0 = A_2^i{}_{jk[m} A_2^k{}_{npq]} + A_1 jk \delta_{[m}^i A_2^k{}_{npq]} - A_1 j[m A_2^i{}_{npq]} + \\ + \frac{1}{24} \varepsilon_{mnpqrstu} \left(A_2 j^{ikr} A_2 k^{stu} + A_1^{ik} \delta_j{}^r A_2 k^{stu} - A_1^{ir} A_2 j^{stu} \right), \quad (5.10)$$

$$0 = A_2^r{}_{ijk} A_2 r^{mnp} - 9A_2^{[m}{}_{r[ij} A_2 k]^{np]r} - 9\delta_{[i}^{[m} A_2^n{}_{|rs|j} A_2 k]^{p]rs} + \\ - 9\delta_{[ij}^{[mn} A_2^{|u|}{}_{k]rs} A_2 u^{p]rs} + \delta_{ijk}{}^{mnp} A_2^u{}_{rst} A_2 u^{rst}. \quad (5.11)$$

These equations are the ones we will employ in our computations, following the procedure already adopted in [50] to find AdS vacua of the maximal supergravity theory.

In order to find Minkowski vacua of the theory, apart from solving the equations (5.9), (5.10), (5.11), we need to impose that the scalar potential

$$V = -\frac{3}{4}g^2 \left(A_{1kl} A_1{}^{kl} - \frac{1}{18} A_{2n}{}^{jkl} A_2{}^n{}_{jkl} \right) \quad (5.12)$$

has a vanishing value, together with the extremization condition of the potential, which can be expressed requiring that the tensor

$$C_{ijkl} = A_2{}^m{}_{[ijk} A_{1l]m} + \frac{3}{4} A_2{}^m{}_{n[ij} A_2{}^n{}_{kl]m} \quad (5.13)$$

becomes anti-self dual

$$C_{ijkl} + \frac{1}{24} \varepsilon_{ijklmnpq} C^{mnpq} = 0. \quad (5.14)$$

In this way, we can find vacuum solutions using as unknowns exclusively the components of the tensors A_1 , A_2 . Since we only have to solve homogeneous quadratic equations in the tensor components, a scaling constant will remain undetermined for any solution we can find. The embedding tensor and the gauge group can then be reconstructed making use of the defining relation for the T -tensor and its expression in terms of A_1 , A_2 .

Due to the high number of independent degrees of freedom involved, in order to find analytical solutions we cannot simply consider the full set of equations and find at once all the possible solutions; instead, we need to divide the problem into smaller subcases, i.e. to put some constraints on the components of tensors A_1 and A_2 . Our strategy is to proceed by considering some specific supersymmetry breaking patterns, starting from cases when the residual supersymmetry is larger (then the computational treatment is easier) and progressively reducing the number of preserved supersymmetry charges.

Here and in the following of this chapter we denote the indices corresponding to unbroken supercharges with capital letters, while the remaining indices are denoted with the first letters of the alphabet in lower cases,

$$I, J, \dots = 1, \dots, \mathcal{N} \quad ; \quad a, b, \dots = \mathcal{N} + 1, \dots, 8, \quad (5.15)$$

where \mathcal{N} denotes now the number of unbroken generators at the vacuum.

The supersymmetry requirement can be imposed at the level of the field transformation rules under supersymmetry, depending on infinitesimal spinorial parameters ϵ_i , with $i = 1, \dots, 8$. In a Minkowski vacuum all fermionic fields must take a vanishing value in order not to break Lorentz invariance, then the transformation rules for bosons, which necessarily involve fermionic fields to be contracted with ϵ parameters, will be

automatically null. What we need to verify are then the fermionic transformation rules, in particular the terms induced by the gauging

$$\begin{aligned}\delta_{\text{gauge}}\psi_{\mu}^i &= +\sqrt{2}gA_1^{ij}\gamma_{\mu}\epsilon_j, \\ \delta_{\text{gauge}}\lambda^{ijk} &= -2gA_{2_I}^{ijk}\epsilon^I.\end{aligned}$$

In order to impose a residual \mathcal{N} supersymmetry at the vacuum, these infinitesimal variations must vanish for every choice of the parameters ϵ^I , which immediately gives

$$A_1^{Ij} = A_1^{jI} = 0 \quad , \quad A_{2_I}^{ijk} = 0 \quad \forall I = 1, \dots, \mathcal{N} \quad \forall i, j, k.$$

In general, if we start from the theory with maximal supergravity (8 supersymmetry generators) and consider a vacuum configuration in which supersymmetry is partially broken, the R-symmetry group $SU(8)$ of the theory splits into

$$SU(8) \longrightarrow SU(\mathcal{N}) \times SU(8 - \mathcal{N}) \times U(1). \quad (5.16)$$

The $SU(8 - \mathcal{N})$ group can be further reduced to smaller subgroups, as we will see. Once we have fixed the R-symmetry group at the vacuum, by requiring invariance under this group we can get additional constraints on the tensors A_1 and A_2 , in order to reduce the number of independent unknowns in the equations we have to solve.

The analysis in terms of A_1 and A_2 is particularly useful for our purposes because they transform according to irreducible representations of $SU(8)$, then we can easily derive their transformation rules under the residual R-symmetry exploiting the branching rules for group representations, as we explain in more detail in appendix B.

5.3 RESIDUAL $\mathcal{N} = 6$ SUPERSYMMETRY

Since there is no supergravity theory with $\mathcal{N} = 7$ (its supermultiplet, in the CPT-completed version, would be equivalent to the $\mathcal{N} = 8$ one), we first look for vacua preserving $\mathcal{N} = 6$ supersymmetries. Imposing that the corresponding supersymmetry variations of the fermionic fields are vanishing, we get

$$A_1^{Ij} = A_1^{jI} = 0 \quad , \quad A_{2_I}^{ijk} = 0 \quad \forall I = 1, \dots, 6 \quad \forall i, j, k,$$

i.e. the A_1 tensor has only 4 non-vanishing components A_1^{ab} , while for A_2 we are left with 40 components of type $A_{2_a}^{IJK}$, 60 of type $A_{2_a}^{bIJ}$ and 12 of type $A_{2_a}^{bcI}$.

To further constrain the form of A_1 and A_2 we can now exploit the R-symmetry group. In particular, we consider the case

$$SU(8) \longrightarrow U(6) \times U(1),$$

where the $U(6)$ factor results from $SU(6) \times U(1)$ (compare with (5.16)), while the $U(1)$ symmetry comes as a subgroup of $SU(2)$, then it is embedded in the original $SU(8)$

group as a rotational symmetry acting on the indices $a, b \dots = 7, 8$. To get invariance of A_1 and A_2 under this additional $U(1)$ factor, the submatrix A_1^{ab} must be diagonal, while among the components of A_2 only the ones with two upper indices of type I, J can be different from zero. In particular, in order to respect both $U(1)$ invariance and the condition of vanishing trace, it must be

$$A_{2a}{}^{bIJ} = \varepsilon^{ab} \Sigma^{IJ} \quad \text{with} \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where the matrix Σ is 6×6 and antisymmetric.

For antisymmetric complex matrices of even dimension, *Youla's theorem* [51] guarantees that they can be "block-diagonalized", i.e. that there always exists a unitary matrix V such that $V^T \Sigma V$ has a block-diagonal structure, each block being proportional to the ε matrix through a real constant. In principle the theorem does not give information about how to find the proper matrix V , but in our case the matrix Σ must respect the underlying $U(6)$ symmetry, then we can always go to the basis in which

$$\Sigma = \begin{pmatrix} 0 & m_1 & & & & \\ -m_1 & 0 & & & & \\ & & 0 & m_2 & & \\ & & -m_2 & 0 & & \\ & & & & 0 & m_3 \\ & & & & -m_3 & 0 \end{pmatrix},$$

once we know that such a basis always exists. In conclusion, the remaining free parameters in A_1 and A_2 are just 4: m_1, m_2 and m_3 for A_2 , α for $A_1^{ab} = \alpha \delta^{ab}$ (among them only α is still a complex parameter).

Having a null potential immediately implies

$$\begin{aligned} A_{1kl} A_1^{kl} &= \frac{1}{18} A_{2n}{}^{jkl} A_2^n{}_{jkl} \quad \Leftrightarrow \\ A_{177} A_1^{77} + A_{188} A_1^{88} &= \frac{1}{18} \cdot 3 (A_{27}{}^{8IJ} A_2^7{}_{8IJ} + A_{28}{}^{7IJ} A_2^8{}_{7IJ}) \quad \Leftrightarrow \\ |\alpha|^2 &= \frac{1}{3} (m_1^2 + m_2^2 + m_3^2). \end{aligned}$$

If we want to find extremal points of V , the totally antisymmetric tensor \mathcal{C}_{ijkl} , considering all the restrictions on the structure of A_1 and A_2 , has non-vanishing components only if all the four indices are of type $I, J, K, L = 1, \dots, 6$ (in this case the second term in the RHS of (5.13) can be different from zero) or if two indices are of this type and the remaining two are of type $a, b = 7, 8$ (then the non-vanishing term is the first one in (5.13)). From the anti-selfduality condition we get

$$\frac{3}{2} A^7{}_{8[IJ} A^8{}_{KL]7} + \frac{1}{2} \varepsilon_{IJKLMN78} \mathcal{C}^{MN78} = 0$$

and assuming without loss of generality to choose for any I, J, K, L the indices M, N ordered in the proper way so that $IJKLMN$ is an even permutation, the equation we obtain is

$$A^7_{8IJ}A^8_{KL7} - A^7_{8IK}A^8_{JL7} + A^7_{8IL}A^8_{JK7} - A_7^{8MN}A^{77} = 0.$$

From all the possible choices of the indices we can derive three similar equations

$$\begin{aligned} m_1m_2 + m_3\alpha &= 0, \\ m_1m_3 + m_2\alpha &= 0, \\ m_2m_3 + m_1\alpha &= 0, \end{aligned}$$

which are simultaneously verified if

$$m_1 = m_2 = m_3 = -\alpha, \quad (5.17)$$

giving a candidate vacuum state which is automatically a Minkowski one.

To verify that this extremal point of the potential actually corresponds to the vacuum of a theory with a consistent gauging, we have just to put the values of the A_1 and A_2 components within equations (5.9), (5.10), (5.11) and verify that the identities are satisfied. In order to do this, it is convenient to analyze for which combinations of indices (in the set $I, J = 1, \dots, 6$ or in the set $a, b = 7, 8$) each term on the RHS of the equation is not vanishing and then verify how, in any case, the full combination gives a zero result if the parameters are constrained by (5.17).

After carrying out this procedure explicitly, we can confirm that, indeed, (5.17) describes a Minkowski vacuum with residual $\mathcal{N} = 6$ supersymmetry.

5.4 RESIDUAL $\mathcal{N} = 5$ SUPERSYMMETRY

We can try to find other vacuum states progressively reducing the number of preserved supersymmetries: the next case to analyze is then the set of possible vacua preserving $\mathcal{N} = 5$.

In analogy to the case $\mathcal{N} = 6$, by imposing supersymmetry at the level of fermionic transformation rules, we can constrain the tensors A_1 and A_2 as follows:

$$A_1^{Ij} = A_1^{jI} = 0, \quad A_2^{ijk} = 0 \quad \forall I = 1, \dots, 5 \quad \forall i, j, k,$$

i.e. the A_1 tensor has only 9 non-vanishing components A_1^{ab} , while for A_2 we are left with 30 components of type A_{2a}^{IJK} , 90 (with 10 constraints) of type A_{2a}^{bIJ} and 45 (with 15 constraints) of type A_{2a}^{bcI} . As for the components A_{2a}^{bcd} , they automatically vanish due to the traceless condition on A_2

$$A_{26}^{678} = A_{2a}^{a78} = 0$$

and analogously for the other two components.

Once we break the original $\mathcal{N} = 8$ supersymmetry leaving only 5 preserved charges, the largest possible residual R-symmetry is

$$\text{SU}(8) \longrightarrow \text{SU}(5) \times \text{SU}(3) \times \text{U}(1) ,$$

but in order to analyze some more general cases we can further break the $\text{SU}(3)$ symmetry acting on the indices 6,7,8 in two different ways

$$\begin{aligned} \text{SU}(3) &\longrightarrow \text{SO}(3) , \\ \text{SU}(3) &\longrightarrow \text{U}(1) \times \text{U}(1) . \end{aligned}$$

5.4.1 $\text{SU}(3) \longrightarrow \text{SO}(3)$

The symmetry under $\text{SO}(3)$, meant as a group acting in its fundamental representation on the contravariant indices 6,7,8, imposes the following constraints on the residual components of A_1 and A_2 :

- A_1^{ab} is invariant only if it is proportional to the identity: $A_1^{ab} = \alpha \delta^{ab}$;
- A_{2a}^{IJK} is never invariant, unless all components vanish;
- A_{2a}^{bIJ} is never invariant, because $\text{SO}(3)$ transformations would leave invariant the identity but traceless condition forces also the diagonal components to be vanishing;
- A_{2a}^{bcI} could be invariant under $\text{SO}(3)$ if $A_{2a}^{bcI} = \epsilon^{abc} V^I$.

In order to verify simultaneously the extremization condition of the potential and the consistency equations (5.9), (5.10), (5.11), the only possibility would be that $A_{2i}^{jkl} = 0$, but in that case a non-vanishing α could never give a Minkowski vacuum. Then, imposing a $\text{U}(5) \times \text{SO}(3)$ R-symmetry, we don't find any vacua besides the trivial one.

5.4.2 $\text{SU}(3) \longrightarrow \text{U}(1)_I \times \text{U}(1)_{II}$

We consider two distinct $\text{U}(1)$ groups, each of them acting on a 3-dimensional representation as independent rotations of the three basis vectors, with proper charge assignments to each vector in order to preserve the unit determinant inherited from the original $\text{SU}(8)$ symmetry group. A possibility is that indices 6,7 transform as a doublet and index 8 as a singlet under the group $\text{U}(1)_I$, while the group action of $\text{U}(1)_{II}$ rotates the doublet components with opposite charges. The charges with respect to the two groups would be, for instance,

$$\begin{aligned} q_I^6 &= 1 , & q_I^7 &= 1 , & q_I^8 &= -2 , \\ q_{II}^6 &= 1 , & q_{II}^7 &= -1 , & q_{II}^8 &= 0 , \end{aligned}$$

but there are also other possible choices of the group action. An inequivalent one could be

$$\begin{aligned} q_1^6 &= 1, & q_1^7 &= 1, & q_1^8 &= -2, \\ q_{1I}^6 &= 1, & q_{1I}^7 &= 0, & q_{1I}^8 &= -1. \end{aligned}$$

In any case, if we require symmetry under the two groups¹,

- A_1^{ab} is never invariant under the action of the two groups simultaneously, unless all the components vanish;
- also $A_{2a}^{IJK} = 0 = A_{2a}^{bcI}$;
- among A_{2a}^{bIJ} components, the invariant ones under the group action are A_{2a}^{aIJ} , for $a = 6, 7, 8$.

The structure of A_2 allows to parametrize the non-vanishing components as

$$A_{26}^{6IJ} = x \Sigma^{IJ}, \quad A_{27}^{7IJ} = y \Lambda^{IJ}, \quad A_{28}^{8IJ} = -\left(A_{26}^{6IJ} + A_{27}^{7IJ}\right),$$

where Σ^{IJ} and Λ^{IJ} are antisymmetric matrices. Exploiting the $U(5)$ symmetry, one of these two matrices could also be rewritten in a block-diagonal form, similarly to the case $\mathcal{N} = 6$.

However, if we try to find Minkowski vacua, it is immediate that we cannot get any non-trivial solution, because the only components which could be non-vanishing are all coming from the A_2 tensor. Indeed, they would appear in the expression of the potential (5.12) as a sum of squared moduli with all positive signs, then the potential can take a zero value only if all the tensor components are identically null.

5.5 RESIDUAL $\mathcal{N} = 4$ SUPERSYMMETRY

If we impose that the number of preserved supersymmetries at the vacuum state reduces to $\mathcal{N} = 4$, from fermionic transformation rules we derive

$$A_1^{Ij} = A_1^{jI} = 0, \quad A_{2I}^{ijk} = 0 \quad \forall I = 1, \dots, 4 \quad \forall i, j, k,$$

which means that the A_1 tensor has at most 16 non-vanishing components A_1^{ab} , while for A_2 we should consider 16 components of type A_{2a}^{IJK} , 96 (with 6 constraints) of type A_{2a}^{bIJ} , 96 (with 16 constraints) of type A_{2a}^{bcI} , and 16 (with 6 constraints) of type A_{2a}^{bcd} .

The largest possible residual R-symmetry is now

$$SU(8) \longrightarrow SU(4) \times SU(4) \times U(1),$$

¹ We are not considering the case in which the three charges with respect to $U(1)_I$ are proportional to the charges of $U(1)_{II}$, because in that case there would be a single $U(1)$ symmetry group.

where clearly the first $SU(4)$ group acts on indices corresponding to the preserved supersymmetry generators (indices 1,2,3,4), while the second $SU(4)$ rotates the remaining indices. As a first step, we suppose to break this second $SU(4)$ symmetry in the following ways:

$$\begin{aligned} SU(4) &\longrightarrow SU(3) \times U(1) , \\ SU(4) &\longrightarrow Sp(4) , \\ SU(4) &\longrightarrow SU(2) \times SU(2) . \end{aligned}$$

5.5.1 $SU(4) \longrightarrow SU(3) \times U(1)$

The $SU(3)$ group acts in its fundamental representation on three of the four indices (wlog 5,6,7), while the $U(1)$ group acts by rotating the basis vectors with charges

$$q^5 = q^6 = q^7 = 1 , \quad q^8 = -3 .$$

Imposing R-symmetry under this subgroup of $SU(4)$ means that

- all A_1^{ab} components must vanish because otherwise the tensor would not be invariant under $U(1)$ (there are no combinations of two charges that cancel each other);
- also A_{2a}^{IJK} , A_{2a}^{bcI} and A_{2a}^{bcd} are forced to vanish (again, it can be obtained by imposing symmetry under $U(1)$ with the above charge assignments);
- among A_{2a}^{bIJ} components, the diagonal ones A_{2a}^{aIJ} are invariant under $U(1)$; to satisfy the $SU(3)$ symmetry and also the traceless condition, they must be

$$A_{25}^{5IJ} = A_{26}^{6IJ} = A_{27}^{7IJ} = \Sigma^{IJ} , \quad A_{28}^{8IJ} = -3 \Sigma^{IJ} ,$$

where Σ^{IJ} is an antisymmetric matrix, which could be led to a block-diagonal form by exploiting the unbroken $U(4)$ symmetry.

Then, only A_2 tensor can have some non-vanishing components, which means that imposing the condition of zero potential we immediately get the trivial solution.

5.5.2 $SU(4) \longrightarrow Sp(4)$

The group of matrices

$$Sp(4) = \{M \in U(4) : M^T \Omega M = \Omega\} ,$$

where Ω is the skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix} ,$$

acts in its 4-dimensional representation on indices 5,6,7,8. Symmetry under $\text{Sp}(4)$ implies that

- all A_1^{ab} components vanish (A_1 must be symmetric, then we cannot have $A_1^{ab} = \Omega^{ab}$);
- all A_{2a}^{IJK} and A_{2a}^{bcI} components must vanish, too;
- A_{2a}^{bIJ} components must vanish due to the traceless condition on A_2 ;
- A_{2a}^{bcd} components could be $\text{Sp}(4)$ invariant if they had the form

$$A_{2a}^{bcd} = \alpha \delta_a^b \Omega^{cd} ,$$

but if $\alpha \neq 0$ they would violate antisymmetry and traceless constraint on A_2 .

In summary, by imposing a residual $\text{U}(4) \times \text{Sp}(4)$ R-symmetry we automatically get that the only possibility is to have no gauging.

5.5.3 $\text{SU}(4) \longrightarrow \text{SU}(2) \times \text{SU}(2)$

The two $\text{SU}(2)$ groups act on orthogonal vector spaces, i.e. for instance one group acts on the subspace generated by basis vectors with indices 5,6 and the other one on the subspace generated by 7,8. If we impose symmetry under the product of the two groups

- all A_1^{ab} components must vanish (due to the symmetry condition on A_1);
- also A_{2a}^{IJK} and A_{2a}^{bcI} are never invariant unless they vanish;
- A_{2a}^{bIJ} components can be $\text{SU}(2) \times \text{SU}(2)$ invariant if they take the form

$$A_{25}^{5IJ} = A_{26}^{6IJ} = \Sigma^{IJ} , \quad A_{27}^{7IJ} = A_{28}^{8IJ} = -\Sigma^{IJ} ,$$

where Σ^{IJ} is an antisymmetric matrix as usual, while all other components cannot be different from zero;

- among A_{2a}^{bcd} , the only possibility would be that the non-vanishing components were

$$\begin{aligned} A_{25}^{578} = A_{26}^{678} = -A_{25}^{587} = -A_{26}^{687} = \alpha , \\ A_{27}^{756} = A_{28}^{856} = -A_{27}^{765} = -A_{28}^{865} = \beta , \end{aligned}$$

but the traceless condition on A_2 forces $\alpha = \beta = 0$.

Again, we obtain that the possibly non-vanishing tensor components all come from A_2 tensor, which prevents non-trivial Minkowski vacua being found.

5.5.4 $SU(4) \longrightarrow U(1) \times U(1)$

This symmetry group can be seen as a subgroup of the $SU(2) \times SU(2)$ analyzed in section 5.5.3. In particular, assuming that the charges with respect to the first $U(1)$ are

$$q^{5+i6} = +1, \quad q^{5-i6} = -1,$$

and analogously for the second $U(1)$

$$q^{7+i8} = +1, \quad q^{7-i8} = -1,$$

symmetry under $U(1) \times U(1)$ implies that

- the non-vanishing components of A_1 are

$$A_1^{\tilde{a}\tilde{b}} = \alpha \delta^{\tilde{a}\tilde{b}} \quad \text{for } \tilde{a}, \tilde{b} = 5, 6, \quad A_1^{\hat{a}\hat{b}} = \beta \delta^{\hat{a}\hat{b}} \quad \text{for } \hat{a}, \hat{b} = 7, 8,$$

- components of the A_2 tensor should have 2 or 4 indices of type 5, 6, 7, 8 in order to be invariant under the symmetry group and they should take the form

$$A_{2\tilde{a}}^{\tilde{b}IJ} = \epsilon^{\tilde{a}\tilde{b}} \Sigma^{IJ} + \delta^{\tilde{a}\tilde{b}} \Gamma^{IJ} \quad \text{for } \tilde{a}, \tilde{b} = 5, 6,$$

$$A_{2\hat{a}}^{\hat{b}IJ} = \epsilon^{\hat{a}\hat{b}} \Lambda^{IJ} - \delta^{\hat{a}\hat{b}} \Gamma^{IJ} \quad \text{for } \hat{a}, \hat{b} = 7, 8,$$

$$A_{2\tilde{a}}^{\tilde{b}\hat{c}\hat{d}} = \gamma \epsilon^{\tilde{a}\tilde{b}} \epsilon^{\hat{c}\hat{d}} \quad \text{for } \tilde{a}, \tilde{b} = 5, 6, \hat{c}, \hat{d} = 7, 8,$$

$$A_{2\hat{a}}^{\hat{b}cd} = \delta \epsilon^{\hat{a}\hat{b}} \epsilon^{cd} \quad \text{for } \hat{a}, \hat{b} = 7, 8, c, d = 5, 6,$$

and Youla's theorem can be applied so that, exploiting the $U(4)$ unbroken symmetry, we go to the basis where

$$\Sigma = \begin{pmatrix} 0 & m_1 & & \\ -m_1 & 0 & & \\ & & 0 & m_2 \\ & & -m_2 & 0 \end{pmatrix}.$$

As a consequence, requiring such a symmetry means that there are 18 parameters to fix: $\alpha, \beta, \gamma, \delta, m_1, m_2$ and 6 independent components for each of the antisymmetric Λ^{IJ} and Γ^{IJ} , which are all in principle complex numbers, apart from m_1 and m_2 .

In order to find a Minkowski vacuum, we have to impose the condition of vanishing potential

$$V = 0 \quad \Leftrightarrow \quad -3(|\alpha|^2 + |\beta|^2) + \left(|\gamma|^2 + |\delta|^2 + m_1^2 + m_2^2 + \sum_{I < J} |\Lambda^{IJ}|^2 + 2 \sum_{I < J} |\Gamma^{IJ}|^2 \right) = 0,$$

together with the extremization condition, which gives

$$\begin{aligned}
& (\gamma^* \alpha^* + \delta^* \beta^*) + m_1 m_2 + \Lambda^{12} \Lambda^{34} - \Lambda^{13} \Lambda^{24} + \Lambda^{14} \Lambda^{23} - 2(\Gamma^{12} \Gamma^{34} - \Gamma^{13} \Gamma^{24} + \Gamma^{14} \Gamma^{23}) = 0, \\
& \alpha^* m_1 + \delta^* (\Lambda^{12})^* + \beta \Lambda^{34} + \gamma m_2 = 0, \\
& \alpha^* m_2 + \delta^* (\Lambda^{34})^* + \beta \Lambda^{12} + \gamma m_1 = 0, \\
& \delta^* (\Lambda^{13})^* - \beta \Lambda^{24} = 0, \\
& \delta^* (\Lambda^{14})^* + \beta \Lambda^{23} = 0.
\end{aligned}$$

These equations, combined with other conditions coming from equation (5.9), give the following constraints on the squared moduli of the parameters:

$$\begin{aligned}
|\alpha|^2 &= |\gamma|^2 = m_1^2 + |\Gamma^{12}|^2 + |\Gamma^{13}|^2 + |\Gamma^{14}|^2, \\
|\beta|^2 &= |\delta|^2 = |\Lambda^{12}|^2 + |\Lambda^{13}|^2 + |\Lambda^{14}|^2 + |\Gamma^{12}|^2 + |\Gamma^{13}|^2 + |\Gamma^{14}|^2, \\
m_1^2 &= m_2^2, \quad |\Lambda^{12}|^2 = |\Lambda^{34}|^2, \quad |\Lambda^{13}|^2 = |\Lambda^{24}|^2, \quad |\Lambda^{14}|^2 = |\Lambda^{23}|^2, \\
|\Gamma^{12}|^2 &= |\Gamma^{34}|^2, \quad |\Gamma^{13}|^2 = |\Gamma^{24}|^2, \quad |\Gamma^{14}|^2 = |\Gamma^{23}|^2,
\end{aligned}$$

which are compatible with the requirement of zero potential, plus some constraints on the phases, coming also from equations (5.10) and (5.11),

$$\begin{aligned}
e^{-i(\theta_\alpha + \theta_\gamma)} &= -e^{i(\theta_{m_1} + \theta_{m_2})}, \\
e^{-i(\theta_\beta + \theta_\delta)} &= -e^{i(\theta_{12} + \theta_{34})} = -e^{i(\theta_{14} + \theta_{23})} = +e^{i(\theta_{13} + \theta_{24})}, \\
e^{i(\varphi_{12} + \varphi_{34})} &= e^{i(\varphi_{14} + \varphi_{23})} = -e^{i(\varphi_{13} + \varphi_{24})},
\end{aligned}$$

where the phase of Λ^{IJ} is denoted by θ_{IJ} , the phase of Γ^{IJ} by φ_{IJ} and clearly $e^{i(\theta_{m_1} + \theta_{m_2})} = \pm 1$ since m_1 and m_2 are real quantities. At this point, we can exploit the unbroken $U(1) \times U(1)$ symmetry to fix the phases of two complex parameters, for instance we can fix α and β to be real quantities.

5.5.5 $SU(4) \longrightarrow U(1)$

The original $SU(4)$ symmetry can be broken to $U(1)$ in different ways, corresponding to different actions of the $U(1)$ group. In particular, three inequivalent choices for the charges of basis vectors are

$$\begin{aligned}
\text{I)} \quad & q^{5+i6} = 1, \quad q^{5-i6} = -1, \quad q^{7+i8} = 0, \quad q^{7-i8} = 0, \\
\text{II)} \quad & q^{5+i6} = 1, \quad q^{5-i6} = -1, \quad q^{7+i8} = 1, \quad q^{7-i8} = -1, \\
\text{III)} \quad & q^{5+i6} = 1, \quad q^{5-i6} = -1, \quad q^{7+i8} = 2, \quad q^{7-i8} = -2,
\end{aligned}$$

Case I

The tensor components invariant under the action of U(1) are

- for the tensor A_1

$$A_1^{55} = A_1^{66} = \alpha, \quad A_1^{77} = \beta, \quad A_1^{88} = \gamma, \quad A_1^{78} = A_1^{87} = \delta;$$

- components $A_{27}^{IJK}, A_{28}^{IJK}$;

- among components of type $A_{2_a}^{bIJ}$

$$\begin{aligned} A_{25}^{5IJ} = A_{26}^{6IJ} = \Sigma^{IJ}, \quad A_{27}^{7IJ} = M_1^{IJ}, \quad A_{28}^{8IJ} = -(2\Sigma^{IJ} + M_1^{IJ}), \\ A_{25}^{6IJ} = -A_{26}^{5IJ} = M_2^{IJ}, \quad A_{27}^{8IJ} = M_3^{IJ}, \quad A_{28}^{7IJ} = M_4^{IJ}, \end{aligned}$$

which correspond to 5 independent antisymmetric matrices; exploiting the U(4) symmetry acting on indices I, J , a basis can be always chosen in such a way that one of these matrices, for instance $A_{25}^{5IJ} = \Sigma^{IJ}$, takes the form

$$\Sigma = \begin{pmatrix} 0 & m_1 & & \\ -m_1 & 0 & & \\ & & 0 & m_2 \\ & & -m_2 & 0 \end{pmatrix},$$

where m_1 and m_2 are real quantities;

- among components of type $A_{2_a}^{bcI}$

$$\begin{aligned} A_{25}^{57I} = A_{26}^{67I} = V_1^I, \quad A_{25}^{67I} = -A_{26}^{57I} = V_2^I, \quad A_{27}^{56I} = V_3^I, \\ A_{25}^{58I} = A_{26}^{68I} = V_4^I, \quad A_{25}^{68I} = -A_{26}^{58I} = V_5^I, \quad A_{28}^{56I} = V_6^I, \\ A_{28}^{87I} = -2V_1^I, \quad A_{27}^{78I} = -2V_4^I; \end{aligned}$$

- among components of type $A_{2_a}^{bcd}$

$$A_{25}^{678} = -A_{26}^{578} = \varepsilon, \quad A_{27}^{756} = -A_{28}^{856} = \zeta, \quad A_{28}^{567} = \eta, \quad A_{27}^{568} = \iota.$$

The condition of having a null potential in this case becomes

$$\begin{aligned} V = 0 \Leftrightarrow 0 = & -3(2|\alpha|^2 + |\beta|^2 + |\gamma|^2 + 2|\delta|^2) + [2|\varepsilon|^2 + 2|\zeta|^2 + |\eta|^2 + |\iota|^2 + \\ & + \sum_{I<J<K} (A_{27}^{IJK} + A_{28}^{IJK}) + \sum_I (6V_1^I + 2V_2^I + V_3^I + 6V_4^I + 2V_5^I + V_6^I) + \\ & + \sum_{I<J} (2|\Sigma^{IJ}|^2 + |M_1^{IJ}|^2 + 2|M_2^{IJ}|^2 + |M_3^{IJ}|^2 + |M_4^{IJ}|^2 + |2\Sigma^{IJ} + M_1^{IJ}|^2)]. \end{aligned}$$

To have an extremal point of the potential, instead, the requirement is

$$\begin{aligned} (-2\varepsilon^* \alpha^* + 2\zeta^* \delta^* + \eta^* \gamma^* - \iota^* \beta^*) + [6m_1 m_2 + 2(m_2 M_1^{12} + m_1 M_1^{34}) + \\ + 2(M_1^{12} M_1^{34} - M_1^{13} M_1^{24} + M_1^{14} M_1^{23}) + 2(M_2^{12} M_2^{34} - M_2^{13} M_2^{24} + M_2^{14} M_2^{23}) + \\ + (M_3^{12} M_3^{34} - M_3^{13} M_3^{24} + M_3^{14} M_3^{23}) + (M_4^{12} M_4^{34} - M_4^{13} M_4^{24} + M_4^{14} M_4^{23})] = 0. \end{aligned}$$

Case II

The tensor components invariant under the action of U(1) are

- for the tensor A_1

$$A_1^{55} = A_1^{66} = \alpha, \quad A_1^{57} = A_1^{68} = \gamma, \quad A_1^{58} = -A_1^{67} = \delta, \quad A_1^{77} = A_1^{88} = \beta;$$

- among components of type $A_{2a}{}^{bIJ}$

$$\begin{aligned} A_{25}{}^{5IJ} &= A_{26}{}^{6IJ} = \Sigma^{IJ}, & A_{27}{}^{7IJ} &= A_{28}{}^{8IJ} = -\Sigma^{IJ}, \\ A_{25}{}^{6IJ} &= -A_{26}{}^{5IJ} = M_1{}^{IJ}, & A_{27}{}^{8IJ} &= -A_{28}{}^{7IJ} = M_2{}^{IJ}, \\ A_{25}{}^{7IJ} &= A_{26}{}^{8IJ} = M_3{}^{IJ}, & A_{27}{}^{5IJ} &= A_{28}{}^{6IJ} = M_4{}^{IJ}, \\ A_{26}{}^{7IJ} &= -A_{25}{}^{8IJ} = M_5{}^{IJ}, & A_{27}{}^{6IJ} &= -A_{28}{}^{5IJ} = M_6{}^{IJ}; \end{aligned}$$

which correspond to 7 independent antisymmetric matrices; again, a basis can be always chosen in such a way that one of these matrices, for instance $A_{25}{}^{5IJ} = \Sigma^{IJ}$, takes the form

$$\Sigma = \begin{pmatrix} 0 & m_1 & & \\ -m_1 & 0 & & \\ & & 0 & m_2 \\ & & -m_2 & 0 \end{pmatrix};$$

- among components of type $A_{2a}{}^{bcd}$

$$\begin{aligned} A_{25}{}^{678} &= -A_{26}{}^{578} = \varepsilon, & A_{27}{}^{856} &= -A_{28}{}^{756} = \zeta, \\ A_{25}{}^{567} &= -A_{28}{}^{867} = A_{26}{}^{568} = -A_{27}{}^{578} = \eta, \\ A_{25}{}^{568} &= -A_{27}{}^{768} = -A_{26}{}^{567} = A_{28}{}^{587} = \iota. \end{aligned}$$

We can impose the condition for having a Minkowski vacuum

$$\begin{aligned} V = 0 \quad \Leftrightarrow \quad & -3\left(|\alpha|^2 + |\beta|^2 + 2|\gamma|^2 + 2|\delta|^2\right) + \\ & + \left(|\varepsilon|^2 + |\zeta|^2 + 2|\eta|^2 + 2|\iota|^2 + 2m_1^2 + 2m_2^2 + \sum_{i=1}^6 \sum_{I<J} |M_i{}^{IJ}|^2\right) = 0 \end{aligned}$$

and the extremization condition, which gives

$$\begin{aligned} & (\varepsilon^* \alpha^* + \zeta^* \beta^* + 2\iota^* \gamma^* - 2\eta^* \delta^*) + [-2m_1 m_2 + \\ & + (M_1{}^{12} M_1{}^{34} - M_1{}^{13} M_1{}^{24} + M_1{}^{14} M_1{}^{23}) + (M_2{}^{12} M_2{}^{34} - M_2{}^{13} M_2{}^{24} + M_2{}^{14} M_2{}^{23}) + \\ & - (M_3{}^{12} M_3{}^{34} - M_3{}^{13} M_3{}^{24} + M_3{}^{14} M_3{}^{23}) - (M_4{}^{12} M_3{}^{34} - M_4{}^{13} M_3{}^{24} + M_4{}^{14} M_3{}^{23}) + \\ & - (M_5{}^{12} M_6{}^{34} - M_5{}^{13} M_6{}^{24} + M_5{}^{14} M_6{}^{23}) - (M_6{}^{12} M_5{}^{34} - M_6{}^{13} M_5{}^{24} + M_6{}^{14} M_5{}^{23})] = 0, \end{aligned}$$

$$\begin{aligned}
& \alpha^*(M_1^{IJ})^* + (\gamma^* + \iota^*)(M_6^{IJ})^* + (\delta^* - \eta^*)(M_4^{IJ})^* + \zeta^*(M_2^{IJ})^* + \\
& + \beta M_2^{LM} - (\gamma + \iota)M_5^{LM} - (\delta - \eta)M_3^{LM} + \varepsilon M_1^{LM} = 0, \\
& (\alpha^* + \zeta^*)(M_3^{IJ})^* - (\beta^* + \varepsilon^*)(M_4^{IJ})^* - 2(\gamma^* - \iota^*)(\Sigma^{IJ})^* + (\delta^* - \eta^*)(M_1^{IJ} + M_2^{IJ})^* + \\
& - \left[(\alpha + \zeta)M_3^{LM} - (\beta + \varepsilon)M_4^{LM} - 2(\gamma - \iota)\Sigma^{LM} + (\delta - \eta)(M_1^{LM} + M_2^{LM}) \right] = 0, \\
& (\alpha^* - \zeta^*)(M_5^{IJ})^* + (\varepsilon^* - \beta^*)(M_6^{IJ})^* - (\gamma^* - \iota^*)(M_1^{IJ} + M_2^{IJ})^* + 2(\delta^* - \eta^*)(\Sigma^{IJ})^* + \\
& - \left[(\alpha - \zeta)M_5^{LM} + (\varepsilon - \beta)M_6^{LM} - (\gamma - \iota)(M_1^{LM} + M_2^{LM}) + 2(\delta - \eta)\Sigma^{LM} \right] = 0,
\end{aligned}$$

where, in the last equations, indices $(IJLM)$ are meant to be any even permutation of (1234) .

From (5.9) we get many additional quadratic constraints

$$\begin{aligned}
& \sum_{I < J} \left(|M_3^{IJ}|^2 - |M_4^{IJ}|^2 \right) + |\varepsilon|^2 - |\zeta|^2 - |\alpha|^2 + |\beta|^2 = 0, \\
& \sum_{I < J} \left(|M_5^{IJ}|^2 - |M_6^{IJ}|^2 \right) + |\varepsilon|^2 - |\zeta|^2 - |\alpha|^2 + |\beta|^2 = 0, \\
& m_1^2 + \sum_{J \neq 1} |M_1^{1J}|^2 + |M_3^{1J}|^2 + |M_5^{1J}|^2 - |\alpha|^2 - |\gamma|^2 - |\delta|^2 = 0, \\
& m_2^2 + \sum_{J \neq 3} |M_1^{3J}|^2 + |M_3^{3J}|^2 + |M_5^{3J}|^2 - |\alpha|^2 - |\gamma|^2 - |\delta|^2 = 0, \\
& m_1^2 + \sum_{J \neq 1} |M_2^{1J}|^2 + |M_4^{1J}|^2 + |M_6^{1J}|^2 - |\beta|^2 - |\gamma|^2 - |\delta|^2 = 0, \\
& m_2^2 + \sum_{J \neq 3} |M_2^{3J}|^2 + |M_4^{3J}|^2 + |M_6^{3J}|^2 - |\beta|^2 - |\gamma|^2 - |\delta|^2 = 0,
\end{aligned}$$

where the last four equations also hold if we replace the matrix indices $I = 1$ with $I = 2$ and $I = 3$ with $I = 4$,

$$\begin{aligned}
& \sum_{\substack{i=1,3,5 \\ \text{or } i=2,4,6}} M_i^{13}(M_i^{23})^* + M_i^{14}(M_i^{24})^* = 0, & \sum_{\substack{i=1,3,5 \\ \text{or } i=2,4,6}} M_i^{12}(M_i^{23})^* - M_i^{14}(M_i^{34})^* = 0, \\
& \sum_{\substack{i=1,3,5 \\ \text{or } i=2,4,6}} M_i^{12}(M_i^{24})^* + M_i^{13}(M_i^{34})^* = 0, & \sum_{\substack{i=1,3,5 \\ \text{or } i=2,4,6}} M_i^{12}(M_i^{13})^* + M_i^{24}(M_i^{34})^* = 0, \\
& \sum_{\substack{i=1,3,5 \\ \text{or } i=2,4,6}} M_i^{12}(M_i^{14})^* - M_i^{23}(M_i^{34})^* = 0, & \sum_{\substack{i=1,3,5 \\ \text{or } i=2,4,6}} M_i^{13}(M_i^{14})^* + M_i^{23}(M_i^{24})^* = 0.
\end{aligned}$$

Case III

The tensor components invariant under the action of $U(1)$ are the same as in the case of a residual $U(1) \times U(1)$ symmetry

- for the tensor A_1

$$A_1^{55} = A_1^{66}, \quad A_1^{77} = A_1^{88};$$

- among components of type $A_{2_a}{}^{bIJ}$

$$\begin{aligned} A_{2_5}{}^{5IJ} &= A_{2_6}{}^{6IJ} , & A_{2_7}{}^{7IJ} &= A_{2_8}{}^{8IJ} = -A_{2_5}{}^{5IJ} , \\ A_{2_5}{}^{6IJ} &= -A_{2_6}{}^{5IJ} , & A_{2_7}{}^{8IJ} &= -A_{2_8}{}^{7IJ} , \end{aligned}$$

which correspond to 3 independent antisymmetric matrices; a basis can be always chosen in such a way that $A_{2_5}{}^{5IJ} = \Sigma^{IJ}$ takes the form

$$\Sigma = \begin{pmatrix} 0 & m_1 & & \\ -m_1 & 0 & & \\ & & 0 & m_2 \\ & & -m_2 & 0 \end{pmatrix} ;$$

- among components of type $A_{2_a}{}^{bcd}$

$$A_{2_5}{}^{678} = -A_{2_6}{}^{578} , \quad A_{2_7}{}^{856} = -A_{2_8}{}^{756} .$$

Then, if we look for Minkowski vacua, the solutions are analogous to the results obtained in section 5.5.4. The only difference is that, since in this case there is no more a $U(1) \times U(1)$ symmetry, but just a single $U(1)$, we can choose arbitrarily the phase of one complex parameter (for instance we can require α to be real) but not of two of them.

CONCLUSIONS AND OUTLOOK

In this thesis, we have reviewed some important ideas about gauged supergravity theories and their deep connection with string compactifications in presence of fluxes of various origin, then we have presented the maximal supergravity theory in 4 dimensions, in order to perform a partial but systematical analysis of its Minkowski vacua.

The results we have obtained can be summarized as follows:

Residual supersymmetry	Residual gauge group
$\mathcal{N} = 6$	$U(6) \times U(1)$
$\mathcal{N} = 5$	no flat vacua
$\mathcal{N} = 4$	$U(4) \times U(1) \times U(1)$
	$U(4) \times U(1)$

The above outcomes are in agreement with the literature on the subject. In particular, an analysis of Minkowski vacua for the spontaneously broken $\mathcal{N} = 8$ supergravity can be found in [52]. Here, all the models considered show supersymmetry breaking patterns to an even number of residual supersymmetry generators $\mathcal{N} = 6, 4, 2, 0$, compatibly with our results for $\mathcal{N} = 6$ and $\mathcal{N} = 4$.

As for the case of 5 residual supersymmetry generators, it was expected that Minkowski vacua could not be found, because this pattern is prevented by kinematical constraints. In general, when supersymmetry is broken from $\mathcal{N} = 8$ to some \mathcal{N}' , the field content of the unique supermultiplet has to be rearranged in smaller multiplets. Among them, there are always one gravity multiplet (with \mathcal{N}' gravitini) and $8 - \mathcal{N}'$ massive gravitino multiplets, but this is not possible in the case of $\mathcal{N}' = 5$ [43], [53].

For the other possible supersymmetry breaking patterns with an odd number of residual generators ($\mathcal{N} = 3$ and $\mathcal{N} = 1$), there are no similar kinematical constraints, but examples of such Minkowski vacua have never been found up to now (if we do not require a vanishing vacuum energy, instead, there are possible solutions resulting in de Sitter spacetimes). On the other hand, there are some hints, coming from string theory calculations, that suggest to not exclude their existence (see the results reported in section 3.3). Then, it would be very interesting to continue our analysis by investigating

these two cases. Since the technique we have used gives complete results once we have imposed a given residual symmetry, if calculations were able to exclude the presence of Minkowski vacua once and for all, this could suggest that there is some yet unknown mechanism at work.

A

COSET MANIFOLDS

We mentioned in section 2.2 that scalar manifolds of extended supergravity theories can be often described as coset manifolds, which simplifies a lot the discussion of gauging procedure. Given the relevance of this geometric structure, in particular for the construction of the maximal supergravity theory in 4 dimensions in chapter 4, we give in this appendix a general introduction to coset manifolds, with particular attention to those definition and properties that are employed in the rest of the thesis. Notation is mainly taken from [54].

A.1 FIRST DEFINITIONS AND CLASSIFICATION

Definition A.1. A metric space \mathcal{M} is *homogeneous* if it admits an isometry group G with a *transitive action* on \mathcal{M} , i.e. if any point of the metric space can be reached starting from any other point and applying the group action

$$\forall x, y \in \mathcal{M} \quad \exists g \in G \quad \text{s.t.} \quad y = gx .$$

Definition A.2. The *isotropy group* of a point x of an homogeneous space is the subgroup $H \subset G$ whose action on x leaves the point invariant

$$hx = x \quad \forall h \in H .$$

Due to the transitive action of G , any point of the homogeneous space is left invariant by a subgroup of type gHg^{-1} , for some $g \in G$:

$$y = gx \quad \Rightarrow \quad (ghg^{-1})y = y \quad \forall h \in H .$$

This property allows to identify the homogeneous space with the coset G/H , which is the set of equivalence classes of G once we have defined the equivalence relation $g \sim g'$ iff $g = g'h$, for some $h \in H$. If G is a Lie group, we call G/H a *coset manifold*; its dimension is $d = \dim(G) - \dim(H)$.

The Lie algebra \mathfrak{g} of G can be written as the direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} , \tag{A.1}$$

where \mathfrak{h} is the Lie algebra of H and \mathfrak{k} contains the *coset generators*. The \mathfrak{g} algebra structure can be described in terms of the generators as follows:

$$\begin{aligned} [t_i, t_j] &= f_{ij}^k t_k, \\ [t_i, t_a] &= f_{ia}^j t_j + f_{ia}^b t_b, \\ [t_a, t_b] &= f_{ab}^i t_i + f_{ab}^c t_c, \end{aligned} \tag{A.2}$$

where we used the convention that $i, j, k \dots$ are \mathfrak{h} indices and $a, b, c \dots$ are \mathfrak{k} indices.

Coset manifolds can be classified according to the form of (A.2) relations. In particular, a coset manifold is said *reductive* if $[\mathfrak{h}, \mathfrak{k}] \subseteq \mathfrak{k}$, which means that

$$f_{ia}^j = 0. \tag{A.3}$$

This is always the case for semi-simple G groups, whose Cartan–Killing metric on \mathfrak{g} $\gamma_{AB} = f_{AC}^D f_{BD}^C$ (where capital letters denote generic \mathfrak{g} indices) is non-degenerate: once a proper basis is chosen so that the metric is diagonal, it can be used to raise and lower the structure constant indices and then to obtain $f_{ia}^j = 0$ from $f_{ij}^a = 0$.

A coset manifold is called *symmetric* if

$$f_{ab}^c = 0, \tag{A.4}$$

which corresponds to requiring that $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{h}$. If G/H is non-compact, reductive and H is the maximal compact subgroup of G , symmetry of the coset manifold is always guaranteed. Indeed, the non-compact \mathfrak{g} algebra can be obtained starting from its compact counterpart and multiplying by the imaginary unit only generators in \mathfrak{k} , then $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{h}$ is needed for the algebra to close with real structure constants. This is what happens for the coset manifold $G/H = E_{7(7)}/SU(8)$ of maximal supergravity in 4 dimensions.

A.2 COSET REPRESENTATIVES AND INFINITESIMAL TRANSFORMATIONS

For each coset in G/H , i.e. for each point ϕ of the homogeneous manifold, one can choose an element $L(\phi) \in G$ belonging to the equivalence class under right H multiplication and use it as a *coset representative*. Once we fix the way of performing this choice, acting on a coset representative with a group element $g \in G$ does not necessarily send it into another representative: in general, we also need to apply a local H transformation

$$g L(\phi) = L(\phi') h(\phi). \tag{A.5}$$

A natural way to choose coset representatives is through exponentiation of the generators spanning the \mathfrak{k} subspace of \mathfrak{g} :

$$L(\phi) = L(y^1, \dots, y^d) = \prod_a e^{(y^a t_a)} \quad \text{with } a = 1, \dots, d = \dim(G/H), \tag{A.6}$$

where the y^1, \dots, y^d are real parameters.

In the case of non-compact spaces, an alternative choice comes from the *Iwasawa decomposition* ([6], [55]). Indeed, having an euclidean non-compact maximal G/H , there is always a solvable subalgebra $\mathfrak{so}\mathfrak{lv} \subset \mathfrak{g}$ having the same dimension as \mathfrak{k} as a vector space and such that the Lie algebra \mathfrak{g} can be written as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{so}\mathfrak{lv} . \quad (\text{A.7})$$

Thus, coset representatives can be taken as exponentials of the generators of the solvable subalgebra

$$L(\phi) = \prod_a e^{(y^a \tilde{t}_a)} \quad \text{with} \quad \langle \tilde{t}_1, \dots, \tilde{t}_d \rangle = \mathfrak{so}\mathfrak{lv} . \quad (\text{A.8})$$

The transformation law (A.5) of coset representatives under the action of constant g can be considered at the infinitesimal level, i.e.

$$\begin{aligned} g &= \mathbb{1} + \epsilon^A t_A , \\ h(\phi, g) &= \mathbb{1} - \epsilon^A w_A^i(\phi) t_i , \\ \phi'^\alpha(\phi, g) &= \phi^\alpha + \epsilon^A \zeta_A^\alpha(\phi) , \end{aligned} \quad (\text{A.9})$$

where we assume to have established a set of local coordinates on the coset manifold $\phi \equiv \{\phi^\alpha\}$, $\alpha = 1, \dots, d$. Remembering that G is the isometry group of G/H , the vector fields $\zeta_A \equiv \zeta_A^\alpha \partial_\alpha$ describing the group action on the coset manifold are the *Killing vectors* associated to the generators $t_A \in \mathfrak{g}$. The ϕ -dependent w_A^i is usually called *H-compensator*. In this formalism, the variation of $L(\phi)$ can be expressed as

$$L(\phi') = L(\phi) + \epsilon^A \zeta_A^\alpha \partial_\alpha L(\phi) , \quad (\text{A.10})$$

so that the transformation rule (A.5) becomes, for infinitesimal transformations,

$$t_A L(\phi) = \zeta_A(\phi) L(\phi) - L(\phi) w_A^i(\phi) t_i . \quad (\text{A.11})$$

It can be shown [54] that the Killing vectors $-\zeta_A$ satisfy the \mathfrak{g} Lie algebra, i.e. that

$$[\zeta_A, \zeta_B] = -f_{AB}^C \zeta_C . \quad (\text{A.12})$$

A.3 LOCAL STRUCTURE: THE MAURER–CARTAN FORM

In the following, we focus on reductive coset manifolds. Starting from coset representatives, the following Lie algebra-valued 1-form can be defined:

$$\Omega(\phi) \equiv L^{-1}(\phi) dL(\phi) , \quad (\text{A.13})$$

called the *Maurer–Cartan form* of a coset manifold, which generalizes the left-invariant 1-form defined on Lie groups. Taking values in \mathfrak{g} , (A.13) can be expressed in terms of the algebra generators

$$\Omega(\phi) = \mathcal{V}^a(\phi) t_a + \omega^i(\phi) t_i , \quad (\text{A.14})$$

where \mathcal{V}^a is a covariant *vielbein* and ω^i is called the *H-connection*. These two objects completely characterize the geometry of a coset manifold.

Due to the coset structure, the Maurer–Cartan form is not invariant under left multiplication by a constant element $g \in G$: from (A.5), which can be rewritten as $L(\phi') = g L(\phi) h^{-1}(\phi)$, we derive

$$\begin{aligned}\Omega(\phi') &= h \Omega(\phi) h^{-1} + h dh^{-1} = \\ &= \mathcal{V}^a(\phi) h t_a h^{-1} + \omega^i(\phi) h t_i h^{-1} + h dh^{-1} .\end{aligned}\tag{A.15}$$

Using the (A.14) decomposition, we can write the transformation rules for the vielbein and the *H-connection* under left G transformations

$$\mathcal{V}^a(\phi') = \mathcal{V}^b(\phi) D_b^a(h^{-1}) ,\tag{A.16a}$$

$$\omega^i(\phi') = \omega^j(\phi) D_j^i(h^{-1}) + (h dh^{-1})^i ,\tag{A.16b}$$

where matrices $D_A^B(g)$ denote the adjoint representation of elements $g \in G$, defined as $D_A^B(g) t_B = g^{-1} t_A g$ and we exploit $D_a^i(h) = 0$, which holds due to the reductivity of the coset manifold (A.3). Transformation rules (A.16b) show that the *H-connection* behaves indeed as a connection under the gauge group H .

From the definition of Maurer–Cartan form and the decomposition (A.14), taking infinitesimal transformations of the coset representatives (A.11) and multiplying both sides by $L^{-1}(\phi)$ from the left, one obtains

$$\begin{aligned}L^{-1}(\phi) t_A L(\phi) &= D_A^B(L(\phi)) t_B = L^{-1}(\phi) \zeta_A^\alpha \partial_\alpha L(\phi) - w_A^i t_i = \\ &= \zeta_A^\alpha(\phi) \mathcal{V}_\alpha^a t_a + \underbrace{\zeta_A^\alpha \omega_\alpha^i t_i - w_A^i t_i}_{-\mathcal{P}_A^i t_i} .\end{aligned}\tag{A.17}$$

The quantities $\mathcal{P}_A^i = -\zeta_A^\alpha \omega_\alpha^i + w_A^i$ are called *momentum maps* or *Killing prepotentials* [12] and generalize the prepotentials constructed for $\mathcal{N} = 2$ supergravity theories [56]; they are particularly important because they enter the definition of the gauged connection, as in (4.41b).

An explicit expression of the Killing vectors can be derived from (A.17) by projecting on the generators $t_a \in \mathfrak{k}$

$$\zeta_A^\alpha(\phi) = D_A^a(L(\phi)) \mathcal{V}_a^\alpha(\phi) ,\tag{A.18}$$

where \mathcal{V}_a^α is defined as the inverse of the vielbein

$$\mathcal{V}_a^\alpha \mathcal{V}_\beta^a = \delta_\beta^\alpha .\tag{A.19}$$

Projecting on the generators $t_i \in \mathfrak{h}$, instead, gives an explicit expression for the *H-compensator*,

$$w_A^i(\phi) = \omega_\alpha^i(\phi) \zeta_A^\alpha(\phi) - D_A^i(L(\phi)) ,\tag{A.20}$$

which can be also interpreted as an identity for the prepotentials

$$\mathcal{P}_A{}^i = -D_A{}^i(L(\phi)) = -(L^{-1}(\phi)t_A L(\phi))^i. \quad (\text{A.21})$$

A left-invariant metric on the coset manifold G/H can be obtained from the Cartan–Killing metric on G using the vielbein

$$g_{\alpha\beta}(\phi) = \gamma_{ab}\mathcal{V}_\alpha{}^a(\phi)\mathcal{V}_\beta{}^b(\phi). \quad (\text{A.22})$$

This metric is insensitive to the choice of coset representatives and allows to convert flat indices $a, b \dots$ into curved indices $\alpha, \beta \dots$ on the coset manifold.

From the H -connection, one can define a *covariant derivative* on the coset manifold

$$\mathcal{D}_\alpha L(\phi) \equiv \partial_\alpha L(\phi) - \omega^i{}_\alpha L(\phi)t_i. \quad (\text{A.23})$$

Given the definition of Maurer–Cartan form and the decomposition (A.14), H -covariant derivatives can be used to write

$$L^{-1}(\phi)\mathcal{D}_\alpha L(\phi) = \mathcal{V}_\alpha{}^a t_a \quad \Leftrightarrow \quad \mathcal{D}_\alpha L(\phi) = L(\phi)\mathcal{V}_\alpha. \quad (\text{A.24})$$

A.3.1 Maurer–Cartan equations

The Maurer–Cartan equations express the differential properties of $\Omega = L^{-1}dL$

$$d\Omega(\phi) + \Omega(\phi) \wedge \Omega(\phi) = 0. \quad (\text{A.25})$$

Projecting on the \mathfrak{k} and \mathfrak{h} subspaces of the Lie algebra \mathfrak{g} , we obtain

$$d\mathcal{V}^a + \mathcal{V}^b \wedge \omega^i f_{bi}{}^a + \frac{1}{2}\mathcal{V}^b \wedge \mathcal{V}^c f_{bc}{}^a = 0, \quad (\text{A.26a})$$

$$d\omega^i + \frac{1}{2}\omega^j \wedge \omega^k f_{jk}{}^i + \frac{1}{2}\mathcal{V}^a \wedge \mathcal{V}^b f_{ab}{}^i = 0. \quad (\text{A.26b})$$

If we assume that the coset manifold, apart from being reductive, is also symmetric (then $f_{ab}{}^c = 0$), equations (A.26) can be written in a more compact notation as

$$\mathcal{D}\mathcal{V} \equiv d\mathcal{V} + \mathcal{V} \wedge \omega + \omega \wedge \mathcal{V} = 0, \quad (\text{A.27a})$$

$$R \equiv d\omega + \omega \wedge \omega = -\mathcal{V} \wedge \mathcal{V}, \quad (\text{A.27b})$$

where we have defined the covariant derivative of the vielbein $\mathcal{D}\mathcal{V}$, while the 2-form R is the H -curvature [12]. In components, equation (A.27b) becomes

$$R_{\alpha\beta}{}^i = \partial_\alpha \omega^i{}_\beta - \partial_\beta \omega^i{}_\alpha + \omega^j{}_\alpha \omega^k{}_\beta f_{jk}{}^i = -\mathcal{V}_\alpha{}^a \mathcal{V}_\beta{}^b f_{ab}{}^i. \quad (\text{A.28})$$

The curvature R can be used to relate Killing vectors and prepotentials

$$\mathcal{D}_\beta \mathcal{P}_A{}^i = \xi_A{}^\alpha R_{\alpha\beta}{}^i, \quad (\text{A.29})$$

which holds not only in the case of coset manifolds. Indeed, for non-homogeneous scalar manifolds, as in the case of some $\mathcal{N} = 2$ supergravity models, momentum maps can be constructed from Killing vectors as solutions to the differential equation (A.29).

B

BRANCHING RULES

In this appendix we give, through a concrete example, some details about the strategy used to select the independent non-vanishing components of the tensors A_1 and A_2 for each chosen supersymmetry breaking pattern. The underlying idea is that the components we cannot automatically set to zero are those transforming trivially under the residual R-symmetry group, i.e. the components that are left invariant by the group action.

In this context, a key element is the knowledge of rules for decomposing any irreducible representation of a group into irreducible representations of its subgroups, the so-called *branching rules*. We will recover information about the branching rules for the groups of our interest from [57], where they are obtained exploiting the Mathematica application LieART, which is specifically designed to study Lie groups and Lie algebras.

B.1 AN EXAMPLE

We consider the case in which we look for vacua with 4 unbroken supersymmetry generators, as in section 5.5. The R-symmetry group splits into

$$\mathrm{SU}(8) \longrightarrow \mathrm{SU}(4) \times \mathrm{SU}(4) \times \mathrm{U}(1) ,$$

then the first step is to take the branching rules of the **36** and **420** representations of $\mathrm{SU}(8)$ and match the irreducible representations with the tensor components. For the **36** representation, corresponding to the A_1 tensor, we have

$$\mathbf{36} \longrightarrow (\mathbf{4}, \mathbf{4})(0) + (\mathbf{10}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{10})(-2) ,$$

where, for each term, the first two numbers denote the representations with respect to the two $\mathrm{SU}(4)$ groups, while the third number is the charge with respect to $\mathrm{U}(1)$. Using again the notation (5.15), we can easily establish that

- the $(\mathbf{4}, \mathbf{4})(0)$ representation corresponds to the tensor components $A_1^{aI} = A_1^{Ia}$;
- the $(\mathbf{10}, \mathbf{1})(2)$ representation corresponds to the components A_1^{IJ} ;
- the $(\mathbf{1}, \mathbf{10})(-2)$ representation corresponds to the components A_1^{ab} .

Due to the supersymmetry requirements, we know that we have to take into account only the last irreducible representation, while the others will be automatically vanishing.

The same can be done for the A_2 tensor: considering the branching rule for the **420** representation, we find that

- the $(\overline{\mathbf{10}}, \mathbf{1})(2)$ representation corresponds to the tensor components $A_{2_I}{}^{JKL}$ traceless, i.e. verifying $A_{2_I}{}^{IJK} = 0$;
- the $(\mathbf{1}, \overline{\mathbf{10}})(-2)$ representation corresponds to the components $A_{2_a}{}^{bcd}$ traceless, i.e. verifying $A_{2_a}{}^{abc} = 0$;
- the $(\mathbf{15}, \mathbf{6})(-2)$ representation corresponds to the components $A_{2_I}{}^{Jab}$ such that $A_{2_I}{}^{Iab} = 0$;
- the $(\mathbf{6}, \mathbf{15})(2)$ representation corresponds to the components $A_{2_a}{}^{IJb}$ s.t. $A_{2_a}{}^{IJa} = 0$;
- the $(\mathbf{20}, \mathbf{4})(0)$ representation corresponds to the components $A_{2_I}{}^{JKa}$ s.t. $A_{2_I}{}^{IJa} = 0$;
- the $(\mathbf{4}, \mathbf{20})(0)$ representation corresponds to the components $A_{2_a}{}^{Ibc}$ s.t. $A_{2_a}{}^{Iab} = 0$;
- the $(\overline{\mathbf{4}}, \overline{\mathbf{4}})(4)$ representation corresponds to the components $A_{2_a}{}^{IJK}$;
- the $(\overline{\mathbf{4}}, \overline{\mathbf{4}})(-4)$ representation corresponds to the components $A_{2_I}{}^{abc}$;
- the $(\mathbf{4}, \mathbf{4})(0)$ representation corresponds to $A_{2_i}{}^{jaK} = \delta_i^j \Lambda^{aK}$ for $i = 1, \dots, 4$ and $A_{2_i}{}^{jaK} = -\delta_i^j \Lambda^{aK}$ for $i = 5, \dots, 8$;
- the $(\mathbf{6}, \mathbf{1})(2)$ representation corresponds to $A_{2_i}{}^{jKL} = \delta_i^j \Lambda^{KL}$ for $i = 1, \dots, 4$ and $A_{2_i}{}^{jKL} = -\delta_i^j \Lambda^{KL}$ for $i = 5, \dots, 8$;
- the $(\mathbf{1}, \mathbf{6})(-2)$ representation corresponds to $A_{2_i}{}^{jab} = \delta_i^j \Lambda^{ab}$ for $i = 1, \dots, 4$ and $A_{2_i}{}^{jab} = -\delta_i^j \Lambda^{ab}$ for $i = 5, \dots, 8$.

Due to the requirement that $A_{2_I}{}^{ijk} = 0$ imposed by supersymmetry at the vacuum, we can keep only the $(\mathbf{1}, \overline{\mathbf{10}})(-2)$, $(\mathbf{6}, \mathbf{15})(2)$, $(\mathbf{4}, \mathbf{20})(0)$ and $(\overline{\mathbf{4}}, \overline{\mathbf{4}})(4)$ representations, which correspond precisely to the non-vanishing tensor components listed at the beginning of section 5.5.

If we further break the $SU(4)$ symmetry associated with the broken supersymmetry generators, we have to exploit the branching rules of $SU(4)$. In particular, we consider the breaking pattern

$$SU(4) \longrightarrow SU(2) \times SU(2) \times U(1) \longrightarrow U(1) \times U(1), \quad (\text{B.1})$$

which leads to the residual R-symmetry analyzed in section 5.5.4. The two $U(1)$ factors in the final symmetry group are meant as subgroups of the two $SU(2)$ respectively.

We use now a notation where indices transforming non-trivially with respect to the first $SU(2)$ factor are denoted with $\tilde{a}, \tilde{b} \dots$, while $\hat{a}, \hat{b} \dots$ are the indices associated to the second $SU(2)$ factor. If we take the A_1 tensor, its non-vanishing components A_1^{ab} sitting in the **10** representation of $SU(4)$ transform under $SU(2) \times SU(2) \times U(1)$ according to the following irreducible representations:

- $(\mathbf{2}, \mathbf{2})(0)$: components $A_1^{\tilde{a}\hat{b}}$;
- $(\mathbf{3}, \mathbf{1})(2)$: components $A_1^{\tilde{a}\tilde{b}}$;
- $(\mathbf{1}, \mathbf{3})(-2)$: components $A_1^{\hat{a}\hat{b}}$.

For the A_2 tensor, instead, we have that

- the representation $\overline{\mathbf{4}}$ (corresponding to the components $A_{2\tilde{a}}^{IJK}$) splits into
 - $(\overline{\mathbf{2}}, \mathbf{1})(-1)$: components $A_{2\tilde{a}}^{IJK}$;
 - $(\mathbf{1}, \overline{\mathbf{2}})(1)$: components $A_{2\hat{a}}^{IJK}$;
- the representation **15** (corresponding to the components $A_{2\tilde{a}}^{bIJ}$) splits into
 - $(\mathbf{1}, \mathbf{1})(0)$: components $A_{2\tilde{a}}^{bIJ} = (\delta_{\tilde{a}}^{\tilde{b}} - \delta_{\tilde{a}}^{\hat{b}})\Lambda^{IJ}$;
 - $(\mathbf{2}, \mathbf{2})(2)$: components $A_{2\hat{a}}^{\tilde{b}IJ}$;
 - $(\mathbf{2}, \mathbf{2})(-2)$: components $A_{2\tilde{a}}^{\hat{b}IJ}$;
 - $(\mathbf{3}, \mathbf{1})(0)$: components $A_{2\tilde{a}}^{\tilde{b}IJ}$ such that $A_{2\tilde{a}}^{\tilde{a}IJ} = 0$;
 - $(\mathbf{1}, \mathbf{3})(0)$: components $A_{2\hat{a}}^{\hat{b}IJ}$ such that $A_{2\hat{a}}^{\hat{a}IJ} = 0$;
- the representation **20** (corresponding to the components $A_{2\tilde{a}}^{bcI}$) splits into
 - $(\mathbf{2}, \mathbf{1})(1)$: components $A_{2\tilde{a}}^{\tilde{b}\hat{c}I}$;
 - $(\mathbf{2}, \mathbf{1})(-3)$: components $A_{2\tilde{a}}^{\hat{b}\hat{c}I}$;
 - $(\mathbf{1}, \mathbf{2})(3)$: components $A_{2\hat{a}}^{\tilde{b}\hat{c}I}$;
 - $(\mathbf{1}, \mathbf{2})(-1)$: components $A_{2\hat{a}}^{\hat{b}\hat{c}I}$;
 - $(\mathbf{3}, \mathbf{2})(-1)$: components $A_{2\tilde{a}}^{\tilde{b}\hat{c}I}$ such that $A_{2\tilde{a}}^{\tilde{a}\hat{c}I} = 0$;
 - $(\mathbf{2}, \mathbf{3})(1)$: components $A_{2\hat{a}}^{\hat{b}\hat{c}I}$ such that $A_{2\hat{a}}^{\hat{a}\hat{c}I} = 0$;
- the representation $\overline{\mathbf{10}}$ (corresponding to the components $A_{2\tilde{a}}^{bcd}$) splits into
 - $(\mathbf{2}, \mathbf{2})(0)$: components $A_{2\tilde{a}}^{b\hat{c}\hat{d}} = (\delta_{\tilde{a}}^{\tilde{b}} - \delta_{\tilde{a}}^{\hat{b}})\Lambda^{\hat{c}\hat{d}}$;
 - $(\mathbf{3}, \mathbf{1})(2)$: components $A_{2\hat{a}}^{\tilde{b}\hat{c}\hat{d}}$ such that $A_{2\hat{a}}^{\tilde{b}\hat{c}\hat{a}} = 0$;

– $(\mathbf{1}, \mathbf{3})(-2)$: components $A_{2\tilde{a}}^{\tilde{b}\hat{c}\hat{d}}$ such that $A_{2\tilde{a}}^{\tilde{a}\hat{c}\hat{d}} = 0$.

At this point, as expressed in (B.1), we want to consider that the symmetry is broken to $U(1) \times U(1)$, where the two $U(1)$ act respectively on indices \tilde{a} and \hat{a} . Due to the well-known structure of the branching rules for $SU(2) \rightarrow U(1)$ (from the n -dimensional representation of $SU(2)$ we obtain n 1-dimensional representations of $U(1)$ that are charged under $U(1)$ with charges going from $-(n-1)$ to $n-1$ in steps of 2), we can immediately select the tensor components of A_1 and A_2 transforming trivially under $U(1) \times U(1)$. Indeed, since only from an odd-dimensional $SU(2)$ representation we can get a $U(1)$ representation with zero charge, we will get a singlet under $U(1) \times U(1)$ for each representation of $SU(2) \times SU(2)$ of the type $(2\mathbf{n} + \mathbf{1}, 2\mathbf{m} + \mathbf{1})$.

Using this procedure and at the same time fixing the action of the $U(1)$ in the chosen basis (i.e. the charges under the $U(1)$ groups), we can find explicitly the A_1 and A_2 components which are singlets under the $U(1) \times U(1)$ symmetry; they correspond to those listed in section 5.5.4.

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