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Stability of periodic systems and Floquet Theory

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Chapter 1: INTRODUCTION

1 Historical Introduction

1.1 Cycling is good

The constant coefficient linear equations are well known in Mathematics; their solution, for both the free and the forced responses, is straightforward with the help of the Laplace Transform and is at the basis for control theory of linear systems.

However, Laplace-transform methods are not applicable when dealing with a nonlinear, time-dependent system. So it is useful to highlight a specific class of time-dependent differential equations which have a behaviour similar to the linear case and are thus easier to solve: differential equations with periodically-varying coefficients. The equations more frequently used in applications are of in second order form, as:

$$\ddot{y} + (a - 2q\psi(t))y = 0 \tag{1.1}$$

where:

- $\psi(t) = \psi(t + \pi)$ is a periodic coefficient, generally of period π;
- a is a content coefficient;
- q is the amplitude of periodic change rate;

Equation *(1.1)*, a fundamental equation in this thesis, is known as *Hill's equation.*

1.2 Historical background

Before describing in mathematical details the periodic differential equations and control theory, in this section we give a short historical excursus to highlight the steps forward that have been done in this field in the last two centuries.

The first studies about the parametric behaviour of systems were carried out by *Michael Faraday*, in 1831.

Fig.1: M. Faraday

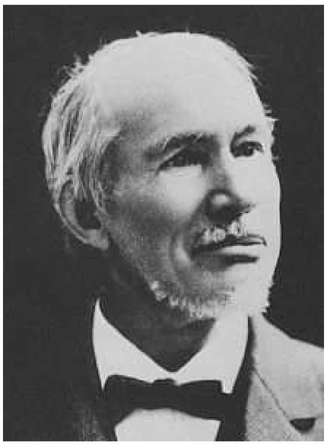


Fig.2: G.W. Hill

Subsequently, the first detailed theory about time-dependent, periodic systems was developed by *Emile Mathieu* in 1868, particularly devoted to the analysis of surface waves in elliptic-shaped lakes. A few years later, in 1883, *G. Floquet*, laid the foundation of his most widely known theory about the parametric behavior of space-distributed systems and transmission lines.

One of the most important contributions was given by *George William Hill,* who, in 1886, published some papers that motivated the rise of the stability analysis for parametric systems [for details, see Section 2]. His most important work has been the determination of the time-dependent gravitational influence of the system sun-moon.

Hill's paper gave the impulse to the rise of *Lord Rayleigh*'s analysis of second order systems (1887).

Fig.3: Lord Rayleigh

Another important scholar working on time-dependent system was *E. Meissner*: in 1918 he published an important paper about Hill's equation where ψ (t) is a rectangular function, so that *(1.1)* is solvable exactly as a couple of constant-coefficient equations, each of them valid in equally-spaced alternated timeslots. The application area of these theories is very wide: two modern examples are described in the fifth chapter.

1.3 An every-day application: children swing

The application fields of periodic differential equations are, as we remarked above, very different: some practical examples are explained in detail in next chapters. Nevertheless, a simple example is noteworthy to be reported: the children swing.

Fig.4: A children swing.

A child playing with a swing can be represented like a fixed-mass variable-length pendulum: the variable-length of the thread depends on the legs extension and on the relative positioning of the body-seat system.

With the hypothesis of well-known initial conditions for position and velocity, the person can increment the amplitude of the oscillation, raising and lowering his legs while synchronizing the movement with the swing instantaneous frequency. In such a way the length of the chains varies with the natural frequency.

This concept is called *system pumping*. More specifically, the person may oscillate at double frequency respect to natural frequency, lowering his gravity center approaching to each end of the swing trajectory and raising it after the middle point.

Alternatively, the body may pulse at the exact system frequency extending or shortening the chord length with hands once in a period timespan.

If the system is pumped with wrong initial phase, the system response will evolve correcting it in a sort of closed-loop control to maximize the energy transfer from source to system: the system is put in a phase-locking condition.

An example of the latter in electronic-control field is the PLL (Phase Locked Loop) circuit: a closed-loop system used to synchronize the output frequency to the input frequency, allowing for easy FM envelope modulation/demodulation, CD speed rotation stabilization at fixed throughput, constant frequency motor control and precise CPU clock setting.

1.4 Example: LC oscillating circuit

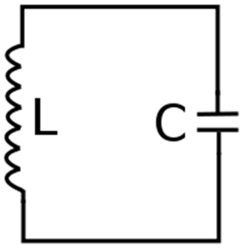


Fig.5: LC oscillating circuit

An LC parallel oscillating circuit is described through a second order equation with variable V (*voltage*); if the capacitance is a time-periodic function, the relation is a periodic-coefficients Hill's equation.

Under the hypothesis that the system is initially energy-charged and the capacitance value may change, the energy of the system oscillates between electrostatic energy (*C)* and electromagnetic energy *(L)* with a fixed frequency:

$$\omega_0 = (LC)^{-\frac{1}{2}}.$$

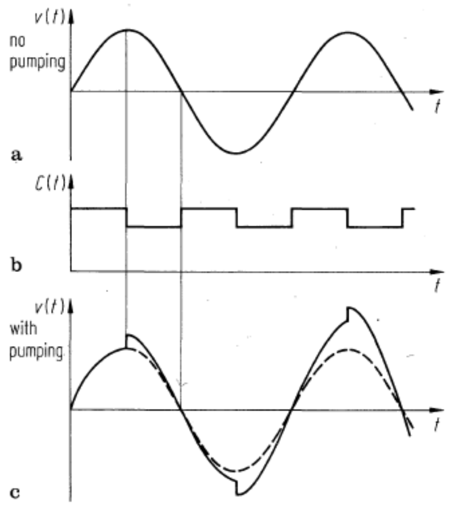


Fig.6: a. Unpumped capacitor voltage
b. Square ware capacitor variation
c. Amplification of the capacitor voltage resulting from pumping the capacitor

When $C = 0$ all the system energy resides in the inductance; vice versa if $C = \pm\, max$ it's entirely contained in the capacitance.

Suppose that when $C = \pm\, max$, there is an external source feeding the system (it does not really matter how it is realized), adding work to the total energy of the circuit.

Assuming that the initial energy resided in the capacitance, we can observe that there's an amplification of the capacitor voltage. In this way the capacitor voltage describes a new sinusoid with higher amplitude.

This process can be repeated each time $C = \pm\, max$, allowing an unbounded periodic voltage amplification. If some resistors are put in parallel to L, C (in this way we have a more complicated circuit, but with a similar functioning), this amplification is resistance-limited.

This effect is called *pumping effect*, different from the forcing effect characterized by a current input in the system, like a source of current.

Pumping the system at double frequency of resonance frequency, we can measure crescent-amplitude oscillations.

The latter suffices to lead to instability the passive system described.

$$\omega_{pumping} = 2\omega_0 \Rightarrow in\ general\ \ \omega_p = \frac{2\omega_0}{n}$$

For increasing values for *n*, we increase the dumping factor; the parametric instability, caused by pumping, is frequently experienced only for $n = 1$. In the presence of resistors, the parametric gain results bigger for the same value of *n*.

In conclusion, a general parametric behaviour results from each storing-energy component pumped above resonance frequency.
Specifically, the parametric effects depend on the natural (or static) frequency variation via a pumping method. Note that not every energy-storing component contributes to the natural frequency of the system and may not lead to parametric behaviour, even though energy-pumped.

1.5 Analytical explanation: the Hill's Equation

In this section the periodic-coefficient differential equations will be shortly described. They can be revealed in very different scientific fields and they take many analytical shapes.
The most general way to denote a periodic equation is the following:

$$g_v(t)\, x^v + g_{v-1}(t)\, x^{v-1} + \dots\dots\dots\dots + g_o(t)x = f(t),$$

The latter is called homogeneous if $f(t) = 0$. Their coefficient may have a constant or time-dependent periodic value, where θ is the period:

$$g_i(t) = g_i(t + \theta)\ .$$

More specifically, the most widely used periodic equation is a second order relation:

$$\ddot{x} + g_1(t)\dot{x} + g_0(t)x = 0$$

A more classical form of the equation is:

$$\ddot{x} + (a - 2q\psi(t))x = 0$$

That is called Hill's Equation where:

- $(t) = \psi\,(t + \pi\,)$ is a periodic coefficient with period π;
- $|\psi\,(t)|_{\max} = 1$;
- a is a scalar coefficient;
- q is the amplitude of the periodic envelope;

The most famous representation of the Hill's Equation, called the *Mathieu's Equation,* is characterized by the sinusoidal form of $\psi(\mathrm{t})$.

Chapter 2: THE FLOQUET THEORY

2.1 Introduction to the general theory and matrix representation

The periodic equation to be considered for our purposes shall be expressed in the state-space representation (or matrix representation). This form lends itself well to the computation of the solution of a system of differential equations. The variables of interest will be arranged in a column vector *y(t),* and their dynamics will be expressed in terms of a vector of variables *x(t)* which we will call the *state vector*. The number *n* of elements composing vector *x(t)* is the system dimension. A state–space model for the system dynamics in continuous time is of the form:

$$x'(t) = A(t)x(t)+B(t)u(t), \tag{2.1}$$

$$y(t) = C(t)x(t)+D(t)u(t), \tag{2.2}$$

where A(·), B(·), C(·), and D(·) are matrices of appropriate dimensions. In particular, A(·) specifies the dynamical relation between the state vectors and is called *dynamic matrix*, *u(t)* is a vector of exogenous variables representing the inputs of the system. The two coupled equations above are known as *state–space models*: (2.1) is called *state equation*, whereas (2.2) is the so-called *output equation*. Notice that the output equation is static, and equation (2.1) completely describe the dynamical behaviour. We shall thus focus on (2.1). The state variable, in general, may or may not have physical meaning.

A state-space model is said to be *periodic* if the real matrices A(·), B(·), C(·), and D(·) are periodic, that is there exists T > 0 such that:

$$\begin{aligned} A(t+T) &= A(t), \\ B(t+T) &= B(t), \\ C(t+T) &= C(t), \\ D(t+T) &= D(t). \end{aligned} \tag{2.3}$$

The smallest T for which these periodicity conditions are met is called the *system period*. Typically, the dimension n is constant, so that the system matrices have constant dimension too.

Let us introduce some relevant cases that will be studied in detail later. If we call, for convenience, $f(t) = B(t)\,u(t)$, we can write the state equation as:

$$x'(t) = A(t)x(t) + f(t), \tag{2.4}$$

where $f(t)$ is a vector describing the effect of the effect of the inputs, also called the *forcing term*. A(t) is a matrix of constant or periodically varying coefficients. When $f(t) = 0$ the equation (2.4) becomes the homogeneous set of v first order periodic equations:

$$\begin{aligned} x'(t) &= A(t)x(t), \\ A(t) &= A(t+\theta). \end{aligned} \tag{2.5}$$

Some interesting examples from the classical literature on periodic systems have this form. In particular, consider a state vector containing a scalar variable x and the ordered set of its first n-1 derivatives:

$$x(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ldots \\ \ldots \\ x^{v-1}(t) \end{bmatrix} , \qquad (2.6)$$

The periodic equations we consider are in the form:

$$g_v(t)\, x^v + g_{v-1}(t)\, x^{v-1} + \ldots\ldots\ldots\ldots + g_o(t)x = f(t), \qquad (2.7)$$

in which the notation x^v implies the *v-th* derivative of *x* with respect to the independent variable *t*, which is usually time. The function *f(t),* which appears in the equation (2.7), generally associated with a periodic forcing function, may also involve periodically varying coefficients. The coefficients $g_i(t)$ may be constant or periodically varying with *t:*

$$g_i(t) = g_i(t + \theta) . \qquad (2.8)$$

for some period θ. When *f(t)* is zero the equation (2.7) is said to be *homogeneous*. A specific case of equation (2.7) that is very useful in physics is:

$$\begin{aligned} &\ddot{y} + (a - 2q\psi(t))y = 0 \\ &\psi(t) = \psi(t + \pi) \end{aligned} , \qquad (2.9)$$

which is known as the *Hill equation*, and will be further described in the following paragraphs. The function ψ(t) has, by convention, a period of T and also |ψ(t)max| = 1; *a* is a constant parameter and *2q* is a parameter which represents the magnitude of the time variation.

The most widely known form of the Hill equation is the *Mathieu equation* in which ψ(t) is sinusoidal:

$$\ddot{y} + (a - 2q\cos(2t))y = 0 \quad . \tag{2.10}$$

This equation will be described in details in the third chapter of this text.
In this chapter, instead, we will study properties of system (2.1) and (2.2), that follow from its periodicity, and present the fundamental theorem of the Floquet theory.

2.2 Solutions and state-transition matrix

The results we will present relating to periodic systems deal with periodic systems evolving autonomously, namely without any exogenous input. This amounts to considering, in the continuous time picture, the homogeneous equation:

$$x'(t) = A(t)x(t), \tag{2.12}$$

$$A(t) = A(t + \theta), \qquad \text{with } \theta > 0.$$

A(t) is considered of constant dimension $n \times n$. Our aim is to study the *Floquet theory*, which is concerned with the problem of finding a periodic state–space change of basis, so that, in the new basis, the state matrix is constant. This clearly would simplify many analysis problems for the dynamics, and in particular the study of its stability properties.

First, the properties of the so-called fundamental matrix are pointed out.

Theorem 2.1 Consider again the linear periodic continuous system in :

$$x' = A(t)\, x, \tag{2.13}$$

$$x(t_0) = x_0,$$

where the elements of the $n \times n$ matrix A are continuous functions of t over the period $[t_0; t_0 + T]$.

x(t) is a solution of the differential equation if and only if it satisfies the integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} A(\tau)x(\tau)d\tau. \tag{2.14}$$

Proof: In order to prove the above, we shall follow the following strategy: we show that the equation (2.14) can be written as the limit of a sequence of approximating vector-valued functions that starts from a constant one, corresponding to the initial conditions. In order to do this, we introduce the sequence, we show that it converges, and lastly that it is exactly the solution of system (2.13).

The successive approximations are defined by:

$$
\begin{aligned}
&x(t) = x_0;\\
&x_1(t) = x_0 + \int_{t_0}^{t} A(\sigma_1)x_1(\sigma_1)d\sigma_1;\\
&x_2(t) = x_0 + \int_{t_0}^{t} A(\sigma_1)x_1(\sigma_1)d\sigma_1 =\\
&\qquad = x_0 + \int_{t_0}^{t} A(\sigma_1)x_0 d\sigma_1 +\\
&\qquad\qquad + \int_{t_0}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)x_0 d\sigma_2 d\sigma_1;\\
&x_k(t) = x_0 + \int_{t_0}^{t} A(\sigma_1)x_{k-1}(\sigma_1)d\sigma_1 + \dots .
\end{aligned}
$$

The second step consists in checking the convergence, because if it was not verified, the solution may not exist. We try to understand if the solution x_k converges as k increases:

$$
\begin{aligned}
x_k(t) &= x_0(t) + x_1(t) - x_0(t) + \dots + x_k(t) - x_{k-1}(t) =\\
&= x_0 + \sum_{j=0}^{k-1}[x_{j+1}(t) - x_j(t)].
\end{aligned}
$$

For $x_f = 0$:

$$\|x_1(t) - x_0(t)\| = \left\| \int_{t_0}^{t} A(\sigma_1) x_0 d\sigma_1 \right\| \leq$$

$$\leq \int_{t_0}^{t} \|A(\sigma_1)\| \cdot \|x_0\| d\sigma_1 \leq \alpha T \|x_0\|$$

for all $t \in [t_0, t_0 + T]$, where $\alpha = \max_{t_0 \leq t \leq t_0 + T} \|A(t)\|$.

For $x_f = 1$:

$$\|x_2(t) - x_1(t)\| = \left\| \int_{t_0}^{t} A(\sigma_1)[x_1(\sigma_1) - x_0(\sigma_1)] d\sigma_1 \right\| \leq$$

$$\leq \int_{t_0}^{t} \|A(\sigma_1)\| \cdot \|x_1(\sigma_1) - x_0(\sigma_1)\| d\sigma_1 \leq$$

$$\leq \int_{t_0}^{t} \alpha \alpha T \|x_0\| d\sigma_1 =$$

$$= \alpha^2 T \|x_0\| (t - t_0).$$

For $x_f = 2$:

$$\|x_3(t) - x_2(t)\| = \left\| \int_{t_0}^{t} A(\sigma_1)[x_2(\sigma_1) - x_1(\sigma_1)] d\sigma_1 \right\| \leq$$

$$\leq \int_{t_0}^{t} \|A(\sigma_1)\| \cdot \|x_2(\sigma_1) - x_1(\sigma_1)\| d\sigma_1 \leq$$

$$\leq \int_{t_0}^{t} \alpha \alpha^2 T \|x_0\| (\sigma_1 - t_0) d\sigma_1 =$$

$$= \alpha T \|x_0\| \frac{\alpha^2 (t - t_0)^2}{2!}.$$

Similarly, for x_f increasing:

$$\|x_{j+1}(t) - x_j(t)\| \leq \alpha T \|x_0\| \frac{\alpha^j (t - t_0)^j}{j!} \leq \alpha T \|x_0\| \frac{\alpha^j T^j}{j!},$$

and reminding that:

$$x_k(t) = x_0 + \sum_{j=0}^{k-1} [x_{j+1}(t) - x_j(t)],$$

we can conclude that:

$$\|x_k(t)\| \le \|x_0\| + \sum_{j=0}^{k-1} \|x_{j+1}(t) - x_j(t)\| \le$$
$$\le \|x_0\| + \alpha T \|x_0\| \cdot \sum_{j=0}^{k-1} \frac{\alpha^j T^j}{j!}.$$

So, we have proved that the limit for k going to infinity exists and it converges. Thus now we ask if actually the series over indicated:

$$x_k(t) = x_0 + \int_{t_0}^{t} A(\sigma_1) x_{k-1}(\sigma_1) d\sigma_1 ,$$

converges to the solution of the system described in (2.6). In order to do so, we proceed with the derivation of each term of the series, taking into account the fundamental theorem of calculus:

$$x'(t) = 0 + A(t)x_0 + A(t)\int_{t_0}^{t} A(\sigma_2) d\sigma_2 x_0 \ + ...$$
$$= A(t)[x_0 + \int_{t_0}^{t} A(\sigma_2) d\sigma_2 x_0 \ + ...] =$$
$$= A(t)x(t).$$

q.e.d.

Notice that the solution x(t) can also be expressed as:

$$x(t) = [I + \int_{t_0}^{t} A(\sigma_1) d\sigma_1 + \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2) d\sigma_2 d\sigma_1 + ...] x_0 .$$

We then ready to introduce one of the fundamental objects of Floquet Theory.

Define the *n x n* matrix $\Phi(t, t_0)$ by:

$$\Phi(t,t_0) := I + \int_{t_0}^{t} A(\sigma_1)d\sigma_1 + \int_{t_0}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1 + ... \tag{2.15}$$

So,

$$x(t) = \Phi(t,t_0)x_0 \tag{2.16}$$

.

$\Phi(t, t_0)$ is called the *state transition matrix*. The solution of the system (2.7) is unique, so also $\Phi(t, t_0)$ is unique as well. This follows from Property 3 that will be proved below.

Alternately, starting from a generic initial condition $x(\tau)$ at time τ, the solution is obtained as:

$$x(t) = \Phi(t,\tau)x(\tau) \tag{2.17}$$

where the transition matrix $\Phi(t,\tau)$ is given by the solution of the differential matrix equation:

$$\Phi'(t, \tau) = A(t)\,\Phi(t, \tau), \qquad \Phi(\tau, \tau) = I, \tag{2.18}$$

in continuous-time. Hence, for any given τ, the matrix $\Phi(t, \tau)$ is then the principal fundamental matrix solution of the system, that is the unique solution of the matrix initial value problem:

$$X' = A(t)\,X$$

$$X(\tau) = I.$$

If it is a solution for a different initial value, then it is simply called a *fundamental solution*.

It is easily seen that the periodicity of the system involves the double periodicity of matrix $\Phi(t, \tau)$, i.e.:

$$\Phi(t + T, \tau + T) = \Phi(t, \tau).$$

Proof: Take the definition of Φ (t,t_0) given in (2.15) and insert Φ(t + T, τ +T) in that equation:

$$\Phi(t,t_0) := I + \int_{t_0}^{t} A(\sigma_1)d\sigma_1 + \int_{t_0}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1 + \ldots.$$

but as A(t+T) = A(t), it follows that:

$$\Phi(t + T, \tau + T) = \Phi(t, \tau). \qquad \text{q.e.d.}$$

The transition matrix over one period is of particular interest, and we will indicate it as:

$$\psi(t) = \Phi(t + T, t). \qquad (2.20)$$

The matrix ψ(t) is known as *monodromy matrix* at time t.

2.3 Properties of Φ(t, τ)

As we will use the transition matrix for the definition and the proof of the fundamental Floquet's theorem, it is proper to list and prove the properties of this matrix for a generic τ .

1.

$$\frac{\partial}{\partial \tau}\Phi(t,\tau) = -\Phi(t,\tau)A(\tau)$$

Proof: the transition matrix is defined as:

$$\Phi(t,\tau) = I + \int_{\tau}^{t} A(\sigma_1)d\sigma_1 + \int_{\tau}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)d\sigma_2\, d\sigma_1 + \ldots.$$

By noticing that time-derivation of the integral terms yields:

$$\frac{\partial}{\partial\tau}\int_{\tau}^{t} A(\sigma_1)d\sigma_1 = -A(\tau),$$

and

$$\frac{\partial}{\partial\tau}\int_{\tau}^{t} A(\sigma_1)\int_{\tau}^{t} A(\sigma_2)d\sigma_2\, d\sigma_1 = -\int_{\tau}^{t} A(\sigma_1)d\sigma_1 A(\tau),$$

and so on for the other terms of the series, we get:

$$\frac{\partial}{\partial\tau}\Phi(t,\tau) = -[I + \int_{t_0}^{t} A(\sigma_1)d\sigma_1 + \int_{t_0}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1 + ...]A(\tau) =$$
$$= -\Phi(t,\tau)A(\tau).$$

q.e.d.

2.

$$\Phi(t,\tau) = \Phi(t,\sigma)\Phi(\sigma,\tau) \quad \forall t,\tau,\sigma \,.$$

Proof: Let x(t) be the solution of the state equation $x' = A(t)\,x$, so:

$$x(\sigma) = \Phi(\sigma,\tau)x(\tau) \quad ,$$
$$x(t) = \Phi(t,\sigma)x(\sigma) = \Phi(t,\sigma)\Phi(\sigma,\tau)x(\tau) \,.$$

On the other hand

$$x(t) = \Phi(t,\tau)x(\tau) \,.$$

By the uniqueness of the solution, demonstrated above:

$$\Phi(t,\sigma)\Phi(\sigma,\tau)x(\tau) = \Phi(t,\tau)x(\tau) \,,$$
$$\Rightarrow \Phi(t,\sigma)\Phi(\sigma,\tau) = \Phi(t,\tau) \,.$$

q.e.d.

3. $\Phi(t, \tau)$ is non-singular for all t, and:

$$\Phi^{-1}(t,\tau) = \Phi(\tau,t).$$

Proof: We know that

$$\Phi(t,\tau)\Phi(\tau,t) = \Phi(t,t) = I \ .$$

This shows that Φ(t ,τ) is nonsingular and:

$$\Phi^{-1}(t,\tau) = \Phi(\tau,t) \, .$$

q.e.d.

2.4 Complete solution

Consider the state equation:

x'(t) = A(t)x(t)+B(t)u(t),

where A(t) and B(t) are continuous functions of t and u(t) is also a continuous function of t. The equation has an unique solution, given by:

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau \, .$$

Proof: We want to verify if:

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau \, ,$$

is really a solution for the state equation. Calculating the first derivate of x(t):

$$x'(t) = \frac{\partial}{\partial t}\Phi(t,t_0)x_0 + \Phi(t,t)B(t)u(t) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau \, ,$$

and using the properties described above:

$$= A(t)[\Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau] + B(t)u(t)$$

$$= A(t)x(t) + B(t)u(t) \, .$$

q.e.d.

We can consider now some special cases in the computation of the general solution is easier.

Special case: A is constant

Remember the definition of $\Phi(t, \tau)$:

$$\Phi(t,\tau) = I + \int_{\tau}^{t} A\, d\sigma_1 + \int_{\tau}^{t} A \int_{\tau}^{\sigma_1} A\, \mathrm{d}\sigma_2\, \mathrm{d}\sigma_1 + $$
$$+ \int_{\tau}^{t} A \int_{\tau}^{\sigma_1} A \int_{\tau}^{\sigma_2} A\, \mathrm{d}\sigma_3\, \mathrm{d}\sigma_2\, \mathrm{d}\sigma_1 + \ldots$$

For the hypothesis of A constant, we can proceed as follows:

$$\int_{\tau}^{t} A\, \mathrm{d}\sigma_1 = (t-\tau)A$$
$$\int_{\tau}^{t} A \int_{\tau}^{\sigma_1} A\, \mathrm{d}\sigma_2\, \mathrm{d}\sigma_1 = \frac{(t-\tau)^2}{2!} A^2$$
$$\int_{\tau}^{t} A \int_{\tau}^{\sigma_1} A \int_{\tau}^{\sigma_2} A\, \mathrm{d}\sigma_3\, \mathrm{d}\sigma_2\, \mathrm{d}\sigma_1 = \frac{(t-\tau)^3}{3!} A^3$$

Until we arrive at the solution:

$$\Phi(t,\tau) = \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} A^k = e^{(t-\tau)A}$$

Special case: n=1

Starting with: $$\dot{x} = a(t)\, x$$

If we derive: $$\frac{dx}{x} = a(t)\, \mathrm{dt}$$

So: $$\int_{x(t_0)}^{x(t)} \frac{dz}{z} = \int_{t_0}^{t} a(\tau)\, \mathrm{d}\tau$$

Applying the natural logarithm: $$\ln\left(\frac{x(t)}{x(t_0)}\right) = \int_{t_0}^{t} a(\tau)\, \mathrm{d}\tau$$

$$x(t) = e^{\int_{t_0}^{t} a(\tau)\, \mathrm{d}\tau} x(t_0)$$

We arrive at: $$\Phi(t,t_0) = e^{\int_{t_0}^{t} a(\tau)\, \mathrm{d}\tau}.$$

Special case: A is diagonal

If the structure of the matrix A is the following:

$$A(t) = \begin{bmatrix} a_1(t) & & & \\ & a_2(t) & & \\ & & \ddots & \\ & & & a_n(t) \end{bmatrix},$$

it's obvious that the $\Phi(t, t_0)$ has the form:

$$\Phi(t,t_0) = \begin{bmatrix} \phi_1(t,t_0) & & & \\ & \phi_2(t,t_0) & & \\ & & \ddots & \\ & & & \phi_n(t,t_0) \end{bmatrix},$$

and the solution is: $$\phi_i(t,t_0) = e^{\int_{t_0}^{t} a(\tau)\,d\tau}.$$

2.5 Floquet Theory

One of the long-standing issues in periodic systems is whether it is possible to reduce the study of the original periodic system to that of a time-invariant one, to simplify all the computations. A first and easy way to transform a time variant system in a time invariant one consists in extending the space using a fictitious variable as:

$$t = x_{n+1}.$$

This method however leads some problems in the study of equilibrium states, so if we deal with stability of periodic systems, it is more efficient to use some other methods.

When focusing on the system (2.11), this is the so-called *Floquet problem*. More precisely, it consists in finding a T-periodic invertible state–space transformation:

$$x^1(t) = S(t)x(t),$$

such that, in the new coordinates, the system is time-invariant:

$$x^1(t)' = A^1\, x^1(t),$$

where matrix A^1, known as Floquet factor is constant. So in this case, $A(\cdot)$ would be algebraically equivalent to a constant matrix.

We begin by presenting two elementary propositions, that will be useful for the following proofs.

Proposition 2.1 If Q is a $n \times n$ matrix with det $Q \neq 0$, then there exists a $n \times n$ complex matrix such that:

$$Q = e^{A0} \quad . \tag{2.21}$$

Proposition 2.2 The matrix A_0 expressed in the theorem 2.1 is not unique.

Proof:

$$e^{A_0 + 2\pi k i I} = e^{A_0} e^{2\pi k i} I_n = e^{A_0} e^{2\pi k i} = e^{A_0},$$

for any integer k.

q.e.d.

Now we are ready, after giving all the necessary theorems and definitions, to state the principal theorem of this paper.

Floquet Theorem If $\Phi(t, \tau)$ is a fundamental matrix solution for the periodic system:

$$x'(t) = A(t)\, x(t),$$

then so is $\Phi(t + T, \tau)$.

Moreover, if A_0 is such that $\psi(\tau) = e^{A_0}$, there exist an invertible matrix P(t) with period T such that:

$$\Phi(t,0) = P(t)\, e^{A0t} \quad . \tag{2.22}$$

Proof: By the properties we have seen before we have:

$$\Phi'(t+T,\ \tau) = A(t+T)\ \Phi(t+T,\ \tau),$$

and, by using the fact that A(t+T)=A(t) for periodic systems, we have that $\Phi(t + T, \tau)$ is a fundamental solution, and is thus invertible.

Consider $Q(t, \tau) = \Phi^{-1}(t,\tau)\Phi(t+T, \tau)$, so that:

$$\Phi(t + T,\ \tau) = \Phi(t,\ \tau)\ Q(t,\ \tau). \qquad \textit{for all } t \in \mathfrak{R}.$$

We need to show that $Q(t,\ \tau)$ is in fact independent of t. Let $Q = Q(\tau,\ \tau) = \Phi(\tau+T,\ \tau) = \psi(\tau)$. Clearly both $\Phi(t + T,\ \tau)$ and $\Phi(t,\ \tau)Q$ are fundamental solutions of the periodic system, and both correspond for $t = \tau$ *to* $Q = \Phi(\tau+T,\ \tau)=\psi(\tau)$, so they must correspond to the same unique *principal fundamental solution* for all times. This means that $\Phi(t,\ \tau)\ Q(t,\ \tau) = \Phi(t,\ \tau)\ Q$, where both Φ are invertible, and hence $Q(t, \tau)$ does not actually depend on t.

To conclude, we can observe that, given Proposition 2.1, there exists a matrix A_0 such that $Q= e^{A_0T}$. For such a matrix B , we take $P(t) := \Phi(t,\ \tau)\ e^{-A_0T}$, that is, $\Phi(t,\ \tau) = P(t)\ e^{A_0T}$.

Then:

$$P(t + T) = \Phi(t + T,\ \tau)\ e^{-A_0(T+t)} = \Phi(t,\ \tau)\ C\ e^{-A_0(t+T)} = \Phi(t,\ \tau)\ e^{-A_0T} = P(t)$$

Therefore is invertible for all $t \in \mathfrak{R}$ and periodic of period T. This concludes the proof.

q.e.d.

Some observations:

A. If we know $\Phi(t\ ,\tau)$ over the period $[t_0,t_0+T]$, then we will know $\Phi(t\ ,\tau)$ for all $t \in \Re$ by Floquet Theorem. This means that $\Phi(t\ ,\tau)$ on $[t_0,t_0+T]$ determines $\Phi(t,\tau)$ for all $t\in \Re$.

Proof: Suppose $\Phi(t\ ,\tau)$ is known over the period $[t_0,t_0+T]$.

Since, for the proposition 2.1

$$\Phi(t + T , \tau) = \Phi(t , \tau)\ Q,$$

we take

$$Q = \Phi(t ,\tau)^{-1}\ \Phi(t + T, \tau),$$

and

$$A_0 = p^{-1}\ln Q,$$

$$P(t) := \Phi(t ,\tau)\ e^{-A_0T}.$$

Since P is periodic for all $t\in \Re$, $\Phi(t, \tau)$ is given over $t\in \Re$ by:

$$\Phi(t, \tau) = P(t)\ e^{-A_0T}$$

So to a $\Phi(t, \tau)$ we associate a matrix A_0 using the Floquet Theorem.

q.e.d.

B. If $\Phi(t, \tau)$ determines e^{A_0t} and so A_0, then any fundamental matrix solution $\psi(t)$ determines a similar matrix $Se^{A_0t}S^{-1}$, or equivalently SA_0S^{-1}.

Proof: For any fundamental matrix solution $\psi(t)$ there exists S with $\det S \neq 0$ such that $\Phi(t , \tau) = \psi(t)\ S$. Since

$$\Phi(t + T, \tau) = \Phi(t, \tau)\ e^{A_0t},$$

we have:

$$\psi(t + T)\, S = \psi(t)\, S\, e^{A0t}$$
$$\Rightarrow \quad \psi(t + T) = \psi(t)\, S\, e^{A0t}\, S^{-1} = \psi(t)\, e^{SA_0 S\text{-}1T}.$$

q.e.d.

C. The solutions of a linear periodic system are not necessarily periodic. That is, in general:

$$\Phi(t + T, \tau) \neq \Phi(t, \tau).$$

Proof: You can check this fact using a counterexample like in [Richards].

We now prove an important theorem, which is useful for the whole theory.

Theorem 2.2 Under the transformation x = P(t)y, which is invertible and periodic, the periodic system:

$$x'(t) = A(t)x(t),$$

is a time invariant system.

Proof: Suppose P(t) and B defined as in the paragraph 2.4 and let x = P(t)y. Then:

$$x' = P'(t)x + P(t)y'$$
$$x' = A(t)x = A(t)P(t)y$$
$$\Rightarrow P'(t)x + P(t)y' = A(t)P(t)y$$
$$\Rightarrow y' = P^{-1}(t)[A(t)P(t) - P'((t)]y$$

If we want to get y, by Floquet Theorem we have:

$$P'(t) = A(t)\Phi(t)e^{-A_0 t} + \Phi(t)e^{-A_0 t}(-A_0) = A(t)P(t) - P(t)A_0\,.$$

It follows that:

$$y' = P^{-1}(t)[A(t)P(t) - P'(t)]y = P^{-1}(t)P(t)A_0 y = A_0 y\,,$$

so the system is time-invariant.

Some notes:

a. $x = P(t)\, y$ is called *Lyapunov transformation* with P(t) which plays an important role. However, it is difficult in general difficult to find it explicitly since the computation depends on a fundamental matrix solution $\Phi\ (t,\tau)$.

b. Since $\Phi\ (t + T, \tau) = \Phi\ (t,\tau)\ e^{A_0 t}$, the eigenvalues ρ of Q, with $Q = e^{A_0}$, are called the *characteristic multipliers* of the periodic linear system. The eigenvalues λ of A_0 are called *characteristic exponents* of the periodic linear system $\rho = e^{\lambda p}$.

c. Since A_0 is not unique, the characteristic exponents are not uniquely defined, but the multipliers ρ are so uniquely defined [Richards]. We always choose the exponents λ as the eigenvalues of A_0 , where is any matrix such that:

$$e^{A0t} = Q\ .$$

d. For the same reasons, A_0 is not necessarily real [see the proof at Richards, Section 3.2].

2.6 Multipliers

The eigenvalues of the monodromy matrix, called *characteristic multipliers* as we have seen in the previous paragraph, do not depend on the particular time tag: see the brief proof in [Bittanti-Colaneri].

Accordingly, if λ is a characteristic exponent, the associated characteristic multiplier ρ is given by:

$$\rho = e^{\lambda T}$$

The first relation is real in continuous time. Since the characteristic multipliers depend only on matrix A(·), one can conclude that the same holds for the characteristic exponents, which are determined by matrix A(·) only. An extended proof is available in [Richards].

The monodromy matrix $\psi(\tau)$ relates the value of the state in free motion at a given time-point τ to the value after one period τ + T:

$$x\,(\tau+T) = \psi(\tau)\, x(\tau)\ .$$

Therefore, the sampled state x(k) = x(τ + kT) is governed in the free motion by the time-invariant discrete-time equation:

$$x\,(k+1) = \psi(\tau)^{\,k}\, x\,(k)\ .$$

This is why the eigenvalues of $\psi(t)$ play a major role in the modal analysis of periodic systems. The monodromy matrix is the basic tool in the stability analysis of periodic systems. Indeed, the free motion goes to zero asymptotically if and only if all characteristic multipliers have modulus lower than one. Hence,

> *a periodic system (in continuous) is stable if and only if its characteristic multipliers belong to the open unit disk in the complex plane.*

Obviously, the previous condition can be stated in terms of characteristic exponents.
In continuous-time a periodic system is stable if and only if the real part of all characteristic exponents is negative.

Notice that there is no direct relation between the eigenvalues of A(t) and the system stability. In particular, it may well happen that all eigenvalues of A(t) belong to the stability region and nevertheless the system be unstable. This fact will be widely described in Chapter 3.

In section 2.2, the notions of transition matrix over one period have been introduced, and their relation is:

$$\psi(t) = \Phi(t+T, t) .$$

As already stated, the eigenvalues of the monodromy matrix at time t are named *characteristic multipliers* at t.

Summarizing, the transition matrix is used to find the solution for the free input periodic system:

$$x(t)' = A(t)x(t),$$

$$x(t) = \psi(t,\tau)\, x(\tau), \qquad t > \tau$$

and is biperiodic, i.e.:

$$\Phi(t + T, \tau + T) = \Phi(t, \tau) \quad , \qquad \forall\, t \geq \tau.$$

The monodromy matrix $\psi(t)$ is periodic:

$$\psi(t) = \psi(t+T) ,$$

and is used in the equation for the discrete case:

$$x(T+kt) = \psi(t)^{k}\, x(t),$$

$$t \in [0, T-1] \textit{ and k positive integer} .$$

2.7 Main properties of monodromy matrix

The characteristic polynomial of $\psi(t)$:

$$p_c(\gamma) = \det[\gamma I - \psi(t)],$$

is independent of t.

Proof: Indeed, consider the monodromy matrices $\psi(t)$ and $\psi(\tau)$, with $\tau, t \in [0,T-1]$.

Following the definition, these matrices can be written as:

$$\psi(\tau) = FG,$$
$$\psi(t) = GF,$$

with:

$$F = \begin{cases} \Phi(\tau,t) & \tau \geq t \\ \Phi(\tau+T,t) & \tau < t \end{cases}$$
$$G = \begin{cases} \Phi(t+T,\tau) & \tau \geq t \\ \Phi(t,\tau) & \tau < t \end{cases}$$

Now, take the singular value decomposition of G, i.e.:

$$G = U\Sigma V,$$

where

$$\Sigma = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\det[G_1] \neq 0$$

and U, V are suitable orthogonal matrices.

It follows that:

$$VFGV' = (VFU)\Sigma$$
$$U'GFU = \Sigma(VFU).$$

We can partition the matrix VFU as:

$$VFU = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}.$$

,

Denoting by n_1 the dimension of the sub-matrix G_1 it is easy to show that:

$$\det[\gamma I_n - FG] = \det[\gamma I_n - VFGV'] = \gamma^{n-n_1}\det[\gamma I_{n_1} - F_1G_1]$$
$$\det[\gamma I_n - GF] = \det[\gamma I_n - U'GFU] = \gamma^{n-n_1}\det[\gamma I_{n_1} - G_1F_1].$$

Since G_1 is invertible, matrices F_1G_1 and G_1F_1 are similar, so:

$$\det[\gamma I_{n_1} - F_1G_1] = \det[\gamma I_{n_1} - G_1F_1].$$

Therefore, the characteristic polynomials of ψ (t) and ψ (τ) coincide.

q.e.d.

An important consequence of the previous observation is that the characteristic multipliers at time t, namely as seen before the eigenvalues of ψ (t), are in fact independent of t. Notice that not only the eigenvalues are constant, but also their multiplicities are time invariant, as it is shown in [Richards].

This is why one can make reference to such eigenvalues as characteristic multipliers without any specification of the time point.

In conclusion, by denoting by r the number of distinct non-zero eigenvalues γ_i of ψ(t), the characteristic polynomial can be written as:

$$p_c(\lambda) = \gamma^{h_c} \prod_{i=1}^{r} (\gamma - \gamma_i)^{k_{ci}} \ ,$$

where γ_i, h_c and k_{ci} do not depend on time. The characteristic multipliers are the γ_i and $\gamma = 0$ if $h_c \neq 0$.

So, n_m characteristic multipliers of A(·) are time-independent and constitute the core spectrum of ψ(·). Among them, there are in general both nonzero and zero multipliers.

Obviously, the multiplicity of a nonzero characteristic multiplier γ_i in the minimal polynomial of ψ(t) does not change with time. Therefore, the minimal polynomial can be given the form:

$$p_m(\gamma, t) = \gamma^{h_m(t)} \prod_{i=1}^{r} (\gamma - \gamma_i)^{k_{mi}} \ ,$$

where the multiplicities k_{mi} are constant.

So, the characteristic polynomial of the monodromy matrix ψ(t) does not depend on t. As for the minimal polynomial, the only possible reason for the dependence upon t is limited to the multiplicity $h_m(t)$ of the null singularities. The multiplicity can change under the constraint:

$$\left|h_m(t)-h_m(\tau)\right|\leq 1 \qquad \forall t,\tau\in[0,T-1]\,.$$

When matrix A(t) is nonsingular for each t, all characteristic multipliers are all different from zero. In such a case, the system is reversible, in that the state x(τ) can be uniquely recovered from x(t) , t > τ .

2.8 An example: The Meissner Equation

Consider the second order equation:

$$\begin{aligned}&\ddot{y}+(a-2q\psi(t))y=0\\&\psi(t)=\psi(t+\pi)\end{aligned}\quad ,$$

in which ψ(t) is unit rectangular waveform as you can see in Fig. 2.1.

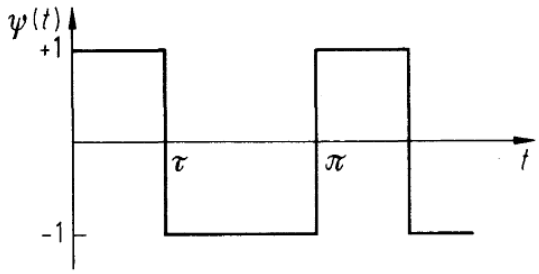


Fig.2.1

It is evident that over one period this equation can be viewed as the pair of constant coefficient equations:

$$\begin{aligned}&\ddot{y}+(a-2q)y=0 \qquad 0\leq t<\tau\\&\ddot{y}+(a+2q)y=0 \qquad \tau\leq t<\pi\end{aligned}$$

for which solutions are easily determined in terms of trigonometric functions. As a result, state transition matrices and exact solutions can be found. To simplify notation the equation can be rewritten as:

$$\ddot{y} + c^2 y = 0,$$

for which linearly independent basis solutions xi(t) are cos(ct) and sin(ct).

Chapter 3: THE MATHIEU EQUATION

3.1 Introduction

The most widely known form of the Hill equation is the Mathieu equation, in which $\psi(t)$ is sinusoidal:

$$\ddot{y} + (a - 2q\cos(2t))y = 0 \,. \tag{3.1}$$

This is the most extensively treated form of Hill equation: in fact, the Mathieu equation is a Hill equation with only one harmonic mode. By association with Fourier series, it may have been assumed that once solutions to the Mathieu equation had been determined, solutions to Hill equations would follow. Unfortunately, this is not the case.

In this chapter we will describe various tools to study the Mathieu equation: in particular, it is shown that the usefulness of the exact Mathieu equation and its solutions for studying the general Hill equation is limited. On the other hand, it is shown that by appropriately modelling the sinusoidal coefficient, solutions to similar equations can be found. So, rather than looking for approximate solutions to the exact Mathieu equation, exact solutions to an approximate Mathieu equation will be pursued.

3.2 Physical applications

In mathematics, the Mathieu functions, introduced by Émile Léonard Mathieu (1868) in the context of vibrating elliptical drumheads, are special functions that are useful for treating a variety of problems in applied mathematics, including:

- quadrupole mass analyzers and quadrupole ion traps for mass spectrometry;
- wave motion in periodic media, such as ultracold atoms in an optical lattice;
- the phenomenon of parametric resonance in forced oscillators;
- exact plane wave solutions in general relativity;
- the Stark effect for a rotating electric dipole;
- in general, the solution of differential equations that are separable in elliptic cylindrical coordinates.

Some physical applications, like the above and others, will be described in details in the fifth chapter.

3.3 Periodic Solutions

The simplest and most widespread method to obtaining solutions to the Mathieu equation, is the series expansion approach: the starting point is to observe that these solutions must reduce to either $cos(mt)$ or $sin(mt)$, $m^2 = a$, for $q \to 0$, i.e. the equation and its solutions reduce to those of simple harmonic motion. As q is increased from zero, the basic solution must be modified to account for the degree of periodic coefficient that has been introduced.

3.3.1 Mathieu sine and cosine

For fixed a, q, the Mathieu cosine $\mathbf{C(a, q, x) = 1}$ is a function of x defined as the unique solution of the Mathieu equation which:

- takes the value **C(a, q, 0) = 1,**
- is an even function, hence **C'(a, q, 0) = 1**.

Similarly, the Mathieu sine **S(a, q, x) = 1** is the unique solution which:

- takes the value **S'(a, q, 0) = 1,**
- is an odd function, hence **S (a,q,0) = 0**.

These are real-valued functions, which are closely related to the Floquet solution:

$$C(a,q,x) = \frac{F(a,q,x) + F(a,q,-x)}{2F(a,q,0)} \ , \tag{3.2}$$

$$S(a,q,x) = \frac{F(a,q,x) - F(a,q,-x)}{2F'(a,q,0)} \ . \tag{3.3}$$

The general solution to the Mathieu equation (for fixed *a* and *q*) is a linear combination of the Mathieu cosine and Mathieu sine functions.

A remarkable special case is:

$$C(a,0,x) = \cos(\sqrt{a}x), \ S(a,0,x) = \frac{\sin(\sqrt{a}x)}{\sqrt{a}}, \tag{3.4}$$

i.e. when the corresponding Helmholtz equation [see Appendix A] problem has circular symmetry. In general, the Mathieu sine and cosine are aperiodic. However, for small values of q, we have approximately:

$$C(a,q,x) \approx \cos(\sqrt{a}x), \ S(a,q,x) \approx \frac{\sin(\sqrt{a}x)}{\sqrt{a}}, \tag{3.5}$$

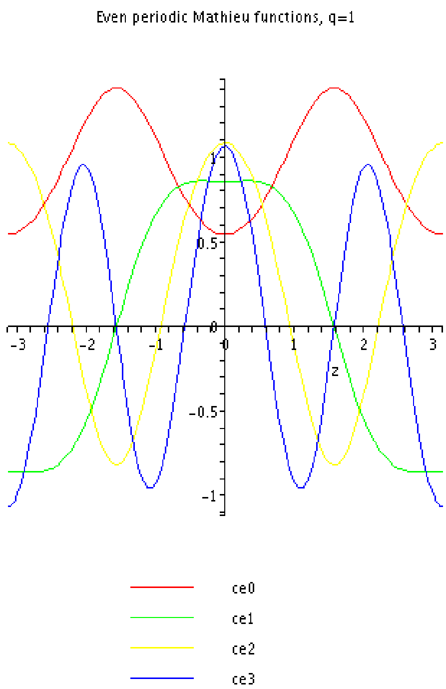
Given *q*, for countable special values of the characteristic values *a*, the Mathieu equation admits solutions which are periodic with period 2π. The characteristic values of the Mathieu cosine and sine functions respectively are written $a_n(q)$ and $b_n(q)$, where *n* is a natural number. The periodic special cases of the Mathieu cosine and sine functions are often written

CE (a, q, x) and *SE (a, q, x)* respectively, although they are traditionally given a different normalization.

Therefore, for positive q, we have:

$$C(a_n(q), q, x) = \frac{CE(n,q,x)}{CE(n,q,0)} \;, \tag{3.6}$$
$$S(b_n(q), q, x) = \frac{SE(n,q,x)}{SE(n,q,0)} \,.$$

In the figure below there are the first few periodic Mathieu cosine functions for $q = 1$:



3.3.2 Even solution for $a = 1$ and $q = 0$

At $a = 1$ and $q = 0$, the basis solutions are exactly *cos(t)* and *sin(t)*. Concentrating upon the even solution as a starting point, a general solution, for q non zero, can be expressed therefore as:

$$x(t) = \cos(t) + qC_1(t) + q^2C_2(t) + \ldots, \tag{3.7}$$

where the functions $C_i(t)$ are to be determined. The first task in the classical theory is to determine solutions that are periodic for all q. To maintain periodicity as q increases above zero the value of a may need to be continuously changed as q changes. To allow for this, a is also expressed as an infinite series in q:

$$a = 1 + \alpha_1 q + \alpha_2 q^2 + \ldots \tag{3.8}$$

The constants α_i and the functions $C_i(t)$ can be determined by substituting equations (3.7) and (3.8) into equation (3.1), equating coefficients of powers of q to zero and by removing non-periodic terms. Doing that, it is found that the first periodic solution of the Mathieu equation is:

$$\begin{aligned} x(t) &= \cos(t) - \frac{1}{8} q \cos(3t) + \frac{1}{64} q^2 \left[-\cos(3t) + \frac{1}{3}\cos(5t) \right] + \ldots \\ &\ldots - \frac{1}{512} q^3 \left[\frac{1}{3}\cos(3t) + \frac{4}{9}\cos(5t) + \frac{1}{18}\cos(7t) \right] + \ldots \end{aligned} \tag{3.9}$$

with a constrained to:

$$a = 1 + q - \frac{1}{8} q^2 - \frac{1}{64} q^3 - \frac{1}{1536} q^4 + \ldots \tag{3.10}$$

The solution described in equation (3.9) is called a *Mathieu Function* (of the first kind) and is denoted $ce_1(t,q)$. For a given value of q, the value of a generated by equation (3.10), at which the corresponding Mathieu function exists, is called a *characteristic number*.

3.3.3 Odd solution for $a = 1$ and $q = 0$

The second step is to analyse the odd solution sin t for $a = 1$ and $q = 0$; another general periodic solution to the Mathieu equation is:

$$se_1(t,q) = \sin(t) - \frac{1}{8}q\sin(3t) + \frac{1}{64}q^2\left(\sin(3t) + \frac{1}{3}\sin(5t)\right) + \\ -\frac{1}{512}q^3\left(\frac{1}{3}\sin(3t) + \frac{4}{9}\sin(5t) + \frac{1}{18}\sin(7t)\right) + \ldots \tag{3.11}$$

provided that:

$$a = 1 - q - \frac{1}{8}q^2 + \frac{1}{64}q^3 - \frac{1}{1536}q^4 + \ldots \tag{3.12}$$

Inspection of equations (3.10) and (3.12) reveals that the solutions given in equations (3.9) and (3.11) do not coexist except when $q = 0$.

3.3.4 Mathieu Functions of fractional order, $q = 0$

If $a \neq m^2$, m integer, for $q = 0$, then the solutions obtained for q non zero will be of the form:

$$ce_v(t,q) = \cos(vz) + \sum_{r=1}^{\infty} q^r C_r(t),$$
$$se_v(t,q) = \sin(vz) + \sum_{r=1}^{\infty} q^r S_r(t), \tag{3.13}$$
$$a = v^2 + \sum_{r=1}^{\infty} \alpha_r q^r .$$

It can be demonstrated that the characteristic number is the same for both $ce_v(t, q)$ and $se_v(t, q)$,so that the solutions of both type can coexist, and thus form a linearly independent pair. A complete solution therefore is of the form:

$$x(t) = A\, ce_v(t, q) + B\, se_v(t, q), \tag{3.14}$$

with A and B constants which will be determined by initial conditions. Generally the functions $ce_v(t, q)$ and $se_v(t, q)$ are non-periodic, although bounded. They are therefore the solution types applicable in the stable regions of the Mathieu equation stability diagram.

3.3.5 Solutions for $q \neq 0$

For $q \neq 0$, the values of a for which the periodic solutions exist are quite different. The second linearly independent solutions for non-zero q are non-periodic. Indeed it can be demonstrated that periodic solutions of the types above exist only on stability boundaries, with the a, q relationships in equations (3.10) and (3.12) therefore being polynomial expressions for those boundaries. In particular equation (3.10) describes the a_1 boundary in the Mathieu equation stability diagram, and equation (3.12) describes the b_1.

In an analogous manner to that above, a complete hierarchy of periodic solutions of odd and even type, along with their associated characteristic numbers, can be established. The first few members in this hierarchy are:

$$\mathrm{ce}_0(t,q) = 1 - \frac{1}{2} q\cos(2t) + \frac{1}{32} q^2 \cos(4t) - \ldots$$

$$\mathrm{ce}_1(t,q) = \cos(t) - \frac{1}{8} q\cos(3t) + \frac{1}{64} q^2\left(-\cos(3t) + \frac{1}{3}\cos(5t)\right) - \ldots$$

$$\mathrm{se}_1(t,q) = \sin(t) - \frac{1}{8} q\cos(3t) + \frac{1}{64} q^2\left(\sin(3t) + \frac{1}{3}\cos(5t)\right) - \ldots$$

$$\mathrm{ce}_2(t,q) = \cos(2t) - \frac{1}{8} q\left(\frac{2}{3}\cos(4t) - 2\right) + \frac{1}{384} q^2 \cos(6t) - \ldots$$

$$\mathrm{se}_2(t,q) = \sin(2t) - \frac{1}{12} q\sin(4t) + \frac{1}{384} q^2 \sin(64t) - \ldots$$

And so on, with the corresponding characteristic numbers:

$$a_0\text{: } a = -\frac{1}{2} q^2 + \frac{7}{128} q^4 - \frac{29}{2304} q^6 + \ldots$$

$$a_1\text{: } a = 1 + q - \frac{1}{8} q^2 - \frac{1}{64} q^3 - \ldots$$

$$b_1\text{: } a = 1 - q - \frac{1}{8} q^2 + \frac{1}{64} q^3 - \ldots$$

$$a_2\text{: } a = 4 + \frac{5}{12} q^2 - \frac{763}{13824} q^4 + \ldots$$

$$b_2\text{: } a = 4 - \frac{1}{12} q^2 + \frac{5}{13824} q^4 - \ldots$$

There exists a table of pairs of a and q that satisfy the characteristic numbers. These can be useful in creating an approximate stability diagram for the Mathieu equation. Solutions of the type $ce_m(t,q)$ and $se_m(t,q)$ are referred to as Mathieu functions of order m and, in addition, if m is integral as in the above formula, they are called *functions of integral order.*

Within the limits of numerical accuracy, the procedures of paragraph 3.3.1 have the appeal that they give exact results, since they do not rely upon algebraic approximations to the solutions of the equation. Their negative aspects, however, lie with their numerical error, especially when solutions near stability boundaries are required.
Accuracy is governed by the number of iteration points chosen for the technique, per period of the periodic coefficient in the Mathieu equation. For solutions which are unreservedly stable or unstable, only a low number of points, about 10 or 20 per period, is necessary to ensure reliable solutions. However, near stability boundaries, upwards of 60 points per period is necessary to give a solution accuracy of 0.1 %, after a total time interval corresponding to 50 complete periods of the time-varying coefficient.
Summarizing, whilst numerical methods are convenient tools with which to solve the Mathieu equation they have two disadvantages: first, they are unable to give quantitative information regarding stability, and secondly they are very time-consuming in evaluation.

3.4 Other forms of Mathieu equation

Closely related to the classical form of Mathieu equation is *Mathieu's modified differential equation:*

$$\frac{d^2y}{du^2} - [a - 2q\cosh(2u)]y = 0 \tag{3.15}$$

which follows on substitution $u = ix$.

The above equation can be obtained from the Helmholtz equation [see the appendix A] in two dimensions, by expressing it in elliptical coordinates, and then separating the two variables. This is why they are also known as *angular and radial Mathieu equation*, respectively.
The substitution $t = cos(x)$ or equivalently $x \rightarrow arccos(x)$ transforms Mathieu's equation to the algebraic (or rational) form:

$$(1-t^2)\frac{d^2y}{dt^2} - t\frac{dy}{dt} + [a + 2q(a-2t^2)]y = 0$$
$$\frac{d^2y}{dx^2} = \frac{x}{1-x^2}\frac{dy}{dx} + \frac{(4x^2-2)q-a}{1-x^2}y \tag{3.16}$$

This has two regular singularities at $t = \pm 1$, and one irregular singularity at infinity, which implies that, in general (unlike many other special functions), the solutions of Mathieu's equation cannot be expressed in terms of hypergeometric functions.
Mathieu's differential equations arise as models in many contexts, including:

- the stability of railroad rails as trains drive over them
- seasonally forced population dynamics
- the four-dimensional wave equation
- the Floquet theory of the stability of limit cycles.

Even some of these themes will be treated in the chapter 5, when we will described practical applications of theorems we had seen in the previous paragraphs.

3.5 Floquet solution for the Mathieu Equation

According to Floquet's theorem, also known as Bloch's theorem, for fixed values of a and q, Mathieu's equation admits a complex valued solution in the following form:

$$F(a,q,x) = \exp(i\mu x)P(a,q,x) \quad , \tag{3.17}$$

where μ is a complex number, the Mathieu exponent, and P is a complex valued function which is periodic in x with period π. However, P is in general not sinusoidal.

In the example plotted below, $a = 1$, $q = 1/5$, $\mu \approx 1+0.0995i$.

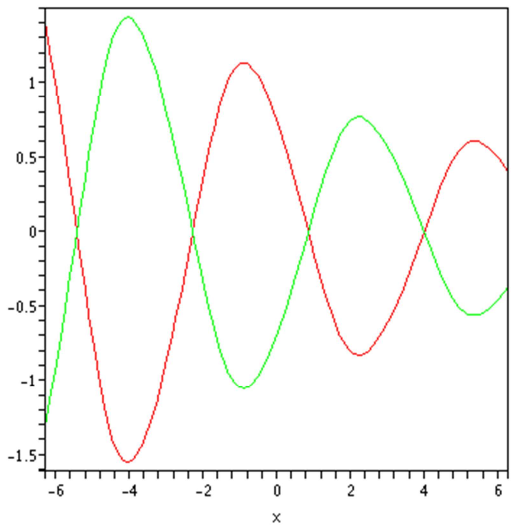


Fig. 3.1: plot in which real part is red, and imaginary part green

3.6 Modelling Techniques for Analysis

The techniques that are well-suited for the analysis of the Mathieu equation, typically focus the frequency of the sinusoidal coefficient. Following these procedures, a suitable combination of steps and ramps is sought as a replacement for the sinusoid, and this combination is manipulated such that its lower frequency spectrum closely resembles that of the sinusoid. Simple Fourier analysis shows that the complex amplitude spectrum of a sinusoid is:

$$|\Psi_{\pm 1}| = 1/2,\ \Psi_i = 0 \qquad \text{for all } i \neq \pm 1. \tag{3.18}$$

Consequently, model coefficients should be chosen such that they have a spectrum which is, as closely as possible, a single component of amplitude 1/2 at the frequency of the sinusoid, i.e. at 2 radians per second (or a period of π). In addition, of course, the positive aspect of the method rests upon being able to solve analytically the resulting substitute equation. Three likely models come to mind. First is that in which the sinusoid is replaced by a simple square waveform, the second is that in which a trapezoidal waveform is employed and the third model is that which utilises a staircase waveform [for detailed descriptions of these methods see Richards at Chapter 6].

3.7 Stability Diagrams for the Mathieu Equation

Stability diagrams are plots of *a* against *q*, that represent regions of *a* and *q* for which the solution to a Mathieu equation is stable, as opposed to those regions for which the solution is unstable. The regions of stability and instability are separated by stability boundaries on which the solution is marginally stable.
If $\Phi(\pi,0)$ is the discrete transition matrix for the Mathieu equation computed, in principle, according to the material of chapter 2, then on stability boundaries:

$$\text{trace}\,(\Phi\,(\pi,0)) = \pm\,2.$$

The eigenvalues of $\Phi(\pi,0)$ are either both *+1* or both *-1,* and the characteristic exponent of the solution of the Mathieu equation is j_m, where *m* is an integer. As a result, the fundamental solution to a Mathieu equation on stability boundaries is purely periodic. Note also there is a second, linearly independent solution that is a t-multiplied version of the first, owing to the degenerate eigenvalues [see the proof in Richards 4.5.7]. In Sect. 3.3 it has been shown that purely periodic solutions occur for values of *a* and *q* related by equations such as equations (3.10) and (3.12): in other words, for the equation's characteristic numbers. There are two sequences of characteristic numbers:

- the first set, designated a_0, a_1 a_2 ..., corresponds to even, periodic solutions of the equation;
- the second set, called b_1 , b_2 , b_3 , ... corresponds to odd, periodic solutions.

The first few are given in the paragraph 3.3.5. Following the previous reasoning, to produce a stability diagram it is necessary only to identify the *a* and *q* functional forms of the characteristic numbers, in the region of *(a, q)* space of interest, and then plot them. Since regions of instability are known to come from points on the *a* axis that are squares of integers [see Richards 6.4], it is easy to label the regions between the stability boundaries. The characteristic numbers have been computed for the standard Mathieu equation without a first derivative (loss) term, and [McLachlan's Appendix II] tabulates from a_0 to a_5 and from b_1 to b_6 . From these values, the stability diagram shown in Fig. 3.2 has been constructed.

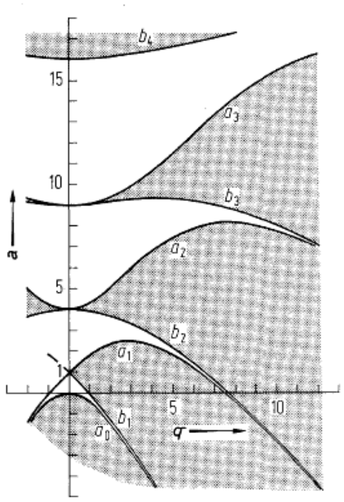


Fig. 3.2: Stability diagram for the Mathieu equation. Blank regions correspond to stable solutions and shaded regions to unstable solutions.

Determination of the sets of characteristic numbers for the Mathieu equation and plotting them is the classical means by which the stability diagram has been determined.

An alternative, and perhaps more convenient method, is to resort to a different modelling technique. The techniques developed in the past are based upon modelling or replacing a particular intractable Hill equation by a counterpart which exhibits very similar solutions and stability properties, and yet which can be handled exactly mathematically.
These modelling procedures are developed by demonstrating, first, that the behaviour of a Hill equation, especially for the ranges of its coefficients encountered in most practical situations, is determined principally by the lower order harmonics in its periodic coefficients, while higher harmonics having a progressively decreasing influence as they become of higher order. Secondly, the approach leans on being able to identify classes of periodic coefficient, in which harmonic content can be quickly adjusted, and for which the corresponding Hill equation is solvable. [For details of this topic see the chapter 5 of Richards].

For this reason, the sinusoid in the Mathieu equation is replaced by a sufficiently accurate similar model that leads to an analytically tractable equation. The Meissner equation of paragraph 2.8 is an example of this way to solve the problem. A diagram for the substitute equation is then generated by computing and testing any of the characteristic exponents, the eigenvalues of the discrete transition matrix or simply the trace of the discrete transition matrix. An approximate stability diagram for the Mathieu equation, produced by this method, is shown in the picture below.

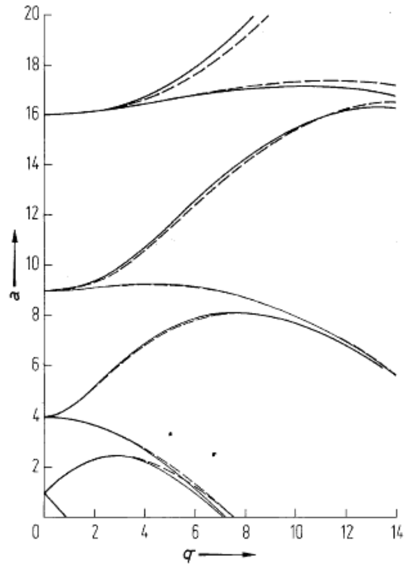


Fig. 3.3: Approximated stability diagram for the Mathieu equation. The real diagram has the boundaries shown in broken lines.

This is, in fact, the diagram for a Hill equation with a simple trapezoidal waveform coefficient, such where n is the period of the waveform. The frequency spectrum of the trapezoid looks closely like the sinusoid's one, consisting of one fundamental, no harmonics that are multiples of two or three, and very small and diminishing fifth, seventh and so on growing higher harmonics. The figure shows clearly that the simulated diagram is very accurate, generally near an axis and certainly for $a \leq q/2$. A more accurate simulated diagram is possible if the amplitude of the trapezoid is adjusted to 0.949703, to make its fundamental magnitude 0.5, thereby better matching the spectrum of the sinusoid in the Mathieu equation.

Chapter 4: STABILIZATION AND CONTROL OF PERIODIC SYSTEMS

4.1 Periodic Control of Time-invariant Systems

A control problem can be formulated according to different objectives and design criteria. The early developments of periodic control were focused on the problem of driving a periodic system in order to improve the performances of an industrial plant (*periodic optimization*). Nowadays, the term *periodic control* has taken a wider sense, and includes the design of control systems where the controller and the plant are described by periodic models.

In this first section we will describe the main results on periodic control of time-invariant systems, while periodic control of periodic systems will be the subject of the reminder of the Chapter. We start by recalling the key problem of output stabilization via periodic control by using various techniques, including the *sampled and hold method*. This technique has the advantage of reducing the problem to an equivalent one in the time-invariant environment. However, it has the effect of disturbances in the inter-sample periods.

Then, in the first part of the chapter, we will consider the problem of stabilization, i.e. the possibility of ensuring stability with a periodic control law. After that, we pass to time invariant representation of periodic systems, and their properties. The second part of the chapter is dedicated first to optimization, following some different techniques, and after to the control of these systems.

4.1.1 Preliminaries: Output Stabilization of Linear Time-invariant Systems

The application of periodic controllers to time-invariant processes has been treated extensively in literature. The aim is to solve problems otherwise unsolvable with time-invariant controllers, or to improve the achievable control performances.

A typical method makes reference to the classical output stabilization problem of finding an algebraic feedback control law, based on the measurements of the output signal in order to stabilize the control system. If the system is of the form:

$$\begin{aligned} \mathrm{d/dt}\, x(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{4.1}$$

with the control law:

$$u(t) = Fy(t), \tag{4.2}$$

the problem is to find a matrix F (if any) such that the closed-loop system:

$$x(t) = (A + BFC)\, x(t) \tag{4.3}$$

is stable. Although a number of necessary and sufficient conditions concerning the existence of a stabilizing matrix F have been provided in the literature, it is not easy to find a reliable algorithm for its determination.

Obviously, stabilization is a preliminary step for the control target. A further step leads to the *regulation problem*, in which the designer has to create a controller to reduce the error, due to exogenous disturbances, with respect to a reference signals. When the *exosystem* (the model generating the disturbances and/or the reference signals) is periodic, it is suitable to design a periodic controller, even if the process under control is time-invariant.

4.1.2 A Note on Periodic Systems in Frequency Domain

The frequency domain representation is a fundamental tool in the analysis and control of time-invariant linear systems. It is related to the property that, for input-output stable systems, sinusoidal inputs result (at least asymptotically) into sinusoidal outputs at the same frequency and different amplitude and phase.

A similar tool can be worked out for periodic systems, by making reference to their response to the *exponentially modulated periodic* (EMP) signals. In fact, given any complex number *s*, a complex signal $u(t),\ t \in \Re$, is said to be *EMP of period T and modulation s*, if:

$$u(t) = \sum_{k \in \mathbb{Z}} u_k e^{s_k t}, \qquad s_k = s + jk\Omega. \tag{4.4}$$

The quantity $\Omega\ /2\pi$ is called *period of the EMP signal.* The class of EMP signals is a generalization of the class of *T*-periodic signals: in fact, an EMP signal with $s = 0$ is just an ordinary time-periodic signal.
Note that, as a time-invariant system subject to a (complex) exponential input admits an exponential regime, a periodic system of period *T:*

$$\begin{aligned} x(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

subject to an EMP input of the same period, admits an EMP asymptotic regime. While the method is potentially interesting, we shall focus on state-space methods, which allow for better computer implementation and are natural in treating optimal control problem.

4.2 Time-invariant Representations

Periodicity is often the result of appropriate operations over time-invariant systems. Here, we deal with the problem of *transforming* the periodic system into a time-invariant one. In such a way, one can use the results already available in the literature regarding time-invariant system.

4.2.1 Generalized Sample and Hold for feedback control

Periodic sampled control offers a practical alternative for the problem of finding an appropriate matrix $F(\cdot)$. Indeed, consider the time-varying control law based on the sampled measurements of $y(\cdot)$:

$$u(t) \;=\; F(t)\, y(kT)\,, \quad t \in [kT,\; kT+T).$$

The modulating function $F(\cdot)$ and the sampling period T have to be selected in order to stabilize the closed-loop system, now governed by the equation:

$$x(kT\; + T) \;=\; A_c\, x(kT)\,,$$

where:

$$A_c = \left[e^{AT} + \int_0^T e^{A(T-s)} B\, F(s) C\, ds\right].$$

The fundamental point is the selection of matrix $F(\cdot)$ for a given period T. A first method is to consider an $F(\cdot)$ given by the following expression:

$$F(t) = B' e^{A'(T-t)} \left[\int_0^T e^{A(T-s)} B\, B' e^{A'(T-s)} ds\right]^{-1} Z$$

with matrix Z still to be specified. Note that the formula above is valid if the matrix inversion can be performed (in the reachability condition). In this way, the closed-loop matrix A_C takes the form:

$$A_c = e^{AT} + ZC.$$

Where C is the matrix of the system (4.1) and Z can be selected so as to stabilize A_C, or equivalently the pair (e^{AT}, C) is detectable.

Some practical application of this *generalized sample and hold* method above described in the problem of stabilization are:

- the problem of simultaneous stabilization of a finite number of plants;
- fixed poles removal in decentralized control;
- the issue of pole and/or zero-assignment;
- adaptive control;
- model matching.

When using generalized sample-data control, however, the inter-sample behaviour can present some critical aspects. Indeed, the action of the *generalized sample and hold function* is a sort of amplitude modulation which, in the frequency domain, may lead to additional high-frequency components centred on multiples of the sampling frequency. Consequently, there are high-frequency components both in the output and control signals. To solve this issue, a possibility is to monitor at each time instant the output signal and to adopt the feedback control strategy:

$$u(t) = F(t)\, y(t)$$

with a periodic gain $F(t)$, in place of the sampled strategy before seen. This leads to the issue of *memoryless output-feedback control* of time-invariant systems. The periodic matrix F is chosen as to stabilize the previous system.

4.2.2 Sample and Hold

The simplest way to achieve stationarity is to resort to a *sample and hold* procedure, similar to the previous method described above, but more general and with a wider field of applications. Indeed, with reference to a continuous or a discrete-time periodic system, suppose that the input is kept constant over a period, starting from an initial time point τ, i.e.,

$$u(t) = u(k), \quad t \in [kT + \tau, \ kT + T + \tau).$$

In this way, the evolution of the system state sampled at $\tau + kT$, i.e., $x_\tau(k) = x(kT + \tau)$, is governed by a time-invariant equation in discrete-time. Precisely,

$$x_\tau(k+1) = \Phi_A(T + \tau, \tau)x_\tau(k) + \Gamma(\tau)\,\tilde{u}(k),$$
$$\Gamma(\tau) = \int_\tau^{T+\tau} \Phi_A(T + \tau, s)B(s)ds.$$

4.2.3 Lifting

The idea underlying the lifting technique is very simple. To be precise, given an analog signal $v(\cdot)$, the corresponding *lifted signal* is obtained by splitting the time axis into intervals of length *T*, and defining the segments v_k, *k* integer, as follows:

$$v_k = u(t), \ t \in [kT, \ kT + T].$$

In this way, the lifted signal v_k is defined in discrete-time. This procedure can be applied to the input $u(\cdot)$ and output $y(\cdot)$ of a continuous-time *T*-periodic system, thus giving the discrete-time signals u_k and y_k, respectively. The dynamic system relating u_k to y_k is called the *lifted system*. It's useful to transform the continuous-time system into a discrete one, and so control the system exploiting the techniques describe in the above paragraphs.

4.3 Periodic Optimization

In continuous-time, the basic *periodic* optimization problem can be stated as follows. Consider the system:

$$x(t) = f(x(t), u(t))$$
$$y(t) = h(x(t))$$

subject to the periodicity constraint

$$x(T) = x(0) \,,$$

The performance index to be maximized is assumed to be in the form:

$$J(u(\cdot), T) = \frac{1}{T}\int_0^T g\big(y(t), u(t)\big)\, dt.$$

We will assume that the function $h(\cdot)$ is differentiable and functions $f\ (\cdot,\cdot)$ and $g(\cdot,\cdot)$ are twice differentiable. If we want only to work in steady-state conditions, then we have an algebraic optimization problem, and it can be solved with mathematical programming techniques. More in detail, consider the problem in steady-state, that is when all variables are constant: $(t) = u\ ,\ x(t) = x,\ y(t) = h(x) = y.$ Then the periodicity constraint $x(T) = x(0)$ is simply satisfied and the performance index becomes:

$$J = g(h(x, u)) \,,$$

to be maximized with the constraint :

$$f\ (x, u) = 0.$$

We will denote by u^0 and x^0 the values of u and x solving such a problem. Hence (u^0, x^0) is the optimal steady-state condition and $y^0 = h(x^0)$ is the corresponding optimal output. The associate performance index is $J^0 = g(h(x^0, u^0))$.

When passing from steady-state to periodic operations, an important preliminary question is whether the optimal steady-state regime can be improved by cycling or not. A problem for which there exists a periodic operation with a better performance is said to be *proper*. To be precise, the optimization problem is proper if there exists a period T ' and a control signal $u'(\cdot)$, such that the periodic solution of period T' of the state equation, denoted by $x'(\cdot)$, is such that:

$$J\,(u'(\cdot), T') > J^0 \,.$$

The issue of proper periodicity can be tackled by calculus of variation method as follows: consider the perturbed signal,

$$u(t) = u^0 + \delta u(t)\,,$$

where the *periodic* variation $\delta u(t)$ is expressed through its Fourier expansion:

$$\delta u(t) = \sum_{k=-\infty}^{\infty} U_k e^{jk\Omega t},\ \ \Omega = 2\pi/T.$$

Define then the Hamiltonian function

$$H\big(x(t), u(t), \lambda(t)\big) = g((h(x(t), u(t)) + \lambda(t)'\, f(x(t), u(t)),$$

where $\lambda(t)$ is a n-dimensional vector. At steady-state, the non-linear programming problem of maximizing J with constraint $f(x, u) = 0$ can be solved as the problem of solving a set of algebraic equations constituted by $f(x, u) = 0$ and

$$H_x(x, u, \lambda) = 0\,, \qquad (4.5)$$
$$H_u(x, u, \lambda) = 0,$$

for some λ . These are the well-known *Lagrange conditions of optimality*. In fact, at the steady-state the Hamiltonian function is the Lagrange function and the elements of λ are the Lagrange multipliers. The whole system $f(x, u) = 0$ and (4.5) is a set of *2n+m* equations in *2n+m* unknowns. A solution is easily obtained with the classical method of substitution.

Once a solution x^0, u^0, λ^0 of this system is obtained, we can compute:

$$A = f_x(x^0, u^0),$$
$$B = f_u(x^0, u^0),$$
$$P = H_{xx}(x_0, u_0, \lambda_0),$$
$$Q = H_{xu}(x_0, u_0, \lambda_0),$$
$$R = H_{uu}(x_0, u_0, \lambda_0).$$

4.4 Periodic Control of Periodic Systems

A typical way to control a plant described by a linear *periodic* model is to impose:

$$u(t) = K(t)x(t) + S(t)v(t)\,,$$

where $K(\cdot)$ is a periodic feedback gain, i.e. $K(t+T) = K(t)$, for each t, $R(\cdot)$ is a periodic feedforward gain, $R(t+T) = R(t)$, for each t, and $v(t)$ is a new exogenous signal. The associated closed-loop system is then

$$x(t) = \big(A(t) + B(t)K(t)\big)x(t) + B(t)S(t)v(t)\,.$$

In particular, the closed-loop dynamic matrix is the periodic matrix $A(t) + B(t)K(t)$. The main problems considered in the literature are described in the following paragraphs.

4.4.1 Stabilization via State-feedback

As a paradigm problem in control, the *stabilization* issue is to understand if the closed-loop system is stable: any $K(\cdot)$ that is T-periodic and fulfills such a requirement is called *stabilizing gain*. In other words, a gain $K(\cdot)$ is said to be *stabilizing* when the characteristic multipliers of this matrix lie in the open unit disk of the complex plane.

A first basic question is to find the class of all stabilizing gains $K(\cdot)$. A general parameterization of all periodic stabilizing gains can be worked out by means of a suitable matrix inequality. To be more precise, by making reference to continuous-time systems, the *Lyapunov inequality condition* enables to conclude the following proposition:

> **Proposition 1**: the closed-loop system associated with a periodic gain $K(\cdot)$ is stable *if and only if there exists a positive definite periodic matrix* $P(\cdot)$ *satisfying* $\forall t$ *the inequality*:
>
> $$P(t+1) > \big(A(t)+B(t)K(t)\big)\cdot P(t)\cdot(A(t)+B(t)K(t))'. \qquad (4.y)$$

Then, it is possible to show that a periodic gain is stabilizing if and only if it can be written in the form:

$$K(t) = W(t)'\,P(t)^{-1}\,,$$

where $W(\cdot)$ and $P(\cdot)$ are periodic matrices (of dimensions *m*×*n* and *n*×*n*, respectively) solving the matrix inequality:

$$P(t+1) > \; A(t)P(t)A(t)' + B(t)W(t)'A(t)' + A(t)W(t)B(t)' + B(t)W(t)'P(t)^{-1}\,W(t)\,B(t)'.$$

Properties: In fact, on the basis of the periodic Lyapunov inequality, it is easy to show that, if there exists a pair of matrices $W(\cdot)$ and $P(\cdot)$ such that, for all t:

1. $W(t+T) + W(t)$;
2. $P(t+T) = P(t)$;

3. $(t) > 0$;
4. $W(\cdot)$ and $P(\cdot)$ satisfy the matrix inequality above described;

Consequences: We note first that $K(t) = W(t)'P(t)^{-1}$ is a stabilizing gain. Conversely, if $K(\cdot)$ is a T-periodic stabilizing gain, then the periodic Lyapunov Lemma guarantees that there exists a periodic and positive definite $P(\cdot)$ satisfying (4.y). Then, letting $W(t) = K(t)P(t)$ it is obvious that the gain $K(t)$ takes the factorized form $K(t) = W(t)'\,P(t)^{-1}$. In conclusion, the class of stabilizing gains is generated by all periodic pairs $W(\cdot)$ and $P(\cdot)$, with $P(t) > 0,\ \forall t$, satisfying the inequality above.

4.4.2 Pole Assignment for Control by state feedback

The basic idea of *pole assignment problem by state-feedback* is to make the system algebraically equivalent to a time-invariant one, by using a first periodic state-feedback (invariantization), and then to resort to the pole assignment theory for time-invariant systems, in order to locate the characteristic multipliers. Thus, the control scheme includes two feedback loops, the inner for invariantization, and the outer for pole placement.

Finding a periodic feedback gain that allows for positioning the closed-loop characteristic multipliers in given locations in the complex plane, is one of the typical problems studied in linear system theory. We will explore two ways to do that: the first one refers to the possibility of shaping a new dynamics by means of a *sample and hold strategy*, as seen in a previous section, of the type:

$$u(t) = K(t)x(nT + \tau),\ t \in [nT + \tau, nT + \tau + T - 1]\,, \tag{4.6}$$

where the gain $K(t)$ is a T-periodic function to be suitably designed and τ is a fixed tag in time. The advantage of this approach is that the pole assignment algorithm developed for time-invariant systems can be eventually applied to the periodic case too, using the techniques described above, that are easily solvable.

However, in this sampled feedback strategy, the input signal is updated only at the beginning of each period, so that the system is operated in open loop in the interperiod instants. This may be a serious issue as important information may be lost due to sparse sampling. In particular, in the interperiod instants, the performance may deteriorate due to the long action of a noise. To prevent this problem, continuous monitoring of the state can be appropriate. This leads to the alternative feedback control law:

$$u(t) = K(t)x(t) \ ,$$

where $K(\cdot)$ is a T-periodic gain matrix. This control strategy will be referred to as *instantaneous feedback*.

4.4.3 Reachability

For time-invariant systems, it is known that reachability is necessary and sufficient for arbitrary pole assignability. Starting from this observation, consider the system with feedback (4.6).

Then it is apparent that:

$$x(nT + \tau + T) = \left[\psi_A(\tau) + \sum_{j=\tau+1}^{\tau+T} \Phi_A(\tau + T, j) B(j-1) K_C(j-1) \right] x(nT + \tau)$$

Hence, the monodromy matrix at τ of the closed-loop system is:

$$\widehat{\psi_A}(\tau) = \left[\psi_A(\tau) + \sum_{j=\tau+1}^{\tau+T} \Phi_A(\tau+T,j)B(j-1)\overline{K}\,(j-1)\right] = F_\tau + G_\tau\widetilde{K},$$

where F_τ and G_τ are the lifted matrices, and :

$$\widetilde{K} = [\,\overline{K}(\tau)\ \ \overline{K}(\tau+1)\ \ \dots\ \ \overline{K}(\tau+T-1)\,]'.$$

Therefore, if the pair (F_τ, G_τ) is reachable (i.e., the periodic system reachable at time τ), it is possible to select K so as to impose the specified set of eigenvalues of $\Psi_A(\tau)$.

4.4.4 Pole Assignment via Instantaneous Feedback

Consider now the memoryless feedback law $u(t) = K(t)x(t)$ applied to the system. The associated closed-loop dynamic matrix is:

$$A_1(t) = A(t) + B(t)K(t).$$

The question is whether it is possible to find, for each set Λ of n complex numbers (in conjugate pairs), a periodic gain $K(\cdot)$ in such a way that $A_1(\cdot)$ has all its characteristic multipliers coincident with the elements of Λ. A first idea is to try to find a memoryless feedback law from the sampled feedback law $u(t) = K(t)\,x(iT+\tau)$ in such a way that the characteristic multipliers of $A_1(\cdot)$ coincide with the eigenvalues of $\Psi_A(\tau)$.
The simplest way to do this is to impose the coincidence of the two state evolutions associated with both control actions. To be precise, this consists into equating the two expressions of $x(iT+\tau+k)$, starting from $k=1$ and prosecuting up to $k=T-1$.

It is easy to conclude that, if:

$$K_c(\tau) = K(\tau),$$
$$K_c(\tau+1) = K(\tau+1)(A(\tau) + B(\tau)K(\tau)),$$
$$K_c(\tau+2) = K(\tau+2)\big(A(\tau+1) + B(\tau+1)K(\tau+1)\big)\big(A(\tau) + B(\tau)K(\tau)\big),$$
$$\dots$$

then the two state evolutions coincide. Notice that this set of equations can be given in the easy form:

$$K_c(\tau+i) = K(\tau+I)\Phi_A(\tau+i,\tau), \qquad i\ =\ 0,1,\cdot\cdot\cdot,T\ -1.$$

Although this pole assignment procedure by instantaneous feedback law is simple, there is no guarantee that the invertibility condition on $A_1(\cdot)$ is satisfied for a certain $K_c(\cdot)$. An interesting question is then whether, among all possible $K_c(\cdot)$ which assign the prescribed characteristic multipliers, there is one for which the invertibility condition is met. However, finding an algorithm of wide applicability is still an open question.

From the decomposition into reachable/unreachable parts, it clearly appears that the characteristic multipliers of the unreachable part $A_{r1}(\cdot)$ cannot be moved by any periodic feedback gain $K(\cdot)$. Hence the problem has to be faced by focusing on the reachable pair $(A_r(\cdot), B_r(\cdot))$ only, and applying the method developed above.

4.4.5 Other design methods: LQ Optimal Control

The classical *finite horizon optimal control* problem is that of minimizing the quadratic performance index over the time interval (t, t_f):

$$J\big(t,t_{f,}x_t\big) = x\big(t_f\big)'Q_{tf}x\big(t_f\big) + \int_t^{tf}\big(z(\tau)'z(\tau)\big)d\tau$$

where x_t is the system initial state at time *t*, $Q_{t_f} \geq 0$ is the matrix weighting the final state $x(t_f)$ [for details see Bittanti 1.8.2]. A first task of the controller is then to determine the actual value of the state $x(t)$ from the past observation of $y(\cdot)$ and $u(\cdot)$ up to time *t*. This leads to the problem of finding an estimate $\hat{x}(t)$ of $x(t)$ as the output of a linear system (*filter*) with as input the available measurements. The design of such a filter can be carried out in a lot of different ways, one of them is the *Kalman filter*. The implementation requires the solution of a matrix Riccati equation with periodic coefficients. Once $\hat{x}(t)$ is available, the control action is typically obtained as:

$$u(\tau) = K_{opt}(\tau)\,\hat{x}(\tau).$$

as shown in the figure below:

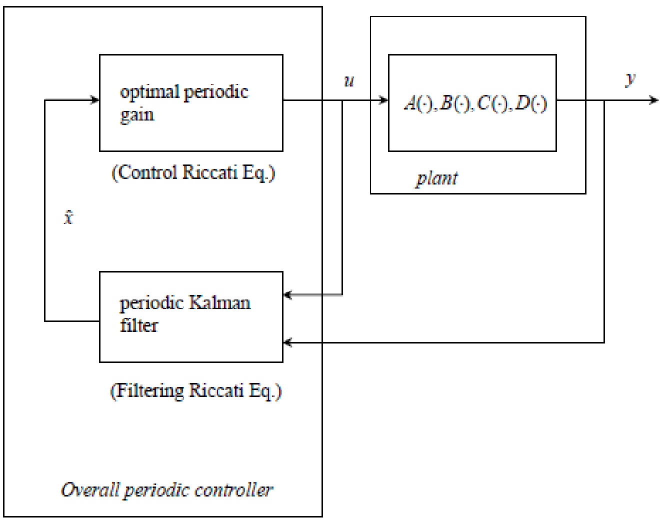


Fig. 4.x: An optimal controller structure

using the solution of a Kalman Filter and the Riccati Equation. The fundamental fact is that the control is periodic, due to the structure of the system.

4.4.6 Other design methods: H∞ Periodic Control

In the classical linear quadratic periodic control problem introduced above, the objective was the minimization of a mean-square performance criterion. An alternative measure of performance, which takes into account the requirement of robustness of the closed-loop system, is the H_∞ *criterion*. To describe such method, the system dynamic equation is conveniently modified by introducing a disturbance $w(\cdot)$, with unspecified characteristics, as follows:

$$x(t) = A(t)x(t) + Bw(t)w(t) + B(t)u(t).$$

Moreover, for an initial time point, denoted again by *t* for simplicity, let the initial state $x(t)$ be zero, i.e., $x(t) = 0$. The focus is to achieve a satisfactory bound for the ratio between the energy of the performance variable $z(\cdot)$ and the energy of the disturbance $w(\cdot)$. So, a way is to consider the following *differential game cost index*:

$$J_\infty(t, t_f) = x(t_f)' P_{tf} x(t_f) + \int_t^{tf} \left(z(\tau)' z(\tau) - \gamma^2 w(\tau)' w(\tau)\right) d\tau$$

where $Q_{t_f} \geq 0$. Moreover, γ is a positive scalar. In optimal control, the design corresponds to a minimization problem. In the H_∞ control, the objective is make $J_\infty(t, t_f) < 0$ for all possible disturbances $w(\cdot)$. It corresponds to attenuating the effect of disturbance $w(\cdot)$ on the performance evaluation variable $z(\cdot)$. In this regard, parameter γ plays a major role in order to achieve a trade-off between the two energies. Actually, the disturbance $w(\cdot)$ is unspecified, so that the real problem is to ensure $J_\infty(t, t_f) < 0$ in the *worst case*, i.e. for the disturbance maximizing the cost. This is a typical situation encountered in differential game theory, where two players have conflicting objectives, leading to the so-called *min-max* techniques. See a detailed explanation in the next chapter.

4.5 Exact Model Matching

The problem of model matching consists in finding a compensator for a given system, so as to obtain an overall dynamics as close as possible to an assigned input–output map. So, we restrict attention to the exact model matching for periodic systems via state feedback controllers. To be precise, given a periodic input–output map S:

$$x(t+1) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t)\,,$$

we try to find a state-feedback controller :

$$u(t) = K(t)x(t) + G(t)v(t)\,,$$

with $K(t)$ and $G(t)$ T-periodic design matrices, in such a way that the closed-loop input–output map coincides with the assigned one. A special case arises when the assigned map is actually time-invariant, so that the controller task is to make invariant the input–output closed-loop behaviour. To simplify the solution of the main problem, we make the assumption that $G(t)$ is square and non-singular for each t. Nowadays there are many studies related to this argument, that will develop in the future: these is only a brief overview which could change with the next discoveries.

At last, the overall scheme is depicted in the following figure.

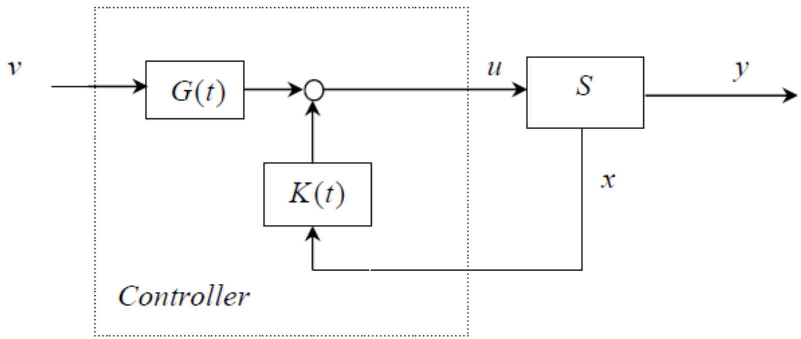


Fig. 4.1: Exact model matching scheme

Chapter 5: APPLICATIONS

5.1 Quadrupole ion trap

A quadrupole ion storage trap (**QUISTOR**) exists in both linear and 3D varieties, and refers to an ion trap that uses constant DC and radio frequency (RF) oscillating AC electric fields to trap ions. It is commonly used as a component of mass spectrometers. The invention of the 3D quadrupole ion trap itself is attributed to Wolfgang Paul, who shared the Nobel Prize in Physics in 1989 for this work.

The 3D trap generally consists of two hyperbolic metal electrodes and a hyperbolic ring electrode, halfway between the other two electrodes. The ions are trapped in the space between these three electrodes by AC (oscillating) and DC (static) electric fields. The AC radio frequency voltage oscillates between the two hyperbolic metal end cap electrodes, if ion excitation is desired; the driving AC voltage is applied to the ring electrode. The ions are first pulled up and down axially while being pushed in radially. The ions are then pulled out radially and pushed in axially (from the top and bottom).

The quadrupole ion trap has two configurations: the three-dimensional form described above and the linear form made of 4 parallel electrodes. A simplified rectilinear configuration has also been used. The advantage of the linear design is in its simplicity, but this leaves a particular constraint on its modeling. The motions of a single ion in the trap are described by Mathieu Equations, which can only be solved numerically via computer simulations.

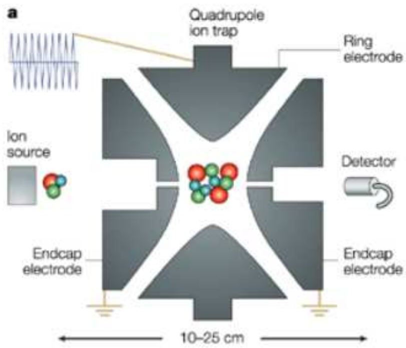


Figure 1: A quadrupole ion trap

5.1.1 Equations of motion

Ions in a quadrupole field are trapped by forces that drive them back toward the center of the trap. The motion of the ions in the field is described by solutions to the Mathieu equation. When written for ion motion in a trap, the equation is:

$$\frac{d^2u}{d\xi^2} + (a_u - 2q_u \cos 2\xi)u = 0\,,$$

where *u* represents either the x, y or the z coordinate, ξ is a dimensionless parameter given by $\xi = \Omega t/2$, and a_u and q_u are dimensionless trapping parameters. The parameter Ω is the radial frequency of the potential applied to the ring electrode. By using the chain rule for computing the derivative of the composition of two functions on the first addend of the previous formula, it can be shown that:

$$\frac{d^2u}{dt^2} = \frac{\Omega^2}{4}\frac{d^2u}{d\xi^2}\,.$$

Substituting Equation 2 into the Mathieu Equation 1 yields:

$$\frac{4}{\Omega^2}\frac{d^2u}{dt^2}+[a_u-2q_u\cos(\Omega t)]u=0.$$

Reorganizing terms shows us that:

$$m\frac{d^2u}{dt^2}+m\frac{\Omega^2}{4}[a_u-2q_u\cos(\Omega t)]u=0.$$

By Newton's laws of motion, the above equation represents the force on the ion: the Floquet theorem describes exactly the propagator's structure.

The forces in each dimension are not coupled, thus the force acting on an ion in, for example, the x dimension is:

$$F_x=ma=m\frac{d^2x}{dt^2}=-e\frac{\partial\phi}{\partial x}$$

Here, ϕ is the quadrupolar potential, given by

$$\phi=\frac{\phi_0}{r_0^2}(\lambda x^2+\sigma y^2+\gamma z^2)$$

where ϕ_0 is the applied electric potential and λ, σ and γ are weighting factors due to the geometry of the device, and r_0 is a size parameter constant. It can be shown that:

$$\lambda+\sigma+\gamma=0.$$

For an ion trap, $\lambda = \sigma = 1$ and $\gamma = -2$ and for a quadrupole mass filter, $\lambda = -\sigma = -1$ and $\gamma = 0$.

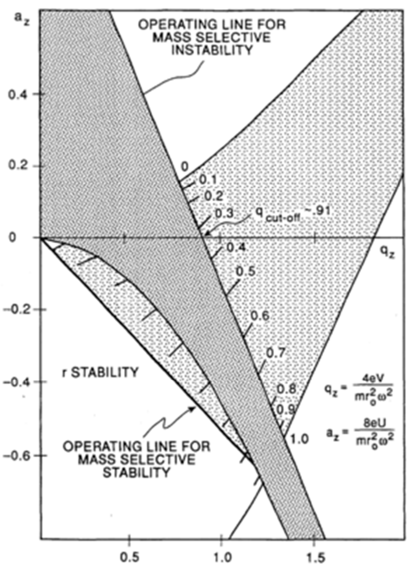


Figure 1: Diagram of the stability regions of a quadrupole ion trap according to the voltage and frequency applied to the ion trap elements.

The trapping of ions can be understood in terms of stability regions in q_u and a_u space.

5.3 Quadrupole mass analyzer

A quadrupole mass analyzer consists of four electrically isolated hyperbolic or cylindrical rods linked to RF (radio frequency) and DC (direct current) voltages. The combination of RF and DC voltages creates a region of strong focusing and selectivity known as a *hyperbolic field*. In simplest terms, the ratio of RF/DC allows for the selective transmission of ions of a narrow range of mass-to-charge ratios from the total population of ions introduced from the ionization source.

The idealized hyperbolic field can be described in terms of Cartesian coordinates (x and y directions toward the rods, and z direction along the rods' axis). Ions of a selected mass-to-charge ratio follow a stable trajectory around the center of the field, and are transmitted in the z direction through the device. Motion of these "stable ions" in the x and y directions are small in amplitude. Ions of other mass-to-charge ratio will have unstable trajectories, with increasing displacement in the x and y directions away from the center of the hyperbolic field, and thus strike the quadrupole rods, where they neutralize upon contact. The application of RF and DC voltages creates a region of stability for transmission of ions of a limited mass-to-charge ratio range. In this regard, the quadrupole analyzer is operating as a mass filter. Ions of increasing mass-to-charge ratio will sequentially achieve stable trajectories and reach the detector in order of increasing mass-to-charge ratio.

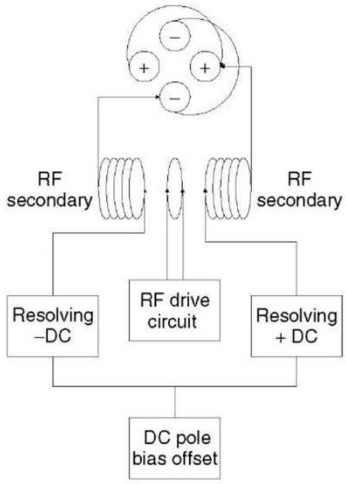


Figure 2: The quadrupole mass filter architecture and its power supply

To understand the behavior of ions in the quadrupole, we give a brief introduction to the mathematics associated with ion motion.

The motion of ions through a quadrupole is described by the second-order linear differential *Mathieu equation*, which can be derived starting from the familiar equation relating force to mass and acceleration, *F* = *ma*, yielding the final parameterized form, with the following substitutions for the parameters *a* and *q*:

$$\frac{d^2u}{d\xi^2} + (a_u - 2q_u \cos 2\xi)u = 0$$

$$a_u = \frac{8eU}{mr_0^2\Omega^2}$$

$$q_u = \frac{4eV}{mr_0^2\Omega^2}$$

The *u* in the above equations represents position along the coordinate axes (x or y), *f* is a parameter, *t* is time, *e* is the charge on an electron, *U* is the applied DC voltage, *V* is the applied RF voltage, *m* is the mass of the ion, r_0 is the effective radius between electrodes, and Ω is the applied RF frequency. The parameters *a* and *q* are proportional to the DC voltage *U* and the RF voltage *V*, respectively.

The analytical solution to this second-order linear differential equation is:

$$u(\xi) = \Gamma \sum_{n=-\infty}^{\infty} C_{2n} \exp(2n+\beta)i\xi + \Gamma' \sum_{n=-\infty}^{\infty} c_{2n} \exp-(2n+\beta)i\xi$$

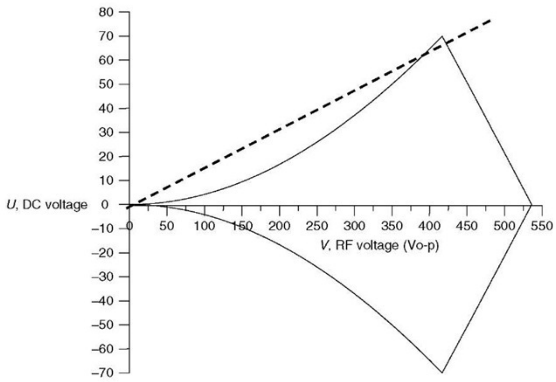


Figure 3: Mathieu stability diagram for an ion of m/z 219 in a quadrupole mass filter with 9.5-mm diameter round rods and an RF frequency of 1.2MHz

The solutions to the Mathieu equation can be presented graphically, as shown in Figure 2 in a so-called *stability diagram*. Points in (U, V) space (DC, RF voltage space) within the lines lead to stable trajectories; points outside the lines will lead to an unstable trajectory. The dotted line is called the scan line: if the line passes just under the apex of the stability region, the mass-to-charge ratio will have a stable trajectory at those voltages, and will be transmitted to the detector.

The selection of RF and DC voltages, V and U, the RF frequency, and the inscribed radius r_0 between the rods determines the performance of the quadrupole mass filter. The mass range of the mass filter can be increased by increasing the RF and DC voltages, or by decreasing the inscribed radius or RF frequency. The mass resolution is a function of the ratio of the RF and DC voltages

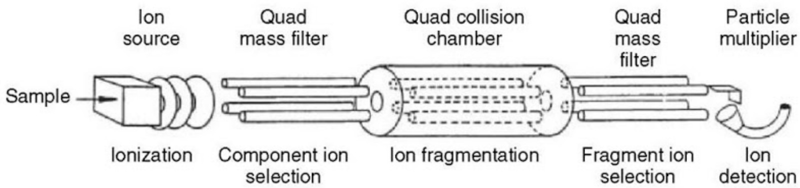


Figure 4: Schematic of a triple quadrupole mass spectrometer (TQMS)

The quadrupole mass analyzer enjoys a central role in the success of mass spectrometric methods in proteomics. Development of an understanding of the theoretical and practical aspects of quadrupoles can improve ability to utilize single and triple quadrupole mass spectrometers effectively to solve important problems in proteomics.

5.4 A program for numerical solution of the Mathieu Equation

In this paragraph we will review the full spectrum of solutions to the *Mathieu differential equation*:

$$\ddot{y} + (a - 2q\cos(2t))y = 0$$

and we will describe a numerical algorithm which allows a flexible approach to the computation of all the Mathieu functions. A compact matrix notation is used, which can be readily implemented on any computing platform. Also there will be some explicit examples (written in the programming language *Scilab*) that provide a ready-to-use package for solving the Mathieu differential equation and related applications in several fields.

Some high-level mathematical language contains Mathieu functions, but these are defined as black-box commands: thus, it is difficult to understand how these commands work and how to extend their use to other cases or specific applications. In this example open-source software is employed, which is characterized by its modularity and simplicity: that allows easy improvements, extensions and applications to different specific cases.

The program is composed of two modules, essentially based on the *Scilab* command `spec` for the diagonalization of a matrix, which turns out to be necessary for elliptical-to-cartesian coordinate transformation when plotting the results. In contrast with polynomial approximations, this one does not accumulate errors as it is based on matrix diagonalization and Fourier transform. Moreover, the modular form of the program makes it more flexible and efficient in determining a whole range of values rather than a single one.

In this paragraph first we briefly recall some known facts about the solutions of the Mathieu equation. The Mathieu equation is a second-order homogeneous linear differential equation of the form:

$$\ddot{y} + (a - 2q\cos(2t))y = 0 \tag{5.1}$$

where the constants *a, q* are often referred as characteristic number and parameter, respectively.

By substituting the independent variable $z \rightarrow iz$ in the above equation we can obtain the so called *modified Mathieu equation*, which is easier to solve:

$$\frac{d^2y}{du^2} - [a - 2q\cosh(2u)]y = 0 \tag{5.2}$$

The most general solution $y(z)$ of the first equation can be written as a linear combination of two independent solutions $y_1(z)$ and $y_2(z)$, i.e.

$$\mathbf{y(z) = Ay_1(z) + By_2(z)}$$

with A, B arbitrary complex constants. According to Floquet's theorem, it is possible to choose $y_1(z)$ and $y_2(z)$ in a very simple and convenient form: indeed, there always exists a solution of (5.1) of the first kind:

$$\mathbf{y_1(z) = e^{i\nu z}p(z)} \tag{5.3}$$

where the characteristic exponent ν depends on *a* and *q*, and $p(z)$ is a periodic function with period π. It is easy to check that a solution of the form (5.3) is bounded for $z \rightarrow \infty$, unless ν is a complex number, for which $y_1(z)$ is unbounded. If ν is real but not a rational number then (5.3) is non-periodic. If ν is a rational number, i.e. ν is a proper fraction as $\nu = s/p$, then (5.3) is periodic of period at most $2\pi p$ (and not π or 2π). Finally if ν is a real integer then y_1 is a periodic function with period π or 2π.

There are two ways of tackling the problem of solving (5.1), according to the physical problem one is interested in:

1) The first case is with *a, q* being independently given constants (e.g. in the case of the parametric oscillator). Then, the general solution may be periodic or not, bounded or not depending on the corresponding values. Such a ν value can be determined by the following method: by introducing in (5.3) the Fourier representation of a periodic function with period π,

$$p(z)=\sum_{k=-\infty}^{+\infty} c_k e^{2ikz}$$

and by inserting the result in eq. (5.1), one obtains the following recurrence equation for c_k:

$$[(2k+\nu)^2-a]c_k+q[c_{k+1}+c_{k-1}]=0$$

which can be written in matrix form:

$$(H_\nu - a\mathrm{I})c=0$$

where I is the identity matrix and H_ν is a symmetric tridiagonal matrix and c is a column vector.

2) The second case is when, from the very beginning, we are interested only to periodic solutions with period π or 2π (e.g. in the case of the angular part of the wave equation in elliptical coordinates). Then *a* and *q* cannot be given independently: they must satisfy the equation (owing the periodicity) of the form $\nu(a; q) = n$, where n is an integer number. All *a*'s values for which $\nu(a; q) = n$, are called *characteristic values*: the corresponding periodic solutions (5.3) are called *Mathieu functions* (or Mathieu functions of the first kind). In this case, the second solution y_2 (Mathieu function of the second kind) is usually rejected since it is not bounded.

It is easy to see from (5.3) that if $\nu(a; q) = n$, then y_1 is periodic with period π for even n and periodic with period 2π for odd n. Since a periodic function with period π it has also period 2π, then by inserting the Fourier representation of a 2π periodic function in (5.3),

$$y_1(z) = \sum_{k=-\infty}^{+\infty} c_k e^{ikz}$$

we can obtain the following recurrence relation,

$$(k^2 - a)^2 c_k + q(c_{k+2} + c_{k-2}) = 0 \quad \text{or} \quad (H - a\mathrm{I})c = 0$$

where H is a symmetric pentadiagonal matrix.

The Mathieu functions so obtained are defined as $c_{en}(z; q)$ (even solutions corresponding to a_n eigenvalues: cosine-elliptic), $s_{en}(z; q)$ (odd solutions corresponding to b_n eigenvalues: sine-elliptic).
The solutions of the modified Mathieu equation can be easily obtained by the following way; for *a*-values corresponding to $c_{em}(z, q)$, $s_{em}(z, q)$ the first solutions of (5.2) are derived by substituting *iz* for *z*, i.e.

$$\mathbf{C_{en}(z, q) = c_{en}(iz, q)}$$

$$\mathbf{S_{en}(z, q) = -is_{en}(iz, q)}$$

C_{en} and S_{en} are called *modified Mathieu functions of the first kind*.

5.4.1 *Scilab* modules

The program is composed of four modules, two for the calculation of periodic Mathieu functions (hence q is given as an input, and the *a*'s and *b*'s are calculated as output), and two for the general aperiodic case (both *a* and *q* are given as arbitrary input):

- [ab,c]=mathieuf(q,[mat_dimension])

  calculates the characteristic values a_k, b_k and the coefficients c_k of the expansion of Mathieu functions, for a given matrix dimension;

- mathieu('kind',order,arg,q,[precision])

  uses the former program to calculate the Mathieu functions with a given precision, by starting with a matrix size (depending on the order *n* and the parameter q) which is increased until the required precision is reached. Some inputs can also be entered interactively.

- [nu,c] =mathieuexp(a,q,[mat_dimension])

  calculates the characteristic exponent *ν(a, q)* of non-periodic solutions of the Mathieu equations and the coefficients c_k of the expansion of the periodic factor.

- mathieus(arg,a,q,[precision])

  uses the former program and calculates solutions of Mathieu Equations with arbitrary *a* and *q* parameters.

- [x,y]=ell2cart(u,v,c)

  has also been written for the transformation from cartesian to elliptical coordinates, which is necessary when plotting the results of a two dimensional calculation (for example the elliptical waveguide).

5.4.2 Some Examples

Figure 5 shows the well-known stability curves, where the white areas show values of *a* and *q* with stable (periodic) solutions, and the coloured contours indicate the unstable solutions (the value expressed by the contour is the exponential envelope factor of the diverging function). The equations in this case might describe a parametric oscillator, i.e. a harmonic oscillator.

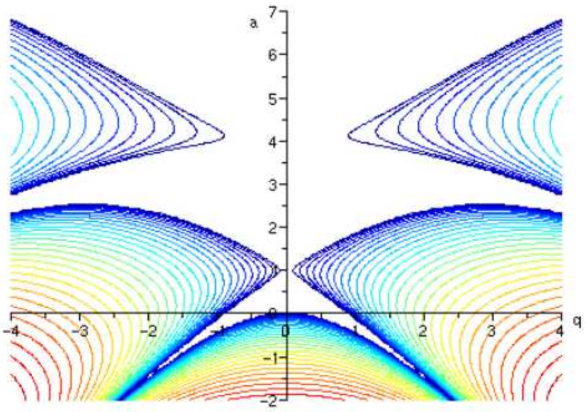


Figure 5: Stability curves of Mathieu Equation obtained by the program

Figure 6 is a plot over one period of $s_{e1}(x, q)$ for various values of q, showing how the function, which is a sinusoid for $q = 0$, deviates from it as q increases.

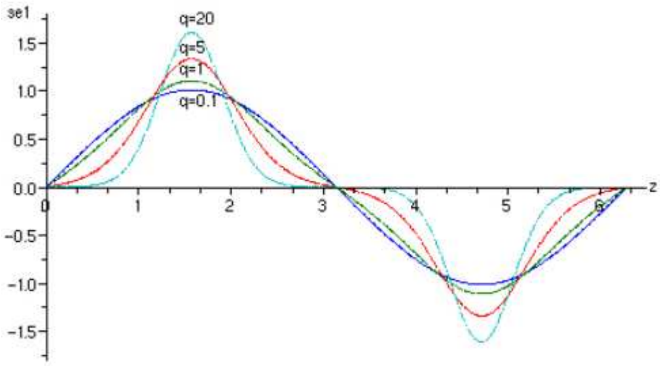


Figure 6: The function se1 for various values of q

Figure 7 is the 3-D plot of the field of a higher-order mode of an elliptical waveguide (or vibrating drum). It was the first case of study of Mathieu, and we show that also this complicated example could be represented with the previous functions.

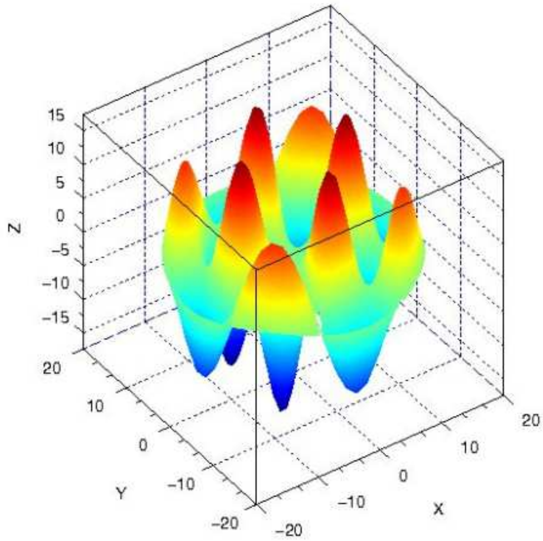


Figure 7: Elliptical waveguide

The limitation of this technique is given by the framework program language, which imposes restrictions both on numerical precision and on memory available.

APPENDIX A
Algebraically equivalent systems

We want to recall how change the structure of a linear system when we make a basis change in in the state space. We will refer to the discrete case: the continuous one leads to the same result.

Consider **(v_1, v_2, ….v_n)** as the state space basis, described by the following equations:

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) + Du(t) \end{aligned}$$

Suppose that we want to see X in a new basis **(v'_1, v'_2, ….v'_n)**: each vector v'_i is described as a linear combination $\sum_j v_j t_{ji}$ of vector v_i, i.e.:

$$(v_1', v_2', \ldots\ldots, v_n') = (v_1, v_2, \ldots\ldots, v_n)\begin{bmatrix} t_{11} & t_{12} & \ldots & t_{1n} \\ t_{21} & t_{22} & \ldots & t_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ t_{n1} & t_{n2} & \ldots & t_{nn} \end{bmatrix} = (v_1, v_2, \ldots\ldots, v_n)T$$

For the vectors $\mathbf{v}_i$ are also expressible as a linear combination of vectors $\mathbf{v}_i$', the matrix T is invertible and the link between two basis could be expressed as follows:

$$(v_1, v_2, \ldots\ldots, v_n) = (v_1', v_2', \ldots\ldots, v_n')T^{-1}$$

The system state, defined by the *n* elements related with the basis $(v_1, v_2, \ldots\ldots, v_n)$:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

Will have as components related with the new base $(v_1', v_2', \ldots\ldots, v_n')$

$$x' = T^{-1}x$$

Proof: It follows directly from:

$$(v_1, v_2, \ldots\ldots, v_n)\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = (v_1', v_2', \ldots\ldots, v_n')\begin{bmatrix} x_1' \\ x_2' \\ \cdots \\ x_n' \end{bmatrix} = (v_1, v_2, \ldots\ldots, v_n)T\begin{bmatrix} x_1' \\ x_2' \\ \cdots \\ x_n' \end{bmatrix}$$

By substituting, we have:

$$\begin{aligned} x'(t+1) &= T^{-1}x(t+1) = T^{-1}FTx'(t) + T^{-1}Gu(t) \\ y(t) &= HTx'(t) + Du(t) \end{aligned}$$

Or:

$$\begin{aligned} x'(t+1) &= F'x'(t) + G'u(t) \\ y(t) &= H'x'(t) + Du(t) \end{aligned}$$

In which we have replaced:

$$\begin{aligned} F' &= T^{-1}FT \\ G' &= T^{-1}G \\ H' &= HT \end{aligned}$$

Definition: *Two systems (F, G, H, D) and (F', G', H', D) are called algebraically equivalent if they meet the previous equations. So they could be considered as the same system referred to different basis in the state space.*

APPENDIX B

Matrix Norm

In what follows, **K** will denote the field of real or complex numbers. Let **K** denote the vector space containing all matrices with *m* rows and *n* columns with entries in **K**.

A *matrix norm* is a vector norm on **K**. That is, if ||A|| denotes the norm of the matrix **A**, then,

- $\|A\| \geq 0$;
- $\|A\| = 0 \Leftrightarrow A = 0$;
- $\|\alpha A\| = |\alpha| \|A\|$ for all α in K and all matrices A in K_{mxn} ;
- $\|A + B\| \leq \|A\| + \|B\|$ for all matrices A and B in K_{mxn} .

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