UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento Di Matematica "Tullio Levi Civita"

Corso Di Laurea Magistrale In Matematica

TOPOLOGICAL FUNDAMENTAL GROUP
AND
ENRICHED MONODROMY EQUIVALENCE

Supervisor:
Dott. PIETRO POLESELLO

Candidate:
LUCA CARRARO
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Abstract

The theory of covering spaces is well-behaved when the base space is locally path connected and semilocally 1-connected. Following works of Brazas, by generalizing the notion of covering to that of semicovering, by defining a topological fundamental group and enriching over **Top** the usual monodromy functor, we get an extended theory which is well-behaved with respect to a wider class of spaces, namely locally wep-connected topological spaces.

A chi mi è sempre stato accanto

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INTRODUCTION

The starting point of this work is the theory of covering spaces:

for a topological space X, consider the following quasi-commutative diagram of functors:

$$\begin{array}{ccc}
\mathbf{Cov}\,(X) & \xrightarrow{\ \Pi_1\ } & \mathbf{CovMor}\,(\Pi_1 X) \\
& \searrow^{\mu} & \downarrow{\scriptstyle \mathscr{R}} \\
& & \mathbf{Fct}\,(\Pi_1 X, \mathbf{Set}),
\end{array}$$

where $\mathscr{R}$ is the standard equivalence between the category $\mathbf{CovMor}\,(\Pi_1 X)$ of covering morphisms of the fundamental groupoid $\Pi_1 X$ of X and the category $\mathbf{Fct}\,(\Pi_1 X, \mathbf{Set})$ of its representations in $\mathbf{Set}$; μ is the monodromy functor between the category $\mathbf{Cov}\,(X)$ of covering maps of X and $\mathbf{Fct}\,(\Pi_1 X, \mathbf{Set})$, and finally Π_1 is induced by the fundamental groupoid functor. It is a classical result that for a locally path connected and semilocally 1-connected topological space X, each arrow in the above diagram is an equivalence.

Following [**2**], [**4**] and [**5**], we build a topologically enriched fundamental groupoid $\Pi_1^\tau X$ and define the notion of semicovering in order to get a topologically enriched version of the previous diagram:

$$\begin{array}{ccc}
\mathbf{SCov}\,(X) & \xrightarrow{\ \Pi_1^\tau\ } & \mathbf{OCovMor}\,(\Pi_1^\tau X) \\
& \searrow^{\mu^\tau} & \downarrow{\scriptstyle \mathscr{R}^\tau} \\
& & \mathbf{TopFct}\,(\Pi_1^\tau X, \mathbf{Set}^e),
\end{array}$$

where each category is a $\mathbf{Top}-$category and each functor is a $\mathbf{Top}-$functor in a sense which will be defined.

Moreover, there exists a class of topological spaces, *locally wep-connected* spaces, for which each arrow is an equivalence: in particular any locally path connected and semilocally 1-connected topological space is locally wep-connected and this diagram reduces to the classical one.

The general structure of this thesis is as follows. Chapter 1 is devoted to classical results: we recall some definitions and properties about the *compact-open topology*; about *groupoids* and their representations; about *fibrations* and their *lifting properties*. Finally we describe with some details the above mentioned functors $\mathscr{R}$, μ and Π_1.

In chapter 2, by exploiting free topological groups, first we define the *quasitopological fundamental group* $\pi_1^{qtop}(X)$ of a based space X, next we define the functor τ to achieve a true topological group $\pi_1^\tau(X)$ which will be called *topological fundamental group.* We go on with the example of the *Hawaiian earring* $\mathbb{H}$, a topological space such that $\pi_1^{qtop}(\mathbb{H}) \neq \pi_1^\tau(\mathbb{H})$. Finally we extend these constructions to define the *topological fundamental groupoid* $\Pi_1^\tau X$*.*

In chapter 3 we define the notion of *semicovering map* $p\colon Y \to X$ to show that such a map always induces an *open covering morphism* of groupoids $\Pi_1^\tau p\colon \Pi_1^\tau Y \to \Pi_1^\tau X$ via the enriched fundamental groupoid functor. We end up describing the enriched functors μ^τ and $\mathscr{R}^\tau$ to prove the quasi-commutativity of the above enriched diagram.

Finally in chapter 4 we introduce and describe the class of *locally wep-connected* spaces, to which the teory is well-behaved: in the final section we show that the enriched functors Π_1^τ, $\mathscr{R}^\tau$ and μ^τ define equivalences on this class of spaces.

In the appendix we give a constructive inductive approximation of the topological fundamental groupoid starting from the quasitopological one.

NOTATION AND CONVENTIONS

Besides classical notions about covering spaces theory, the reader will be considered familiar with basic definitions and results on category theory. We will use the standard bold notation to denote usual categories: **Set** for the category of sets, **Grp** for groups, **Grpd** for groupoids, **Top** for topological spaces... other categories will be defined later. For a category $\mathcal{C}$, the hom-set between the objects $a, b \in \mathcal{C}$ will be always denoted by $\mathcal{C}(a, b)$ and for a fixed object $a \in \mathcal{C}$ the *vertex hom-set* at a is $\mathcal{C}(a, a) := \mathcal{C}(a)$. A star as low index denotes a pointed category: for example $\mathbf{Top}_*$ is the category of topological spaces with selected base point and base point-preserving morphisms. For a category $\mathcal{C}$ and an object $x \in \mathcal{C}$, $\mathcal{C}_x$ denotes the set of arrows starting from x. For a couple of small categories $\mathcal{A}$ and $\mathcal{B}$, the category of functors with natural transformations is denoted with $\mathbf{Fct}(\mathcal{A}, \mathcal{B})$.

We will make use also of these notations: a **Top**-category $\mathcal{C}$ is a category enriched in **Top**, i.e. its hom-sets are all equipped with a topology such that all composition maps are continuous; in particular the multiplication $\mathcal{C}(c_1, c) \times \mathcal{C}(c, c_2) \to \mathcal{C}(c_1, c_2)$, given by $(f, g) \mapsto g \circ f$, is continuous if the domain is equipped with the product topology. For instance, **Set** is a **Top**-category if we endow each set with the discrete topology and giving hom-sets the topology of point-wise convergence (cfr. Theorems 1.1.5 and 1.1.6 and see also [**19**, VII] for basic definitions). We will denote by $\mathbf{Set}^e$ the so enriched category of sets.

Given two **Top**-categories $\mathcal{A}$ and $\mathcal{B}$, a **Top**-functor is a functor $F \colon \mathcal{A} \to \mathcal{B}$ such that each function $\mathcal{A}(a_1, a_2) \to \mathcal{B}(F(a_1), F(a_2))$ is continuous. A **Top**-natural transformation of **Top**-functors is a natural transformation of the underlying functors. The category of **Top**-functors $\mathcal{A} \to \mathcal{B}$ and **Top**-natural transformations is denoted $\mathbf{TopFct}(\mathcal{A}, \mathcal{B})$.

When the underlying category of a **Top**-category $\mathcal{C}$ is a groupoid and each inversion map $\mathcal{C}(a, b) \to \mathcal{C}(b, a)$ is continuous, we call $\mathcal{C}$ a **Top**-groupoid. The category of **Top**-groupoids is denoted **TopGrpd**. It must be emphasized that the notion of **Top**-groupoid is different from that of topological groupoid which refers to a groupoid internal to **Top**.

Chapter 1

CLASSICAL RESULTS

In this introductory chapter, first we remember how the *compact-open topology* is build and briefly describe some of its properties: it will be an important tool to deal with spaces of continuous paths. Next, we list some known results in standard algebraic topology: we give the definition of *groupoid* and its *representations* and we introduce the central notion of *covering morphism of groupoids;* in particular we show the equivalence $\mathscr{R}\colon \mathbf{CovMor}(\mathcal{G}) \simeq \mathbf{Fct}(\Pi_1 X, \mathbf{Set})$ between the category of covering morphisms of a groupoid $\mathcal{G}$ and its representations. The final section is devoted to *monodromy*: we define the category of *fibrations with unique path lifting* over a space X and recall the definition of *fundamental groupoid* $\Pi_1 X$ of a space; second, we show that for any topological sapce X the monodromy $\mu\colon \mathbf{Fib}^!(X) \to \mathbf{Fct}(\Pi_1 X, \mathbf{Set})$ and the fundamental groupoid functor $\Pi_1\colon \mathbf{Fib}^!(X) \to \mathbf{CovMor}(\Pi_1 X)$ provide a quasi-commuting diagram of categories; finally we restrict our study to a special kind of fibrations, *covering spaces,* and we end up with the classifications theorems for coverings of locally path connected and semilocally 1-connected topological spaces.

1.1 THE COMPACT-OPEN TOPOLOGY

In general, sets of continuous functions $\mathbf{Top}(X, Y)$ of topological spaces will be endowed with the *compact-open* topology, which we are to define.

Given topological spaces X, T and Y and a function $h\colon X \times T \to Y$ which is continuous in x for each fixed t, there is associated with h the function $h^*\colon T \to \mathbf{Top}(X, Y)$ defined by $h^*(t) =: h_t$ where $h_t(x) =: h(x, t)$ for every $x \in X$. The correspondence between h and h^* is clearly one-to-one.

Although the continuity of any particular h depends only on the given topological spaces X, T and Y, the topology of the function space $\mathbf{Top}(X, Y)$ is involved in the continuity of h^*. It would be desirable to so topologize $\mathbf{Top}(X, Y)$ that the functions h^* which are continuous are precisely those which correspond to continuous functions h. It is worth emphasizing that the problem is motivated by the special case in which T is the unit interval: in this case h is a homotopy and h^* is a path in the functional space $\mathbf{Top}(X, Y)$.

Definition 1.1.1. In the above notations, for any two sets $K \subseteq X$ and $W \subseteq Y$ define $M(K, W) :=$ $\{f \in \mathbf{Top}(X, Y) : f(K) \subseteq W\}$. The *compact-open* topology is defined by choosing as a subbasis for the open sets, the family $M(K, W)$ where K ranges over the compact subsets of X and W ranges over the open subsets of Y. In particular the family of finite intersections of sets of the form $M(K, W)$ is a base for this topology.

Lemma 1.1.2. *If* **Top** (X, Y) *has the compact-open topology, then continuity of* h *implies continuity of* h^* *under no restrictions on the topological spaces* X*,* T *and* Y*.*

Proof. Let K be a compact set in X, W an open set in Y and let t_0 be a point in $h^{*-1}(M(K, W))$. Then $K \times \{t_0\} \subseteq h^{-1}(W)$. Since $h^{-1}(W)$ is open, it is the union of open sets $U_\alpha \times V_\alpha$. Since K is compact, $K \times t_0$ is contained in a finite union $\bigcup_{i=1}^n U_i \times V_i$ with each V_i a neighborhood of t_0. Then $\bigcap_{i=1}^n V_i$ is an open neighborhood of t_0 and is contained in $h^{*-1}(M(K, W))$. So $h^{*-1}(M(K, W))$ is open since it is a neighborhood of each of its points. □

Lemma 1.1.3. *Let* X, Y *be topological spaces, where* X *is locally compact and Hausdorff and let* $\mathcal{P}$ *be a subbasis of the topology on* Y*. Then the familiy* $M(K, U)$ *is a subbasis of the compact-open topology on* **Top** (X, Y)*, where* K *ranges over the compact subsets of* X *and* $U \in \mathcal{P}$*.*

Proof. We have to show that for each continuous $f\colon X \to Y$, for each compact set $K \subseteq X$ and for each open set $U \subseteq Y$ such that $f(K) \subseteq U$, there exist compact sets $K_1, \dots, K_n \subseteq X$ and open sets $U_1, \dots, U_n \in \mathcal{P}$ such that $f \in M(K_1, U_1) \cap \dots \cap M(K_n, U_n) \subseteq M(K, U)$. Let $\mathcal{B}$ be the familiy of finite intersections of elements of the subbasis $\mathcal{P}$. Then, by definition, $\mathcal{B}$ is a basis for the topology on Y and, since $f(K)$ is compact, we can find $V_1, \dots, V_m \subseteq \mathcal{B}$ such that $f(K) \subseteq V_1 \cup \dots \cup V_m \subseteq U$.

Each point $x \in K$ has a compact neighborhood K_x such that $f(K_x) \subseteq V_i$ for some i: since the family $\left\{\mathring{K}_x\right\}_{x \in K}$ of the interiors of K_x's is an open covering of K, there exists a finite subset $S \subseteq K$ such that $K \subseteq \bigcup_{x \in S} K_x$. Let us denote $K_i \coloneqq \bigcup \{K_x \colon x \in S,\, f(K_x) \subseteq V_i\}$, hence each K_i is compact and $f \in M(K_1, V_1) \cap \dots \cap M(K_m, V_m) \subseteq M(K, U)$. For each index i, there exist $U_{i_1}, \dots, U_{i_s} \in \mathcal{P}$ such that $V_i = U_{i_1} \cap \dots \cap U_{i_s}$ and by noticing that $M(K_i, U_{i_1}) \cap \dots \cap M(K_i, U_{i_s}) = M(K_i, V_i)$, we end the proof. □

Theorem 1.1.4 (Exponential law)**.** *Let* X, T, Y *be topological spaces and let all spaces* **Top** $(\cdot, \cdot)$ *be endowed with the compact-open topology. Then:*

1. *The map* $(\cdot)^* : \mathbf{Top}(X \times T, Y) \to \mathbf{Top}(X, \mathbf{Top}(T, Y))$ *is injective.*
2. *If* T *is locally compact and Hausdorff, then* $(\cdot)^*$ *is bijective.*
3. *If* X *is locally compact and Hausdorff, then* $(\cdot)^*$ *is continuous.*
4. *If* X, T *are locally compact and Hausdorff, then* $(\cdot)^*$ *is a homeomorphism.*

Proof. 1. This is essentially Lemma 1.1.2.

2. We have to show that for each continuous $g\colon X \to \mathbf{Top}(T, Y)$, the function $f\colon X \times T \to Y$, given by $f(x, t) = g(x)(t)$, is continuous. Take an open set $U \subseteq Y$ and a point $(x, t) \in f^{-1}(U)$; the map $g(x)$ is continuous and T is locally compact and Hausdorff, thus there exists a compact neighborhood B of t in T such that $g(x)(B) \subseteq U$ and so $g(x) \in M(B, U)$. The map g is continuous, thus there exists a compact neighborhood A of x in X such that $g(A) \subseteq M(B, U)$. So $f(A \times B) \subseteq U$.

3. By Lemma 1.1.3, if X is locally compact and Hausdorff, a subbasis for the compact-open topology on $\mathbf{Top}(X, \mathbf{Top}(T, Y))$ is given by the family $M(H, M(K, U))$ where H and K range over the compact subsets of X and T respectively and U over the open subsets of Y. Since the map $(\cdot)^*$ identifies $M(H \times K, U)$ with $M(H, M(K, U))$, it follows that $(\cdot)^*$ is continuous.

4. It is enough to show that the family of open sets of the form $M(H \times K, U)$ gives a subbasis for the compact-open topology on $\mathbf{Top}(X \times T, Y)$. So let $S \subseteq X \times T$ be compact and $f \in M(S, U)$.

For each $s \in S$, there are two compact sets $H_s \subseteq X$ and $K_s \subseteq T$ such that $f(H_s \times K_s) \subseteq U$ and s is in the interior of $H_s \times K_s$. Thanks to compactness, we can find two finite sequences of compact sets $H_1, \ldots, H_n \subseteq X$ and $K_1, \ldots, K_n \subseteq T$ such that $S \subseteq \bigcup_i H_i \times K_i$ and $f(H_i \times K_i) \subseteq U$; so $f \in M(H_1 \times K_1, U) \cap \ldots \cap M(H_n \times K_n, U) \subseteq M(S, U)$. $\square$

It may be of some interest to compare the compact-open topology with other topologies commonly defined on function spaces: there are classical results which show that if the topological space X has some reasonable properties, the compact open topology is closely related to the topologies of pointwise convercence and of uniform convergence on compact sets. Hereunder such results are stated without proofs (see [**20**, VII, Th. 1-13]).

Theorem 1.1.5. *The compact-open topology $\mathcal{C}$ contains the topology of pointwise convergence. The topological space* $\mathbf{Top}(X, Y)$ *is Hausdorff if Y is Hausdorff.*

Theorem 1.1.6. *Let F be a family of continuous functions from a topological space X to a metric space Y. Then the topology of uniform convergence on compact sets is the compact-open topology.*

Let $I = [0,1]$ be the unit interval, let $\mathcal{P}X = \mathbf{Top}(I, X)$ denote the space of paths $\alpha\colon I \to X$ and c_x denote the constant path at x. Exploiting Definition 1.1.1, if $\mathscr{B}$ is a basis for the topology of X which is closed under finite intersection, sets of the form $\bigcap_{j=1}^{n} M\left(K_n^j, U_j\right)$, where $K_n^j = \left[\frac{j-1}{n}, \frac{j}{n}\right]$ and $U_j \in \mathscr{B}$, form a basis for the compact-open topology on $\mathcal{P}X$. For any fixed, closed subinterval $A \subseteq I$, let $T_A\colon I \to A$ be the unique increasing homeomorphism. For a path $\alpha \in \mathcal{P}X$ the restricted path of α to A is $\alpha_A := \alpha|_A \circ T_A\colon I \to A \to X$. As a convention if $A = \{t\}$, we let $\alpha_A = c_{\alpha(t)}$. Clearly if $0 = t_0 \leq t_1 \leq \ldots \leq t_n = 1$, knowing the paths $\alpha_{[t_{i-1}, t_i]}$ for $i = 1, \ldots, n$, uniquely determines α. With these notations one can easily define concatenations of paths: if $\alpha_1, \ldots, \alpha_n \in \mathcal{P}X$ are such that $\alpha_j(1) = \alpha_{j+1}(0)$ for all $j = 1, \ldots, n-1$, the *n-fold concatenation* of this sequence is the unique path $\beta = \alpha_1 * \ldots * \alpha$ such that $\beta_{K_n^j} = \alpha_j$ for all $j = 1, \ldots, n$. Since the space $\mathcal{P}X$ is endowed with the compact-open topology, the *concatenation morphism* $\mathcal{P}X \times_X \mathcal{P}X := \{(\alpha, \beta) : \alpha(1) = \beta(0)\} \to \mathcal{P}X$, $(\alpha, \beta) \mapsto \alpha * \beta$ is continuous. If $\alpha \in \mathcal{P}X$, $\alpha^{-1}(t) := \alpha(1-t)$ is the *reverse* of α; the *reversing morphism* $\mathcal{P}X \to \mathcal{P}X$, $\alpha \mapsto \alpha^{-1}$ is a self-homeomorphism of $\mathcal{P}X$.

We will use also the following notations: if $A \subseteq X$ and $B \subseteq Y$, $M(X, A; Y, B) \subseteq M(X, Y)$ is the subspace of maps $f\colon X \to Y$ such that $f(A) \subseteq B$ and if (X, x) and (Y, y) are based spaces, $M_*(X, x; Y, y)$ is the subspace of $M(X, Y)$ preserving base points. With this notation we can define $\Omega(X, x) := M_*\left(\mathbb{S}^1, (1,0); X, x\right)$, which is nothing but the space of loops based at x. It is clear also that $\Omega(X, x) = M_*(I, \{1, 0\}; X, \{x\})$.

Example

Since we will exploit the compact-open topology all along this article, it is worth describing explicitly the shape of an open neighborhood of a point $p \in M(I, X)$. Such a point is a path $p\colon I \to X$ and an open neighborhood of p must be the intersection of finitely many sets of the form $M\left(K_n^j, U_j\right)$ where K_n^j and U_j have been defined above. For instance, fix $n = 4$ and consider $p\colon I \to \mathbb{R}^2$, $t \mapsto e^{2\pi i t}$ (it is just the circle in $\mathbb{R}^2$). Choose U_j, $j = 1, \ldots, 4$, to be the squares of sides with lenght 2, of centers $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$ and with sides parallel to the axes. For sure $U_1 \ldots, U_4$ belong to a basis for the standard topology of $\mathbb{R}^2$. Then $\mathcal{U} := \bigcap_{j=1}^{4} M\left(K_n^j, U_j\right)$ is a (very large) open neighborhood of p in $M(I, X)$: for example $p'\colon t \mapsto \frac{1}{2}e^{2\pi i t}$ lies in this neighborhood, while $p''\colon t \mapsto e^{2\pi i t} + 10 + 10i$ does not. Making a drawing of all this may help in having an intuitive idea of this topology.

Let $\mathcal{U} = \bigcap_{j=1}^{n} M\left(K_n^j, U_j\right)$ be a basic open neighborhood of a path p in $M(I,X)$. Then $\mathcal{U}_A := \bigcap_{A\cap K_n^j\neq\emptyset} M\left(T_A^{-1}\left(A\cap K_n^j\right), U_j\right)$ is a basic open neighborhood of p_A, called the *restricted neighborhood* of $\mathcal{U}$ to A. If $A=\{t\}$ is a singleton, then $\mathcal{U}_A = \bigcap_{t\in K_n^j} M(I,U_j) = M\left(I, \bigcap_{t\in K_n^j} U_j\right)$. On the other hand if $p=q_A$ for some path $q\in M(I,X)$, then $\mathcal{U}^A := \bigcap_{j=1}^{n} M\left(T_A\left(K_n^j\right), U_j\right)$ is a basic open neighborhood of q called *induced neighborhood* of $\mathcal{U}$ on A. If $A=\{t\}$ is a singleton, then p_A is the constant path and we let $\mathcal{U}^A = \bigcap_{j=1}^{n} M(\{t\}, U_j)$.

1.2 COVERINGS OF GROUPOIDS

Definition 1.2.1. A small category $\mathcal{G}$ is called a *groupoid* whenever all its morphisms are invertible, in particular each hom-set of the form $\mathcal{G}(x,x) =: \mathcal{G}(x)$ is a group. A groupoid $\mathcal{G}$ with no empty hom-sets is called *connected.*

Remark 1.2.2. Any group G may be seen as a groupoid $\mathcal{G}$ with one object $\{\star\}$ and arrows $\mathcal{G}(\star) := G$; such a groupoid $\mathcal{G}$ is connected. Conversely any connected groupoid is equivalent to a groupoid with one object.

The following lemma will be implicitly used throughout the whole article:

Lemma 1.2.3. *Let $a\colon x\to x'$ and $b\colon y\to y'$ be arrows in a groupoid $\mathcal{G}$. Then there exists a bijection $\phi\colon \mathcal{G}(x,y)\to\mathcal{G}(x',y')$ which, if $x=y$ and $x'=y'$, can be chosen to be an isomorphism of groups.*

Proof. We define:

$$\phi\colon \mathcal{G}(x,y)\to\mathcal{G}(x',y'),\ c\mapsto bca^{-1}$$
$$\psi\colon \mathcal{G}(x',y')\to\mathcal{G}(x,y),\ d\mapsto b^{-1}da.$$

Clearly $\phi\circ\psi$ and $\psi\circ\phi$ are the identities and so ϕ is a bijection. If $x=y$ and $x'=y'$, let $a=b$ so that ϕ sends c to aca^{-1}. If $c,c'\in\mathcal{G}(x,x)$, then $\phi c\circ\phi c' = aca^{-1}ac'a^{-1} = acc'a^{-1} = \phi(cc')$, thus ϕ is an isomorphism. □

In particular the vertex hom-sets $\mathcal{G}(x)$ of any connected groupoid $\mathcal{G}$ are all isomorphic as groups.

Remark 1.2.4. For a groupoid $\mathcal{G}$, the *category of representations of $\mathcal{G}$ in* **Set** is

$$\mathbf{Fct}(\mathcal{G},\mathbf{Set}).$$

Identifying a group G with a groupoid as in Remark 1.2.2, we recover the usual category of representations of G in **Set** (i.e. of sets with left G-actions), indeed: $\mathbf{Fct}(\mathcal{G},\mathbf{Set})$ may conveniently be seen as the category whose objects are pairs of the form (S,μ_s), where S is a set and $\mu_S\in\mathbf{Grp}\left(G,S^S\right)$ is a map such that $\mu_S(1)(s)=s$ for all $s\in S$ and $\mu_S(g_1g_2)(x)=\mu_S(g_1)(\mu_S(g_2)(x))$; morphisms in $\mathbf{Fct}(\mathcal{G},\mathbf{Set})$ are of the form $\mu_f\colon (S,\mu_S)\to(T,\mu_T)$ where $f\colon S\to T$ is a map of sets such that $\mu_T(g)\circ f = f\circ\mu_S(g)$ for all $g\in G$. Notice that $\mathbf{Fct}(\mathcal{G},\mathbf{Set})$ always contains the full subcategory **Set**, identified to the *trivial* representations of G (i.e. representations of the form (S,id_S) where $id_S(g)(s)=s$ for all $g\in G$).

Let $\mathcal{G}$ be a groupoid. For each object $x\in\mathcal{G}$, let the *star* of x in $\mathcal{G}$, denoted by $\mathcal{G}_x$, be the union of the hom-sets $\mathcal{G}(x,y)$ for all objects $y\in\mathcal{G}$. Thus $\mathcal{G}_x$ consists of all arrows of $\mathcal{G}$ starting from the object x.

Definition 1.2.5. Let $p\colon \mathcal{H} \to \mathcal{G}$ be a morphism of groupoids. We say that p is a *covering morphism* if the map $\mathcal{H}_x \to \mathcal{G}_{p(x)}$, $h \mapsto p(h)$, is a bijection for each object $x \in \mathcal{H}$. If $g \in \mathcal{G}_{p(x)}$, then $\tilde{g}_x$ denotes the unique morphism in $\mathcal{H}_x$ such that $p(\tilde{g}_x) = g$. If $\mathcal{G}$ is a groupoid, $\mathbf{CovMor}(\mathcal{G})$ denotes the category of covering morphisms of $\mathcal{G}$; morphisms in $\mathbf{CovMor}(\mathcal{G})$ are the obvious commuting triangles of functors.

For example if $\mathcal{G}$ is a groupoid with just one element (which makes it into a group) the only covering morphisms are isomorphisms.

Remark 1.2.6. It is immediate from the definition that if $ob(\mathcal{H}) \neq \emptyset$, $\mathcal{G}$ is connected and $p\colon \mathcal{H} \to \mathcal{G}$ is a coverig morphism of groupoids, than p must be surjective on objects.

Remark 1.2.7. Let $p\colon \mathcal{H} \to \mathcal{G}$ be any morphism of groupoids: it is clear that $p(\mathcal{H}(x))$ is a subgroup of $\mathcal{G}(p(x))$ for each object $x \in \mathcal{H}$, but if p is a covering morphism, $p(\mathcal{H}(x))$ is mapped isomorphically onto $\mathcal{G}(p(x))$. On the other hand, if p is a covering morphism and $x, y \in ob(\mathcal{H})$, $x \neq y$, it follows straightforward from the definition that the induced map $p\colon \mathcal{H}(x,y) \to \mathcal{G}(p(x), p(y))$ is injective, but in general it is not surjective as shown in the following example.

Example: a covering morphism of groupoids

Let I be the groupoid with two objects $\{0_I, 1_I\}$ and just two morphisms other than identities, namely $I(0_I, 1_I) = \{i\}$ and $I(1_I, 0_I) = \{i^{-1}\}$; let $\mathbb{Z}_2$ be the groupoid with one object $\{\star\}$ and two morphisms $\mathbb{Z}_2(\star,\star) = \{0,1\}$ (actually, $\mathbb{Z}_2$ is the usual additive cyclic group with two elements). Finally define the functor $p\colon I \to \mathbb{Z}_2$ by $p(0_I) = p(1_I) = \star$ on objects and $p(id_{0_I}) = p(id_{1_I}) = 0$, $p(i) = p(i^{-1}) = 1$ on morphisms. One checks easily that:

- $I_{0_I} = \{id_{0_I}, i\} \to \{0,1\} = \mathbb{Z}_{2_0}$ is bijective
- $I_{1_I} = \{id_{1_I}, i^{-1}\} \to \{0,1\} = \mathbb{Z}_{2_1}$ is bijective

and hence p is a covering morphism of groupoids. But $I(0_I, 1_I) = \{i\} \to \{0,1\} = \mathbb{Z}_2(\star,\star)$ is clearly not surjective.

The following is a standard result (for details, see [**16**, Prop. 30]):

Theorem 1.2.8 ($\mathscr{R}$-EQUIVALENCE)**.** *For any groupoid $\mathcal{G}$, there is an equivalence of categories*

$$\mathscr{R}\colon \mathbf{CovMor}(\mathcal{G}) \simeq \mathbf{Fct}(\mathcal{G}, \mathbf{Set}).$$

Let us recall briefly how this equivalence is build: a covering morphism $F\colon \mathcal{H} \to \mathcal{G}$ corresponds to the functor $\mathscr{R}F\colon \mathcal{G} \to \mathbf{Set}$ given by $\mathscr{R}F(x) = F_{ob}^{-1}(x)$ on objects $x \in ob(\mathcal{G})$ and for $g \in \mathcal{G}(x_1, x_2)$, $\mathscr{R}F(g)$ is the function $F_{ob}^{-1}(x_1) \to F_{ob}^{-1}(x_2)$, $y \mapsto t(\tilde{g}_y)$ where $t\colon \mathcal{H} \to ob(\mathcal{H})$ is the target map of $\mathcal{H}$.

The inverse $\mathscr{R}^{-1}\colon \mathbf{Fct}(\mathcal{G}, \mathbf{Set}) \to \mathbf{CovMor}(\mathcal{G})$ may be described as follows: given a functor $N\colon \mathcal{G} \to \mathbf{Set}$, let $\mathcal{H}$ be the groupoid with objects set $\bigsqcup_{x \in ob(\mathcal{G})} N(x)$ and if $y_i \in N(x_i)$, $i = 1,2$, then $\mathcal{H}(y_1, y_2) = \{g \in \mathcal{G}(x_1, x_2) : N(g)(y_1) = y_2\}$. The functor $\mathscr{R}^{-1}N\colon \mathcal{H} \to \mathcal{G}$ taking $y \in N(x)$ to x and which is the inclusion on hom-sets is the corresponding covering morphism.

1.3 MONODROMY OF FIBRATIONS

Definition 1.3.1. Let X be a topological space. Let us define $(\mathcal{P}X)_x := \{\alpha \in \mathcal{P}X : \alpha(0) = x\}$, $(\mathcal{P}X)^y := \{\alpha \in \mathcal{P}X : \alpha(1) = y\}$, $\mathcal{P}X(x,y) = (\mathcal{P}X)_x \cap (\mathcal{P}X)^y = \{\alpha \in \mathcal{P}X : \alpha(0) = x \text{ and } \alpha(1) = y\}$ and $\Omega(X,x) :=$

$\mathcal{P}X(x,x)$.

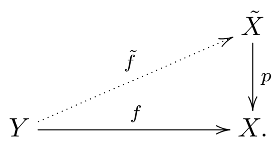
In what follows we heavily exploit the well known notions of *homotopy* of paths and maps: as usual the writing $\alpha \sim \beta$ means that α and β are homotopic paths or maps (for definitions see, for example, [**24**, II]) and $[\alpha]_\sim$ is the equivalence class of α; unless otherwise specified, we will not distinguish a path α from its equivalence class $[\alpha]_\sim$.

Definition 1.3.2. Let X be a topological space. The set of endpoint preserving homotopies of paths starting at x is denoted by $(\mathcal{H}X)_x$.

Both $\mathcal{P}$ and $\mathcal{H}$ may be seen as functors $\mathbf{Top}_* \to \mathbf{Top}_*$: for example $(X,x) \mapsto ((\mathcal{P}X)_x, c_x)$ is a functor which is $\mathcal{P}f(\alpha) = f \circ \alpha$ on morphisms.

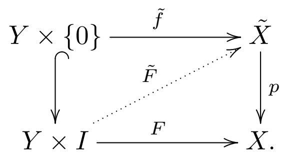
1.3.1 Fibrations and liftings

Let $p\colon \tilde{X} \to X$ and $f\colon Y \to X$ be continuous maps. The *lifting problem* for f is to determine whether there is a continuous map $\tilde{f}\colon Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$, i.e. if the dotted arrow of the following diagram corresponds to a continuous map making the diagram commutative:



If it exists, $\tilde{f}$ is called a *lifting* of f.

Definition 1.3.3 (HOMOTOPY LIFTING PROPERTY)**.** A map $p\colon \tilde{X} \to X$ is said to have the *homotopy lifting property* with respect to a space Y if, given maps $\tilde{f}\colon Y \to \tilde{X}$ and $F\colon Y \times I \to X$ such that $F(y,0) = p \circ \tilde{f}(y)$ for each $y \in Y$, there is a map $\tilde{F}\colon Y \times I \to \tilde{X}$ such that $\tilde{F}(y,0) = \tilde{f}(y)$ for each $y \in Y$ and $p \circ \tilde{F} = F$. Pictorially, we are asking the existence of the dotted arrow making the following diagram commutative:



Remark 1.3.4. If $f, g\colon Y \to X$ are homotopic maps and $p\colon \tilde{X} \to X$ has the homotopy lifting property, f can be lifted if and only if g can be lifted. Hence, whether or not a map $Y \to X$ can be lifted is a property of the homotopy class of the map.

In what follows we denote by I^n the topological closed cube of dimension n.

Definition 1.3.5. The map $p\colon \tilde{X} \to X$ is called a *(Serre or weak) fibration* if p satisfies the Homotopy Lifting Property 1.3.3 with respect to cubes I^n of all dimensions.

For example if $p\colon \tilde{X} \to X$ is a Serre fibration, any path $\alpha \subseteq p\left(\tilde{X}\right)$ can be lifted to a path in $\tilde{X}$, indeed: α can be seen as a homotpy $\alpha\colon P \times I \to X$ where P is a one-point space (in practise $P = I^0$) and a point $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = \alpha(0)$ corresponds to a map $\tilde{f}\colon P \to \tilde{X}$ such that $p \circ \tilde{f}(P) = \alpha(P,0)$. Since p is a Serre fibration, there exists a path $\tilde{\alpha}$ in $\tilde{X}$ such that $\tilde{\alpha}(0) = \tilde{x}_0$ and $p \circ \tilde{\alpha} = \alpha$. So $\tilde{\alpha}$ is the lifting of α.

As usual, for each point $x \in X$, the set $p^{-1}(x) \subseteq \tilde{X}$ is called *fiber*.

We are interested in these objects since both covering and semicovering maps, which we are to define, are Serre fibrations.

Definition 1.3.6. A continuous map $p\colon \tilde{X} \to X$ is said to have *unique path lifting* if, given paths α and α' in $\tilde{X}$ such that $p \circ \alpha = p \circ \alpha'$ and $\alpha(0) = \alpha'(0)$, then $\alpha = \alpha'$.

Lemma 1.3.7. *Let $p\colon \tilde{X} \to X$ be a continuous map with unique path lifting and Y be a path connected space. If $f, g\colon Y \to \tilde{X}$ are maps such that $p \circ f = p \circ g$ and $f(y_0) = g(y_0)$ for some $y_0 \in Y$, then $f = g$ for all $y \in Y$.*

Proof. With the above hypothesis, let $y \in Y$ and $\alpha \in \mathcal{P}Y(y_0, y)$. Then $f \circ \alpha$ and $g \circ \alpha$ are paths in $\tilde{X}$ that are liftings of the same path in X (namely, of the path $p \circ f \circ \alpha = p \circ g \circ \alpha$) and have the same origin. Because p has unique path lifting, $f \circ \alpha = g \circ \alpha$. So $f(y) = (f \circ \alpha)(1) = (g \circ \alpha)(1) = g(y)$. □

The following theorem characterizes fibrations with unique path lifting, the proof may be found in [**30**, II.2.5]:

Theorem 1.3.8. *For a fibration $p\colon \tilde{X} \to X$, the following are equivalent:*

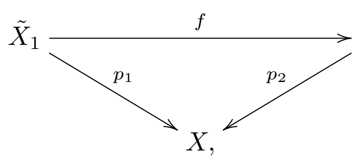
1. *p has unique path lifting.*
2. *The fibers of p are totally path-disconnected (i.e. the path connected components of the fibers are the points).*

Moreover ([**30**, II.2.6]):

Theorem 1.3.9. *The composite of fibrations (with unique path lifting) is a fibration (with unique path lifting).*

Thanks to this theorem we gain the following:

Definition 1.3.10. For any topological space X, we denote by $\mathbf{Fib}^{!}(X)$ the category whose objects are fibrations with unique path lifting over X and whose morphisms are the commuting triangles



where p_i are fibrations with unique path lifting and f is required to be continuous.

1.3.2 The fundamental groupoid

Let X be a topological space: recall from Section 1.1 that each path in X admits an inverse, so the following definition makes sense.

Definition 1.3.11. Given a topological space X, the *fundamental groupoid* $\Pi_1 X$ of X is the category whose objects are the points of X and whose morphisms $\Pi_1 X(x, y)$ are the path classes from x to y; composition is defined by the opposite of concatenation of paths.

If $f\colon X\to Y$ is a continuous map between topological spaces, by defining $\Pi_1 f\colon \Pi_1 X\to\Pi_1 Y$, $[\alpha]\mapsto[f\circ\alpha]$, it follows straightforward from the definition that the fundamental groupoid $\Pi_1\colon \mathbf{Top}\to\mathbf{Grpd}$ is a functor (cfr. [**8**, Th. 6.4.2] for the details). Let us remark also that if we choose $x=y$, then $\Pi_1 X(x,x)$ is a group which is usually referred to as *fundamental group* of X in x and is denoted by $\pi_1(X,x)$ or $\pi_1(X)$ if fundamental groups are isomorphic for all $x\in X$.

Examples

1. If X consists of a single point x, then $\Pi_1 X$ has a single object and $\Pi_1 X(x,x)$ consists only of the zero-path class. More generally , if the path-components of X consist of single points, then

$$\Pi_1 X(x,y)=\begin{cases}\emptyset & \text{if } x\neq y\\ \{c_x\} & \text{if } x=y\end{cases}.$$

A groupoid with this property is called *discrete.*

2. Let X be a convex subset of a normed vector space and let α,β be two paths in X from x to y. Then α,β are homotopic since $h\colon I\times I\to X$, $(t,s)\mapsto(1-s)\,\alpha(t)+\beta(t)$ is a homotopy $\alpha\sim\beta$. So any two paths from x to y are equivalent and $\Pi_1 X(x,y)$ has exactly one element for all x,y in X. A groupoid with this property is called *1-connected.* If $\Pi_1 X$ is a 1-connected groupoid, the space X is said 1-connected too. Clearly if X is 1-connected it is also path-connected.

3. If X has more than one connected component, the hom-set $\Pi_1 X(x,y)$ is possibly empty. So the fundamental groupoid of X is connected if and only if X is path connected.

Remark 1.3.12. If X is path connected, by applying Lemma 1.2.3 to the fundamental groupoid $\Pi_1 X$, we get that if α is a path from x to y in X, then α determines an isomorphism $\alpha_X\colon\pi_1(X,x)\to\pi_1(X,y)$, $\beta\mapsto\alpha*\beta*\alpha^{-1}$ of fundamental groups (but such an isomorphism is not canonical). Hence all groups $\Pi_1 X(x)$, $x\in X$, are isomorphic, and the groupoid $\Pi_1 X$ is equivalent to the group $\pi_1(X,x_0)$ identified to the category with one object $\{x_0\}$ and morphisms $\pi_1(X,x_0)$ (cfr. Remark 1.2.2).

We sum up in the following theorem two classical results linking homotopies and fundamental groupoids (see [**8**, VI.5, *passim*] for proofs):

Theorem 1.3.13. *Let X,Y be topological spaces. If $X\to Y$ is a homotopy equivalence of topological spaces, then $\Pi_1 f\colon\Pi_1 X\to\Pi_1 Y$ is an equivalence of groupoids. In particular for each $x\in X$ the map $\pi_1(f):\pi_1(X,x)\to\pi_1(Y,f(x))$ is a group homomorphism.*

1.3.3 Monodromy representations

In this section we recall some classical results in order to describe the covering groupoid functor Π_1 and the monodromy functor μ: lacking proofs may be found in [**30**, II.3 and II.4, *passim*].

Lemma 1.3.14. *Let $p\colon\tilde X\to X$ be a fibration. If $\tilde A$ is any path component of $\tilde X$, then $p\left(\tilde A\right)$ is a path component of X and $p|_{\tilde A}\colon\tilde A\to p\left(\tilde A\right)$ is a fibration.*

Proposition 1.3.15. *Let $p\colon\tilde X\to X$ be a fibration with unique path lifting, let A be a subset of X and let $\tilde A=p^{-1}(A)$. Then the induced morphism $\Pi_1 p\colon\Pi_1\tilde A\to\Pi_1 A$ is a covering morphism of groupoids.*

Proof. Let $\tilde{x} \in \tilde{A}$ and let $p(\tilde{x}) = x$. For each path α in X with initial point x, let $\tilde{\alpha}$ denote the unique path of $\tilde{X}$ with initial point $\tilde{x}$. If the final point of α is in A, then the final point of $\tilde{\alpha}$ is in $\tilde{A}$. Also, the equivalence class of $\tilde{\alpha}$ depends only on the equivalence class of of α by the unique lifting property. So the mapping $[\alpha] \mapsto [\tilde{\alpha}]$ is inverse to the restriction of p which maps $\left(\Pi_1 \tilde{A}\right)_{\tilde{x}} \to (\Pi_1 A)_x$; this last map being a bijection, $\Pi_1 p$ is a covering morphism. □

Remark 1.3.16. For any topological space X, Proposition 1.3.15 provides us of a well defined functor

$$\Pi_1 \colon \mathbf{Fib}^! (X) \to \mathbf{CovMor} (\Pi_1 X) .$$

Lemma 1.3.17. *Let $p \colon \tilde{X} \to X$ be a fibration with unique path lifting. If α and α' are paths in $\tilde{X}$ such that $\alpha(0) = \alpha'(0)$ and $p \circ \alpha = p \circ \alpha'$, then $\alpha \sim \alpha'$.*

Remark 1.3.18. With this lemma we get that if $p \colon \tilde{X} \to X$ is a fibration with unique path lifting, for any two objects $\tilde{x}_0, \tilde{x}_1 \in \Pi_1 \tilde{X}$, the induced map $p_* \colon \Pi_1 \tilde{X}(\tilde{x}_0, \tilde{x}_1) \to \Pi_1 X (p(\tilde{x}_0), p(\tilde{x}_1))$ is injective and for $\tilde{x}_0 = \tilde{x}_1$ the group homomorphism $p_* \colon \pi_1 \left(\tilde{X}, \tilde{x}_0\right) \to \pi_1 (X, p(\tilde{x}_0))$ is a monomorphism.

Lemma 1.3.19. *Let $p \colon \tilde{X} \to X$ be a fibration with unique path lifting and assume $\tilde{X}$ is a nonempty connected space. If $\tilde{x}_0, \tilde{x}_1 \in \tilde{X}$, there is a path $\alpha \in \mathcal{P}X (p(\tilde{x}_0), p(\tilde{x}_1))$ such that*

$$p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0\right)\right) = \alpha * p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_1\right)\right) * \alpha^{-1}.$$

Conversely, given a path $\alpha \in \mathcal{P}X (p(\tilde{x}_0), x_1)$ there is a point $\tilde{x}_1 \in p^{-1}(x_1)$ such that

$$\alpha * p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_1\right)\right) * \alpha^{-1} = p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0\right)\right).$$

Proof. First, let $\tilde{\alpha} \in \mathcal{P}\tilde{X}(\tilde{x}_0, \tilde{x}_1)$. Then $\pi_1 \left(\tilde{X}, \tilde{x}_0\right) = \tilde{\alpha} * \pi_1 \left(\tilde{X}, \tilde{x}_1\right) * \tilde{\alpha}^{-1}$. Therefore

$$p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0\right)\right) = (p \circ \tilde{\alpha}) * p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_1\right)\right) * (p \circ \tilde{\alpha})^{-1}$$

and so $p \circ \tilde{\alpha}$ is the path α we looked for.

Conversely, given a path $\alpha \in \mathcal{P}X (p(\tilde{x}_0), x_1)$, let $\tilde{\alpha}$ be a path in $\tilde{X}$ such that $\tilde{\alpha}(0) = \tilde{x}_0$ and $p \circ \tilde{\alpha} = \alpha$. If $\tilde{x}_1 = \tilde{\alpha}(1)$, then

$$\alpha * p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_1\right)\right) * \alpha^{-1} = p_* \left(\tilde{\alpha} * \pi_1 \left(\tilde{X}, \tilde{x}_1\right) * \tilde{\alpha}^{-1}\right) = p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0\right)\right).$$

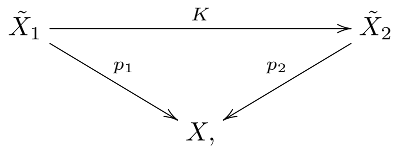
□

Thanks to Lemma 1.3.19 one immediately shows that:

Proposition 1.3.20. *Let $p \colon \tilde{X} \to X$ be a fibration with unique path lifting and assume $\tilde{X}$ is a nonempty connected space. For $x_0 \in p\left(\tilde{X}\right)$ the family $\left\{p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0\right)\right) : \tilde{x}_0 \in p^{-1}(x_0)\right\}$ is a conjugacy class in $\pi_1 (X, x_0)$.*

Suppose that $p \colon \tilde{X} \to X$ is a fibration with unique path lifting and let $\alpha \in \Pi_1 X (x_0, x_1)$. It is a standard exercise to show that the map $\mu(p)(\alpha) : p^{-1}(x_0) \to p^{-1}(x_1)$ defined by $\tilde{x}_0 \mapsto \tilde{\alpha}(1)$, where $\tilde{\alpha}$ is the unique lifting of α starting at $\tilde{x}_0$, is well defined. Hence we get:

Theorem 1.3.21. *Let* $p\colon \tilde{X} \to X$ *be a fibration with unique path lifting. Then there is a functor from* $\Pi_1 X$ *to the category* **Set***: it takes a point* $x \in X$ *to its fiber* $\mu(p)(x) = p^{-1}(x)$ *and it takes a path homotopy class* $\alpha \in \Pi_1 X(x_0, x_1)$ *to the function* $\mu(p)(\alpha) : p^{-1}(x_0) \to p^{-1}(x_1)$ *defined above. Concerning morphisms, if* K *is a morphism of fibrations* p_1, p_2*, i.e. this triangle commutes*



then $\mu(K)$ *is the induced natural transformation of functors defined by* $\mu(K)(x) \coloneqq K|_{p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(x)$ *which maps* $z \in p_1^{-1}(x)$ *to* $K(z) \in p_2^{-1}(x)$*.*

Remark 1.3.22. It is possible to show a stronger version of previous theorem:

For a fibration with unique path lifting $p\colon \tilde{X} \to X$, there is a functor $\Pi_1 X$ to the category **Top**: it takes a point $x \in X$ to its fiber $\mu(p)(x) = p^{-1}(x)$ and it takes a path homotopy class $\alpha \in \Pi_1 X(x_0, x_1)$ to the continuous function $\mu(p)(\alpha) : p^{-1}(x_0) \to p^{-1}(x_1)$ defined above.

Since we will consider fibrations with unique path lifting and discrete fibers, we will not need to take care of the topology induced on the fibers.

Definition 1.3.23. Theorem 1.3.21 provides the *monodromy functor*

$$\mu\colon \mathbf{Fib}^{!}(X) \to \mathbf{Fct}(\Pi_1 X, \mathbf{Set}).$$

For a given fibration $p \in \mathbf{Fib}^{!}(X)$, $\mu(p)$ is called *monodromy of p*.

Theorem 1.2.8 immediately gives the equivalence

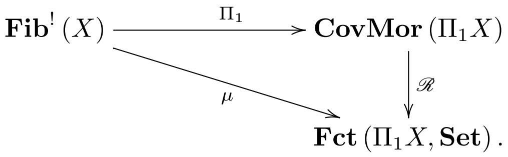
$$\mathscr{R}\colon \mathbf{CovMor}(\Pi_1 X) \simeq \mathbf{Fct}(\Pi_1 X, \mathbf{Set}).$$

In this case a covering morphism $F\colon \mathcal{H} \to \Pi_1 X$ corresponds to the functor $\mathscr{R}F\colon \Pi_1 X \to \mathbf{Set}$ given by $\mathscr{R}F(x) = F_{ob}^{-1}(x)$ on points $x \in X$ and for a path $\alpha \in \Pi_1 X(x_1, x_2)$, $\mathscr{R}F(\alpha)$ is the function $F_{ob}^{-1}(x_1) \to F_{ob}^{-1}(x_2)$, $y \mapsto t(\tilde{\alpha}_y)$ where $\tilde{\alpha}_y$ is the unique morphism in $\mathcal{H}$ whose source is y and such that $F(\tilde{\alpha}_y) = \alpha$ (it exists and is unique because F is a covering morphism of groupoids).

Conversely, given a functor $N\colon \Pi_1 X \to \mathbf{Set}$, let $\mathcal{H}$ be the groupoid with objects set $\bigsqcup_{x \in X} N(x)$ and if $y_i \in N(x_i)$, $i = 1, 2$, then $\mathcal{H}(y_1, y_2) = \{\alpha \in \Pi_1 X(x_1, x_2) : \ N(\alpha)(y_1) = y_2\}$. The functor $\mathscr{R}^{-1}N\colon \mathcal{H} \to \Pi_1 X$ taking $y \in N(x)$ to x and which is the inclusion on hom-sets is the corresponding covering morphism.

Hence we get:

Theorem 1.3.24. *For any topological space* X*, the following diagram quasi-commutes:*



Thus any fibration $p\colon \tilde{X} \to X$ is characterized either by its corresponding covering morphism $\Pi_1 \tilde{X} \to \Pi_1 X$ or by its corresponding monodromy functor $\Pi_1 X \to \mathbf{Set}$.

Definition 1.3.25. If X and Y are topological spaces, a *covering map* is a continuous function $p\colon Y \to X$ with the property that each point of X has an open neighborhood N such that $p^{-1}(N)$ is a disjoint union of open sets, each of which is mapped homeomorphically by p onto N (if N is connected, these must be the components of $p^{-1}(N)$). Such a N is called *canonical* (or *fundamental*) *neighborhood* and Y is called *covering space* of X. A *morphism* between coverings $p\colon Y \to X$, $p'\colon Y' \to X$ is a continuous map $\phi\colon Y \to Y'$ such that $p' \circ \phi = p$.

With such definitions there remains defined the category $\mathbf{Cov}(X)$ of coverings of X. Two coverings are called *equivalent* if they are isomorphic in this category. A covering $p\colon Y \to X$ is *connected* if Y is non-empty and path-connected: let $\mathbf{Cov_0}(X)$ be the full subcategory of connected coverings of X. If p is an initial object in $\mathbf{Cov_0}(X)$, we call p a *universal covering* of X.

It is possible to give another (equivalent) definition of covering space which better explains the *local triviality* of such a map:

Let S be a set endowed with the discrete topology. Then $X \times S \simeq \bigsqcup_{s\in S} X_s$ where $X_s := X \times \{s\}$ is a copy of X and each X_s is open.

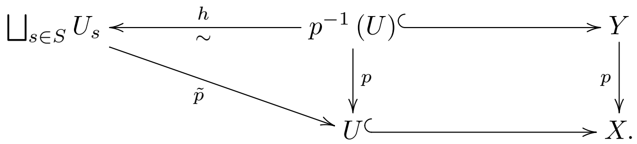
Definition 1.3.26.

1. A continuous map $p\colon Y \to X$ is a *trivial covering* if there exist a non-empty set S (endowed with the discrete topology) and a homeomorphism $h : Y \to X \times S$ such that $p = \tilde{p} \circ h$ where $\tilde{p}\colon X \times S \to X$ is the standard projection.

2. A continuous map $p\colon Y \to X$ is a *covering* if p is surjective and any $x \in X$ has an open neighborhood U such that $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$ is a trivial covering.

3. If $p\colon Y \to X$ is a covering, a *section* u of p is a continuous map $u\colon X \to Y$ such that $p \circ u = id_X$. A *local section* is defined in the obvious way in an open subset U of X.

Remark 1.3.27. So, roughly speaking, a covering is always locally isomorphic to a trivial covering. Pictorially, a covering is visualized by this commuting diagram:



Remark 1.3.28. A covering map is always a local homeomorphism and hence it has always discrete fibers ([**23**, 12.3 and 12.12]).

As shown in [**30**, II.2-5], we have the following:

Theorem 1.3.29. *If $p\colon Y \to X$ is a covering map, then p is a fibration with unique path lifting.*

Notice that if $p\colon Y \to X$ is a covering and X is locally path connected, then so is Y since p is a local homeomorphism. The proof of the following theorem may be found in [**30**, II.4.10]:

Theorem 1.3.30. *Every fibration with unique path lifting $p\colon Y \to X$ with X locally path connected and semilocally 1-connected and Y locally path connected is a covering projection.*

Moreover, Theorem 1.3.24 becomes:

Theorem 1.3.31. *If X is locally path connected and semilocally 1-connected, all arrows in the following quasi-commutative diagram are equivalences:*

$$\begin{array}{ccc} \mathbf{Cov}(X) & \xrightarrow[\sim]{\Pi_1} & \mathbf{CovMor}(\Pi_1 X) \\ & \searrow^{\sim}_{\mu} & \wr\downarrow \mathscr{R} \\ & & \mathbf{Fct}(\Pi_1 X, \mathbf{Set}). \end{array}$$

We devote the remaing of this section to prove it; in next chapters we will develop tools exactly to generalize it.

In order to show the Π_1-equivalence we need some preliminary considerations.

Proposition 1.3.32. *Let $p\colon Y \to X$ be a covering map, let A be a subset of X and let $\tilde{A} = p^{-1}(A)$. Then the induced morphism $\Pi_1 p\colon \Pi_1\tilde{A} \to \Pi_1 A$ is a covering morphism of groupoids.*

Proof. Since p is a fibration, the result follows from Proposition 1.3.15. □

Let $q\colon \mathcal{G} \to \Pi_1 X$ be a covering morphism of the fundamental groupoid $\Pi_1 X$. Define $\tilde{X} \coloneqq ob(\mathcal{G})$ and $p \coloneqq q_{ob}\colon \tilde{X} \to X$. We use q to "lift" the topology of X to a topology on $\tilde{X}$. Let $\mathcal{U}$ be the family of all open, path connected subsets of X. If $U \in \mathcal{U}$, consider the diagram

$$\begin{array}{ccc} & & \mathcal{G} \\ & \overset{\tilde{i}}{\nearrow} & \downarrow q \\ \Pi_1 U & \xhookrightarrow{i} & \Pi_1 X, \end{array}$$

where i is induced by inclusion. Then i lifts to a unique morphism $\tilde{i}\colon \Pi_1 U \to \mathcal{G}$ (see [**8**, 10.3.3]). The set $\tilde{i}(U)$ is a subset $\tilde{U}$ of $\tilde{X}$ which we call *lifting of U*. As shown in [**8**, 10.5.2-4]:

Lemma 1.3.33. *With the above notations:*

1. *The set $\tilde{\mathcal{U}}$ of all liftings $\tilde{U}$ for $U \in \mathcal{U}$ is a base for the topology of $\tilde{X}$.*

2. *If $f\colon Z \to X$ is a continuous map and $\Pi_1 f\colon \Pi_1 Z \to \Pi_1 X$ lifts to a morphism of groupoids $f'\colon \Pi_1 Z \to \mathcal{G}$, then $\tilde{f} = f'_{ob}\colon Z \to \tilde{X}$ is continuous and is a lifting of f.*

3. *If $p\colon \tilde{X} \to X$ is a covering map, the topology of $\tilde{X}$ is that of X lifted by $\Pi_1 p\colon \Pi_1\tilde{X} \to \Pi_1 X$.*

Moreover:

Proposition 1.3.34. *The* lifted topology *is the only topology on $\tilde{X}$ such that:*

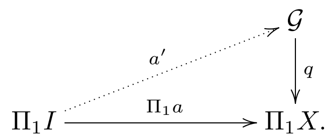
1. *$p\colon \tilde{X} \to X$ is a covering map.*

2. *There is an isomorphism $r_{\mathcal{G}}\colon \mathcal{G} \to \Pi_1\tilde{X}$ which is the identity on objects and such that the following diagram quasi-commutes:*

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{r_{\mathcal{G}}} & \Pi_1\tilde{X} \\ & \searrow_{q} & \downarrow \Pi_1 p \\ & & \Pi_1 X. \end{array}$$

In particular $r_{\mathcal{G}}$ itself is a covering morphism.

For the complete proof, see [**8**, 10.5.5]. Here we only remember how $r_{\mathcal{G}}$ is built: it is the identity on objects; concerning morphisms, let $\alpha \in \mathcal{G}(\tilde{x}, \tilde{y})$ and suppose $q(\alpha) \in \Pi_1 X(x, y)$. Let $a\colon I \to X$ be a path in the homotopy class of $q(\alpha)$; then a induces a morphism $\Pi_1 a\colon \Pi_1 I \to \Pi_1 X$ such that $(\Pi_1 a)(\star) = q(\alpha)$, where $\star$ is the unique element of $\Pi_1 I(0, 1)$. Since I is 1-connected, $\Pi_1 a$ lifts uniquely to a pointed morphism $a'\colon (\Pi_1 I, 0) \to (\mathcal{G}, \tilde{x})$ (see [**8**, 10.3.3]); notice that $a'(\star) = \alpha$: pictorially one has



By (2) of Lemma 1.3.33, $\tilde{a} := a'_{ob}\colon I \to \tilde{X}$ is continuous and we can define $r_{\mathcal{G}}(\alpha) = [\tilde{a}]$.

Theorem 1.3.35 (Π_1-EQUIVALENCE). *Let X be a locally path connected and semilocally 1-connected space. Then the fundamental groupoid functor Π_1 induces an equivalence of categories*

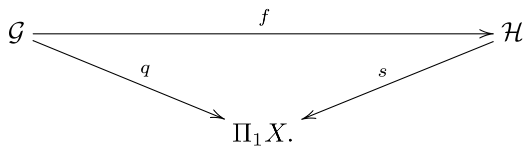
$$\Pi_1\colon \mathbf{Cov}(X) \simeq \mathbf{CovMor}(\Pi_1 X).$$

Proof. Recall that two categories $\mathcal{C}, \mathcal{D}$ are equivalent when there are two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G \simeq id_{\mathcal{D}}$ and $G \circ F \simeq id_{\mathcal{C}}$, i.e. there are natural transformations between $F \circ G$, $G \circ F$ and $id_{\mathcal{D}}$, $id_{\mathcal{C}}$ respectively.

Il $p\colon Y \to X$ is a covering map of topological spaces, then $\Pi_1 p\colon \Pi_1 Y \to \Pi_1 X$ is a covering morphism of groupoids by Proposition 1.3.32. To show Π_1 is an equivalence, we build a functor $\rho\colon \mathbf{CovMor}(\Pi_1 X) \to \mathbf{Cov}(X)$ and prove that we can find natural transformations of functors $id_{\mathbf{CovMor}(\Pi_1 X)} \simeq \Pi_1 \circ \rho$ and $id_{\mathbf{Cov}(X)} \simeq \rho \circ \Pi_1$.

Let $q\colon \mathcal{G} \to \Pi_1 X$ be a covering morphism of groupoids. As above, let $\tilde{X} := ob(\mathcal{G})$ and let $p := q_{ob}\colon \tilde{X} \to X$. By Proposition 1.3.34, there is a topology on $\tilde{X}$ making p into a covering map and there is an isomorphism $r_{\mathcal{G}}\colon \mathcal{G} \to \Pi_1 \tilde{X}$. This defines ρ on objects.

Now let $f\colon \mathcal{G} \to \mathcal{H}$ be a morphism in $\mathbf{CovMor}(\Pi_1 X)$, i.e. we have covering morphisms q, s such that this diagram quasi-commutes:



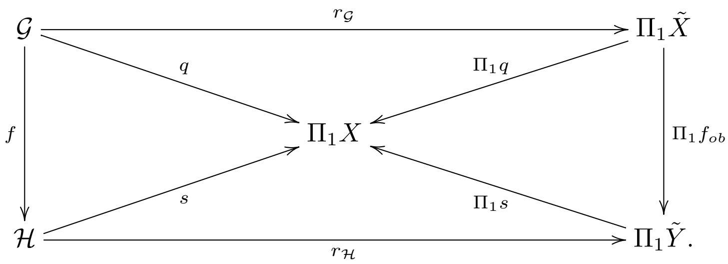
Let $\mathcal{U}$ be the family of all open, path connected subsets U of X such that the inclusion $\Pi_1 U \hookrightarrow \Pi_1 X$ is trivial (recall that X is semilocally 1-connected). As explained above, the cover $\mathcal{U}$ lifts to covers $\tilde{\mathcal{U}}_q$ and $\tilde{\mathcal{U}}_s$ of $\tilde{X} = ob(\mathcal{G})$ and $\tilde{Y} = ob(\mathcal{H})$ respectively. Let $\tilde{x} \in \tilde{X}$ and suppose $\tilde{U}_s \in \tilde{\mathcal{U}}_s$ is an element containing $f(\tilde{x})$. Then $U = s\left(\tilde{U}_s\right)$ belongs to $\mathcal{U}$ and U lifts to an element $\tilde{U}_q \in \tilde{\mathcal{U}}_q$ containing $\tilde{x}$. Moreover $f\left(\tilde{U}_q\right) = \tilde{U}_s$ by definition. So $f\colon \tilde{X} \to \tilde{Y}$ is continuous and the topology given to $\tilde{X}$ is compatible with all morphisms in $\mathbf{CovMor}(\Pi_1 X)$. This defines ρ on morphisms.

The natural transformation $id_{\mathbf{CovMor}(\Pi_1 X)} \simeq \Pi_1 \circ \rho$ is essentially given by the functor $r_{\mathcal{G}}\colon \mathcal{G} \to \Pi_1 \tilde{X}$

described in Proposition 1.3.34: we need to show that the following diagram commutes

$$\begin{array}{ccc} id_{\mathbf{CovMor}(\Pi_1 X)}\,(q) & \xrightarrow{\quad r_{\mathcal{G}} \quad} & \Pi_1 \circ \rho\,(q) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Pi_1 \circ \rho(f)} \\ id_{\mathbf{CovMor}(\Pi_1 X)}\,(s) & \xrightarrow{\quad r_{\mathcal{H}} \quad} & \Pi_1 \circ \rho\,(s)\,. \end{array}$$

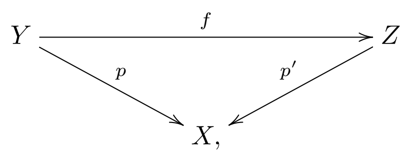
By applying the definitions, it reduces to show that this other diagram quasi-commutes (we are interested in the external square):



And indeed: let $\alpha \in \mathcal{G}_{\tilde{x}}$ and let $a\colon I \to X$ such that $a \in [q\,(\alpha)]$, where $q\,(\alpha) \in \Pi_1 X$ and the source of $q\,(\alpha)$ is x. Then a induces a morphism $\Pi_1 a\colon \Pi_1 I \to \Pi_1 X$ such that $(\Pi_1 a)\,(\star) = q\,(\alpha)$ (it is the same construction made also in Proposition 1.3.34). Moreover $\Pi_1 a$ lifts uniquely to a pointed morphism $a'\colon (\Pi_1 I, 0) \to (\mathcal{G}, \tilde{x})$. Then $r_{\mathcal{G}}\,(\alpha)$ is the path $ob\,(a') : I \to \tilde{X}$. Next, let $\beta = f\,(\alpha)$ and apply the same argument to find $b'\colon (\Pi_1 I, 0) \to (\mathcal{H}, f\,(\tilde{x}))$ where $b = f_{ob}\,(a)$. Since b' is uniquely determined by b it follows that $r_{\mathcal{H}}\,(f\,(\alpha)) = \Pi_1 f_{ob}\,(r_{\mathcal{G}}\,(\alpha))$.

Actually we have only shown that there is a natural transformation $id_{\mathbf{CovMor}(\Pi_1 X)} \Longrightarrow \Pi_1 \circ \rho$, but $r_{\mathcal{G}}$ and $r_{\mathcal{H}}$ are isomorphisms and those arrows may be reversed giving the inverse transformation.

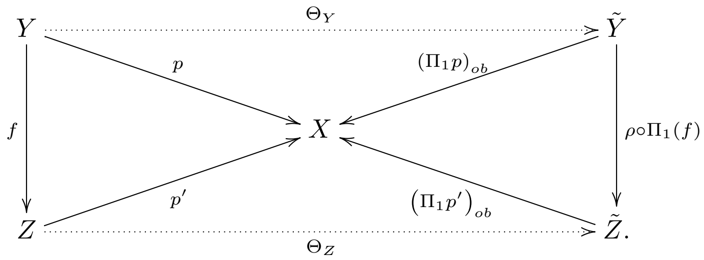
Finally we need a natural transformation $\Theta\colon id_{\mathbf{Cov}(X)} \simeq \rho \circ \Pi_1$. But $\tilde{X} = ob\left(\Pi_1 \tilde{X}\right)$ and the topology of $\tilde{X}$ is precisely the topology induced by a covering in $\mathbf{Cov}\,(X)$: indeed, if f is a morphism of covering maps of X, i.e. we have two covering maps p, p' making this diagram commute



notice that $\Pi_1 p\colon \Pi_1 Y \to \Pi_1 X$ is a covering morphism of groupoids, so $\rho\,(\Pi_1 p) = (\Pi_1 p)_{ob} : Y \to X$. Hence one wants a commuting diagram of the form

$$\begin{array}{ccc} id_{\mathbf{Cov}(X)}\,(p) & \xrightarrow{\quad \Theta_Y \quad} & \rho \circ \Pi_1\,(p) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \rho \circ \Pi_1(f)} \\ id_{\mathbf{Cov}(X)}\,(p') & \xrightarrow{\quad \Theta_Z \quad} & \rho \circ \Pi_1\,(p')\,. \end{array}$$

By applying the definitions, it becomes (look at the external square):



Recalling that $\tilde{Y} = ob(\Pi_1 Y) = Y$ and that $\tilde{Z} = ob(\Pi_1 Z) = Z$, this diagram commutes exactly when the dotted arrows are the identities and $\rho \circ \Pi_1(f) = f$. Thus in this case we do not only have $id_{\mathbf{Cov}(X)} \simeq \rho \circ \Pi_1$ but in fact $id_{\mathbf{Cov}(X)} = \rho \circ \Pi_1$. □

Finally we need to prove the $\mu-$equivalence.

Theorem 1.3.36 ($\mu-$EQUIVALENCE)**.** *Let X be locally path connected and semilocally 1-connected. Then the monodromy μ induces an equivalence of categories*

$$\mu\colon \mathbf{Cov}(X) \simeq \mathbf{Fct}(\Pi_1 X, \mathbf{Set}).$$

Proof. We have already described μ in Definition 1.3.23. To define the inverse transformation we just exploit the inverse of the $\mathscr{R}-$equivalence 1.2.8 and the inverse of the Π_1-equivalence 1.3.35: the composition $\rho \circ \mathscr{R}^{-1}$ does as μ^{-1}. □

For a direct construction of μ^{-1}, i.e. without exploiting Π_1 and $\mathscr{R}$, see [**23**, 12.35] or [**25**, 7.3].

Example: failure of Theorem 1.3.35 when X is not semilocally locally 1-connected

In the following example, due to E. C. Zeeman (see [**17**, Ex. 6.6.14]), we describe a planar topological space Z which is not locally 1-connected, that admits non-equivalent coverings but which induces equivalent covering morphisms. In particular the functor Π_1 of Theorem 1.3.35 fails to be an equivalence of categories.

Let $Z \subseteq \mathbb{R}^2$ be the space given by the union $Z = Z_0 \cup Z_1 \cup Z_2$:

- Z_0 is the unit circle $x^2 + y^2 = 1$;
- Z_1, the "fingers" of Z, is the union of the segments joining the point $(1, 0) \in Z_0$ to the family of points (the "nails") $(3, 1/n)$, $n = 1, 2, \ldots$;
- Z_2, the "thumb" of Z, is the arc $(x-2)^2 + y^2 = 1$ with $y \leq 0$; its extremities are the points $z_0 = (1, 0)$ and $(3, 0)$.

Next, consider the space $X \subseteq \mathbb{R}^2$ given by the union $\bigcup_{n=0}^{4} X_n$:

- $X_0 = Z_0$, $X_1 = Z_1$;
- X_2 is the union of the segments joining the point $(-1, 0)$ with the family of points $(-3, -1/n)$, $n = 1, 2, \ldots$;
- X_3 is the arc $(x-1)^2 + y^2 = 4$ with $y \leq 0$; its extremities are the points $(-1, 0)$ and $(3, 0)$;

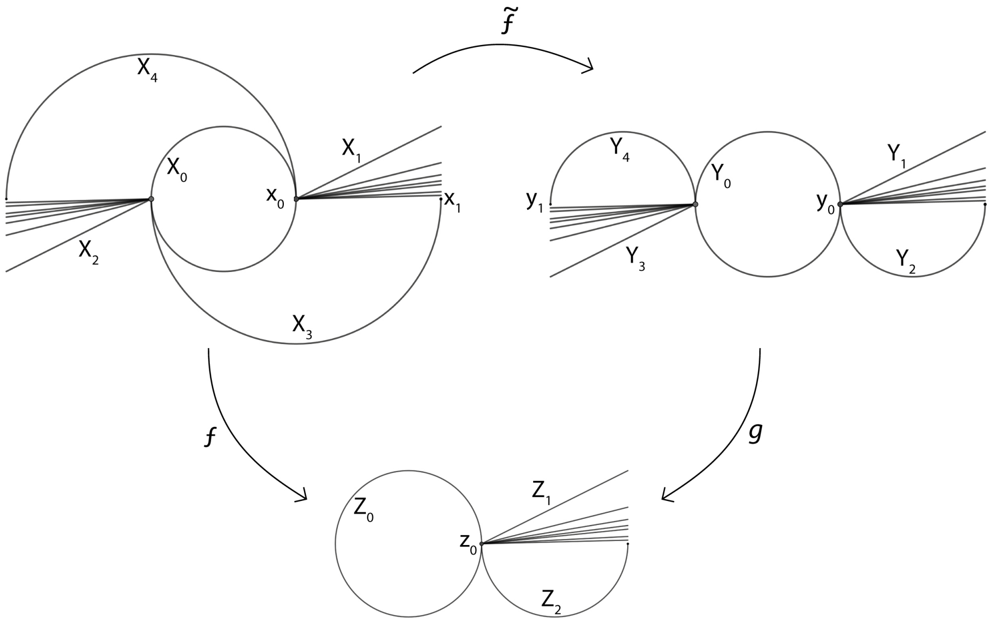


Figure 1.3.1: Zeeman example

- X_4 is the arc $(x+1)^2 + y^2 = 4$ with $y \geq 0$; its extremities are the points $(1,0)$ and $(-3,0)$.

Finally, consider the space $Y \subseteq \mathbb{R}^2$ given by the union $\bigcup_{n=0}^{4} Y_n$:

- $Y_0 = Z_0$, $Y_1 = Z_1$, $Y_2 = Z_2$, $Y_3 = X_2$;
- Y_4 is the arc $(x+1)^2 + y^2 = 1$ with $y \geq 0$; its extremities are the points $(-1,0)$ and $(-3,0)$.

Figure 1.4.1 perfectly gives the idea. Remark that overlapping the three spaces, the three points x_0, y_0, z_0 coincide.

Now we define two coverings $f\colon X \to Z$ and $g\colon Y \to Z$:

- under f, X_0 is wrapped twice around Z_0; both right and left fingers of X are sent isometrically to the fingers of Z; both right and left thumb of X are sent by some homeomorphism to the thumb of Z;
- under g, Y_0 is wrapped twice around Z_0; both right and left fingers of Y are sent isometrically to the fingers of Z; both right and left thumb of Y are sent isometrically to the thumb of Z.

It is possible to write down explicitly all involved homeomorphisms but the intuitive idea is even clearer. It is also clear that both f and g are covering maps as defined in 1.3.25 and that $f_*(\pi_1(X, x_0)) = g_*(\pi_1(Y, y_0))$: with a bit more work it is possible to show that the covering morphisms of the fundamental groupoid $\Pi_1 Z$ induced by f and g are equivalent in **CovMor** $(\Pi_1 Z)$. Moreover we can find a function $\tilde{f}\colon (X, x_0) \to (Y, y_0)$ such that $g \circ \tilde{f} = f$: let $x \in X$ and let γ be a path in X from x_0 to x. Then $f \circ \gamma$ is a path in Z from z_0 to $f(x)$. By the unique path lifting property (Lemma 1.3.29), we can lift this path to a path in Y from y_0 to some point $\tilde{y}$ such that $g(\tilde{y}) = f(x)$. Now let γ' be another path in X from x_0 to x. Then $\gamma \cdot \gamma'^{-1}$ is a loop in X based at x_0, hence $[f \circ \gamma] \cdot [f \circ \gamma'] \in f_*(\pi_1(X, x_0))$. Since $f_*(\pi_1(X, x_0)) = g_*(\pi_1(Y, y_0))$, it follows that the path $(f \circ \gamma) \cdot (f \circ \gamma')^{-1}$ lifts to a loop in Y based at

y_0. This implies that, if we lift $f \circ \gamma'$ to a path in Y starting at y_0, then the final point of this path is also $\tilde{y}$. Thus $\tilde{y}$ depends only on x (and not on the choice of γ) and the transformation $\tilde{f} \colon X \to Y$, $x \mapsto \tilde{y}$ is well defined. Then certainly $g \circ \tilde{f} = f$ (and $\tilde{f}(x_0) = y_0$).

But $\tilde{f}$ fails to be continuous and so f, g cannot be isomorphic, indeed: in X the sequence of right-hand finger nails converges to x_1, but their images under $\tilde{f}$ do not converge to y_1 which is easily seen to be the image of x_1.

It is worth noticing that if Z had been locally 1-connected, the construction above would have given a necessarily continuous function $\tilde{f} \colon X \to Y$ such that $g \circ \tilde{f} = f$, see for example [**24**, Th. V.5.1].

Chapter 2

THE TOPOLOGICAL FUNDAMENTAL GROUP

The question we focus on this chapter is the following: is it possible to endow the fundamental group of a topological space X with a group topology?

For the first attempt, one may try to use the quotient topology induced by the quotient map $\Omega(X,x) \to \pi_1(X,x)$. For sure this makes $\pi_1(X,x)$ into a "group with topology" but it turns out that it is not enough to get a "topological group" (i.e. a group with a topology making both inversion and multiplication continuous). Examples of such patological spaces are quite complicated but can be found in literature: the classical example is the *Hawaiian Earring* (see Section 2.2 hereunder) or the ones in [**4**]. To have a true topological group, one has to work harder. We will get to this object in three steps: first we introduce the category of *quasitopological* groups (i.e. groups with a topology making inversion continuous and multiplication continuous just component-wise). Next, we define the *quasitopological fundamental group* $\pi_1^{qtop}(X,x)$ and finally by adjunction we will obtain the *topological fundamental group* $\pi_1^{\tau}(X,x)$.

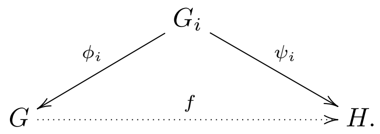
In the final section we introduce the notion of **Top**−groupoid and **Top**−functor to extend the above construction to groupoids getting the *topological fundamental groupoid* $\Pi_1^{\tau}X$.

2.1 A TOPOLOGY FOR THE FUNDAMENTAL GROUP

2.1.1 The free (Markov) topological group and the functor τ

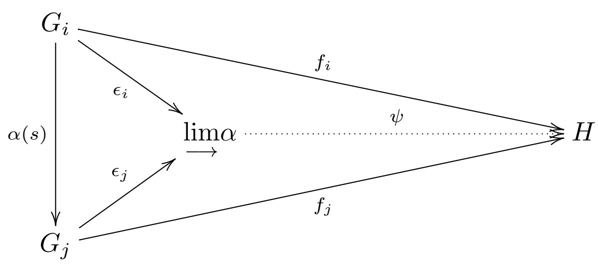
Let us recall briefly some basic definitions about free products of groups and free groups. For proofs and details we refer to [**24**, III.4-5] and [**22**, III.3].

Definition 2.1.1. Let $\{G_i : i \in I\}$ be a collection of groups and assume there is given for each index i a homomorphism $\phi_i : G_i \to G$, where G is a fixed group. We call G the *free product* of the groups G_i's (with respect to the homomorphisms ϕ_i's) if and only if ϕ satisfies this universal property: for any group H and any homomorphism $\psi_i : G_i \to H$, $i \in I$, there exists a unique homomorphism $f : G \to H$ making the following diagram commutative, i.e. $f \circ \phi_i = \psi_i$ for all $i \in I$:



It is a standard result that the free product exists and is unique up to isomorphism. It is customary to denote the free product of $\{(G_i, \phi_i)\}_{i\in I}$ by $\coprod_{i\in I} G_i$. In particular one should keep in mind how elements of $\coprod G_i$ are effectively done: an element of the free product is called a *word* and is nothing but a string $x_1 * \cdots * x_n$ of finite lenght where each x_i, $i = 1, \ldots, n$, belongs to one of the G_k's for some $k \in I$ and no x_i is an identity element; if $x_i, x_{i+1} \in G_k$, the substring $x_i * x_{i+1}$ is substituted with the element $x_i \cdot x_{i+1}$; if, after the previous contraction, there appears the substring $x_{i-1} * 1 * x_{i+1}$, it is substituted with $x_{i-1} * x_{i+1}$. Product in $\coprod G_i$ is given simply by juxtaposing two words; inverse elements are the obvious ones. With such a construction, the homomorphisms $\phi_i \colon G_i \to G$ may conveniently be seen as immersions.

Actually, the notion of free product of groups may be restated in categorical language. In such a case let I be a category of indeces (for instance, a pre-ordered set), **Grp** the category of groups and $\alpha \colon I \to \mathbf{Grp}$ a functor. Then the free product of the family of groups $\{\alpha(i) = G_i : i \in I\}$ is the inductive limit of α, i.e. the element $\varinjlim \alpha$ satisfying this universal property: for all $H \in \mathbf{Grp}$ and all family of compatible morphisms $\{f_i \colon \alpha(i) \to X\}_{i\in I}$ such that $f_i = f_j \circ f(s)$ for all $s \in I(i,j)$, then there exists a unique morphism of group $\psi \colon \varinjlim \alpha \to H$ such that f_i factorizes through $\varinjlim \alpha$ for each i. Pictorially one gets this commutative diagram:

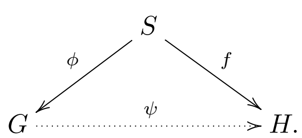


,

where ϵ_i's are the immersion homomorphisms. Let us remark also that if I is a discrete category, the compatibility condition trivially holds for all f_i, f_j.

It is possible to generalize the construction of the free product of groups in order to obtain the free group over an arbitrary set S:

Definition 2.1.2. Let S be an arbitrary set. The *free group* over the set S is the couple (F, ϕ) where F is a group and $\phi \colon S \to F$ is a function such that this universal property holds: for any group H and any function $f \colon S \to H$ there exists a unique homomorphism $\psi \colon F \to H$ making the following diagram commute, i.e. $\psi \circ \phi = f$:



As above, showing such an F exists and is unique up to isomorphism is a standard routine matter. For instance, let us point out just how one builds F: let $S = \{x_i : i \in I\}$, hence $S = \bigcup_{i\in I} \{x_i\}$. Consider

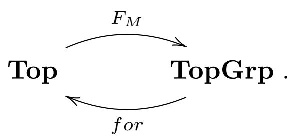
$F_i \coloneqq \{x_i^n : n \in \mathbb{Z}\}$ which is nothing but the infinite cyclic group generated by x_i, and consider for all $i \in I$ the obvious inclusion map $\phi_i \colon \{x_i\} \hookrightarrow F_i$. Then (F_i, ϕ_i) is the free group on the set $\{x_i\}$. Finally, one gets the free group F by taking the free product of the family of groups $\{F_i\}_{i \in I}$. Hence an element of F can be expressed uniquely in the form $x_1^{n_1} * \cdots * x_k^{n_k}$ where $x_1, \ldots, x_k$ are elements of S such that any two successive elements are different and $n_1, \ldots, n_k$ are all non-zero integers. Usually S is just a subset of F and ϕ reduces to be the inclusion.

Again, the notion of *free group* may be restated in categorical language (see [**22**, VI.1] for details). In such a case let $for \colon \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor and define $L \colon \mathbf{Set} \to \mathbf{Grp}$ to be the left-adjoint to for. Hence, there is a bijection $\mathbf{Set}\,(S, for\,(G)) \cong \mathbf{Grp}\,(L\,(S)\,, G)$ functorial in $S \in \mathbf{Set}$ and $G \in \mathbf{Grp}$; clearly nothing ensures that L exists, but, in this case, building such a bijection is easy: group homomorphisms from the free group $L\,(S)$ (as constructed above) and a group G correspond precisely to maps from the set S to the set $for(G)$ because each homomorphism $L\,(S) \to G$ is fully determined by its action on generators. So far, for each set $S \in \mathbf{Set}$, the free group generated by S is defined to be $L\,(S)$.

Now we build a free *topological* group.

The idea is to build a left-adjoint $F_M \colon \mathbf{Top} \to \mathbf{TopGrp}$ to the forgetful functor $for \colon \mathbf{TopGrp} \to \mathbf{Top}$ from the category of topological groups to the category of topological spaces. Unfortunately, things are by far harder than the case of the free group over a set S previously discussed. To prove such a left-adjoint exists, we have to apply the General Adjoint Function Theorem (GAFT), a powerful (and difficult) result which may be found in [**22**, V.6 Th. 2] (the reader may find of some help also [**9**]). Thanks to the GAFT, it is possible to show that:

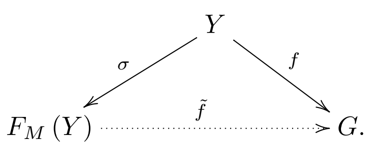
Theorem. *The forgetful functor* $for \colon \mathbf{TopGrp} \to \mathbf{Top}$ *admits a left-adjoint* $F_M \colon \mathbf{Top} \to \mathbf{TopGrp}$*:*



This theorem is in no way constructive: see [**19**], [**26**], [**27**].

Definition 2.1.3. For any topological space Y, the topological group $F_M\,(Y)$ is the *free (Markov) topological group* generated by Y.

Remark 2.1.4. Beyond the technical difficulties on the existence of F_M, if Y is any topological space, $F_M\,(Y)$ is the unique topological group endowed with a continuous map $\sigma \colon Y \to F_M\,(Y)$ satisfying this universal property: for any map $f \colon Y \to G$ to a topological group G there is a unique continuous morphism $\tilde{f} \colon F_M\,(Y) \to G$ making the following diagram commute, i.e. $\tilde{f} \circ \sigma = f$:



In particular, any continuous mapping from a topological space Y to a topological group G factors uniquely through $F_M\,(Y)$, i.e. f can be extended to a continuous homomorphism $\tilde{f} \colon F_M\,(Y) \to G$. The underlying set of $F_M\,(Y)$ is the free group generated by Y seen as a set and σ is the (continuous) canonical injection of generators (see [**26**]).

For the proof of the following lemma, see [**31**, 3.9].

Lemma 2.1.5. *Let X, Y be topological spaces.*

1. *If $\psi\colon X \to Y$ is a quotient map, then so is $F_M(\psi) : F_M(X) \to F_M(Y)$.*
2. *$F_M(Y)$ is discrete if and only if Y is discrete.*

A *group with topology* is a group G with a topology but no restrictions are made on the continutity of the operations. Let $\mathcal{T}_G$ the topology of such a group and denote **GrpwTop** the category of groups with topology and continuous homomorphisms. Given $G \in$ **GrpwTop**, let $m_G\colon F_M(G) \to G$ be the obvious multiplication epimorphism and give G the quotient topology with respect to m_G; hence the resulting group, denoted $\tau(G)$, is a group with topology: the underlying group is G and a subset $U \subseteq G$ is open exactly when $m_G^{-1}(U)$ is open. But actually we have:

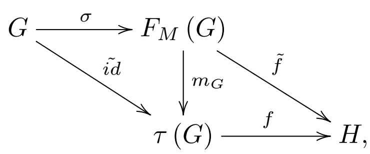
Lemma 2.1.6. *With the above construction $\tau(G)$ is a topological group.*

Proof. The proof follows from this general fact: if $p\colon H \to G$ is an epimorphism of groups and H is a topological group, the group G becomes a topological group when it is endowed with the quotient topology induced by p, indeed: it is a well known result that $\operatorname{Ker}(p)$ is a normal subgroup of H and $H/\operatorname{Ker}(p) \cong \operatorname{Im}(p) = G$; hence G is a topological group (cfr. [**1**, 1.5.3 and 1.5.13]). □

Remark 2.1.7. The identity $\tilde{id}\colon G \to \tau(G)$ is continuous since it is the composition of continuous maps $m_G \circ \sigma\colon G \to F_M(G) \to \tau(G)$.

Lemma 2.1.8. *The topological group $\tau(G)$ enjoys this universal property: if $f\colon G \to H$ is any continuous homomorphism to a topological group H, then $f\colon \tau(G) \to H$ is also continuous.*

Proof. The homomorphism $f\colon G \to H$ induces the homomorphism $\tilde{f}\colon F_M(G) \to H$ which is continuous by the universal property of the free (Markov) groups. Hence one gets this commutative diagram (be careful: f denotes both functions from G and $\tau(G)$):

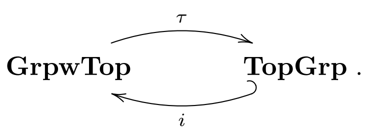


since m_G is continuous, $f\colon \tau(G) \to H$ is continuous too. □

It follows straightforward from this Lemma that **TopGrp** is a full subcategory of **GrpwTop**. Moreover:

Lemma 2.1.9.

1. *The functor $\tau\colon$ **GrpwTop** $\to$ **TopGrp** is left adjoint to the natural inclusion functor $i\colon$ **TopGrp** $\hookrightarrow$ **GrpwTop**, i.e.* **TopGrp** $(\tau(G), H) \cong$ **GrpwTop** $(G, i(H))$ *functorially in $G \in$* **GrpwTop** *and $H \in$* **TopGrp**:



2. *By setting $H = \tau(G)$, the adjunction bijection assigns to the continuous identity $id\colon \tau(G) \to \tau(G)$ the* reflection map $r_G\colon G \to \tau(G)$ *which is the continuous identity homomorphism.*

Proof. 1. Let $f\colon G \to H$ be any continuous map in **GrpwTop**. Define τ to be the identity on underlying groups and homomorphisms. To show τ is a well defined functor, it is enough to check that $\tau(f) : \tau(G) \to \tau(H)$ is continuous. Indeed, remark that $F_M(f) : F_M(G) \to F_M(H)$ is a continuous homomorphism by the universal property of the free (Markov) group and this square of continuous homomorphisms commutes:

$$\begin{array}{ccc} F_M(G) & \xrightarrow{F_M(f)} & F_M(H) \\ \downarrow{\scriptstyle m_G} & & \downarrow{\scriptstyle m_H} \\ \tau(G) & \xrightarrow{\tau(f)} & \tau(H). \end{array}$$

Since m_G and m_H are quotient and $F_M(f)$ is continuous, $\tau(f)$ must be continuous too.

2. The natural bijection characterizing the adjunction is built after the diagram of Lemma 2.1.8: $\mathbf{TopGrp}(\tau(G), H) \xrightarrow{\cong} \mathbf{GrpwTop}(G, i(H))$, $f \mapsto f \circ r_G$. By letting $H = \tau(G)$ and $f = id\colon \tau(G) \to \tau(G)$, r_G turns out to be the continuous identity morphism. □

Remark that we can define also the identity map $id^*\colon \tau(G) \to G$, but it does not need to be continuous: we can think to τ as a functor which makes a group with topology into a topological group by removing the smallest number of open sets from $\mathcal{T}_G$ in order to obtain a topological group.

Proposition 2.1.10. *The functor τ has the following properties:*

1. *It preserves quotient maps.*

2. *If $G \in$ **GrpwTop**, then G is a topological group if and only if $G = \tau(G)$.*

3. *If $G \in$ **GrpwTop**, then G is discrete if and only if $\tau(G)$ is discrete.*

Proof. 1. Suppose $f\colon G \to H$ is a homomorphism which is also a topological quotient map between groups with topology. Since F_M preserves quotient maps (Lemma 2.1.5), $F_M(f)$ is a quotient map, hence top and vertical arrows of diagram in the above Proposition are quotient; this forces $\tau(f)$ to be quotient too.

2. If G is a topological group, the identity $id\colon G \to G$ is continuous and it induces the continuous identity $id^*\colon \tau(G) \to G$ (as above, choose $H = G$ and $f = id$ in the diagram of Lemma 2.1.8). Since $r_G\colon G \to \tau(G)$ is also the continuous identity homomorphism by Lemma 2.1.9, r_G is the continuous inverse of id^*. Hence G and $\tau(G)$ are homeomorphic and the result follows. The other direction is trivial.

3. Since the identity $r_G\colon G \to \tau(G)$ is continuous, G is discrete whenever $\tau(G)$ is. Conversely, if G is discrete, then so is $F_M(G)$ and its quotient $\tau(G)$ (Lemma 2.1.5). □

2.1.2 Quasitopological groups

So far we have only built $\tau(G)$ as the quotient of a free topological group starting from any group with topology; but we have not given an explict description of $\tau(G)$ yet and actually we have already observed that it may be quite complicated (for example, in **[29]** there are given many different descriptions of such a group).

Things are a bit easier if we limit our attention to a special class of groups with topology, namely: quasitopological groups. If G is a quasitopological group (i.e. inversion is continuous and multiplication is continuous component-wise), let $c\,(G)$ be the underlying group of G with the quotient topology with respect to multiplication $\mu_G\colon G\times G\to G$ so that $\mu_G\colon G\times G\to c\,(G)$ is continuous.

Proposition 2.1.11. *Let* **qTopGrp** *be the category of quasitopological groups (with the obvious morphisms). Then* $c\colon$ **qTopGrp** $\to$ **qTopGrp** *is a functor when defined to be the identity on morphisms.*

Proof. For $G\in$ **qTopGrp**, consider the diagram:

$$\begin{array}{ccc} G\times G & \longrightarrow & G\times G \\ {\scriptstyle \mu_G}\downarrow & & \downarrow{\scriptstyle \mu_G} \\ c\,(G) & \longrightarrow & c\,(G)\,. \end{array}$$

Let $g\in c\,(G)$ and define the top map as $(a,b)\mapsto\left(b^{-1},a^{-1}\right)$. This function is continuous since G is a quasitopological group. The diagram commutes when the bottom map is inversion and since the vertical maps are quotients, this operation in $c\,(G)$ is continuous.

Using as top maps $(a,b)\mapsto(ga,b)$ and $(a,b)\mapsto(a,bg)$, the diagram commutes when the bottom map is left multiplication by g and right multiplication by g respectively and since the vertical maps are quotients, these operations in $c\,(G)$ are continuous too.

Similarly one shows that $c(f)=f\colon c\,(G)\to c\,(G')$ is continuous for each continuous homomorphism $f\colon G\to G'$ of quasitopological groups. □

Thus by applying c to any quasitopological group, we get again a quasitopological group.

Proposition 2.1.12. *Let G be a quasitopological group. Then:*

1. *The identity homomorphisms $G\to c\,(G)\to\tau\,(G)$ are continuous.*
2. *Then $\tau\,(c\,(G))=\tau\,(G)$.*
3. *Then G is a topological group if and only if $G=c\,(G)$.*

Proof. 1. Let e be the identity of G. Consider the commuting diagram

$$\begin{array}{ccccc} G\times\{e\} & \hookrightarrow & G\times G & \xrightarrow{r_G\times r_G} & \tau\,(G)\times\tau\,(G) \\ {\scriptstyle \mu_G}\downarrow{\scriptstyle \cong} & & {\scriptstyle \mu_G}\downarrow & & {\scriptstyle \mu_{\tau(G)}}\downarrow \\ G & \longrightarrow & c\,(G) & \longrightarrow & \tau\,(G)\,. \end{array}$$

Left vertical map is a homeomorphism; central and right vertical maps are quotient maps; top identities are continuous. Hence the identities in the bottom are continuous too by the universal property of quotient spaces.

2. Applying τ to $id\colon G\to c\,(G)$ gives $id_1\colon\tau\,(G)\to\tau\,(c\,(G))$ and id_1 is bijective and continuous.

Next, notice that by Lemma 2.1.9 we have a pair of adjoint functors

$$\textbf{qTopGrp}\ \underset{i}{\overset{\tau}{\rightleftarrows}}\ \textbf{TopGrp}\,,$$

so we get the bijection

$$\mathbf{qTopGrp}\left(c\left(G\right),i\left(\tau\left(G\right)\right)\right)=\mathbf{qTopGrp}\left(c\left(G\right),\tau\left(G\right)\right)\cong\mathbf{TopGrp}\left(\tau\left(c\left(G\right)\right),\tau\left(G\right)\right).$$

By (1), $id_2\colon c\left(G\right)\to\tau\left(G\right)$ is continuous and this map corresponds by adjunction to $id_3\colon\tau\left(c\left(G\right)\right)\to\tau\left(G\right)$, which is clearly bijective and continuous. So id_1 and id_3 are both continuous, bijective and are inverse of each other. Hence $\tau\left(G\right)$ and $\tau\left(c\left(G\right)\right)$ are homeomorphic through the identity, i.e. $\tau\left(G\right)=\tau\left(c\left(G\right)\right)$.

3. If G is a topological group, then $G=\tau\left(G\right)$ by (2) of Lemma 2.1.10 and by (1) both identities $\tau\left(G\right)\to c\left(G\right)\to\tau\left(G\right)$ are continuous; moreover they are inverse of each other so $c\left(G\right)$ is homeomorphic through the identity to $\tau\left(G\right)$, i.e. $\tau\left(G\right)=c\left(G\right)$. Conversely, if $G=c\left(G\right)\in\mathbf{qTopGrp}$, then $\mu_G\colon G\times G\to c\left(G\right)=G$ is continuous and so G is a topological group. □

Proposition 2.1.13. *If $\{G_\lambda\}$ is a family of quasitopological groups, each with underlying group G and topology $\mathcal{T}_{G_\lambda}$, then the topology $\bigcap_\lambda\mathcal{T}_{G_\lambda}$ on G makes G into a quasitopological group.*

Proof. It is a well known result in point-set topology that $\bigcap_\lambda\mathcal{T}_{G_\lambda}$ is actually a topology on G. Now let us take $A\in\bigcap_\lambda\mathcal{T}_{G_\lambda}$ and show that inversion $i\colon G\to G$, $g\mapsto g^{-1}$, is continuous: $A\in\mathcal{T}_{G_\lambda}$ for all λ, hence $i^{-1}\left(A\right)$ is open in G for all λ; so $i^{-1}\left(A\right)\in\bigcap_\lambda\mathcal{T}_{G_\lambda}$. The same argument works for both right and left multiplications. □

It is interesting to notice that

Corollary 2.1.14. *If $G\in\mathbf{qTopGrp}$, then G and $\tau\left(G\right)$ have the same open subgroups.*

For the proof we refer to [**6**, 3.9].

2.1.3 The quasitopological fundamental group $\pi_1^{qtop}\left(X\right)$

For a topological space X, recall that $\pi_0\left(X\right)$ denotes the set of path components of X. The *path component space* of X, denoted by $\pi_0^{qtop}\left(X\right)$, is the set $\pi_0\left(X\right)$ with the quotient topology induced by the equivalence relation $\sim$, where, for any two points in X, one sets $x\sim y$ if and only if they belong to the same path component of X.

Remark 2.1.15. It immediately follows from the definition that $\pi_0^{qtop}\left(X\right)$ is discrete whenever the path components of X are open.

Since any map $f\colon X\to Y$ induces a continuous map $f_*\colon\pi_0^{qtop}\left(X\right)\to\pi_0^{qtop}\left(Y\right)$ taking the path component of x in X to the path component of $f\left(x\right)$ in Y, we obtain a functor $\pi_0^{qtop}\colon\mathbf{Top}\to\mathbf{Top}$.

Definition 2.1.16. The path component space $\pi_1^{qtop}\left(X,x_0\right):=\pi_0^{qtop}\left(\Omega\left(X,x_0\right)\right)$ is the *quasitopological fundamental group* of the based space (X,x_0). It is characterized by the canonical map $h\colon\Omega\left(X,x_0\right)\to\pi_1^{qtop}\left(X,x_0\right)$ identifying homotopy classes of loops.

Remark 2.1.17. Since multiplication and inversion in the fundamental group are induced by the continuous operations $(\alpha,\beta)\mapsto\alpha*\beta$ and $\alpha\mapsto\alpha^{-1}$ in the loop space $\Omega\left(X,x_0\right)$, it follows from the universal property of quotient spaces that $\pi_1^{qtop}\left(X,x_0\right)$ is a quasitopological group; for instance, let us check that the inversion

$\tilde{i}\colon \pi_1^{qtop}(X,x_0) \to \pi_1^{qtop}(X,x_0)$ is continuous: one gets the diagram

$$\begin{array}{ccc} \Omega(X,x_0) & \xrightarrow{\;i\;} & \Omega(X,x_0) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \pi_1^{qtop}(X,x_0) & \xrightarrow{\;\tilde{i}\;} & \pi_1^{qtop}(X,x_0)\,. \end{array}$$

Clearly it commutes and since all vertical and top arrows are continuous, $\tilde{i}$ must be continuous too. Analogously, one checks component-wise continuity for multiplication.

Moreover, any based map $f\colon\ (X,x_0) \to (Y,y_0)$ induces a continuous group homomorphism

$$f_* = \pi_0^{qtop}(\Omega(f)) : \pi_1^{qtop}(X,x_0) \to \pi_1^{qtop}(Y,y_0)\,,$$

then we obtain a functor $\pi_1^{qtop}\colon \mathbf{Top}_* \to \mathbf{qTopGrp}$.

The isomorphism class of the quasitopological fundamental group is independent of the choice of the basepoint (see below Lemma 2.1.20), so we can write $\pi_1^{qtop}(X)$ if there is no risk of confusion. Unfortunately $\pi_1^{qtop}(X)$ may fail to be a topological group (see the example of the *Hawaiian Earring* exposed in Section 2.2); but **qTopGrp** is a full subcategory of **GrpwTop**, hence the functor $\pi_1^{\tau} := \tau \circ \pi_1^{qtop}\colon \mathbf{Top}_* \to \mathbf{TopGrp}$ is well defined. This new functor assigns to a based space X a topological group $\pi_1^{\tau}(X)$ whose underlying group is $\pi_1(X)$; remark also that since the identity morphism $\pi_1^{qtop}(X) \to \pi_1^{\tau}(X)$ is continuous, so is $\Omega(X) \to \pi_1^{qtop}(X) \to \pi_1^{\tau}(X)$.

2.1.4 The topological fundamental group $\pi_1^{\tau}(X)$

Definition 2.1.18. The *topological fundamental group* of the space X is the topological group

$$\pi_1^{\tau}(X) := \tau\left(\pi_1^{qtop}(X)\right).$$

According to our previous construction of τ, the topological fundamental group is the quotient of the free topological group $F_M\left(\pi_1^{qtop}(X)\right)$ with respect to the multiplication morphism; but actually it does not give an explicit characterization of $\pi_1^{\tau}(X)$. Next propositions describe some properties of this group.

Proposition 2.1.19. *The topology of $\pi_1^{\tau}(X,x_0)$ is the finest group topology on $\pi_1(X,x_0)$ such that $h\colon \Omega(X,x_0) \to \pi_1(X,x_0)$ is continuous.*

Proof. Suppose $\pi_1(X,x_0)$ is endowed with a topology making it into a topological group and such that $h\colon \Omega(X,x_0) \to \pi_1(X,x_0)$ is continuous. According to the diagram, the identity $id_1\colon \pi_1^{qtop}(X,x_0) \to \pi_1(X,x_0)$ is continuous by the universal property of quotient spaces:

$$\begin{array}{ccc} \Omega(X,x_0) & & \\ {\scriptstyle \pi_1^{qtop}}\downarrow & \searrow{\scriptstyle h} & \\ \pi_1^{qtop}(X,x_0) & \xrightarrow{\;id_1\;} & \pi_1(X,x_0)\,; \end{array}$$

moreover $\pi_1(X,x_0)$ is a topological group and thus we can exploit the adjunction bijection

$$\mathbf{qTopGrp}\left(\pi_1^{qtop}(X,x_0), \pi_1(X,x_0)\right) \cong \mathbf{TopGrp}\left(\tau\left(\pi_1^{qtop}(X,x_0)\right), \pi_1(X,x_0)\right),$$

so that id_1 corresponds to the the continuous identity $id_2\colon \pi_1^{\tau}(X,x_0)=\tau\left(\pi_1^{qtop}(X,x_0)\right)\to\pi_1(X,x_0)$; then the topology of $\pi_1^{\tau}(X,x_0)$ is finer than that of $\pi_1(X,x_0)$. □

Proposition 2.1.20. *If $\gamma\colon I\to X$ is a path, then $\pi_1^{\tau}(X,\gamma(1))\to\pi_1^{\tau}(X,\gamma(0))$, $[\alpha]\to\left[\gamma*\alpha*\gamma^{-1}\right]$ is an isomorphism of topological groups. Consequently if x_0, x_1 lie in the same path component of X, then $\pi_1^{\tau}(X,x_0)\cong\pi_1^{\tau}(X,x_1)$.*

Proof. The continuous operation $c_\gamma\colon\Omega(X,\gamma(1))\to\Omega(X,\gamma(0))$ given by $\alpha\to\gamma*\alpha*\gamma^{-1}$ induces the continuous isomorphism $\Gamma\colon\pi_1^{qtop}(X,\gamma(1))\to\pi_1^{qtop}(X,\gamma(0))$ given by $[\alpha]\mapsto\left[\gamma*\alpha*\gamma^{-1}\right]$. The inverse is continuous since $c_{\gamma^{-1}}$ is continuous. Thus Γ is a continuous isomorphism of quasitopological groups and $\tau(\Gamma)$ is a continuous isomorphism of topological groups. □

Remark 2.1.21. We have shown in Proposition 2.1.12 that $\pi_1^{\tau}(X)=\pi_1^{qtop}(X)$ if and only if $\pi_1^{qtop}(X)$ is already a topological group. Thus if $\pi_1^{qtop}(X)$ fails to be a topological group, the topology of $\pi_1^{\tau}(X)$ is strictly coarser than that of $\pi_1^{qtop}(X)$; however thanks to Corollary 2.1.14, $\pi_1^{qtop}(X)$ and $\pi_1^{\tau}(X)$ have the same open subgroups.

The following proposition characterizes the discreteness of $\pi_1^{\tau}(X)$ for path connected spaces:

Proposition 2.1.22. *For any path connected space X, the following are equivalent:*

1. *$\pi_1^{\tau}(X)$ is a discrete group.*

2. *$\pi_1^{qtop}(X)$ is a discrete group.*

3. *For every null-homotopic loop $\alpha\in\Omega(X)$ there is an open neighborhood $\mathcal{U}$ of α in $\Omega(X)$ containing only null-homotopic loops.*

Proof. $1\Leftrightarrow 2$ is a special case of 2.1.10. $2\Leftrightarrow 3$ follows since $\pi_1^{qtop}(X)$ is a quotient space over $\Omega(X)$. □

It follows straightforward that if $\pi_1^{qtop}(X)$ is discrete, then $\pi_1^{qtop}(X)=\pi_1^{\tau}(X)$ and thus the quasitopological fundamental group is actually a topological group: this is the case of "simple" (i.e. semilocally 1-connected) topological spaces one usually finds in everyday life.

Even though this last proposition gives a link between $\pi_1^{\tau}(X)$ and $\pi_1^{qtop}(X)$, at least when one of these groups is discrete, one usually has explicitly neither $\pi_1^{\tau}(X)$ nor $\pi_1^{qtop}(X)$. Then it is useful the following theorem which characterizes discreteness in terms of local properties of X itself; the proof may be found in [**10**]:

Theorem 2.1.23. *Suppose X is path connected. The following are equivalent:*

1. *$\pi_1^{\tau}(X)$ is discrete.*

2. *X is locally path connected and semilocally 1-connected.*

3. *X admits universal covering.*

Unfortunately $\pi_1^{qtop}(X)$ does not need to preserve finite products: troubles arise because the product of quotient maps is not necessarily a quotient map (see [**24**, A.3] for a simple counterexample or Theorem 3.1.19 hereunder for a harder one). On the other hand one has:

Proposition 2.1.24. *For any based spaces X, Y there is a natural isomorphism*

$$\Phi\colon \pi_1^\tau (X \times Y) \to \pi_1^\tau (X) \times \pi_1^\tau (Y)$$

of topological groups.

Proof. The projections $X \times Y \to X$ and $X \times Y \to Y$ induce the continuous group isomorphism $\Phi\colon \pi_1^\tau (X \times Y) \to \pi_1^\tau (X) \times \pi_1^\tau (Y)$ given by $\Phi([(\alpha, \beta)]) = ([\alpha], [\beta])$. Now we need to prove that the inverse of Φ is continuous also. Let x_0 and y_0 be the basepoints of X and Y respectively and c_{x_0}, c_{y_0} the constant loops. The maps $i\colon X \to X \times Y$, $x \mapsto (x, y_0)$, and $j\colon Y \to X \times Y$, $y \mapsto (x_0, y)$ induce the continuous homomorphisms $i_*\colon \pi_1^\tau (X) \to \pi_1^\tau (X \times Y)$, $[\alpha] \mapsto [(\alpha, c_{y_0})]$, and $j_*\colon \pi_1^\tau (Y) \to \pi_1^\tau (X \times Y)$, $[\beta] \mapsto [(c_{x_0}, \beta)]$. Let μ be the (continuous) group multiplication of the topological group $\pi_1^\tau (X \times Y)$. Then the composition $\mu \circ (i_* \times j_*)$ is the continuous inverse of Φ. □

2.2 THE HAWAIIAN EARRING

In this this section we give a (quite long) example of a "wild" topological space $\mathbb{H}$ whose quasitopological fundamental group $\pi_1^{qtop}(\mathbb{H})$ fails to be a topological group.

The *Hawaiian Earring* $\mathbb{H} \subseteq \mathbb{C}$ is a topological space which is a countably infinite union of circles that are all tangent to a single line at the same point and whose radii tend to zero. This space is paradigmatic in the sense that it is an example of all those patological phoenomena which usual theory is buil to prevent. In Figure 2.2.1 there are represented the first n circles.

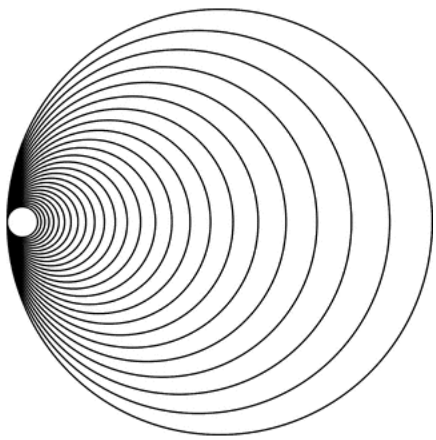


Figure 2.2.1: The first n circles of the Hawaiian Earring

For example, it is a standard exercise to show that the Hawaiian Earring is compact and locally path connected, but it is not semilocally 1-connected, hence it admits no universal covering; in **[18]** one can find an example of two coverings over this space whose composition is not a covering (but it is a semicovering, cfr. the second example in Section 3.1.5, Definition 3.1.5 and Theorem 3.1.11).

In this section we first find the (not at all obvious) fundamental group of $\mathbb{H}$, next we show it is not a free group and finally we prove that the topology induced by the quotient projection $h\colon \Omega(\mathbb{H}, 0) \to \pi_1^{qtop}(\mathbb{H})$ does not make $\pi_1^{qtop}(\mathbb{H})$ into a topological group.

Definition 2.2.1. For each positive integer n, let $C_n \subseteq \mathbb{C}$ be the circle $\{z \in \mathbb{C} \colon |z - 1/n| = 1/n\}$. The topological space $\mathbb{H} = \bigcup_{n=1}^{\infty} C_n$ is called *Hawaiian Earring*.

Our aim is to provide a description of $\pi_1(\mathbb{H},0)$, where 0 is the "wild point" where all circles meet. The base point of $\mathbb{H}$ and of all subspaces of $\mathbb{H}$ considered in our discussion will always be 0, so in the sequel we will write for brevity $\pi_1(\mathbb{H})$ meaning $\pi_1(\mathbb{H},0)$, $\pi_1(C_n)$ meaning $\pi_1(C_n,0)$ and so on.

It is a known fact that if X is the wedge of a finite number of circles, say n, each loop can only go around finitely many circles (because a loop has compact image) and consequently $\pi_1(X)$ is the free product on the obvious generators (see, for example, [**24**, Ex. IV.3.4]), in particular $\pi_1(X) \simeq \coprod_{i=1}^{n} \mathbb{Z}$. It may be tempting to think that the fundamental group of $\mathbb{H}$ is isomorphic to the free product of infinitely many copies of $\mathbb{Z}$, but it is not correct. Indeed, let $f\colon I \to \mathbb{H}$ be a loop that goes around C_1 on the middle third part of the unit interval, around C_2 on the middle third part of each of the remaining intervals and so on: this defines a continuous map $I \to \mathbb{H}$ and in particular such a loop wraps at least one time on each circle of $\mathbb{H}$ (remark that $f^{-1}(0)$ is the triadic Cantor set on the unit interval).

Let $\mathbb{H}_n = \bigcup_{i=1}^{n} C_n$, then $\pi_1(\mathbb{H}_n) =: F_n$ is isomorphic to the free product on n generators $l_1,\dots,l_n$, where l_i is the homotopy class of a loop that wraps counterclockwise around C_i once, so that l_i is a generator of $\pi_1(C_i)$; generators $l_1,\dots,l_n$ will also be called "symbols" or "letters".

For each $n > 0$ we have a retraction $\mathbb{H} \to \mathbb{H}_n$ collapsing the loops C_i's with $i > n$ to the point 0. Therefore we may view F_n as a subgroup of $\pi_1(\mathbb{H})$, and the retraction induces a map from $\pi_1(\mathbb{H})$ to F_n. Moreover the restricted retraction $\mathbb{H}_n \to \mathbb{H}_{n-1}$ induces a group homomorphism $F_n \to F_{n-1}$ fixing l_i for $i < n$ and mapping l_n to 1: in practice, such a homomorphism simply deletes each occurence of the letter l_n.

Thus the family of groups $\{F_n\}$ forms a projective system of groups and we get a canonical group homomorphism

$$\phi\colon \pi_1(\mathbb{H}) \to F_\infty := \varprojlim_n F_n.$$

Recall (see, for example, [**22**, III.4]) that the *projective limit* F_∞ is the subgroup of the product $\prod_{i=1}^{\infty} F_i$ consisting of those sequences $(f_i)_i$ for which the map $F_i \to F_{i-1}$ sends f_i to f_{i-1} for all $i > 1$; for instance, one checks that $\left(l_1, l_1 l_2^2, l_1 l_2^2 l_3^3, \dots\right) \in F_\infty$, while $(l_1,1,1,\dots) \notin F_\infty$.

Moreover notice that there is a natural embedding $F_n \hookrightarrow F_\infty$, $l_i \mapsto (1,\dots,1,l_i,l_i,\dots)$ for each $i \le n$.

It will turn out that ϕ maps $\pi_1(\mathbb{H})$ isomorphically to a subgroup π of F_∞ which can be rouhgly described as the subgroup consisting of those sequences $(f_i)_i \in F_\infty$ for which the following condition holds for each $j \ge 1$: the number of times l_j occurs in the reduced word representation of f_i is a bounded function of i. More precisely:

Definition 2.2.2. Define the *j-weight* $w_j(x)$ of an element $x \in F_i$ as follows: first write x as a reduced word $x = g(1)^{a_1} g(2)^{a_2} \cdots g(s)^{a_s}$, where each a_k is a non-zero integer and g is a map $\{1,2,\dots,s\} \to \{l_1,\dots,l_i\}$ with $g(k) \ne g(k+1)$ for any k. Finally put $w_j(x) = \sum_{g(k)=l_j} |a_k|$.

For example take $x = l_1 l_2^3 l_3^{-5} l_2 \in F_3$: then $w_1(x) = 1$, $w_2(x) = 4$ and $w_3(x) = 5$, while $w_k(x) = 0$ for all $k > 3$.

Notice that $w_j(x)$ is well defined since the representation of x as a reduced word is unique.

Definition 2.2.3.

1. Let π be the subgroup of F_∞ consisting of all sequences $x = (x_i)_i \in F_\infty$ such that for every integer $j \ge 1$, the function $\mathbb{N}^{\ge 1} \to \mathbb{N}$ defined by $i \mapsto w_j(x_i)$ is bounded.

2. For $j > 0$ define the group $F_{\ge j} = \{(x_i)_i \in F_\infty \colon w_k(x_i) = 0 \text{ for all } i \text{ and all } k < j\}$ and the group $\pi_{\ge j} = \pi \cap F_{\ge j}$.

Actually, what we have defined in (1) is nothing but a family of sequences in $\mathbb{N}$: for each j, there remains defined the sequence $(w_j(x_i))_i$ where the integer $w_j(x_i)$ tells how many times the letter l_j appears in the i-th word of the sequence x. By definition all such sequences are non-decreasing and the subgroup π contains the elements $x \in F_\infty$ for which the non-decreasing sequences $w_j(x)$ are eventually constant for all j.

For example:

- $\left(l_1, l_1 l_2^2, l_1 l_2^2 l_3^3, \ldots\right) \in \pi$ because for $j = 1$ one gets the sequence $(1, 1, 1, \ldots)$ which is eventually constant, for $j = 2$ one gets the sequence $(0, 2, 2, \ldots)$ which is eventually constant, for $j = 3$ one gets the sequence $(0, 0, 3, 3, \ldots)$, and so on;
- $\left(1, l_1 l_2 l_1^{-1}, l_1 l_2^2 l_1 l_3 l_1^{-1} l_2^{-1} l_1^{-1}, l_1 l_2^2 l_1 l_3^2 l_1 l_4 l_1^{-1} l_3^{-1} l_1^{-1} l_2^{-1} l_1^{-1}, \ldots\right) \in F_\infty \setminus \pi$ because for $j = 1$ one gets the sequence $(0, 2, 4, 6, \ldots)$ which is not bounded.

Notice also that in the sequences contained in $F_{\geq j}$ the symbols $l_1, \ldots, l_{j-1}$ never appear and consequently the first $j-1$ terms of these elements are trivially 1. Moreover, by definition, $F_{j-1} \cap \pi_{\geq j} = \emptyset$ for all j.

Lemma 2.2.4. *The group π is the free product $F_{j-1} * \pi_{\geq j}$ of its subgroups F_{j-1} and $\pi_{\geq j}$.*

Proof. Before the proof, it is worth noticing that, in the statement, F_{j-1} is identified with its image in the embedding $F_{j-1} \hookrightarrow F_\infty$.

The subgroup of π generated by F_{j-1} and $\pi_{\geq j}$ is their free product because of the unique reduced word representation in any free group F_i and because $F_{j-1} \cap \pi_{\geq j} = \emptyset$. So $F_{j-1} * \pi_{\geq j} \subseteq \pi$. Conversely, if $x = (x_i)_i \in \pi$, then the number of uccurrences of elements of F_{j-1} in the reduced representation of x_i is a bounded function of i. Again by uniqueness of the reduced word representation, it follows that x is a finite poduct of elements of F_{j-1} and of $\pi_{\geq j}$. □

Theorem 2.2.5. *The map $\phi \colon \pi_1(\mathbb{H}) \to \pi$ is a group isomorphism.*

Proof. First step: the image of the homomorphism $\phi \colon \pi_1(\mathbb{H}) \to F_\infty$ is π.

Let $f \colon I \to \mathbb{H}$ be a loop at 0. By van Kampen theorem ([**24**, IV.2]), $\pi_1(\mathbb{H}) = F_1 * \pi_1(C_{\geq 2})$, where $C_{\geq k} = \bigcup_{n \geq k} C_n$. So, after a homotopy, we may assume f is a composition of loops $f_0 e_1 f_1 e_2 f_2 \cdots e_s f_s$ where e_i are loops in C_1 and f_i are loops in $C_{\geq 2}$. We have $w_1(\phi(f_i)) = 0$ and for $n > 1$ this implies $w_1(\phi(f)_n) \leq \sum_{i=1}^{s} w_1(\phi(e_i))$; since this last term does not depend on n, the sequence $(w_1(\phi(f)_n))_n$ is eventually constant. Repeating the same argument shows that for every i, the sequence $(w_2(\phi(f_i)_n))_n$ is eventually constant. By induction, we obtain that $\phi(f) \in \pi$, so the image of ϕ lies in π.

Conversely, let $x = (x_i)_i \in \pi$. We will build a loop $f \colon I \to \mathbb{H}$ such that the homotopy class of f is mapped to x by ϕ. By Lemma 2.2.4, we can write x as $y_0 e_1 y_1 e_2 y_2 \cdots e_s y_s$ with $e_i \in \left\{l_1, l_1^{-1}\right\}$ and $y_i \in \pi_{\geq 2}$. Acordingly, we divide the unit interval I in $2s+1$ intervals $I_i := \left[\frac{i}{2s+1}, \frac{i+1}{2s+1}\right]$, $0 \leq i \leq 2s$, and define a map $f_1 \colon I \to \mathbb{H}$ to be zero on the intervals I_{2i} and goes around C_1 in the appropriate direction on the intervals I_{2i-1}, so that the homotopy class of the loops $f_1|_{I_{2i-1}}$ is e_i for $i = 1, \ldots, s$.

Next we break up $y_i \in \pi_{\geq 2}$ into elements $\left\{l_2, l_2^{-1}\right\}$ and elements in $\pi_{\geq 3}$ and divide the interval I_{2i} consequently. So we define a map f_2 that only differs from f_1 on subitervals of I_{2i}, where it gives l_2-paths in y_i. This way we get a uniformly convergent sequence $(f_i)_i$ of loops in $\mathbb{H}$ which converges to a continuous loop f in $\mathbb{H}$. By construction the image of $\phi(f)$ in F_n is x_n, so $\phi(f) = x$.

Second step: the map $\phi \colon \pi_1(\mathbb{H}) \to F_\infty$ is injective.

The proof of this result is somewhat non-trivial and we skip it: it may be found in [**11**]. □

Thus, by Theorem 2.2.5 we obtain the intere description of the features of $\pi_1(\mathbb{H})$: its elements may be seen as irreducible infinite words $l_{i_1}^{n_{i_1}} \cdots l_{i_k}^{n_{i_k}} \cdots$ such that no generator l_{i_j} appears more than finitely many times. Remark also that $\pi_1(\mathbb{H})$ contains a subgroup canonically isomorphic to the free group over the symbols $\{l_1, l_2, \dots\}$.

Finally, notice that, contrary to the behaviour of "simple" spaces commonly studied, we have that:

Corollary 2.2.6. *The group $\pi_1(\mathbb{H})$ is uncountable.*

Proof. It is a standard fact that the product $\prod_{n=1}^{\infty} \mathbb{Z}_2$ of infinitely many copies of the cyclic group with two elements is uncountable. For any sequence $s = (a_n) \in \prod_{n=1}^{\infty} \mathbb{Z}_2$ we build a loop $\alpha_s \colon I \to \mathbb{H}$ by defining α_s to be constant on $\left[\frac{n-1}{n}, \frac{n}{n+1}\right]$ if $a_n = 0$ and α_s to be l_n on $\left[\frac{n-1}{n}, \frac{n}{n+1}\right]$ if $a_n = 1$. We also define $\alpha_s(1) = 0$. This way, we obtain an uncountable family of homotopy classes $[\alpha_s] \in \pi_1(\mathbb{H})$. It is enough to show $[\alpha_s] \neq [\alpha_t]$ whenever $s \neq t$. Suppose $s = (a_n) \neq (b_n) = t$. Then, without loss of generality, we have $a_N = 1$ and $b_N = 0$ for some N. Consider the retraction $q_N : \mathbb{H} \to C_N$ which collapses all circles C_j to 0 for $j > N$ and all circles C_i to C_N for $i < N$. If $[\alpha_s] = [\alpha_t]$, then $[q_N \circ \alpha_s] = [q_N \circ \alpha_t]$ in $\pi_1(C_N) \simeq \mathbb{Z}$. But $[q_N \circ \alpha_t] = 0 \in \mathbb{Z}$ is trivial, while $[q_N \circ \alpha_s] = 1 \in \mathbb{Z}$ is not trivial, contradiction. Therefore $[\alpha_s] \neq [\alpha_t]$, showing that $\prod_{n=1}^{\infty} \mathbb{Z}_2$ injects into $\pi_1(\mathbb{H})$. □

Next results show that $\pi_1(\mathbb{H})$ is not free.

An uncountable free group G is free on uncoutably many generators and so $Hom(G, \mathbb{Z})$ is also uncountable. To prove that $\pi_1(\mathbb{H})$ is not free, it then suffices so show that $Hom(\pi_1(\mathbb{H}), \mathbb{Z})$ is countable.

Lemma 2.2.7. *For each positive integer j let $x^{(j)} = \left(x_i^{(j)}\right)_{i,j}$ be a sequence in F_∞ such that $x_i^{(j)} = 1$ for all $i < j$. Then there is a homomorphism $f \colon F_\infty \to F_\infty$ sending l_j (identified with its immersion $(1, \dots, 1, l_j, l_j, \dots)$) to $x^{(j)}$ for all j. Moreover, if $x^{(j)} \in \pi_{\geq j}$ for all j, then $f(\pi) \subseteq \pi$.*

Proof. As F_n is a free group on $l_1, \dots, l_n$, there are unique group homomorphisms $f_n \colon F_n \to F_n$ sending l_j to $x_n^{(j)}$ for $j \leq n$. Since we have $x_i^{(j)} = 1$ for $i < j$, we get the commutative diagram

$$\begin{array}{ccccccc} F_1 & \twoheadleftarrow & F_2 & \twoheadleftarrow & F_3 & \twoheadleftarrow & \cdots \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\ F_1 & \twoheadleftarrow & F_2 & \twoheadleftarrow & F_3 & \twoheadleftarrow & \cdots \end{array}$$

and taking the (projective) limit in both rows one finds a homomorphism $f \colon F_\infty \to F_\infty$ that satisfies the conditions.

Now suppose $y = (y_i)_i \in F_\infty$ and fix $k \geq 1$,then

$$w_k\left(f(y)_n\right) = w_k\left(f_n(y_n)\right) \leq \sum_{j=1}^{n} w_j(y_n)\, w_k\left(f_n(l_j)\right) =$$
$$= \sum_{j=1}^{n} w_j(y_n)\, w_k\left(x_n^{(j)}\right)$$

and the inequality must be right there because when we substitute the letter l_g of y_n with $f_n(y_j)$, there could be some word reduction to do.

If $x^{(j)} \in \pi_{\geq j}$ for each j, then the terms with $j > k$ on the right-hand side vanish. If in addition $y \in \pi$, then the remaining k terms are all bounded functions of n, so that $w_k\left(f(y)_n\right)$ is a bounded funtion of n and $f(y) \in \pi$, which is the second part of the statement. □

Proposition 2.2.8. *Let $f\colon \pi \to \mathbb{Z}$ be a homomorphism. Then $f(l_i) = 0$ for all sufficiently large i.*

Proof. Suppose that $f(l_{i_1}), f(l_{i_2}), \ldots$ are non-zero for some $i_1 < i_2 < \ldots$. By Lemma 2.2.7, there is a homomorphism $g\colon \pi \to \pi$ mapping l_j to $l_{i_j}^{\pm 3}$ where the sign is chosen to be the sign of $f\left(l_{i_j}\right)$. By replacing f with $f \circ g$ we may now assume $f(l_i) \geq 3$. Put $a_i = f(l_i)$. For every $j \geq 1$, define the element $x^{(j)} = \left(x_i^{(j)}\right)_i \in \pi$ by $x_i^{(j)} = 1$ if $i < j$ and for $i \geq j$: $x_i^{(j)} = l_j\left(l_{j+1}\left(\cdots\left(l_{i-1}l_i^{a_{i-1}}\right)^{a_{i-2}}\cdots\right)^{a_{j+1}}\right)^{a_j}$. The integer $f\left(x^{(1)}\right)$ now satisfies congruence conditions that cannot hold for any integer. Indeed, first note that $x^{(j)} = l_j\left(x^{(j+1)}\right)^{a_j}$, so $f\left(x^{(j)}\right) = a_j + a_j \cdot f\left(x^{(j+1)}\right)$.

It follows that $f\left(x^{(1)}\right) = a_1 + a_1a_2 + \cdots + a_1a_2\cdots a_n + a_1a_2\cdots a_n \cdot f\left(x^{(n)}\right)$. Put $b_n = a_1 + a_1a_2 + \cdots + a_1a_2\cdots a_{n-1}$ and $c_n = a_1a_2\cdots a_n$. Then $f\left(x^{(1)}\right) \equiv b_n \bmod c_n$. Moreover $b_n < c_n$ and b_n tends to infinity. So:

- if $f\left(x^{(1)}\right) \geq 0$, then $f\left(x^{(1)}\right) \geq b_n$ for all n, which is a contradiction because b_n is increasing and unbounded;
- if $f\left(x^{(1)}\right) < 0$, then $f\left(x^{(1)}\right) \leq b_n - c_n$ and we get a contradiction too, because we have chosen $a_i \geq 3$ and so $b_n - c_n$ also tends to infinity.

Then there must be an integer m such that $f(l_j) = 0$ for all $j \geq m$. □

Proposition 2.2.9. *Let $g\colon \pi \to \mathbb{Z}$ be a homomorphism such that $g(l_i) = 0$ for all i. Then $g \equiv 0$.*

Proof. Suppose $g(x) \neq 0$ for some $x \in \pi$, then we can write $x = f_1y_1f_2y_2\cdots f_sy_s$ with $f_i \in F_{j-1}$ and $y_i \in \pi_{\geq j}$. Now put $x_{\geq j} := y_1y_2\cdots y_s$, then $g(x_{\geq j}) = g(x) \neq 0$ as $g(f_i) = 0$ for $i = 1, \ldots, s$. By Lemma 2.2.7, there exists a homomorphism $\pi \to \pi$ mapping l_j to $x_{\geq j}$. Composing with g, we get a homomorphism $\pi \to \mathbb{Z}$ mapping all l_i to non-zero integers, contradicting Proposition 2.2.8. □

Finally let $Hom(\pi, \mathbb{Z}) \to \prod_{i=1}^{n} \mathbb{Z}$ be the homomorphism that sends $Hom(\pi, \mathbb{Z}) \ni g$ to the tuple $(g(l_1), g(l_2), \ldots)$:

Corollary 2.2.10. *The above homomorphism maps the group $Hom(\pi, \mathbb{Z})$ injectively to $\bigoplus_{i=1}^{\infty} \mathbb{Z}$, which is countable. In particular $\pi_1(\mathbb{H}) \cong \pi$ is not free.*

Now we show that multiplication in $\pi_1^{qtop}(\mathbb{H})$ is discontinuous.

Let $\Omega(\mathbb{H}, 0)$ be the space of loops on $\mathbb{H}$ based at 0 and, as usual, endow $\Omega(\mathbb{H}, 0)$ with the compact-open topology (which in this case coincides with the topology of uniform convergence, see Theorem 1.1.6). Let $h\colon \Omega(\mathbb{H}, 0) \to \pi_1^{qtop}(\mathbb{H})$ denote the canonical quotient map identifying homotopic loops in $\Omega(\mathbb{H}, 0)$ (cfr. Definition 2.1.16 and Remark 2.1.17).

Definition 2.2.11. Let h_n be the point $(2/n, 0) \in C_n$. Define the *oscillation number* $O_n\colon \Omega(\mathbb{H}, 0) \to \mathbb{N}$ to be the maximum number m such that there exists a set $T = \{0, t_1, \ldots, t_{2m}\} \subseteq [0, 1]$ such that $0 < t_1 < \cdots < t_{2m} = 1$ with $f(t_{2i}) = 0$ and $f(t_{2i+1}) = h_n$, where $f \in \Omega(\mathbb{H}, 0)$.

Roughly speaking, $O_n(f)$ tells how many times f wraps around the n-th circle; $O_n(f)$ can be seen as the weight of the symbol l_n into the infinite word f.

We now list some results which will be exploited in Theorem 2.2.14 , the reader may find the proofs in [**12**].

Lemma 2.2.12.

1. *The number $O_n(f)$ is finite for all $n \in \mathbb{N}$ and $f \in \Omega(\mathbb{H}, 0)$ (this is obvious if one remembers how elements of π are made up).*

2. *Suppose $f_k \xrightarrow{k\to\infty} f$ uniformly and $O_n(f_k) \geq m$. Then $O_n(f) \geq m$.*

3. *Suppose f and g are in the same path component of $\Omega(\mathbb{H}, 0)$ and suppose $g\colon I \to H_m$ corresponds to a maximally reduced finite word w in the free group F_m on m letters. Then $O_n(f) \geq O_n(g)$.*

Lemma 2.2.13.

1. *The path components of $\Omega(\mathbb{H}, 0)$ are closed subspaces of $\Omega(\mathbb{H}, 0)$.*

2. *If Z is a metric space such that each path component of Z is a closed subspace of Z, then each path component of $Z \times Z$ is a closed subspace of $Z \times Z$.*

Theorem 2.2.14. *With the above notations:*

1. *The product of quotient maps $h \times h\colon \Omega(\mathbb{H}, 0) \times \Omega(\mathbb{H}, 0) \to \pi_1^{qtop}(\mathbb{H}) \times \pi_1^{qtop}(\mathbb{H})$ fails to be a quotient map.*

2. *The standard multiplication of path $\mu\colon \pi_1^{qtop}(\mathbb{H}) \times \pi_1^{qtop}(\mathbb{H}) \to \pi_1^{qtop}(\mathbb{H})$ is discontinuous and so $\pi_1^{qtop}(\mathbb{H})$ fails to be a topological group.*

Proof. 1. Let $x_n \in \Omega(\mathbb{H}, 0)$ be the loop that orbits C_n once counterclockwise, so that $\pi_1(C_n)$ is generated by $[x_n] = l_n$. Applying path concatenation, for integers $n \geq 2$ and $k \geq 2$, let $a(n,k) \in \Omega(\mathbb{H}, 0)$ be a based loop corresponding to the finite word $\left(l_n l_k l_n^{-1} l_k^{-1}\right)^{n+k}$ and let $w(n,k) \in \Omega(\mathbb{H}, 0)$ be a based loop corresponding to the finite word $\left(l_1 l_k l_1^{-1} l_k^{-1}\right)^n$. Let $D \subseteq \pi_1^{qtop}(\mathbb{H}) \times \pi_1^{qtop}(\mathbb{H})$ denote the set of all doubly indexed ordered pairs $([a(n,k)], [w(n,k)])$. Let $c_0 \in \Omega(\mathbb{H}, 0)$ be the constant path at 0.

To prove $h \times h$ fails to be a quotient map, it is enough to show D is not closed in $\pi_1^{qtop}(\mathbb{H}) \times \pi_1^{qtop}(\mathbb{H})$, but that $(h \times h)^{-1}(D)$ is closed in $\Omega(\mathbb{H}, 0) \times \Omega(\mathbb{H}, 0)$.

To prove D is not closed in $\pi_1^{qtop}(\mathbb{H}) \times \pi_1^{qtop}(\mathbb{H})$, we will prove that $([c_0], [c_0]) \notin D$ but $([c_0], [c_0])$ is a limit point of D. Recall from Section 2.2.1 that $\phi\colon \pi_1(\mathbb{H}) \to \pi$ is a bijection and that we have chosen $k \geq 2$. Hence $[c_0] \neq [a(n,k)]$ and $[c_0] \neq [w(n,k)]$ for all $k \geq 2$. So $([c_0], [c_0]) \notin D$.

Suppose $[c_0] \in U$ and U is open in $\pi_1^{qtop}(\mathbb{H})$. Let $V := h^{-1}(U)$. Then V is open in $\Omega(\mathbb{H}, 0)$ because, by definition, h is continuous. In particular $c_0 \in V$, thus there exist N and K such that if $n \geq N$ and $k \geq K$, then $a(n,k) \in V$ (for instance, notice that for large n and k, $a(n,k) \longrightarrow c_0$). Notice also that $\left(l_1 l_1^{-1}\right)^N \in V$; moreover for some suitable parametrization on $[0,1]$, $w(N,k) \xrightarrow{k\to\infty} \left(l_1 l_1^{-1}\right)^N$ uniformly in $\Omega(\mathbb{H}, 0)$. Thus there exists $K_2 > K$ such that if $k > K_2$, then $w(N,K) \in V$. Hence $([a(N,K_2)], [w(N,K_2)]) \in U \times U$. This shows $([c_0], [c_0])$ is a limit point of D and thus D is not closed in $\pi_1^{qtop}(\mathbb{H}) \times \pi_1^{qtop}(\mathbb{H})$.

To prove $(h \times h)^{-1}(D)$ is closed in $\Omega(\mathbb{H}, 0) \times \Omega(\mathbb{H}, 0)$, suppose $(f_m, g_m) \xrightarrow{m\to\infty} (f,g)$ uniformly and $(f_m, g_m) \in (h \times h)^{-1}(D)$ for all m. Notice that $O_1(w(n,k)) = 2n$ and $O_N(a(N,k)) \geq 2(N+k)$. Let $a(n_m, k_m)$ and $w(n_m, k_m)$ be path homotopic to respectively f_m and g_m. By (3) of Lemma 2.2.12, $O_1(g_m) \geq O_1(w(n_m, k_m)) = 2n_m$. Thus if $\{n_m\}$ contains an unbounded subsequence, then, by (2) of Lemma 2.2.12, $O_1(g) \geq \limsup O_1\left(w\left(n_m, k_{n_m}\right)\right) = \infty$ and we have a contradiction since $O_1(g) < \infty$. Thus $\{n_m\}$ is bounded and so the sequence $\{n_m\}$ takes on finitely many values. Similarly, if $\{k_m\}$ is unbounded, then there exist N and a subsequence $\{k_{m_l}\}$ such that $O_N\left(a\left(N, k_{m_l}\right)\right) \to \infty$. Thus $O_N(f) \geq \limsup O_N\left(a\left(N, k_{m_l}\right)\right) = \infty$ contradicting the fact that $O_N(f) < \infty$. Thus both $\{n_m\}$ and $\{k_m\}$ are bounded and hence there exists a path component $A \subseteq \Omega(\mathbb{H},0) \times \Omega(\mathbb{H},0)$ containing a subsequence (f_{m_l}, g_{m_l}). Thus, by Lemma 2.2.13, $(f,g) \in A$. So $(h \times h)^{-1}(D)$ is closed and $h \times h$ fails to be a quotient map.

2. Now let us see that the group multiplication $\mu\colon \pi_1^{qtop}(\mathbb{H}) \times \pi_1^{qtop}(\mathbb{H}) \to \pi_1^{qtop}(\mathbb{H})$ is discontinuous. To achieve this result, we build a closed set $A \subseteq \pi_1^{qtop}(\mathbb{H})$ such that $\mu^{-1}(A)$ is not closed in $\pi_1^{qtop}(\mathbb{H}) \times \pi_1^{qtop}(\mathbb{H})$.

Consider the doubly indexed set $A = \mu(D) \subseteq \pi_1^{qtop}(\mathbb{H})$ such that each element of A is of the form $[\gamma_{n,k}] = [a(n,k)] * [w(n,k)]$.

First, observe that by definition $[\gamma_{n,k}] \neq [c_0]$ for all n, k (remember that $k \geq 2$). So $[c_0] \notin A$ and $([c_0], [c_0]) \notin \mu^{-1}(A)$. Notice that $D \subseteq \mu^{-1}(A)$ and as shown above, $([c_0], [c_0])$ is a limit point of D. So $\mu^{-1}(A)$ is not closed in $\pi_1^{qtop}(\mathbb{H}) \times \pi_1^{qtop}(\mathbb{H})$. On the other hand, A is closed in $\pi_1^{qtop}(\mathbb{H})$: we show it by proving $h^{-1}(A)$ is closed in $\Omega(\mathbb{H},0)$ (remember that h is continuous). Suppose $f_m \longrightarrow f \in \Omega(\mathbb{H},0)$ and $f_m \in h^{-1}(A)$. Find n_m, k_m such that $f_m \in [a(n,k)] * [w(n,k)]$. Similarly as in (1), if $\{n_m\}$ is unbounded we get the contradiction $O_1(f) \geq \text{limsup} O_1(f_m) = \infty$. If $\{n_m\}$ is bounded and $\{k_m\}$ is unbounded we find N and a subsequence $\{k_{m_l}\}$ and again the contradiction $O_N(f) \geq \text{limsup} O_N(f_{m_l}) = \infty$. Thus both $\{n_m\}$ and $\{k_m\}$ are bounded. It follows that some path component $B \subseteq \Omega(\mathbb{H},0)$ contains a subsequence $\{f_{m_l}\}$ and it follows from (1) of Lemma 2.2.13 that $f \in B$. Hence $h^{-1}(A)$ is closed in $\Omega(\mathbb{H},0)$ and thus A is closed in $\pi_1^{qtop}(\mathbb{H})$. □

2.3 FROM GROUPS TO GROUPOIDS

In what follows we will make use **Top**-groupoids and **Top**-functors. For notations and definitions see the initial section “Notation and conventions”.

2.3.1 Quasitopological groupoids

Definition 2.3.1. A **qTop**-*groupoid* is a groupoid $\mathcal{G}$ whose hom-sets $\mathcal{G}(x,y)$ are equipped with topologies such that multiplications $\mathcal{G}(x,y) \times \mathcal{G}(y,x) \to \mathcal{G}(x,z)$ are continuous in each variable and each inversion function $\mathcal{G}(x,y) \to \mathcal{G}(y,x)$ is continuous. A morphisim of **qTop**-groupoids is a functor $F\colon \mathcal{G} \to \mathcal{G}'$ such that each function $F\colon \mathcal{G}(x,y) \to \mathcal{G}'(F(x), F(y))$, $f \to F(f)$, is continuous. Let **qTopGrpd** be the category of **qTop**-groupoids. Since every **Top**-groupoid is a **qTop**-groupoid, **TopGrpd** is a full subcategory of **qTopGrpd**.

Definition 2.3.2. Let $\mathcal{G}$ be a **qTop**-groupoid. For each $g \in \mathcal{G}(x,y)$ we define the *left-translation* $\lambda_g\colon \mathcal{G}(w,x) \to \mathcal{G}(w,y)$ by $f \mapsto g \circ f$ and the *right translation* $\rho_g\colon \mathcal{G}(y,z) \to \mathcal{G}(x,z)$ by $h \mapsto h \circ g$.

Lemma 2.3.3. *The multiplication $\mathcal{G}(x,y) \times \mathcal{G}(y,x) \to \mathcal{G}(x,z)$ is continuous in each variable if and only if λ_g and ρ_g are homeomorphisms.*

Proof. Let the multiplication be continuous in each variable and fix $g \in \mathcal{G}(x,y)$. Then the map λ_g as defined above is continuous. Moreover it is invertible, having $\lambda_{g^{-1}}$ as inverse (notice that g^{-1} exists since $\mathcal{G}$ is a groupoid). So λ_g is continuous, bijective and with continuous inverse, hence a homeomorphism. Analogously one shows ρ_g is a homeomorphism.

Conversely, let λ_g and ρ_g be homeomorphisms for all g. Then the multiplication $\mu\colon \mathcal{G}(x,y) \times \mathcal{G}(y,x) \to \mathcal{G}(x,z)$ is continuous in the first component because if $h \in \mathcal{G}(y,z)$ is fixed, for all $f \in \mathcal{G}(x,y)$ one has

$\mu(f,h) = h \circ f = \lambda_h(f)$ which is continuous by hypothesis. Analogously for the second component using right translations. □

Definition 2.3.4. We denote by $\Pi_1^{qtop} X$ the *quasitopological fundamental groupoid* of X, where, for each $x_1, x_2 \in X$, $\Pi_1^{qtop} X(x_1, x_2)$ is the quotient space $\pi_0^{qtop}(\mathcal{P}X(x_1, x_2))$ (see Definitions 1.3.11 and 2.1.16).

Proposition 2.3.5. *The quasitopological fundamental groupoid has the structure of a* **qTop**-*groupoid. Moreover* Π_1^{qtop}: **Top** $\to$ **qTopGrpd** *is a functor.*

Proof. We easily get convinced that $\Pi_1^{qtop} X$ is a **qTop**-groupoid: for instance, consider the commuting diagram

$$\begin{array}{ccc} \mathcal{P}X(x_1,x_2) & \longrightarrow & \pi_0^{qtop}(\mathcal{P}X(x_1,x_2)) \\ \downarrow & & \downarrow \\ \mathcal{P}X(x_2,x_3) & \longrightarrow & \pi_0^{qtop}(\mathcal{P}X(x_2,x_3)) \end{array} \quad \text{such that} \quad \begin{array}{ccc} \alpha & \longmapsto & [\alpha] \\ \downarrow & & \downarrow \\ \alpha * \beta & \longmapsto & [\alpha * \beta] = [\alpha] * [\beta] \end{array}$$

where horizontal arrows are the (continuous) quotient maps, the left vertical arrow is the (continuous) right concatenation of paths (for some $\beta \in \mathcal{P}X(x_2, x_3)$ fixed) and the right vertical arrow is the (continuous) right translation. One gets similar diagrams when considering left concatenation or inversion of paths.

Analogously, any map $f\colon X \to Y$ induces the map

$$\Pi_1^{qtop}(\mathcal{P}(f)) : \Pi_1^{qtop} X(x_1, x_2) \to \Pi_1^{qtop} Y(f(x_1), f(x_2))$$

given by $[\alpha] \mapsto [f \circ \alpha]$. □

Finally we can extend the definition of τ from groups (cfr. Lemma 2.1.6) to groupoids: it is enough to apply the group-valued functor τ to all vertex groups (i.e. to all hom-sets $\Pi_1^{qtop} X(x,x)$) and extend via translation. Next lemma gives the details:

Lemma 2.3.6. *The forgetful functor* for: **TopGrpd** $\to$ **qTopGrpd** *has a left adjoint* τ: **qTopGrpd**$\to$ **TopGrpd** *which is the identity on the underlying groupoids.*

Proof. Let $\mathcal{G}$ be a **qTop**-groupoid. For each $x \in ob(\mathcal{G})$, let $\tau(\mathcal{G})(x)$ be the topological group $\tau(\mathcal{G}(x))$. If $x \neq y$ and $\mathcal{G}(x,y) \neq \emptyset$, let $\tau(\mathcal{G})(x,y)$ have the topology generated by sets of the form $g \circ U = \{g \circ u\colon u \in U\}$ where $g \in \mathcal{G}(x,y)$ and U is open in $\tau(\mathcal{G})(x)$. Since $g_2^{-1} \circ g_1 \circ U$ is open in $\tau(\mathcal{G})(x)$ for all $g_1, g_2 \in \mathcal{G}(x,y)$, all left translations $\lambda_g\colon \tau(\mathcal{G})(x) \to \tau(\mathcal{G})(x,y)$, $f \mapsto g \circ f$, are homeomorphisms. Notice that if $g \in \mathcal{G}(x,y)$, then $\lambda_g \circ \rho_{g^{-1}}\colon \mathcal{G}(x) \to \mathcal{G}(y)$, $h \mapsto g \circ h \circ g^{-1}$, is an isomorphism of quasitopological groups (indeed, the inverse is $\lambda_{g^{-1}} \circ \rho_g\colon \mathcal{G}(y) \to \mathcal{G}(x)$, $f \mapsto g^{-1} \circ f \circ g$). We already know that τ: **qTopGrp** $\to$ **TopGrp** is a functor at level of groups, hence $\tau(\mathcal{G})(x) \to \tau(\mathcal{G})(y)$ is an isomorphism of topological groups, in particular it is a continuous map. Thus also all right translations $\rho_g\colon \tau(\mathcal{G})(y) \to \tau(\mathcal{G})(x,y)$, $h \mapsto h \circ g$, are homeomorphisms; pictorially, one has the commuting diagram

$$\begin{array}{ccc} \tau(\mathcal{G})(x) & \xrightarrow{\lambda_g \circ \rho_{g^{-1}}} & \tau(\mathcal{G})(y) \\ \Big\downarrow \lambda_g & \swarrow{\scriptstyle \rho_g} & \\ \tau(\mathcal{G})(x,y) & & \end{array}$$

where horizontal and vertical arrows are homeomorphisms: this forces ρ_g to be a homeomorphism too.

All vertex groups are topological groups and thanks to the homeomorphisms we have just built, we get the commuting diagram

$$\begin{array}{ccc} \tau(\mathcal{G})(x,y) \times \tau(\mathcal{G})(y,z) & \dashrightarrow & \tau(\mathcal{G})(x,z) \\ \downarrow \cong & & \cong \uparrow \\ \tau(\mathcal{G})(y) \times \tau(\mathcal{G})(y) & \longrightarrow & \tau(\mathcal{G})(y) \cong \tau(\mathcal{G})(x) \cong \tau(\mathcal{G})(z) \end{array}$$

where vertical arrows are homeomorphisms one gets by translations and the bottom arrow is the multiplication, which is jointly continuous in both entries; this proves that $\tau(\mathcal{G})$ is a **Top**-groupoid at least at level of objects.

Concerning morphisms, a morphism $F\colon \mathcal{G} \to \mathcal{G}'$ of **qTop**-groupoids induces a morphism $\tau(F) : \tau(\mathcal{G}) \to \tau(\mathcal{G}')$ of **Top**-groupoids since the group-valued τ on vertex groups gives continuous homomorphisms $\tau(\mathcal{G})(x) \to \tau(\mathcal{G}')(F(x))$ and continuity extends to all hom-sets via translations. □

Let us remark also that the vertex groups $\tau(\mathcal{G})(x)$ of $\tau(\mathcal{G})$ are the topological groups $\tau(\mathcal{G}(x))$. Since each identity $\mathcal{G}(x) \to \tau(\mathcal{G})(x)$ is continuous (cfr. Remark 2.1.7), it follows that the identity functor $\tilde{id}\colon \mathcal{G} \to \tau(\mathcal{G})$ is a morphism of **qTop**-groupoids. Similar to the situation with groups, a **qTop**-groupoid $\mathcal{G}$ is a **Top**-groupoid if and only if $\mathcal{G} = \tau(\mathcal{G})$ (cfr. (2) of Proposition 2.1.10).

Finally, by applying the functor τ to the quasitopological fundamental groupoid, we can define:

Definition 2.3.7. The *fundamental* **Top**-*groupoid* of a topological space X is the **Top**-groupoid $\Pi_1^\tau X := \tau\left(\Pi_1^{qtop} X\right)$.

So far we have shown that $\Pi_1^\tau X$ exists but we have no idea about how it is done; for an inductive construction of this object, see Appendix A.

Informally, we could say that in general $\Pi_1^{qtop} X(x,y)$ has "too many" open sets: the induction argument works by removing step by step some of these open sets untill a true topological group is reached. So as a corollary, we have also:

Corollary 2.3.8. *For each $x, y \in X$, the canonical map $h\colon \mathcal{P}X(x,y) \to \Pi_1^\tau X(x,y)$ identifying homotopy classes of paths are continuous.*

Proof. The topology of $\Pi_1^\tau X(x,y)$ is coarser than that of $\Pi_1^{qtop} X(x,y)$ and $h\colon \mathcal{P}X(x,y) \to \Pi_1^\tau X(x,y)$ is continuous by definition. □

Chapter 3

SEMICOVERINGS AND ENRICHED MONODROMY

In this chapter we introduce the central notions of *semicovering* and describe some sufficient conditions for a semicovering to be a classical covering. Next we give the definition of *open covering morphism* of **Top**−groupoids to show that a semicovering always induces such a functor via the enriched fundamental groupoid functor Π_1^τ; after describing the enriched monodromy μ^τ and the enriched equivalence $\mathscr{R}^\tau$, we obtain the topologically enriched version of Theorem 1.3.24.

3.1 SEMICOVERINGS

3.1.1 Definition and properties of semicoverings

Definition 3.1.1. A map $p\colon Y \to X$ has:

1. *continuous lifting of paths* if $\mathcal{P}p\colon (\mathcal{P}Y)_y \to (\mathcal{P}X)_{p(y)}$ is a homeomorphism for each $y \in Y$.

2. *continuous lifting of homotopies* if $\mathcal{H}p\colon (\mathcal{H}Y)_y \to (\mathcal{H}X)_{p(y)}$ is a homeomorphism for each $y \in Y$.

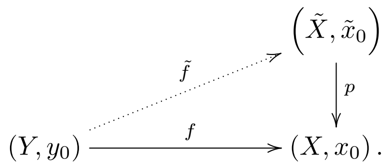
In such a case $\mathcal{P}p$ and $\mathcal{H}p$ are homeomorphisms with respect to compact-open topology. Certainly, every map with continuous lifting of paths has unique path lifting (Lemma 1.3.29): indeed, unique path lifting is equivalent to the injectivity of each map $\mathcal{P}p$ in the above definition. The condition that $\mathcal{P}p$ be a homeomorphism is much stronger than the existence and uniqueness of lifts of paths since each inverse $L_p\colon (\mathcal{P}X)_{p(y)} \to (\mathcal{P}Y)_y$, which we call *lifting homeomorphism,* taking a path α to the unique lift $\tilde{\alpha}_y$ starting at y, is required to be continuous.

As for paths, if we drop the continuity request on $\mathcal{H}p$, we find that every map with continuous lifting of homotopies has also the unique homotopy lifting property (Lemma 1.3.29).

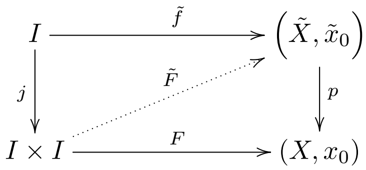
The notion of *semicovering*, first introduced by Jeremy Brazas in [**5**], is the core of this section: in three different papers ([**5**], [**7**] and [**21**]) there are given three different definitions which turn out to be equivalent. We skip the technical details but we will give all three definitions and show their equivalence.

First, we need two introductory lemmas: proofs may be found in [**13**] and [**21**].

Lemma 3.1.2. *Let $p\colon \tilde{X} \to X$ be a local homeomorphism, Y a connected space, $\tilde{X}$ a Hausdorff space and let $f\colon (Y, y_0) \to (X, x_0)$ be a continuous map. Given $\tilde{x}_0 \in p^{-1}(x_0)$, there exists at most one lifting $\tilde{f}\colon (Y, y_0) \to \left(\tilde{X}, \tilde{x}_0\right)$ of f, i.e. there is at most one continuous map $\tilde{f}$ making the following diagram commute:*



Lemma 3.1.3. *Let $p\colon \tilde{X} \to X$ be a local homeomorphism with unique path lifting property. Consider this diagram of continuous maps*



where $j(t) = (t, 0)$ for all $t \in I$. Then there exists a unique continuous map $\tilde{F}\colon I \times I \to \tilde{X}$ making the diagram commute.

Roughly speaking, Lemma 3.1.3 tells that each local homeomorphism enjoying the unique path lifting property, has also the unique lifting homotopy property (see Lemma 1.3.29).

Theorem 3.1.4. *Let $p\colon Y \to X$ be a local homeomorphism. Then the following are equivalent:*

1. *p has continuous lifting of paths.*

2. *p has continuous lifting of paths and homotopies.*

3. *p is a fibration with unique path lifting.*

Proof. $1 \Rightarrow 2$. We have to show that the continuous lifting of paths implies the continuous lifting of homotopies. So let $D^2 \subseteq \mathbb{R}^2$ be the closed unit disk with base point $d_0 = (1, 0)$ and $(\mathcal{H}X)_{x_0}$ be the space of based (continuous) maps $\mathbf{Top}\left(\left(D^2, d_0\right), (X, x_0)\right)$ with the compact-open topology. Now we show that $(\mathcal{H}X)_{x_0}$ is homeomorphic to the space $\Omega\left((\mathcal{P}X)_{x_0}, c_{x_0}\right)$ of loops $\mathbb{S}^1 \to (\mathcal{P}X)_{x_0}$ based at the constant loop c_{x_0} at x_0: if $\lambda_t\colon I \to D^2$ is the linear path from d_0 to a point $t \in \mathbb{S}^1$, then $\Psi\colon (\mathcal{H}X)_{x_0} \to \Omega\left((\mathcal{P}X)_{x_0}, c_{x_0}\right)$ given by $\Psi(f)(t) = f \circ \lambda_t$ is a homeomorphism. Hence thanks to functoriality of $\mathcal{H}$ and $\mathcal{P}$, we get that $\mathcal{H}p\colon (\mathcal{H}Y)_y \to (\mathcal{H}X)_{p(y)}$ is a homeomorphism whenever $\mathcal{P}p\colon (\mathcal{P}Y)_y \to (\mathcal{P}X)_{p(y)}$ is a homeomorphism.

$2 \Rightarrow 3$. This is obvious as noticed in describing the continuous lifting of paths property.

$3 \Rightarrow 1$. Let $p\colon Y \to X$ be a local homeomorphism with unique path lifting, fix $x \in X$ and $y \in p^{-1}(x)$. The map $\mathcal{P}p\colon (\mathcal{P}Y)_y \to (\mathcal{P}X)_x$ is bijective because p has unique path lifting. It is known that $\mathcal{P}p$ is continuous, so it is enough to prove that $\mathcal{P}p$ is open.

We know from the description of the compact-open topology in Section 1.1 that a basic open set in $(\mathcal{P}Y)_y$ is of the form $\tilde{U} = \bigcap_{j=1}^{n} M\left(K_n^j, \tilde{U}_j\right) \cap (\mathcal{P}Y)_y$, where $K_n^j = \left[\frac{j-1}{n}, \frac{j}{n}\right]$ and $\tilde{U}_j$ is an open set in Y such that $p|_{\tilde{U}_j}\colon \tilde{U}_j \to p\left(\tilde{U}_j\right)$ is a homeomorphism for each $j = 1, \ldots, n$. Suppose that

$$U = \bigcap_{j=1}^{n} M\left(K_n^j, p\left(\tilde{U}_j\right)\right) \cap \bigcap_{j=1}^{n-1} M\left(K_n^j \cap K_n^{j+1}, p\left(\tilde{U}_j \cap \tilde{U}_{j+1}\right)\right) \cap (\mathcal{P}X)_x .$$

We show that $\mathcal{P}p\left(\tilde{U}\right) = U$. Since p is an open map, $p\left(\tilde{U}_j\right)$ is open in X for $j = 1, \dots, n$. Moreover it is obvious that $\mathcal{P}p\left(\tilde{U}\right) \subseteq U$. Conversely, let $\alpha \in U$. Since p has unique path lifting, we can find a lift $\tilde{\alpha} \in (\mathcal{P}Y)_y$. Choose $t \in K_n^j$. Since $p|_{\tilde{U}_j}$ and $p|_{\tilde{U}_j \cap \tilde{U}_{j+1}}$ are homeomorphisms and $\alpha\left(K_n^j \cap K_n^{j+1}\right) \subseteq p\left(\tilde{U}_j \cap \tilde{U}_{j+1}\right)$, we have $p|_{\tilde{U}_j}(\tilde{\alpha}(t)) = \alpha(t)$, therefore $\tilde{\alpha}(t) \in \tilde{U}_j$. So $\tilde{\alpha} \in \tilde{U}$ and hence $\mathcal{P}p\left(\tilde{U}\right) = U$. Since U is open in X and $\mathcal{P}p\left(\tilde{U}\right) = U$, $\mathcal{P}p$ is a homeomorphism as we wanted. □

Definition 3.1.5. A *semicovering map* $p\colon Y \to X$ is a local homeomorphism enjoying one of the equivalent properties in Theorem 3.1.4. In such a case Y is called *semicovering space.* If $p'\colon Y' \to X$ is another semicovering of X, a *morphism of semicoverings* is a continuous map $f\colon Y \to Y'$ such that $p' \circ f = p$.

With such definitions there remains defined the category $\mathbf{SCov}(X)$ of semicoverings of X. Two semicoverings are called *equivalent* if they are isomorphic in this category. A semicovering $p\colon Y \to X$ is *path connected* if Y is non-empty and path-connected: let $\mathbf{SCov_0}(X)$ be the full subcategory of path connected semicoverings of X. If p is an initial object in $\mathbf{SCov_0}(X)$, we call p a *universal semicovering* of X.

Let us remark that if $p\colon Y \to X$ is a semicovering and $y \in Y$ and $x \in X$, any path $\alpha \in \mathcal{P}X(p(y), x)$ lifts to a path $\tilde{\alpha}_y$ such that $p(\tilde{\alpha}_y(1)) = x$. Hence if X is path-connected, p is surjective. Moreover, every fiber of a local homeomorphism is discrete ([**23**, 12.3]), so:

Proposition 3.1.6. *Each fiber of any semicovering is discrete.*

As shown in [**30**, II.2] we have also (cfr. also Propositions 1.3.15 and 1.3.32):

Proposition 3.1.7. *Semicoverings share the same lifting properties of classical covering maps, so semicoverings are always fibrations. In particular any semicovering map $p\colon Y \to X$ induces a covering morphism $\Pi_1 p\colon \Pi_1 Y \to \Pi_1 X$ of fundamental groupoids.*

Since every semicovering map is a local homeomorphism, we immediately get the following:

Corollary 3.1.8. *If $p\colon Y \to X$ is a semicovering map and X is locally path connected, then so is Y.*

The following Lemma is an easy generalization of [**24**, V.4-5]:

Lemma 3.1.9. *Let $p\colon Y \to X$ be a semicovering map. If $p(y_i) = x_i$, $i = 1, 2$, and $\beta \in \mathcal{P}X(x_1, x_2)$, then $[\beta]$ lies in the image of $\Pi p\colon \Pi Y(y_1, y_2) \to \Pi X(x_1, x_2)$ if and only if $\tilde{\beta}_{y_1}(1) = y_2$.*

It is a known result ([**15**, 13.20]) that if one does not restrict to spaces with universal covering, the composition of two connected covering maps is not necessarily a covering map. On the other hand we now show that composition of any two semicoverings is always a semicovering:

Lemma 3.1.10. *Let $p\colon X \to Y$, $q\colon Y \to Z$ and $r = q \circ p$ be surjective maps. If two of p, q, r are local homeomorphisms, then so is the third. If two of p, q, r have continuous lifting of paths and homotopies, then so does the third.*

Proof. [Just a sketch.]

Let p, q be local homeomorphisms and $x \in X$. Then there exists an open neighborhood U of x such that $p|_U\colon U \to p(U)$ is a homeomorphism. In particular $p(x) \in P(U)$; since q is a local homeomorphism, there exists an open neighborhood V of $p(x)$ such that $V \subseteq p(U)$ and $q|_V\colon V \to q(V)$ is a homeomorphism. By noticing that $p|_U^{-1}(V)$ is an open neighborhood of x, one gets that r is a local homeomorphism.

Let p, r be local homeomorphisms and $y \in Y$. Since p is a surjective local homeomorphism, there exists an open neighborhood U of $x \in p^{-1}(y)$ such that $p|_U : U \to p(U)$ is a homeomorphism. Since r is a local homeomorphism, there exists an open neighborhood U' of x such that $r|_{U'} : U' \to r(U')$ is a homeomorphism. Define $U'' = U \cap U'$; then $p|_{U''} : U'' \to p(U'') \ni y$ and $r|_{U''} : U'' \to r(U'') \ni q(y)$. But $r(U'') = q(p(U''))$ and $p(U'')$ are open, hence q is a homeomorphism.

Let p, q have continuous lifting of paths and homotopies. Then for each $x \in X$, $(\mathcal{P}X)_x \to (\mathcal{P}Y)_{p(x)}$ and $(\mathcal{P}Y)_{p(x)} \to (\mathcal{P}Z)_{r(x)}$ are homeomorphisms and so their composition is a homeomorphism too.

The remaining cases are easily shown as these ones. □

Theorem 3.1.11. *Let $p\colon X \to Y$, $q\colon Y \to Z$ and $r = q \circ p$ be maps of connected spaces. If two of p, q, r are semicoverings, so is the third.*

Proof. By the above lemma it is enough to show that if two of p, q, r are semicoverings, the third map is surjective. Fix $x_0 \in X$ and define $y_0 = p(x_0)$, $z_0 = q(y_0) = r(x_0)$. If p, q are semicoverings, they are both surjective since Y and Z are path-connected; thus r is surjective.

If q, r are semicoverings and $y \in Y$, take $\alpha \in (\mathcal{P}Y)_{y_0}$ with $\alpha(1) = y$. Then $q \circ \alpha \in (\mathcal{P}Z)_{z_0}$ has unique lift $\widetilde{q \circ \alpha}_{x_0}$ (with respect to r) with endpoint $x = \widetilde{q \circ \alpha}_{x_0}(1)$. Since $q \circ p \circ \widetilde{q \circ \alpha}_{x_0} = q \circ \alpha$ and q has unique path lifting, we must conclude that $p \circ \widetilde{q \circ \alpha}_{x_0} = \alpha$. So $p(x) = \alpha(1) = y$ and p is surjective. Finally suppose p, r are semicoverings. Since Z is path-connected r is surjective; since $r = q \circ p$, r is surjective and p is surjective, q must be surjective too. □

Theorem 3.1.12. *For any topological space X, $\mathbf{Cov}(X)$ and $\mathbf{Cov_0}(X)$ are full subcategories of $\mathbf{SCov}(X)$ and $\mathbf{SCov_0}(X)$ respectively.*

So any covering is a semicovering; the proof may be found in [**5**, 3.7].

3.1.2 When is a local homeomorphism a semicovering map?

In this section we give some sufficient conditions for a local homeomorphism to be a semicovering map. We prove only the main results: proofs of preliminary lemmas may be found in [**21**].

Lemma 3.1.13. *Let $p\colon Y \to X$ be a local homeomorphism. Suppose that α is an arbitrary path in X and $y_0 \in p^{-1}(\alpha(0))$ such that there is no lifting of α starting at y_0. Set*

$$A_\alpha := \left\{ t \in I\colon\ \alpha|_{[0,t]} \text{ has a lifting } \hat{\alpha}_t \text{ on } [0,t] \text{ with } \hat{\alpha}_t(0) = y_0 \right\}.$$

Then A_α is open and connected. Moreover there exists an $s \in I$ such that $A_\alpha = [0, s)$.

Lemma 3.1.14. *Let $p\colon Y \to X$ be a local homeomorphism with at most one lifting for each path $\beta\colon I \to X$; let α be an arbitrary path in X and $y_0 \in p^{-1}(\alpha(0))$ such that there is no lifting of α starting at y_0. Then, using the same notations as in previous lemma, there exists a unique continuous map $\tilde{\alpha}_s\colon A_s = [0, s) \to Y$ such that $p \circ \tilde{\alpha}_s = \alpha|_{[0,s)}$.*

Definition 3.1.15. The path $\tilde{\alpha}_s$ defined in Lemma 3.1.14 is called *incomplete lifting* of α by p starting at y_0.

Theorem 3.1.16. *If Y is Hausdorff and sequential compact and $p\colon Y \to X$ is a local homeomorphism, then p is a semicovering map.*

Proof. Let $\alpha\colon I \to X$ be a path which has no lifting starting at $y_0 \in p^{-1}(\alpha(0))$. Let $\tilde{\alpha}\colon A_s = [0, s) \to Y$ be the incomplete lifting of α at y_0. Suppose $\{t_n\}_{n\in\mathbb{N}}$ is a sequence in A_s which tends to t_0 and $t_n \leq t_0$ for all n. Since Y is sequential compact, there exists a convergent subsequence of $\{\tilde{\alpha}(t_n)\}_{n\in\mathbb{N}}$, call it $\{\tilde{\alpha}(t_{n_k})\}_{k\in\mathbb{N}}$, such that $\tilde{\alpha}(t_{n_k})$ tends to some l. Now define

$$g(t) = \begin{cases} \tilde{\alpha}(t) & 0 \leq t < t_0 \\ l = \lim_{k\to\infty} \tilde{\alpha}(t_{n_k}) & t = t_0 \end{cases}.$$

We have $p(l) = p(\lim_{k\to\infty} \tilde{\alpha}(t_{n_k})) = \lim_{k\to\infty} p(\tilde{\alpha}(t_{n_k})) = \lim_{k\to\infty} \alpha(t_{n_k}) = \alpha(t_0)$ and so $p \circ g = \alpha$.

Now we show that g is continuous at t_0 (in $[0, t_0)$ we already know it is). Since p is a local homeomorphism, there exists a neighborhood W at l such that $p|_W\colon W \to p(W)$ is a homeomorphism. Hence there is $a \in I$ such that $\alpha([a, t_0]) \subseteq p(W)$. Let V be a neighborhood at l and $W' = V \cap W$, then $p(W') \subseteq p(W)$ is an open set. Put $U = \alpha^{-1}(p(W')) \cap (a, t_0]$, which is open in $[0, t_0]$. It is enough to show that $g(U) \subseteq W'$. Since $\alpha(U) \subseteq p(W')$ and p is a homeomorphism on W', $(p|_W)^{-1}(\alpha(U)) \subseteq (p|_W)^{-1}(p(W'))$ and so $(p|_W)^{-1} \circ \alpha = \tilde{\alpha}$ on $[a, t_0)$ since $p(l) = \alpha(t_0)$. Hence $(p|_W)^{-1} \circ \alpha = g$ on $[a, t_0]$. Thus $g(U) \subseteq g\left(\alpha^{-1}(p(W'))\right) = (p|_W)^{-1} \circ \alpha\left(\alpha^{-1}(p(W'))\right) \subseteq (p|_W)^{-1} \circ p(W') \subseteq W' \subseteq V$, so g is continuous. Hence $t_0 \in A_s$, which is a contradiction because A_s is open. Thus p has at least one lifting for each path α in X, and by Lemma 3.1.2 we conclude that p has unique path lifting and so is a semicovering. □

Corollary 3.1.17. *If $p_i\colon X_i \to X_{i-1}$, $i = 1, 2$, are local homeomorphisms and X_2 is a Hausdorff and sequential compact space, then $p_1 \circ p_2$ is a a semicovering map.*

We have also the following result:

Theorem 3.1.18. *Let $p\colon Y \to X$ be a closed local homeomorphism with Y a Hausdorff space. Then p is a semicovering map.*

Proof. By Theorem 3.1.4 it is enough to show that p has unique path lifting. By Lemma 3.1.2, p has at most one lifting for each path α in X. To prove the unique path lifting property for p, suppose there exists a path α in X such that it has no lifting starting at $y_0 \in p^{-1}(\alpha(0))$. Let $g\colon A_s \to Y$ be the incomplete lifting of α at y_0. Suppose $\{t_n\}_{n\in\mathbb{N}}$ is a sequence which tends to s. Put $B := \{t_n\colon n \in \mathbb{N}\}$, then $\overline{g(B)}$ is closed in Y and so $p\left(\overline{g(B)}\right)$ is closed in X, because p is a closed map. Since $\overline{\alpha(B)} \subseteq p\left(\overline{g(B)}\right)$, $\alpha(s) \in p\left(\overline{g(B)}\right)$ and so there exists $z \in \overline{g(B)}$ such that $p(z) = \alpha(s)$. Since p is a local homeomorphism, there exists a neighborhood W_z of z such that $p|_{W_z}\colon W_z \to p(W_z)$ is a homeomorphism. Since $z \in \overline{g(B)}$, there exists an $n_k \in \mathbb{N}$ such that $g(t_{n_r}) \in W_z$ for every $n_r \geq n_k$. Since $\alpha(s) \in p(W_z)$, there exists $k_l \in \mathbb{N}$ such that $\alpha\left(\left[t_{n_{k_l}}, s\right]\right) \subseteq p(W_z)$. Put $h = \left((p|_{W_z})^{-1} \circ \alpha\right)|_{\left[t_{n_{k_l}}, s\right]}$, then $p \circ g = \alpha = p \circ h$ on $\left[t_{n_{k_l}}, s\right)$. Hence $p \circ g = p \circ h$ on $\left[t_{n_{k_l}}, s\right)$. Since $p|_{W_z}$ is a homeomorphism, $g\left(t_{n_{k_l}}\right) = h\left(t_{n_{k_l}}\right)$, so $g = h$ on $\left[t_{n_{k_l}}, s\right)$. Therefore the map $\overline{g}(t)\colon [0, s] \to Y$ defined by

$$\overline{g}(t) = \begin{cases} g(t) & 0 \leq t < s \\ h(s) & t = s \end{cases}$$

is continuous and $p \circ \overline{g} = \alpha$ on $[0, s]$. Thus $s \in A_s$, which is a contradiction because A_s is open. □

3.1.3 When is a semicovering map a covering map?

We end this section by giving some sufficient conditions for a semicovering to be a covering. If $p\colon Y \to X$ is a semicovering map, $\pi_1(X, x_0)$ acts on $p^{-1}(x_0)$ by $\alpha \cdot y_0 = \tilde{\alpha}(1)$ where $y_0 \in p^{-1}(x_0)$ and $\tilde{\alpha}$ is the (unique) lifting of α starting at y_0 (actually this corresponds to the classical construction made also for coverings, for example see [**24**, V.7]; see also Section 4.2 of this article). So the stabilizer of y_0, call it $\pi_1(X, x_0)_{y_0}$ (i.e. the subgroup $\{[\alpha] \in \pi_1(X, x_0) : \alpha \cdot y_0 = y_0\}$) is equal to $p_*(\pi_1(Y, y_0))$ for all $y_0 \in p^{-1}(x_0)$ and so $|p^{-1}(x_0)| = [\pi_1(X, x_0) : p_*(\pi_1(Y, y_0))]$ (cfr. [**24**, V.7-10] for details). Thus if $x_0, x_1 \in X$ and Y is path connected, then $|p^{-1}(x_0)| = |p^{-1}(x_1)|$. So we can define the concept of *sheet* for a semicovering map similarly as for coverings.

Theorem 3.1.19. *Suppose $p\colon Y \to X$ is a semicovering map and Y is a Hausdorff space such that $[\pi_1(X, x_0) : p_*(\pi_1(Y, y_0))]$ is finite; then p is a finite sheeted covering map.*

Proof. Let $x \in X$. Since p is a semicovering map and $[\pi_1(X, x_0) : p_*(\pi_1(Y, y_0))] = m$, p is an m-sheeted semicovering map and so we have $y_0 \in p^{-1}(x_0) = \{y_1, \ldots, y_m\}$. Since p is Hausdorff and p is a local homeomorphism, we can find an open neighborhood V of x_0 and disjoint open neighborhoods U_j such that for every $j = 1, \ldots, m$, $y_j \in U_j$ and $p|_{U_j} : U_j \to V$ is a homeomorphism. Since p is a semicovering map, we have $|p^{-1}(x)| = m$ for each point $x \in X$ and so $p^{-1}(V) = \bigcup_{j=1}^{m} U_j$: but these are exactly the conditions for p to be an m-sheeted covering map. □

Corollary 3.1.20. *Every finite sheeted semicovering map from a Hausdorff space is a finite sheeted covering map.*

Corollary 3.1.21. *If $p\colon Y \to X$ is a local homeomorphism from a Hausdorff first-countable compact space Y to a Hausdorff space X, then p is a finite sheeted covering map.*

Proof. Since Y is compact and first countable, it is sequential compact (see [**23**, 6.21]), so by Theorem 3.1.16, p is a semicovering. Moreover any singleton $\{x_0\}$ is closed in X, so the fiber $p^{-1}(\{x_0\})$ is closed and discrete in Y, which is compact, so $p^{-1}(\{x_0\})$ is finite. Thus p is a semicovering with finite fiber, hence a covering by Corollary 3.1.20. □

Remark 3.1.22. There is a result similar to Theorem 3.1.16 for coverings: let Y be compact and Hausdorff and X be Hausdorff; if $p\colon Y \to X$ is a local homeomorphism, then p is a covering map. See [**24**, Ex. V.2.4].

In general a continuous map $p\colon Y \to X$ between any two topological spaces is called *proper* when it is closed and its fibers are compact and relatively Hausdorff (i.e. two distinct points in the fiber have disjoint neighborhoods in Y). This notion gets easier in some cases, for example ([**3**, 10.2]):

- if X is locally compact and Hausdorff or
- if Y is Hausdorff and X is locally compact, then

the map f is *proper* whenever $f^{-1}(H)$ is compact for any compact subset $H \subseteq X$.

Theorem 3.1.23. *If p is a proper local homeomorphism from a Hausdorff space Y onto a Hausdorff and locally compact space X, then p is a finite sheeted covering map.*

Proof. The map p is open because every local homeomorphism is an open map. Also, every proper map is closed by definition. Hence by Theorem 3.1.18, p is a semicovering map. Since singletons in X are compact and p is proper, every fiber of p is compact in X; since fibers of a local homeomorphism are discrete, fibers of p turn out to be closed and compact and thus they are finite. So p is a finite sheeted semicovering map and by Corollary 3.1.20 p is a finite sheeted covering map. □

Theorem 3.1.24. *If X is locally path connected and semilocally 1-connected, then each semicovering of X with locally path connected total space is a covering.*

Proof. It follows from Theorem 1.3.30 that in this case we have the following chain of inclusions where first and last terms coincide:

$$\mathbf{Cov}\,(X) \subseteq \mathbf{SCov}\,(X) \subseteq \mathbf{Fib}^{!}\,(X) = \mathbf{Cov}\,(X)\,.$$

□

It is worth giving some examples.

3.1.4 Examples

While reading this section, making some drawings may be of great help.

1. A local homeomorphism which is not a semicovering

In view of Theorem 3.1.4 we have to look for a map without unique path lifting: it means that it may have no lifting at all or more than one lifting. The following example has been found in [**21**].

Let $Y := ((0,1) \times \{0\}) \cup \left(\left\{\frac{1}{2}\right\} \times \left[\frac{1}{2}, \frac{3}{4}\right)\right)$ be endowed with a topology whose base sets are of the form

$$\left\{\left(\left(a, \tfrac{1}{2}\right) \times \{0\}\right) \cup \left(\left\{\tfrac{1}{2}\right\} \times \left[\tfrac{1}{2}, b\right)\right) : \, a \in \left(0, \tfrac{1}{2}\right),\, b \in \left(\tfrac{1}{2}, \tfrac{3}{4}\right)\right\}$$
$$\{(a,b) \times \{0\} : \, a, b \in (0,1)\,,\, a < b\}$$
$$\left\{\left\{\tfrac{1}{2}\right\} \times (a,b) : \, a, b \in \left(\tfrac{1}{2}, \tfrac{3}{4}\right),\, a < b\right\}.$$

Let X be the open interval $(0,1)$ and define $p\colon Y \to X$ by

$$p\,(s,t) = \begin{cases} s & t = 0 \\ t & t \neq 0 \end{cases}.$$

For sure p is a local homeomorphism (it is possible to write down explicitly the proof, but intuition should suffice), but it has not the unique path lifting property: for instance consider the path $\alpha\colon I \to X$ given by $\alpha\,(r) = \frac{1}{4} + \frac{1}{2}r$, then one immediately checks that α has at least two different liftings.

2. Two coverings whose composition is not a covering

Topological spaces which are found in real life usually have all those good properties which allow them to have a universal covering. If the universal covering exists, the composition of two coverings is a covering. Hence to find a space admitting two coverings whose composition is not a covering, one has to work with some kind of imagination. The example here exposed has been found in [**30**, II, Ex. 1.15 and 2.8].

Let us define these topological spaces:

- $Z = \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$ is the product of countably many copies of the unit circle;
- $Y = \mathbb{N} \times Z$, where $\mathbb{N}$ is endowed with the discrete topology;
- $X_n = \mathbb{R}^n \times Z$;
- $X = \bigcup_{n\in\mathbb{N}} X_n$, which is clearly a disjoint union.

As usual, all these spaces are endowed with the product topology.

Let us define a covering $p\colon X \to Y$ in this way: for each fixed $n \in \mathbb{N}$, consider the map $p_n : X_n \to Z$ given by $(x_1, \dots, x_n, w_1, w_2, \dots) \mapsto \left(e^{2\pi i x_1}, \dots, e^{2\pi i x_n}, w_1, w_2, \dots\right)$ where $x_i \in \mathbb{R}$ and $w_i \in \mathbb{S}^1$ for all $i \in \mathbb{N}$. In practise, p_n acts as the standard exponential covering $\mathbb{R} \to \mathbb{S}^1$ on the first n factors and as the identity on the countably many remaining ones. So we can define $p\colon X \to Y$ by $x \mapsto (n, p_n(x))$ if $x \in X_n$. One easily gets convinced it actually is a covering, indeed: let $Q = (m, v_1, \dots, v_n, \dots) \in \mathbb{N} \times Z$. Then its preimage is the discrete set of points $x \in \mathbb{R}^m \times \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$ where $x = (x_1, \dots, x_m, w_1, \dots)$ is such that $e^{2\pi i x_j} = v_j$ for $j = 1, \dots, m$ and $w_j = v_{j+m}$ for all $j \in \mathbb{N}$. To build a fundamental neighborhood of Q, consider the open set $U := \{m\} \times U_1 \times \cdots \times U_m \times \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \subseteq \mathbb{N} \times Z$ where $v_i \in U_i$ for $i = 1, \dots, m$ and U_i is an open connected neighborhood of v_i in $\mathbb{S}^1$. It is clear that such a U is an open neighborhood of Q. Then $p^{-1}(U) = p^{-1}(U_1) \times \cdots \times p^{-1}(U_m) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \subseteq X_m$. This set has countably many disjoint connected components and each one is readily seen to be homeomorphic through p to the set U; to keep ideas: if $Q = (1, (0,1), (0,1), \dots) \in \mathbb{N} \times Z$ and $U = \{1\} \times U_1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$, connected components of $p^{-1}(U)$ are for example $(-\epsilon, \epsilon) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$ or $(1-\epsilon, 1+\epsilon) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$.

Second, a simple covering $q\colon Y \to Z$ of Z is given by the standard projection $(n, y) \mapsto y$.

However the composition $r := q \circ p\colon X \to Z$ is not a covering of Z since there is no point of Z admitting a fundamental neighborhood. For example take the point $Q = ((0,1), (0,1), \dots) \in Z$ (but the same idea holds for each point of Z). A neighborhood V of Q is the intersection of finitely many sets of a base for the product topology: if $U \subseteq \mathbb{S}^1$ is a connected open neighborhood of $(0,1)$, one finds $V = \prod_{j\in\mathbb{N}} V_j$ where $V_j = U$ for finitely many j and $V_j = \mathbb{S}^1$ for the remaining ones. For instance, one could choose $V = U_1 \times U_2 \times \cdots \times U_m \times \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$ where $U_i = U$ for $i = 1, \dots, m$. Computing the preimage through r, we get the disjoint union $r^{-1}(V) = \bigcup_{n\in\mathbb{N}} p_n^{-1}(V)$. But the connected components of $r^{-1}(V)$ can not be all mapped homeomorphically by r onto V. For example if $m = 2$ (again: this applies for each m), among all possible connected components of $r^{-1}(V)$, there are $W_1 = (-\epsilon, \epsilon) \times U_2 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$ (for $n = 1$), $W_2 = (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$ (for $n = 2$), $W_3 = (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times \mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$ (for $n = 3$) and clearly $r|_{W_3}\colon W_3 \to V$ is not injetive.

Thus $r\colon X \to Z$ is not a covering; on the other hand, thanks to Theorem 3.1.11, r provides us of an example of a semicovering which is not a covering.

3. A non-trivial covering of the space $\mathbb{S}^1 \times \mathbb{S}^1$

We show that $p\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ defined by $(x, y) \mapsto (x^n y^m, x^s y^t)$ is a covering map whenever $m, n, s, t \in \mathbb{N}$ and $n/s \neq m/t$.

We will exploit the theorems of previous sections.

Let $\exp(\theta) = e^{2\pi i\theta}$ so that p may be written as $p(\exp(\alpha), \exp(\beta)) = (\exp(n\alpha + m\beta), \exp(s\alpha + t\beta))$; as a notation put also $\exp(\gamma, \delta) := \left\{\exp(\theta) \in \mathbb{S}^1 \colon \gamma \leq \theta \leq \delta\right\}$. Define $l = \max\{m, n, s, t\}$ and consider the open set $U = \left(\exp\left(\alpha - \frac{\pi}{2l}, \alpha + \frac{\pi}{2l}\right)\right) \times \left(\exp\left(\beta - \frac{\pi}{2l}, \beta + \frac{\pi}{2l}\right)\right)$, which is an open neighborhood of the point $(\exp(\alpha), \exp(\beta)) \in \mathbb{S}^1 \times \mathbb{S}^1$.

We show that

$$p|_U : U \to \exp\left(n\left(\alpha - \frac{\pi}{2l}\right) + m\left(\beta - \frac{\pi}{2l}\right), n\left(\alpha + \frac{\pi}{2l}\right) + m\left(\beta + \frac{\pi}{2l}\right)\right)$$
$$\times \exp\left(s\left(\alpha - \frac{\pi}{2l}\right) + t\left(\beta - \frac{\pi}{2l}\right), s\left(\alpha + \frac{\pi}{2l}\right) + mt\beta + \frac{\pi}{2l}\right)$$

is a homeomorphism.

Computations give that

$$\begin{gathered}\left(m\left(\alpha + \tfrac{\pi}{2l}\right) + n\left(\alpha + \tfrac{\pi}{2l}\right)\right) - \left(m\left(\alpha - \tfrac{\pi}{2l}\right) + n\left(\alpha - \tfrac{\pi}{2l}\right)\right) = \\ = \tfrac{m}{l}\left(\tfrac{\pi}{2} + \tfrac{\pi}{2}\right) + \tfrac{n}{l}\left(\tfrac{\pi}{2} + \tfrac{\pi}{2}\right) < 2\pi \\ \text{and} \\ \left(s\left(\beta + \tfrac{\pi}{2l}\right) + t\left(\beta + \tfrac{\pi}{2l}\right)\right) - \left(s\left(\beta - \tfrac{\pi}{2l}\right) + t\left(\beta - \tfrac{\pi}{2l}\right)\right) = \\ = \tfrac{s}{l}\left(\tfrac{\pi}{2} + \tfrac{\pi}{2}\right) + \tfrac{t}{l}\left(\tfrac{\pi}{2} + \tfrac{\pi}{2}\right) < 2\pi.\end{gathered}$$

Therefore, if $p(\exp(\alpha_1), \exp(\beta_1)) = p(\exp(\alpha_2), \exp(\beta_2))$, then

$$\begin{cases} n\alpha_1 + m\beta_1 = n\alpha_2 + m\beta_2 \\ s\alpha_1 + t\beta_1 = s\alpha_2 + t\beta_2 \end{cases} \quad \text{and so} \quad \begin{cases} n(\alpha_1 - \alpha_2) = m(\beta_2 - \beta_1) \\ s(\alpha_1 - \alpha_2) = t(\beta_2 - \beta_1). \end{cases}$$

Since $n/s \neq m/t$, we must have $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Thus p is a local homeomorphism.

Now notice that $\mathbb{S}^1 \times \mathbb{S}^1$ is a compact metric space and so it is also sequential compact. Hence by Theorem 3.1.16, p is a semicovering map. In view of Theorem 1.3.30 or Theorem 3.1.24, since $\mathbb{S}^1 \times \mathbb{S}^1$ is locally path connected and semilocally 1-connected, p is also a covering map. Alternatively, we could observe that p is a proper map because $\mathbb{S}^1 \times \mathbb{S}^1$ is compact and Hausdorff, thus p is a finite sheeted covering map by Theorem 3.1.19. The same conclusion can be reached also by Theorem 3.1.23. Remark that despite knowing that p is a finite sheeted covering, it is not at all easy to find an evenly covered neighborhood by p for an arbitrary point of $\mathbb{S}^1 \times \mathbb{S}^1$.

3.2 OPEN COVERING MORPHISMS

Definition 3.2.1. A **Top**-functor $F: \mathcal{A} \to \mathcal{B}$ is *open* if each map $F: \mathcal{A}(x,y) \to \mathcal{B}(F(x), F(y))$ is an open map. Thus an *open covering morphism* $F: \mathcal{H} \to \mathcal{G}$ of **Top**-groupoids is a covering morphism such that each map $F: \mathcal{H}(x,y) \to \mathcal{G}(F(x), F(y))$ is an open embedding (cfr. Remark 1.2.7).

If $\mathcal{G}$ is a **Top**-groupoid, **OCovMor**$(\mathcal{G})$ denotes the category of open covering morphisms $\mathcal{H} \to \mathcal{G}$. A morphism of open covering morphisms p, p' is a **Top**-functor F making this diagram quasi-commute, i.e. $p' \circ F = p$:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{F} & \mathcal{H}' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & \mathcal{G} & \end{array} .$$

The following lemma reminds us of Theorems 1.3.9 and 3.1.11: every morphism of open covering morphisms is an open covering morphism itself.

Lemma 3.2.2. *Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are* **Top***-groupoids and $p: \mathcal{A} \to \mathcal{B}$, $q: \mathcal{B} \to \mathcal{C}$, $r \coloneqq q \circ p$ are functors. Then:*

1. *If p and q are open covering morphisms, then so is r.*

2. *If q and r are open covering morphisms, then so is p.*

3. *If p and r are open covering morphisms and p is surjective on objects, then q is an open covering morphism.*

Proof. Let us remove for a moment each appearance of "open" in the statement; it remains true, for: let $x \in ob(\mathcal{A})$ and consider the composite induced by p and q given by $\mathcal{A}_x \xrightarrow{p'} \mathcal{B}_{p(x)} \xrightarrow{q'} \mathcal{C}_{q(p(x))}$. Clearly if any two of p', q', $q' \circ p'$ are bijections, then so is the third; for (3), one also needs that any $y \in ob(\mathcal{B})$ is $p(x)$ for some x.

To show the statement in the case of *open* covering morphisms, one has to notice that, given

$$\mathcal{A}(x,y) \xrightarrow{\quad p' \quad} \mathcal{B}(p(x), p(y)) \xrightarrow{\quad q' \quad} \mathcal{C}(q(p(x)), q(p(y))):$$

1. the composition of open maps is open;

2. if q' and $q' \circ p'$ are open maps and $A \subseteq \mathcal{A}(x,y)$ is open, $p'(A) = q'^{-1}(q' \circ p'(A))$ is open (for instance, remember that q' is injective and continuous);

3. if p' and $q' \circ p'$ are open maps and $B \subseteq \mathcal{B}(p(x), p(y))$, p' being surjective, there exists at least one open set $A \subseteq \mathcal{A}(x,y)$ such that $p'(A) = B$. Since $q' \circ p'(A)$ is open and $B = q'^{-1}(q' \circ p'(A))$, B is open too.

□

3.2.1 The enriched functors Π_1^τ, μ^τ and $\mathscr{R}^\tau$

In this section we give an enriched version of Theorem 1.3.24: since we will need maps which are both fibrations with unique path lifting and local homeomorphisms, for any topological space X, we cannot consider the whole category $\mathbf{Fib}^!(X)$ but infact we have to restrict to $\mathbf{SCov}(X)$.

Lemma 3.2.3. *If $p\colon Y \to X$ is a semicovering such that $p(y_i) = x_i$, $i = 1, 2$, the map $\mathcal{P}p\colon \mathcal{P}Y(y_1, y_2) \to \mathcal{P}p(x_1, x_2)$ is an open embedding.*

Proof. Since $\mathcal{P}p\colon \mathcal{P}Y(y_1, y_2) \to \mathcal{P}X(x_1, x_2)$ is the restriction of the homeomorphism $\tilde{\mathcal{P}}p\colon (\mathcal{P}Y)_{y_1} \to (\mathcal{P}X)_{x_1}$, $\mathcal{P}p$ is continuous and injective. So it is enough to show the image of $\mathcal{P}p$ is open in $\mathcal{P}X(x_1, x_2)$. Let $\alpha \in \mathcal{P}X(x_1, x_2)$ such that $\tilde{\alpha}_{y_1} \in \mathcal{P}Y(y_1, y_2)$. Let $\mathcal{U} = \bigcap_{j=1}^n M\left(K_n^j, U_j\right)$ be an open neighborhood of $\tilde{\alpha}_{y_1} \in (\mathcal{P}Y)_{y_1}$ such that $p|_{U_n}\colon U_n \to p(U_n)$ is a homeomorphism (such a U_n exists because p is a local homeomoprhism). Since $\tilde{\mathcal{P}}p\colon (\mathcal{P}Y)_{y_1} \cong (\mathcal{P}X)_{x_1}$, $\mathcal{W} \coloneqq \tilde{\mathcal{P}}p(\mathcal{U}) \cap \mathcal{P}X(x_1, x_2)$ is an open neighborhood of α in (the induced topology of) $\mathcal{P}X(x_1, x_2)$. If $\beta \in \mathcal{W}$, then $\mathcal{U}$ is an open neighborhood of $\tilde{\beta}_{y_1}$ in $(\mathcal{P}Y)_{y_1}$. In particular $\tilde{\beta}_{y_1} \in \mathcal{P}Y(y_1, y_2)$, giving the desired inclusion $\mathcal{W} \subseteq \operatorname{Im}(\mathcal{P}p)$. □

Theorem 3.2.4. *If $p\colon Y \to X$ is a semicovering map, then $\Pi_1^\tau p\colon \Pi_1^\tau Y \to \Pi_1^\tau X$ is an open covering morphism.*

Proof. It has already been observed in Lemma 1.3.15 that $\Pi_1^\tau p$ is a covering morphism and in Lemma 2.3.6 that it is a **Top**-functor, so it is enough to show $\Pi_1^\tau p$ is open. Remark that each function $\Pi_1 p\colon \Pi_1 Y(y_1, y_2) \to \Pi_1 X(p(y_1), p(y_2))$ is injective because $\Pi_1 p$ is a covering morphism of groupoids (cfr. Remark 1.2.7). This injectivity is independent of topologies on the hom-sets.

We work by transfinite induction following the constructive approach of the Appendix.

For simplicity, let $\mathcal{H}_0 \coloneqq \Pi_1^{qtop} Y$ and $\mathcal{G}_0 \coloneqq \Pi_1^{qtop} X$ and inductively take $\mathcal{H}_\zeta$ and $\mathcal{G}_\zeta$ to be the approximating **qTop**-groupoids of $\tau(\mathcal{H}_0) = \Pi_1^\tau Y$ and $\tau(\mathcal{G}_0) = \Pi_1^\tau X$ respectively.

For the first inductive step we have to show that whenever $p(y_i) = x_i$, $i = 1, 2$, then the map $\Pi_1 p \colon \mathcal{H}_0(y_1, y_2) \to \mathcal{G}_0(x_1, x_2)$, $[\alpha] \mapsto [p \circ \alpha]$ is open. So let $\mathcal{U}$ be an open set in $\mathcal{H}_0(y_1, y_2)$: the following diagram commutes when the vertical arrows are the canonical quotient maps

$$\begin{array}{ccc} \mathcal{P}Y(y_1, y_2) & \xrightarrow{\mathcal{P}p} & \mathcal{P}X(x_1, x_2) \\ \downarrow{\scriptstyle h_Y} & & \downarrow{\scriptstyle h_X} \\ \mathcal{H}_0(y_1, y_2) & \xrightarrow{\Pi_1 p} & \mathcal{G}_0(x_1, x_2). \end{array}$$

The top map is open by previous Lemma and h_X is quotient so it suffices to show the equality $h_X^{-1}(\Pi_1 p(\mathcal{U})) = \mathcal{P}p\big(h_Y^{-1}(\mathcal{U})\big)$. If $\alpha \in h_X^{-1}(\Pi_1 p(\mathcal{U}))$, then $[\alpha] \in \Pi_1 p(\mathcal{U})$ and the lift $\tilde{\alpha}_{y_1}$ ends at y_2 by Lemma 3.1.9. It is clear that $\Pi_1 p([\tilde{\alpha}_{y_1}]) = [\alpha] \in \Pi_1 p(\mathcal{U})$ and the injectivity of $\Pi_1 p$ gives $[\tilde{\alpha}_{y_1}] \in \mathcal{U}$. Moreover $\alpha = p \circ \tilde{\alpha}_{y_1} = \mathcal{P}p(\tilde{\alpha}_{y_1})$ for $\tilde{\alpha}_{y_1} \in h_Y^{-1}(\mathcal{U})$, hence $\alpha \in \mathcal{P}p\left(h_Y^{-1}(\mathcal{U})\right)$. For the other inclusion, if $\alpha = p \circ \tilde{\alpha}_{y_1}$ is such that $[\tilde{\alpha}_{y_1}] \in \mathcal{U}$, then $[\alpha] = \Pi_1 p([\tilde{\alpha}_{y_1}]) \in \Pi_1 p(\mathcal{U})$ and so $\alpha \in h_X^{-1}(\Pi_1 p(\mathcal{U}))$.

For the second inductive step, suppose ζ is an ordinal and that for each $\eta < \zeta$, $\Pi_1 p \colon \mathcal{H}_\eta(y_1, y_2) \to \mathcal{G}_\eta(p(y_1), p(y_2))$ is open for each $y_1, y_2 \in Y$. Fix $y_1, y_2 \in Y$ and let $p(y_i) = x_i$. Clearly, if ζ is a limit ordinal, then $\Pi_1 p \colon \mathcal{H}_\zeta(y_1, y_2) \to \mathcal{G}_\zeta(x_1, x_2)$ is an open embedding (indeed, cfr. the construction of Theorem in Appendix). If ζ is a successor ordinal, consider the following quasi-commutative diagram:

$$\begin{array}{ccc} \bigsqcup_{b \in Y} \mathcal{H}_{\zeta-1}(y_1, b) \times \mathcal{H}_{\zeta-1}(b, y_2) & \xrightarrow{\mathscr{P}_{\zeta-1}} & \bigsqcup_{a \in X} \mathcal{G}_{\zeta-1}(x_1, a) \times \mathcal{G}_{\zeta-1}(a, x_2) \\ \downarrow{\scriptstyle \mu_Y} & & \downarrow{\scriptstyle \mu_X} \\ \mathcal{H}_\zeta(y_1, y_2) & \xrightarrow{\Pi_1 p} & \mathcal{G}_\zeta(x_1, x_2), \end{array}$$

the vertical multiplication maps are quotients by definition, the top map $\mathscr{P}_{\zeta-1}$ is, on each summand, the product of open embeddings (by inductive hypothesis): in particular $\Pi_1 p \times \Pi_1 p \colon \mathcal{H}_{\zeta-1}(y_1, b) \times \mathcal{H}_{\zeta-1}(b, y_2) \to \mathcal{G}_{\zeta-1}(x_1, p(b)) \times \mathcal{G}_{\zeta-1}(p(b), x_2)$ is given by $([\alpha], [\beta]) \mapsto ([p \circ \alpha], [p \circ \beta])$ and so $\mathscr{P}_{\zeta-1}$ is continuous and open.

Now suppose $\mathcal{U}$ is open in $\mathcal{H}_\zeta(y_1, y_2)$. We have to show that $\Pi_1 p(\mathcal{U})$ is open or, equivalently, that $\mu_X^{-1}(\Pi_1 p(\mathcal{U}))$ is open. If $([\delta], [\epsilon]) \in \mu_X^{-1}(\Pi_1 p(\mathcal{U}))$, then $[\delta * \epsilon] \in \Pi_1 p(\mathcal{U})$. Let $a_0 = \delta(1)$. Consequently, the lift of $\delta * \epsilon$ starting at y_1 ends at y_2. This lift is $\tilde{\delta}_{y_1} * \tilde{\epsilon}_{b_0}$, where $b_0 = \tilde{\delta}_{y_1}(1) \in p^{-1}(a_0)$. Since $\Pi_1 p\left(\left[\tilde{\delta}_{y_1} * \tilde{\epsilon}_{b_0}\right]\right) = \left[p \circ \left(\tilde{\delta}_{y_1} * \tilde{\epsilon}_{b_0}\right)\right] = \left[\left(p \circ \tilde{\delta}_{y_1}\right) * (p \circ \tilde{\epsilon}_{b_0})\right] = [\delta * \epsilon] \in \Pi_1 p(\mathcal{U})$ and $\Pi_1 p$ is injective, hence $\left[\tilde{\delta}_{y_1} * \tilde{\epsilon}_{b_0}\right] \in \mathcal{U}$. Therefore $\mu_Y^{-1}(\mathcal{U})$ is an open neighborhood of $\left(\left[\tilde{\delta}_{y_1}\right], [\tilde{\epsilon}_{b_0}]\right)$ and $\mathscr{P}_{\zeta-1}\left(\mu_Y^{-1}(\mathcal{U})\right)$ is an open neighborhood of $([\delta], [\epsilon])$ in $\bigsqcup_{a \in X} \mathcal{G}_{\zeta-1}(x_1, a) \times \mathcal{G}_{\zeta-1}(a, x_2)$. So we have shown that $\mathscr{P}_{\zeta-1}\left(\mu_Y^{-1}(\mathcal{U})\right) \supseteq \mu_X^{-1}(\Pi_1 p(\mathcal{U}))$. The other inclusion follows by noticing that if $([\alpha], [\beta]) \in \mu_Y^{-1}(\mathcal{U})$, then $[\alpha * \beta] \in \mathcal{U}$ and $\mu_X(\mathscr{P}_{\zeta-1}([\alpha], [\beta])) = [p \circ \alpha] * [p \circ \beta] = [p \circ (\alpha * \beta)] = \Pi_1 p([\alpha * \beta]) \in \Pi_1 p(\mathcal{U})$. □

Corollary 3.2.5. *If $p \colon Y \to X$ is a semicovering such that $p(y_0) = x_0$, the induced homomorphism of groups $p_* \colon \pi_1^\tau(Y, y_0) \to \pi_1^\tau(X, x_0)$ is an open embedding of topological groups.*

Corollary 3.2.6. *Thanks to Theorem 3.2.4, the map*

$$\Pi_1^\tau \colon \mathbf{SCov}(X) \to \mathbf{OCovMor}(\Pi_1^\tau X),$$

which assigns to a given semicovering of X an open covering morphism of the topological fundamental groupoid of X, is a **Top***-functor.*

Similarly, we enrich the functor μ (Definition 1.3.23), which is well defined since any semicovering is a fibration. The monodromy of a semicovering becomes an enriched functor when we use $\Pi_1^\tau X$ instead of $\Pi_1 X$ and view **Set** as a **Top**-category by giving each set the discrete topology and endowing each hom-set $\mathbf{Set}(A,B)$ with the compact-open topology. In what follows $\mathbf{Set}^e$ denotes the so enriched category **Set**.

Theorem 3.2.7. *The monodromy $\mu^\tau(p) : \Pi_1^\tau X \to \mathbf{Set}^e$ of a semicovering $p\colon Y \to X$ is a* **Top***-functor. Moreover,*

$$\mu^\tau : \mathbf{SCov}(X) \to \mathbf{TopFct}(\Pi_1^\tau X, \mathbf{Set}^e)$$

is a well defined functor.

Proof. Suppose $p(y_i) = x_i$, $i = 1,2$. The set $\{[\alpha] \in \Pi_1^\tau X(x_1,x_2) :\ y_1 \cdot [\alpha] = y_2\}$ is open in $\Pi_1^\tau X(x_1,x_2)$ since it is the image of the open embedding $\Pi_1^\tau p\colon \Pi_1^\tau Y(y_1,y_2) \to \Pi_1^\tau X(x_1,x_2)$ (cfr. Theorem 3.2.4). Moreover each fiber $p^{-1}(X)$ is discrete. Therefore each action map

$$p^{-1}(x_1) \times \Pi_1^\tau X(x_1,x_2) \to p^{-1}(x_2),\ \ (y,[\alpha]) \mapsto y\cdot[\alpha]$$

is continuous. Since discrete spaces are locally compact and Hausdorff, the map

$$\mu^\tau(p) : \Pi_1^\tau X(x_1,x_2) \to \mathbf{Set}^e\left(p^{-1}(x_1), p^{-1}(x_2)\right),\ \text{ where } \mu^\tau(p)([\alpha])(y) = y\cdot[\alpha],$$

is continuous by the Exponential Law 1.1.4. Thus $\mu^\tau(p) : \Pi_1^\tau X \to \mathbf{Set}^e$ is a **Top**-functor.

Finally, a morphism $f\colon Y \to Y'$ of semicoverings $p\colon Y \to X$ and $p'\colon Y' \to X$ induces the natural transformation $\mu^\tau f\colon \mu^\tau p \to \mu^\tau p'$ with the obvious components $f\colon p^{-1}(x) \to p'^{-1}(x)$. □

As an aside we get the following:

Corollary 3.2.8. *For each $x_0 \in X$, the monodromy of a semicovering $p\colon Y \to X$ restricts to a continuous group homomorphism $\pi_1^\tau(X,x_0) \to homeo\left(p^{-1}(x_0)\right)$.*

Finally we enrich the equivalence $\mathscr{R}$ (Theorem 1.2.8) getting:

Theorem 3.2.9 (ENRICHED $\mathscr{R}-$EQUIVALENCE)**.** *For a* **Top***-groupoid $\mathcal{G}$, there is an equivalence of categories*

$$\mathscr{R}^\tau \colon \mathbf{OCovMor}(\mathcal{G}) \simeq \mathbf{TopFct}(\mathcal{G}, \mathbf{Set}^e).$$

Proof. We build $\mathscr{R}^\tau$ and its inverse in such a way that when topological structures are forgotten we retain the construction of the $\mathscr{R}-$Equivalence 1.2.8.

First, a subbasis set for the topology of $\mathbf{Set}^e\left(F_{ob}^{-1}(x_1), F_{ob}^{-1}(x_2)\right)$ is of the form $M(\{y_1\},\{y_2\})$ (because on each set $F_{ob}^{-1}(x)$ there is the discrete topology and the topology on $\mathbf{Set}^e\left(F_{ob}^{-1}(x_1), F_{ob}^{-1}(x_2)\right)$ is the compact-open one). Thus if $F\colon \mathcal{H} \to \mathcal{G}$ is an open covering morphism of **Top**-groupoids, then $\mathscr{R}^\tau F\colon \mathcal{G}(x_1,x_2) \to \mathbf{Set}^e\left(F_{ob}^{-1}(x_1), F_{ob}^{-1}(x_2)\right)$ is continuous since each set $(\mathscr{R}^\tau F)^{-1}(M(\{y_1\},\{y_2\})) =$

$\{g \in \mathcal{G}(x_1, x_2) : t(\tilde{g}_{y_1}) = y_2\} = \mathrm{Im}\,(F \colon \mathcal{H}(y_1, y_2) \to \mathcal{G}(x_1, x_2))$ is open in $\mathcal{G}(x_1, x_2)$ (indeed, F is an *open* covering morphism).

Conversely, let $\mathcal{G}$ be a **Top**-groupoid and $N \colon \mathcal{G} \to \mathbf{Set}^e$ be a **Top**-functor. We define $\mathcal{H}$ and $\mathscr{R}^{\tau -1}N \colon \mathcal{H} \to \mathcal{G}$ as in Theorem 1.2.8 and we give $\mathcal{H}(y_1, y_2)$ the subspace topology of $\mathcal{G}(x_1, x_2)$ when $y_i \in N(x_i)$ (it makes sense because $ob(\mathcal{H}) \subseteq ob(\mathcal{G})$). This makes $\mathcal{H}$ into a **Top**-groupoid and $\mathscr{R}^{\tau -1}N$ into a **Top**-functor.

Finally, $N \colon \mathcal{G}(x_1, x_2) \to \mathbf{Set}^e(N(x_1), N(x_2))$ is continuous and each $N(x_i)$ is a discrete space; so $\mathcal{H}(y_1, y_2)$ is open in $\mathcal{G}(x_1, x_2)$ whenever $y_i \in N(x_i)$. Thus $\mathscr{R}^{\tau -1}N$ is open. □

The above constructions of Π_1^τ, μ^τ and $\mathscr{R}^\tau$ prove that:

Theorem 3.2.10. *For any topological space X, the following diagram quasi-commutes:*

$$\begin{array}{ccc} \mathbf{SCov}(X) & \xrightarrow{\Pi_1^\tau} & \mathbf{OCovMor}(\Pi_1^\tau X) \\ & \searrow^{\mu^\tau} & \downarrow \wr\, \mathscr{R}^\tau \\ & & \mathbf{TopFct}(\Pi_1^\tau X, \mathbf{Set}^e)\,. \end{array}$$

Thus any semicovering $p \colon Y \to X$ is characterized either by its corresponding open covering morphism $\Pi_1^\tau Y \to \Pi_1^\tau X$ or by its corresponding enriched monodromy functor $\Pi_1^\tau X \to \mathbf{Set}^e$.

Chapter 4

CLASSIFICATION THEOREMS

This final chapter is devoted to classification theorems. We define the class of *locally wep-connected* spaces and prove that for such a space X the enriched functors $\Pi_1^\tau : \mathbf{SCov}(X) \to \mathbf{OCovMor}(\Pi_1^\tau X)$ and $\mu^\tau : \mathbf{SCov}(X) \to \mathbf{TopFct}(\Pi_1^\tau X, \mathbf{Set})$ are equivalences of categories: this gives the enriched version of Theorem 1.3.31.

4.1 WEP-CONNECTED SPACES

Definition 4.1.1. Let X be a topological space.

1. A path $\alpha : I \to X$ is *well-ended* if for every open neighborhood $\mathcal{U}$ of α in $\mathcal{P}X$ there are open neighborhoods V_0, V_1 of $\alpha(0), \alpha(1)$ in X respectively such that for every $a \in V_0$, $b \in V_1$, there is a path $\beta \in \mathcal{U}$ with $\beta(0) = a$ and $\beta(1) = b$.

2. A path $\alpha : I \to X$ is *well-targeted* if for every open neighborhood $\mathcal{U}$ of α in $(\mathcal{P}X)_{\alpha(0)}$ there is an open neighborhood V_1 of $\alpha(1)$ such that for each $b \in V_1$, there is a path $\beta \in \mathcal{U}$ with $\beta(1) = b$.

3. A space is *wep-connected* if each pair of points can be connected by a well-ended path.

Remark 4.1.2. By definition, every wep-connected space is also path connected.

It is worth giving some intuition of these notions: since a basis for the topology of $\mathcal{P}X$ is given by neighborhoods of the form $\bigcap_{j=1}^{n} M\left(K_n^j, U_j\right)$, where U_j is open in X, we get that a path $\alpha \in \mathcal{P}X$ is well-ended if and only if for each neighborhood $\bigcap_{j=1}^{n} M\left(K_n^j, U_j\right)$ of α, there are open neighborhoods $V_0 \subseteq U_1$, $V_1 \subseteq U_n$ of $\alpha(0), \alpha(1)$ in X respectively such that for every $a \in V_0$, $b \in V_1$ there is a path $\beta \in \bigcap_{j=1}^{n} M\left(K_n^j, U_j\right)$ with $\beta(0) = a$ and $\beta(1) = b$; all this is resumed in the following figure:

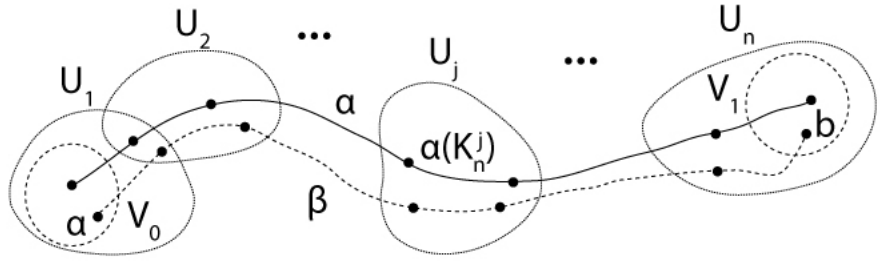


Figure 4.1.1: A well-ended path α

A similar proof as in [**7**, 4.11] shows that the Hawaiian Earring $\mathbb{H}$ introduced in Section 2.2 is locally wep-connected. A non-wep-connected space is for example the space X described in the example at the end of Section 1.3.3: it is easily seen to be non-wep-connected at the point x_1.

Next proposition describes some basic properties of wep-connected spaces:

Proposition 4.1.3. *Let X be a topological space.*

1. *If $\alpha\colon I \to X$ is a path and X is locally path connected at $\alpha(0)$ and $\alpha(1)$, then α is well-ended.*
2. *All path connected and locally path connected spaces are wep-connected.*
3. *Fix any $x_0 \in X$. Then X is wep-connected if and only if for each $x \in X$ there is a well-targeted path from x_0 to x.*

Proof. 1. Suppose $\mathcal{U} = \bigcap_{j=1}^{n} M\left(K_n^j, U_j\right)$ is a basic open neighborhood of α in $\mathcal{P}X$. Fix a path connected neighborhood V_0, V_1 of $\alpha(0)$, $\alpha(1)$ respectively such that $V_0 \subseteq U_1$, $V_1 \subseteq U_n$. For points $a \in V_0$, $b \in V_1$ take paths $\gamma\colon I \to V_0$ from a to $\alpha(0)$ and $\delta\colon I \to V_1$ from $\alpha(1)$ to b. Now define a path $q \in \mathcal{U}$ by setting

$$q_{K_{2n}^1} = \gamma,\ q_{K_{2n}^2} = \alpha_{K_n^1},\ q_{\left[\frac{1}{n},\frac{n-1}{n}\right]} = p_{\left[\frac{1}{n},\frac{n-1}{n}\right]},\ q_{K_{2n}^{2n-1}} = \alpha_{K_n^{n-1}} \text{ and } q_{K_{2n}^{2n}} = \delta.$$

Clearly q is a path in $\mathcal{U}$ from a to b.

2. It is a direct consequence of (1) .

3. Evidently if X is wep-connected it is also well-targeted. For the converse, pick $a, b \in X$. By assumption there are paths $\alpha, \beta \in (\mathcal{P}X)_{x_0}$ ending at a, b respectively, each satisfying the definition of being well-targeted to a, b respectively. We claim $\alpha * \beta^{-1}$ is well-ended. Let $\mathcal{U} = \bigcap_{j=1}^{n} M\left(K_n^j, U_j\right)$ be a basic open neighborhood of $\alpha * \beta^{-1}$ in $\mathcal{P}X$. Since $\mathcal{A} := \mathcal{U}_{\left[0,\frac{1}{2}\right]} \cap (\mathcal{P}X)_{x_0}$ and $\mathcal{B} := \mathcal{U}_{\left[\frac{1}{2},1\right]} \cap (\mathcal{P}X)_{x_0}$ are open neighborhoods of α and β respectively, there are open neighborhoods A of a and B of b such that for any $a' \in A$ (resp. $b' \in B$) there is a path $\alpha' \in \mathcal{A}$ (resp. $\beta' \in \mathcal{B}$) from x_0 to a' (resp. x_0 to b'). Finally $\alpha' * (\beta')^{-1} \in \mathcal{U}$ is the path a' to b' whose existence was required. □

Proposition 4.1.4. *If X is wep-connected and $x_0 \in X$, then the evaluation map $ev_1\colon (\mathcal{P}X)_{x_0} \to X$, $\alpha \mapsto \alpha(1)$ is a quotient map (i.e. the topology of X coincides with the quotient topology induced by ev_1).*

Proof. Suppose $x \in U \subseteq X$ such that $ev_1^{-1}(U)$ is open in $(\mathcal{P}X)_{x_0}$. Since X is wep-connected, there is a well-targeted path $\gamma \in (\mathcal{P}X)_{x_0}$ ending at x. Since $ev_1^{-1}(U)$ is an open neighborhood of γ, there exists an open neighborhood V of x in X such that for each $v \in V$ there is a path $\alpha \in ev_1^{-1}(U)$ from x_0 to v. Thus $V \subseteq U$, showing that U is a neighborhood of all its points, hence U is open. □

Proposition 4.1.5. *If the path components of X are wep-connected, then $\pi_0^{qtop}X$ is discrete (i.e. the path components of X are open).*

Proof. Let $x \in X$ and α any well-ended path such that $\alpha(0) = x$. Since $\mathcal{P}X$ is a (trivial) open neighborhood of α, there are open neighborhoods V_0, V_1 of x, $\alpha(1)$ respectively such that for each $a \in V_0$, $b \in V_1$ there is a path γ from a to $\alpha(1)$; then V_0 is contained in the path component of x, so the path component of x is a neighborood of all its points, hence it is open. □

For further use it is necessary to slightly strenghten the notion of wep-connectedness:

Definition 4.1.6. Let X be a topological space.

1. A path $\alpha\colon I \to X$ is *locally well-ended* if for every open neighborhood $\mathcal{U}$ of α in $\mathcal{P}X$ there are open neighborhoods V_0, V_1 of $\alpha(0), \alpha(1)$ in X respectively such that for every $a \in V_0$, $b \in V_1$, there is a well-ended path $\beta \in \mathcal{U}$ with $\beta(0) = a$ and $\beta(1) = b$.

2. A path $\alpha\colon I \to X$ is *well-targeted* if for every open neighborhood $\mathcal{U}$ of α in $(\mathcal{P}X)_{\alpha(0)}$ there is an open neighborhood V_1 of $\alpha(1)$ such that for each $b \in V_1$, there is a well-targeted path $\beta \in \mathcal{U}$ with $\beta(1) = b$.

3. A space is *locally wep-connected* if each pair of points can be connected by a locally well-ended path.

It is straightforward from the definition that every locally wep-connected space is wep-connected. Moreover one can show that the local version of Proposition 4.1.3 still holds.

Well-ended and well-targeted paths enjoy also the following properties (the proofs may be found in [**5**]):

Proposition 4.1.7. *Let X be a topological space and $\alpha\colon I \to X$ a path.*

1. *If there is a $0 \leq t \leq 1$ such that $\alpha_{[t,1]}$ is well-targeted (resp. locally well-targeted), then α is well-targeted (resp. locally well-targeted) itself.*

2. *If there are $0 \leq s \leq t \leq 1$ such that $\alpha_{[t,1]}$ and $\left(\alpha_{[0,s]}\right)^{-1}$ are well-targeted (resp. locally well-targeted), then α is well-ended (resp. locally well-ended).*

3. *The reverse of a well-ended (resp. locally well-ended) path is well-ended (resp. locally well-ended).*

4. *The concatenation of well-ended (resp.locally well-ended) paths is well-ended (resp. locally well-ended).*

The following corollary allows us to replace any path in a locally wep-connected space by a homotopic locally well-targeted path with the same endpoints.

Corollary 4.1.8. *Let X be wep-connected (resp. locally wep-connected) and $x_1, x_2 \in X$. For each class $[\alpha] \in \pi_1 X(x_1, x_2)$ there is a well-targeted (resp. locally well-targeted) path $\beta \in [\alpha]$.*

Proof. If X is wep-connected (resp. locally wep-connected), there is a well-targeted (resp. locally well-targeted) path γ from x_1 to x_2. Let $\beta := \alpha * \gamma^{-1} * \gamma$. Clearly $[\alpha] = [\beta]$ and (1) of the above lemma implies that β is well-targeted (resp. locally well-targeted). □

It is also interesting to recall the following partial generalization of the van Kampen Theorem to the topological fundamental group, for the proof see [**6**, 4.23]:

Theorem 4.1.9 (VAN KAMPEN THEOREM). *Let (X, x_0) be a based space and $\{U_1, U_2, U_1 \cap U_2\}$ an open cover of X consisting of path connected neighborhoods each containing x_0. If $U_1 \cap U_2$ is wep-connected, there is a canonical isomorphism*

$$\pi_1^\tau(X) \cong \pi_1^\tau(U_1) *_{\pi_1^\tau(U_1 \cap U_2)} \pi_1^\tau(U_2).$$

In other words if $k_i \colon U_1 \cap U_2 \hookrightarrow U_i$ and $j_i \colon U_i \hookrightarrow X$ are the inclusions and $U_1 \cap U_2$ is wep-connected, the induced diagram of continuous homomorphisms

$$\begin{array}{ccc} \pi_1^\tau (U_1 \cap U_2) & \xrightarrow{(k_1)_*} & \pi_1^\tau (U_1) \\ {\scriptstyle (k_2)_*} \downarrow & & \downarrow {\scriptstyle (j_1)_*} \\ \pi_1^\tau (U_2) & \xrightarrow{(j_2)_*} & \pi_1^\tau (X) \end{array}$$

is a pushout square in the category of topological groups.

Recall that if there are given two group homomorphisms $\phi \colon F \to G$ and $\psi \colon F \to H$, the *amalgamated product* of G and H (with respect to ϕ and ψ), denoted by $G *_F H$, is the quotient group $(G * H)/N$, where N is the normal subgroup of $G * H$ generated by all words of the form $\phi(x)\,\psi(x)^{-1}$ for $x \in F$; in our case ϕ and ψ are the continuous homomorphisms induced by inclusions $U_i \hookrightarrow X$. For the definition of *pushout* see [**22**, III.3].

Moreover ([**6**, 4.10]):

Theorem 4.1.10. *Every topological group G is isomorphic to the fundamental topological group $\pi_1^\tau (X)$ of a suitable locally wep-connected space X.*

4.1.1 Semicoverings of wep-connected spaces

We end this section with two statements motivated by the desire to lift properties of a space to its semicovering spaces. Drawing down some pictures while reading may be of some help.

Proposition 4.1.11. *Let $p \colon Y \to X$ be a semicovering map such that $p(y_0) = x_0$. If $\alpha \in (\mathcal{P}X)_{x_0}$ is (locally) well-targeted, then so is the lift $\tilde{\alpha}_{y_0}$.*

Proof. Suppose α is well-targeted and let $\mathcal{W}$ be an open neighborhood of $\tilde{\alpha}_{y_0}$ in $(\mathcal{P}Y)_{y_0}$. Since p is a local homeomorphism, there is an open neighborhood U of $\tilde{\alpha}_{y_0}(1)$ mapped homeomorphically by p onto an open subset of X. Let $\mathcal{U} \coloneqq \mathcal{W} \cap M(\{1\}, U)$ and remark that $\tilde{\alpha}_{y_0} \in \mathcal{U}$. Since $\mathcal{P}p \colon (\mathcal{P}Y)_{y_0} \to (\mathcal{P}X)_{x_0}$ is a homeomorphism, $\mathcal{V} \coloneqq \mathcal{P}p(U)$ is an open neighborhood of α. By assumption, there is an open neighborhood V of $\alpha(1)$ (which may be taken contained in $p(U)$) such that for each $v \in V$ there is a path $\gamma \in \mathcal{V}$ from x_0 to v. Now, $W \coloneqq p^{-1}(V) \cap U$ is a homeomorphic copy of V in U. If $w \in W$, then $p(w) \in V$ and there is a path $\gamma \in \mathcal{V}$ from x to $p(w)$. Since $p \circ \tilde{\gamma}_{y_0}(1) = p(w)$ and $\tilde{\gamma}_{y_0}(1) \in p^{-1}(p(w)) \cap U = \{w\}$, we have $\tilde{\gamma}_{y_0}(1) = w$. This ends the proof of the well-targeted case.

Since the lift of well-targeted paths is still well-targeted, the local version follows from the same argument starting with a locally well-targeted path α. □

Corollary 4.1.12. *Let $p \colon Y \to X$ be a semicovering map. If X is wep-connected (resp. locally wep-connected), then so is every path component of Y.*

Proof. Let $p(y_0) = x_0$: we show the path component of y_0 in Y is wep-connected (resp. locally wep-connected). Suppose $y \in Y$, $p(y) = x$ and $\tilde{\gamma}_{y_0}$ is a path from y_0 to y so that $\gamma = p \circ \tilde{\gamma}_{y_0}$ is a path from x_0 to x. We have to show that $\tilde{\gamma}_{y_0}$ is well-ended. By Corollary 4.1.8, there is a well-targeted (resp. locally well-targeted) path α from x_0 to x homotopic to γ. This homotopy lifts to a homotopy of paths $\tilde{\gamma}_{y_0} \simeq \tilde{\alpha}_{y_0}$; in particular $\tilde{\alpha}_{y_0}(1) = \tilde{\gamma}_{y_0}(1) = y$ and $\tilde{\alpha}_{y_0}$ is well-targeted (resp. locally well-targeted) by Proposition 4.1.11. Repeating the same argument for $(\tilde{\gamma}_{y_0})^{-1}$ one finds that $(\tilde{\alpha}_{y_0})^{-1}$ is well-targeted too and by (2) of Proposition 4.1.7, we conclude that $\tilde{\alpha}_{y_0}$ is well-ended (resp. locally well-ended). So any two points in the path component of y_0 in Y can be connected by a well-ended (resp. locally well-ended) path so any path component of Y is wep-connected (resp. locally wep-connected). □

4.2 THE ENRICHED EQUIVALENCES Π_1^τ, μ^τ AND $\mathscr{R}^\tau$

The aim of this final section is the proof of the following theorem, which is the generalization of Corollary 1.3.31:

Theorem 4.2.1. *If X is locally wep-connected, all arrows in the following quasi-commutative diagram are equivalences:*

$$\begin{array}{ccc} \mathbf{SCov}(X) & \xrightarrow[\sim]{\Pi_1^\tau} & \mathbf{OCovMor}(\Pi_1^\tau X) \\ & \underset{\mu^\tau}{\overset{\sim}{\searrow}} & \wr\downarrow \mathscr{R}^\tau \\ & & \mathbf{TopFct}(\Pi_1^\tau X, \mathbf{Set}^e)\,. \end{array}$$

We start by proving the Π_1^τ−equivalence.

In this case the main difficult is the existence of a semicovering p whose induced covering morphism $\Pi_1^\tau p$ is equivalent to a given open covering morphism $\mathcal{H} \to \Pi_1^\tau X$: roughly speaking, if we are given an open covering morphism, we need a way to build a semicovering which induces exactly that open covering morphism (or an equivalent one).

Let X be a path-connected space and $F\colon \mathcal{H} \to \Pi_1 X$ be a covering morphism of groupoids: since we are interested in genuine coverings, we may assume $ob(\mathcal{H}) \neq \emptyset$ and so F is sujective on objects (Remark 1.2.6).

For each $y \in F_{ob}^{-1}(x)$ and $\alpha \in (\mathcal{P}X)_x$, let $y \cdot [\alpha]$ denote the target of the unique morphism $\widetilde{[\alpha]}_y \in \mathcal{H}_y$ such that $F\left(\widetilde{[\alpha]}_y\right) = [\alpha] \in (\Pi_1^\tau X)_x$.

Definition 4.2.2. Let $\tilde{X}_F$ be the space obtained by giving the set $ob(\mathcal{H})$ the quotient topology with respect to the map

$$\Theta_F\colon \bigsqcup_{x\in X} F_{ob}^{-1}(x) \times (\mathcal{P}X)_x \to ob(\mathcal{H}) \text{ where } \Theta_F(y,\alpha) = y \cdot [\alpha]\,.$$

It is understood that each fiber $F_{ob}^{-1}(x)$ is given the discrete topology. A generic element of $\tilde{X}_F$ will be denoted by $y \cdot [\alpha]$. Finally define the map $p_F\colon \tilde{X}_F \to X$ by $p_F(y \cdot [\alpha]) = \alpha(1)$.

Notice that $ob(\mathcal{H})$ and $\tilde{X}_F$ are the same set: we are just giving $ob(\mathcal{H})$ the topology induced by Θ_F. Moreover, if $h \in ob(\mathcal{H})$, then

$$\Theta_F^{-1}(h) = \left\{(a,\alpha) : \ a \in F_{ob}^{-1}(\alpha(0)),\ \alpha \in (\mathcal{P}X)_{\alpha(0)} \text{ and } \widetilde{[\alpha]}_a \text{ has target } h\right\}.$$

Remark 4.2.3. The diagram

$$\begin{array}{ccc} \bigsqcup_{x\in X} F_{ob}^{-1}(x) \times (\mathcal{P}X)_x & \xrightarrow{\Theta_F} & \tilde{X}_F \\ & \overset{ev}{\searrow} & \downarrow p_F \\ & & X \end{array}$$

commutes when $ev(y,\alpha) = \alpha(1)$. Moreover ev is continuous (Proposition 4.1.4) so p_F is continuous too by the universal property of quotient spaces. Remark also that p_F is necessarily surjective.

Proposition 4.2.4. *If $F' \colon \mathcal{H}' \to \Pi_1 X$ is another covering morphism and $K \colon \mathcal{H} \to \mathcal{H}'$ is a functor such that $F' \circ K = F$, then $K_{ob} \colon \tilde{X}_F \to \tilde{X}_{F'}$ is continuous.*

Proof. For each $x \in X$, let $K_x \colon F_{ob}^{-1}(x) \to F'^{-1}_{ob}(x)$ be the restriction of K_{ob} to $F_{ob}^{-1}(x)$. Since F, F' are covering morphisms and $F' \circ K = F$, we have $K(y \cdot [\alpha]) = K_x(y)$ whenever $F(x) = y$ and $\alpha \in (\mathcal{P}X)_x$. Thus the following diagram commutes:

$$\begin{array}{ccc} \bigsqcup_{x\in X} F_{ob}^{-1}(x) \times (\mathcal{P}X)_x & \xrightarrow{\bigsqcup_{x\in X} K_x \times id} & \bigsqcup_{x\in X} F'^{-1}_{ob} \times (\mathcal{P}X)_x \\ \downarrow \Theta_F & & \downarrow \Theta_{F'} \\ \tilde{X}_F & \xrightarrow{K_{ob}} & \tilde{X}_{F'}. \end{array}$$

The top map is trivially continuous because the topology on each fiber $F_{ob}^{-1}(x)$ and $F'^{-1}_{ob}$is discrete; vertical arrows are continuous by definition; so the continuity of the bottom map follows from the universal property of quotient spaces. □

Next results show that, under appropriate hypothesis, p_F provides us of a semicovering map.

Lemma 4.2.5. *If X is wep-connected, then $p_F \colon \tilde{X}_F \to X$ is an open map.*

Proof. Suppose W is open in $\tilde{X}_F$ and $x = p_F(y) \in p_F(W)$ for $y \in W$. Notice that $y \cdot [c_x] = y$. By Corollary 4.1.8, there exists a well-targeted, null-homotopic loop $\beta \in \Omega(X,x)$. So $y \cdot [\beta] = y \cdot [c_x] = y \in W$. Now, for some open neighborhood $\mathcal{U}$ of β in $(\mathcal{P}X)_x$, we have $\{y\} \times \mathcal{U} = \Theta_F^{-1}(W) \cap (\{y\} \times (\mathcal{P}X)_x)$. Since β is well-targeted, there is an open neighborhood V of $x = \beta(1)$ in X such that for every $v \in V$, there is a path $\delta \in \mathcal{U}$ such that $\delta(1) = v$. Thus $y \cdot [\delta] \in W$ and $p_F(y \cdot [\delta]) = \delta(1) = v$. So $V \subseteq p_F(W)$, showing that $p_F(W)$ is a neighborhood of all its points and so it is open. □

Lemma 4.2.6 (Canonical lifts of paths)**.** *Let X be a wep-connected space and $F \colon \mathcal{H} \to \Pi_1 X$ a covering morphism. Then p_F admits a canonical lift of paths.*

Proof. For $y \in p_F^{-1}(x)$ we need a way to build a continuous section of the map $\mathcal{P}p_F \colon \left(\mathcal{P}\tilde{X}_F\right)_y \to (\mathcal{P}X)_x$, i.e. a map $L_F \colon (\mathcal{P}X)_x \to \left(\mathcal{P}\tilde{X}_F\right)_y$ such that $\mathcal{P}p_F \circ L_F = id_{(\mathcal{P}X)_x}$.

Multiplication $\mu \colon I \times I \to I$ of real numbers is continuous, so the function

$$\mu^{\#} \colon (\mathcal{P}X)_x \to \mathbf{Top}\left((I \times I, \{0\} \times I \cup I \times \{0\}), (X, \{x\})\right), \ \beta \mapsto \beta \circ \mu,$$

is continuous also by the Exponential Law 1.1.4 (see also Section 1.1 for notations); actually, $\beta \circ \mu$ just takes a couple of real numbers $(a,b) \in I \times I$ to the point $\beta(\mu(a,b)) \in X$. Additionally, the map

$$r \colon \mathbf{Top}\left((I \times I, \{0\} \times I \cup I \times \{0\}), (X, \{x\})\right) \to \left(\mathcal{P}(\mathcal{P}X)_x\right)_{c_x}$$

defined by $r(\phi)(s)(t) = \phi(s,t)$ (where $(s,t) \in I \times I$ may be visualized as a "path of paths") is a homeomorphism: indeed, by definition, we have that $(\mathcal{P}X)_x = \mathbf{Top}\left((I, \{0\}), (X, \{x\})\right)$ and $\left(\mathcal{P}(\mathcal{P}X)_x\right)_{c_x} =$

$\mathbf{Top}((I,\{0\}),\mathbf{Top}((I,\{0\}),(X,\{x\})))$ and again by the Eponential Law, this last space is homeomorphic to $\mathbf{Top}((I\times I,\{0\}\times I\cup I\times\{0\}),(X,\{x\}))$. Notice that $r(\beta\circ\mu)(s)(t)=\beta(st)$ and therefore $r(\beta\circ\mu)(s)=\beta_{[0,s]}$. Lastly, the map

$$\mathcal{P}\Theta_F\colon\ (\mathcal{P}(\mathcal{P}X)_x)_{c_x}\to\left(\mathcal{P}\tilde{X}_F\right)_y$$

is obtained by applying $\mathcal{P}$ to the restriction of Θ_F to $\{y\}\times(\mathcal{P}X)_x$, and clearly $\mathcal{P}\Theta_F$ is continuous (because Θ_F was).

Now let $L_F\colon(\mathcal{P}X)_x\to\left(\mathcal{P}\tilde{X}_F\right)_y$ be the composition $\mathcal{P}\Theta_F\circ r\circ\mu^{\#}$ which takes β to the path $\tilde{\beta}_y(s)=y\cdot\left[\beta_{[0,s]}\right]$. Since $p_F\left(y\cdot\left[\beta_{[0,s]}\right]\right)=\beta(s)$, $\tilde{\beta}_y$ is a lift of β starting at y as we wanted. So L_F is the desired section. □

Remark 4.2.7. For $y_1,y_2\in\tilde{X}_F$, let $F\mathcal{H}(y_1,y_2)$ be the image of the injection $F\colon\mathcal{H}(y_1,y_2)\to\Pi_1X(x_1,x_2)$. Since F is a covering morphism, $F\mathcal{H}(y_1,y_2)=\{[\alpha]\in\Pi_1X(x_1,x_2):\ y_1\cdot[\alpha]=y_2\}$ whenever $p_F(y_i)=x_i$. The special case $y_1=y=y_2$ gives that $y\cdot[\alpha]=y\cdot[\beta]$ if and only if $\left[\alpha*\beta^{-1}\right]\in F\mathcal{H}(y)$.

From this moment on, X will be a locally wep-connected space, $\mathcal{H}$ a **Top**-groupoid and the covering morphism $F\colon\mathcal{H}\to\Pi_1^\tau X$ an open **Top**-functor. Our next goal is to obtain a simple basis for the topology of $\tilde{X}_F$. Since, by definition, each map $F\colon\mathcal{H}(y_1,y_2)\to\Pi_1^\tau X(x_1,x_2)$ is an open embedding, $F\mathcal{H}(y_1,y_2)$ is open in $\Pi_1^\tau X(x_1,x_2)$. Moreover, each map $h\colon\mathcal{P}X(x_1,x_2)\to\Pi_1^\tau X(x_1,x_2)$ identifying homotopy classes of paths is continuous (Proposition 2.1.19), so the pre-image $h^{-1}(F\mathcal{H}(y_1,y_2))$ is open in $\mathcal{P}X(x_1,x_2)$.

Suppose $p_F(y_0)=x_0$, $\alpha\in(\mathcal{P}X)_{x_0}$ and U is an open neighborhood of $y_0\cdot[\alpha]$ in $\tilde{X}_F$. By Corollary 2.1.19, we may assume that α is locally well-targeted. Notice also that $h^{-1}(F\mathcal{H}(y_0))$ is an open neighborhood of the null-homotopic loop $\alpha*\alpha^{-1}$ in $\Omega(X,x_0)$ because $\Omega(X,x_0)\subseteq(\mathcal{P}X)_{x_0}$ and since $F\mathcal{H}(y_0)\subseteq\Pi_1^\tau X(x_0)$, we get $h^{-1}(F\mathcal{H}(y_0))\subseteq h^{-1}(\Pi_1^\tau X(x_0))=\Omega(X,x_0)$. This means we can find a neighborhood $\mathcal{U}=\bigcap_{i=1}^m M\left(K_m^i,A_i\right)$ of α in $(\mathcal{P}X)_{x_0}$ such that:

1. $\{y_0\}\times\mathcal{U}\subseteq\Theta_F(U)$: indeed, for sure $y_0\in F_{ob}^{-1}(x_0)$ and if $\mathcal{U}$ is sufficiently small, for all $\beta\in\mathcal{U}$, β is a path from x_0 to $\beta(1)$ and $\beta(1)$ is a point in a small neighborhood of $\alpha(1)$ so that $y_0\cdot[\beta]$ results in U (remember that p_F is continuous);

2. $\alpha*\alpha^{-1}\in\mathcal{U}\cdot\mathcal{U}^{-1}\cap\Omega(X,x_0)\subseteq h^{-1}(F\mathcal{H}(y_0))$: indeed, we know that $h^{-1}(F\mathcal{H}(y_0))\subseteq\Omega(X,x_0)$ and if $\mathcal{U}$ is sufficiently small we can suppose $\mathcal{U}\cdot\mathcal{U}^{-1}\subseteq h^{-1}(F\mathcal{H}(y_0))$, so that $\mathcal{U}\cdot\mathcal{U}^{-1}\cap\Omega(X,x_0)\subseteq h^{-1}(F\mathcal{H}(y_0))$;
   it is undestood that $\mathcal{U}{\cdot}\mathcal{U}^{-1}=\left\{\gamma_1*\gamma_2\in\Omega(X,x_0):\ \gamma_1\in\mathcal{U},\ \gamma_2\in\mathcal{U}^{-1}\text{ and }\gamma_1*\gamma_2\text{ makes sense}\right\}$;

Since α is locally well-targeted, there is an open neighborhood V of $\alpha(1)$ contained in A_m such that for each $v\in V$, there is a well-targeted path $\delta\in\mathcal{U}$ from x_0 to v.

Let $B(y_0\cdot[\alpha],\mathcal{U},V):=\Theta_F(\{y_0\}\times(\mathcal{U}\cap M(\{1\},V)))$; roughly speaking, $\mathcal{U}\cap M(\{1\},V)$ are the paths of $\mathcal{U}$ which end in V.

Lemma 4.2.8. *Sets of the form $B(y_0\cdot[\alpha],\mathcal{U},V)$ give a basis for the topology of $\tilde{X}_F$. Moreover the open set $B(y_0\cdot[\alpha],\mathcal{U},V)$ is mapped homeomorphically onto V by p_F. In particular p_F is a local homeomorphism.*

Proof. The reader is invited to look at Figure 4.1.1 while reading this proof.

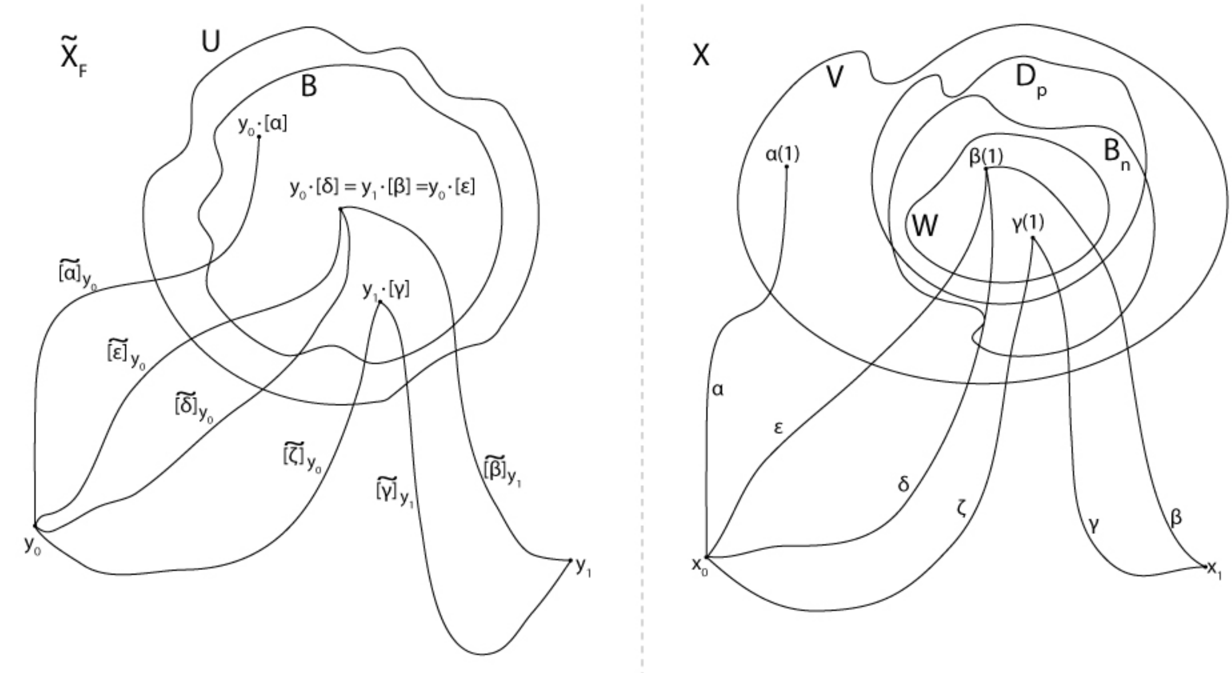


Figure 4.2.1: Lemma 4.2.8

Since U is arbitrary and $B(y_0 \cdot [\alpha], \mathcal{U}, V) \subseteq U$ (by point (1) above) it is enough to show that $B(y_0 \cdot [\alpha], \mathcal{U}, V)$ is open in $\tilde{X}_F$ or, equivalently, that $\Theta_F^{-1}(B(y_0 \cdot [\alpha], \mathcal{U}, V))$ is open. If $(y_1, \beta) \in \Theta_F^{-1}(B(y_0 \cdot [\alpha], \mathcal{U}, V)) \cap \left(\{y_1\} \times (\mathcal{P}X)_{x_1}\right)$, then $y_1 \cdot [\beta] = y_0 \cdot [\epsilon]$ for $\epsilon \in \mathcal{U} \cap M(\{1\}, V)$ and this implies that $\beta(1) = \epsilon(1)$. By assumption there is a well-targeted path $\delta \in \mathcal{U}$ such that $\delta(1) = \epsilon(1)$ because $\epsilon \in M(\{1\}, V)$. Since $\delta * \epsilon^{-1} \in \mathcal{U} \cdot \mathcal{U}^{-1} \cap \Omega(X, x_0) \subseteq h^{-1}(F\mathcal{H}(y_0))$, we have $y_0 \cdot [\delta] = y_0 \cdot [\epsilon] = y_1 \cdot [\beta]$. Since $\left[\delta * \beta^{-1}\right] \in F\mathcal{H}(y_0, y_1)$, the set $h^{-1}(F\mathcal{H}(y_0, y_1))$ is an open neighborhood of $\delta * \beta^{-1}$ in $\mathcal{P}X(x_0, x_1)$. This observation guarantees that there are open neighborhoods $\mathcal{B} = \bigcap_{j=1}^{n} M\left(K_n^j, B_j\right)$ of β in $(\mathcal{P}X)_{x_1}$ and $\mathcal{D} = \bigcap_{k=1}^{p} M\left(K_p^k, D_k\right)$ of δ in $(\mathcal{P}X)_{x_0}$ such that:

1. $\mathcal{D} \subseteq \mathcal{U}$, because $\delta \in \mathcal{U}$ so it is enough to choose a neighborhood of δ contained in $\mathcal{U}$;

2. $\delta * \beta^{-1} \in \mathcal{D} \cdot \mathcal{B}^{-1} \cap \mathcal{P}X(x_0, x_1) \subseteq h^{-1}(F\mathcal{H}(y_0, y_1))$, by construction;

3. $B_n \cup D_p \subseteq V$ (remember that $\beta(1) = \delta(1) \in V$).

Since δ is well-targeted, there is an open neighborhood W of $\beta(1) = \delta(1)$ of $B_n \cap D_p$ such that for each $w \in W$, there is a path $\zeta \in \mathcal{D}$ from x_0 to w. We claim the neighborhood $\{y_1\} \times (\mathcal{B} \cap M(\{1\}, W))$ of (y_1, β) is contained in $\Theta_F^{-1}(B(y_0 \cdot [\alpha], \mathcal{U}, V)) \cap \left(\{y_1\} \times (\mathcal{P}X)_{x_1}\right)$.

If $\gamma \in \mathcal{B} \cap M(\{1\}, W)$, there is a path $\zeta \in \mathcal{D}$ from x_0 to $\gamma(1)$. This gives $\zeta * \gamma^{-1} \in \mathcal{D} \cdot \mathcal{B}^{-1} \cap \mathcal{P}X(x_0, x_1) \subseteq h^{-1}(F\mathcal{H}(y_0, y_1))$ and therefore $y_0 \cdot [\zeta] = y_1 \cdot [\gamma]$. Since $y_1 \cdot [\gamma] = y_0 \cdot [\zeta]$, for $\zeta \in \mathcal{D} \cap M(\{1\}, W) \subseteq \mathcal{U} \cap M(\{1\}, V)$, we have $y_1 \cdot [\gamma] \in B(y_0 \cdot [\alpha], \mathcal{U}, V)$ as we wanted.

For the second part of the statement, since p_F is open by Lemma 4.2.5, we have that the restriction $\bar{p}_F \colon B(y_0 \cdot [\alpha], \mathcal{U}, V) \to V$ of p_F is a homeomorphism if p_F is bijective.

If $v \in V$, there is a path $\delta \in \mathcal{U}$ such that $\delta(1) = v$ which gives $p_F(y_0 \cdot [\delta]) = v$ so $\bar{p}_F$ is surjective. If $\delta, \epsilon \in \mathcal{U} \cap M(\{1\}, V)$ such that $p_F(y_0 \cdot [\delta]) = \delta(1) = \epsilon(1) = p_F(y_0 \cdot [\epsilon])$, then $\delta * \epsilon^{-1} \in \mathcal{U} \cdot \mathcal{U}^{-1} \cap \Omega(X, x_0) \subseteq h^{-1}(F\mathcal{H}(y_0))$ and so $\left[\delta * \epsilon^{-1}\right] \in F\mathcal{H}(y_0)$ and this is possible if and only if $y_0 \cdot [\delta] = y_0 \cdot [\epsilon]$ (i.e. the target of δ is the source of ϵ^{-1}), thus $\bar{p}_F$ is injective. □

Remark 4.2.9. The rough drawing in Figure 4.1.1 makes this last proof seem artificious and redundant: in fact, if V is path connected, $B(y_0\cdot[\alpha],\mathcal{U},V)=\left\{y_0\cdot[\alpha*\zeta]:\ \zeta\in(\mathcal{P}V)_{\alpha(1)}\right\}$. So if X is locally path connected (as it is the sheet of our drawing...) the construction of $\tilde{X}_F$ agrees with the usual construction of coverings of locally path connected spaces (cfr., for example, [**23**, V.7-10]).

By Lemma 4.2.8, if $p_F(y)=x$, the set $y\cdot(\Pi_1^\tau X)_x=\left\{y\cdot[\alpha]\in\tilde{X}_F:\ \alpha\in(\mathcal{P}X)_x\right\}$ is open in $\tilde{X}_F$; moreover X is path-connected, so $(\mathcal{P}X)_x$ is path connected and therefore $y\cdot(\Pi_1^\tau X)_x$ is path connected too.

Proposition 4.2.10. *The path components of $\tilde{X}_F$ are the open sets $y\cdot(\Pi_1^\tau X)_x$.*

Proof. If $y\in y_1\cdot(\Pi_1^\tau X)_{x_1}\cap y_2\cdot(\Pi_1^\tau X)_{x_2}$, then $y_1\cdot[\alpha_1]=y=y_2\cdot[\alpha_2]$ for some paths α_i satisfying the conditions $\alpha_i(0)=x_i$ and $\alpha_1(1)=\alpha_2(1)$. We claim that $y_1\cdot(\Pi_1^\tau X)_{x_1}=y_2\cdot(\Pi_1^\tau X)_{x_2}$. If $y_1\cdot[\beta]\in y_1\cdot(\Pi_1^\tau X)_{x_1}$, where $\beta(0)=x_1$, then $y_1\cdot[\beta]=y_1\cdot[\alpha_1]\left[\alpha_1^{-1}*\beta\right]=y_2\cdot[\alpha_2][\alpha_1*\beta]=y_2\cdot\left[\alpha_2*\alpha_1^{-1}*\beta\right]$ giving $y_1\cdot[\beta]\subseteq y_2\cdot(\Pi_1^\tau X)_{x_2}$. The other inclusion is shown similarly. □

The following theorem shows that p_F is actually a semicovering map.

Theorem 4.2.11. *If X is locally wep-connected and $F\colon\mathcal{H}\to\Pi_1^\tau X$ is an open covering morphism of* **Top**-*groupoids, then $p_F\colon\tilde{X}_F\to X$ is a semicovering map.*

Proof. By Lemmas 4.2.8 and 4.2.6 p_F is a local homeomorphism which has the path lifting property; so it remains to be shown that p_F has unique path lifting (recall Definition 3.1.5). Let $f,g\colon I\to\tilde{X}_F$ be paths in $\tilde{X}_F$ such that $p_F\circ f=p_F\circ g$. We show that $\{t\in I:\ f(t)=g(t)\}$ is either empty or the whole I. By Proposition 4.2.10, we may assume that f and g have image in a path component $y_0\cdot(\Pi_1^\tau X)_{x_0}$ where $p_F(y_0)=x_0$. Exploiting Corollary 4.1.8, we have $f(t)=y_0\cdot[\alpha_t]$ and $g(t)=y_0\cdot[\beta_t]$ for some locally well-targeted paths $\alpha_t,\beta_t\in(\mathcal{P}X)_{x_0}$. The condition $p_F\circ f=p_F\circ g$ means $\alpha_t(1)=\beta_t(1)$ for each $t\in I$ and thus $\alpha_t*\beta_t^{-1}\in\Omega(X,x_0)$. Let $\lambda_t=\left[\alpha_t*\beta_t^{-1}\right]$ so that $h^{-1}(F\mathcal{H}(y_0)*\lambda_t)$ is an open neighborhood of $\alpha_t*\beta_t^{-1}$ in $\Omega(X,x_0)$. Since $\alpha_t*\alpha_t^{-1}$ and $\beta_t*\beta_t^{-1}$ are null-homotopic, there is an open neighborhood $\mathcal{A}_t=\bigcap_{j=1}^{n_t}M\left(K_{n_t}^j,A_j^t\right)$ of α_t and $\mathcal{B}_t=\bigcap_{i=1}^{m_t}M\left(K_{m_t}^i,B_i^t\right)$ of β_t in $(\mathcal{P}X)_{x_0}$ such that $\left(\left(\mathcal{A}_t\cdot\mathcal{A}_t^{-1}\right)\cup\left(\mathcal{B}_t\cdot\mathcal{B}_t^{-1}\right)\right)\cap\Omega(X,x_0)\subseteq h^{-1}(F\mathcal{H}(y_0))$ and $\mathcal{A}_t\cdot\mathcal{B}_t^{-1}\cap\Omega(X,x_0)\subseteq h^{-1}(F\mathcal{H}(y_0)*\lambda_t)$. Since α_t,β_t are locally well-targeted, there is an open neighborhood $U_t\subseteq A_{n_t}^t$ of $\alpha_t(1)$ (resp. $V_t\subseteq B_{m_t}^t$ of $\beta_t(1)$) such that for each $u\in U_t$ (resp. $v\in V_t$) there is a well-targeted path $\delta\in\mathcal{A}_t$ with $\delta(1)=u$ (resp. $\gamma\in\mathcal{B}_t$ with $\gamma(1)=v$).

According to Lemma 4.2.8, for each $t\in I$, $B(y_0\cdot[\alpha_t],\mathcal{A}_t,U_t)$ and $B(y_0\cdot[\beta_t],\mathcal{B}_t,V_t)$ are open neighborhoods of $y_0\cdot[\alpha_t]$ and $y_0\cdot[\beta_t]$ in $y_0\cdot(\Pi_1^\tau X)_{x_0}$ respectively. Suppose now that there are $r,s\in I$ such that $y_0\cdot[\alpha_r]\neq y_0\cdot[\beta_r]$ and $y_0\cdot[\alpha_s]=y_0\cdot[\beta_s]$ and assume $r<s$. Let z be the greatest lower bound of $A=\{t\in[r,s]:\ y_0\cdot[\alpha_t]=y_0\cdot[\beta_t]\}=\left\{t\in[r,s]:\ \left[\alpha_t*\beta_t^{-1}\right]\in F\mathcal{H}(y_0)\right\}$ (in practice $z=\inf(A)$).

Since f,g are continuous, there exists an $\epsilon>0$ such that $y_0\cdot[\alpha_t]\in B(y_0\cdot[\alpha_z],\mathcal{A}_z,U_z)$ and $y_0\cdot[\beta_t]\in B(y_0\cdot[\beta_z],\mathcal{B}_z,V_z)$ for all $t\in(z-\epsilon,z+\epsilon)\cap I$. Now there are two cases:

1. If $z\in A$ (i.e. $\left[\alpha_z*\beta_z^{-1}\right]\in F\mathcal{H}(y_0)$), then $r<z\leq s$ and $F\mathcal{H}(y_0)*\lambda_z=F\mathcal{H}(y_0)$. Pick any $t_0\in(r,z)\cap(z-\epsilon,z)$. We have $y_0\cdot[\alpha_{t_0}]\in B(y_0\cdot[\alpha_z],\mathcal{A}_z,U_z)$ and $y_0\cdot[\beta_{t_0}]\in B(y_0\cdot[\beta_z],\mathcal{B}_z,V_z)$ and therefore $y_0\cdot[\alpha_{t_0}]=y_0\cdot[\zeta]$ for $\zeta\in\mathcal{A}_z$ and $y_0\cdot[\beta_{t_0}]=y_0\cdot[\eta]$ for $\eta\in\mathcal{B}_z$. Since $\zeta(1)=\alpha_{t_0}(1)=\beta_{t_0}(1)=\eta(1)$, we have $\zeta*\eta^{-1}\in\mathcal{A}_z\cdot\mathcal{B}_z^{-1}\cap\Omega(X,x_0)\subseteq h^{-1}(F\mathcal{H}(y_0)*\lambda_z)=h^{-1}(F\mathcal{H}(y_0))$ and $y_0\cdot[\alpha_{t_0}]=y_0\cdot[\zeta]=y_0\cdot[\eta]=y_0\cdot[\beta_{t_0}]$. But $t_0<z$ and $t_0\in A$ contradicting that z is a lower bound of A.
2. If $z\notin A$ (i.e. $\left[\alpha_z*\beta_z^{-1}\right]\notin F\mathcal{H}(y_0)$), then $r\leq z<s$ and $F\mathcal{H}(y_0)*\lambda_z\cap F\mathcal{H}(y_0)=\emptyset$. Pick any $t_0\in(z,s)\cap(z,z+\epsilon)$ so that, as above, $y_0\cdot[\alpha_{t_0}]=y_0\cdot[\zeta]$ for $\zeta\in\mathcal{A}_z$ and $y_0\cdot[\beta_{t_0}]=y_0\cdot[\eta]$ for

$\eta \in \mathcal{B}_z$. If $y_0 \cdot [\alpha_{t_0}] = y_0 \cdot [\beta_{t_0}]$, then $\left[\zeta * \eta^{-1}\right] \in F\mathcal{H}(y_0)$. But this is not possible because $\zeta * \eta^{-1} \in \mathcal{A}_z \cdot \mathcal{B}_z^{-1} \cap \Omega(X, x_0) \subseteq h^{-1}(F\mathcal{H}(y_0) * \lambda_z)$ and $F\mathcal{H}(y_0) * \lambda_z \cap F\mathcal{H}(y_0) = \emptyset$. Thus $y_0 \cdot [\alpha_t] \neq y_0 \cdot [\beta_t]$ for all $t \in [z, s) \cap [z, z + \epsilon)$. Then any $y \in (z, s) \cap (z, z + \epsilon)$ is a lower bound greater than z, and again we gain a contradiction. $\square$

Finally, we are able to give a proof of the following:

Theorem 4.2.12 (ENRICHED Π_1–EQUIVALENCE)**.** *Let X be locally wep-connected. Then the enriched fundamental groupoid functor Π_1^τ induces an equivalence of categories*

$$\Pi_1^\tau \colon \mathbf{SCov}(X) \simeq \mathbf{OCovMor}(\Pi_1^\tau X).$$

Proof. To show Π_1^τ is an equivalence, let us define the inverse equivalence $\mathscr{S} \colon \mathbf{OCovMor}(\Pi_1^\tau X) \to \mathbf{SCov}(X)$: for an open covering morphism $F \colon \mathcal{H} \to \Pi_1^\tau X$, we set $\mathscr{S}(F) := p_F \colon \tilde{X}_F \to X$; for a morphism K in $\mathbf{OCovMor}(\Pi_1^\tau X)$ such that this diagram quasi-commutes

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{K} & \mathcal{H}' \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle F'} & \\ & \Pi_1^\tau X & \end{array}$$

in view of Proposition 4.2.4, we set $\mathscr{S}(K) := K_{ob} \colon \tilde{X}_F \to \tilde{X}_{F'}$. So $\mathscr{S}$ is well defined both on objects and on morphisms.

First, we show that $\mathscr{S} \circ \Pi_1^\tau \simeq id_{\mathbf{SCov}(X)}$.

Suppose $p \colon Y \to X$ is a semicovering of X and $F = \Pi_1^\tau p \colon \Pi_1^\tau Y \to \Pi_1^\tau X$. Certainly, $\mathscr{S}(\Pi_1^\tau p) = \mathscr{S}(F) = p_F \colon \tilde{X}_F \to X$ and p agree as functions because $p = (\Pi_1^\tau p)_{ob} = F_{ob} = p_F$. But to complete this part of the proof, at least at level of objects, we have also to check that the topologies on Y and $\tilde{X}_F$ agree; so consider the following commutative diagram:

$$\begin{array}{ccc} \bigsqcup_{x \in X} p^{-1}(x) \times (\mathcal{P}X)_x & \xrightarrow{\Theta_F} & \tilde{X}_F \\ {\scriptstyle \bigsqcup_{y \in Y} L_p}\downarrow & & \downarrow{\scriptstyle id} \\ \bigsqcup_{y \in Y} (\mathcal{P}Y)_y & \xrightarrow{\bigsqcup_y ev_1} & Y. \end{array}$$

The left vertical map takes $(y, \alpha) \in p^{-1}(x) \times (\mathcal{P}X)_x$ to the lift $\tilde{\alpha}_y \in (\mathcal{P}Y)_y$ and is a homeomorphism, since p, by definition, has continuous lifting of paths; in particular $\bigsqcup_y L_p$ is open. The bottom map is the evaluation at 1 on each element of the disjoint union and is a quotient map by Corollary 4.1.12 and Proposition 4.1.4, in particular ev_1 is continuous. Finally, Θ_F is continuous by definition. So $id \colon \tilde{X}_F \to Y$ is continuous and open, hence a homeomorphism.

Now we check the equivalence holds at level of morphisms: suppose $p' \colon Y' \to X$ is another semicovering of X, $F' := \Pi_1^\tau p' \colon \Pi_1^\tau Y' \to \Pi_1^\tau X$ and $f \colon Y \to Y'$ is a map such that $p' \circ f = p$. Then $\mathscr{S}(\Pi_1^\tau f)$ is the map $f = (\Pi_1^\tau f)_{ob} \colon \tilde{X}_F \to \tilde{X}_{F'}$, but above we have shown that $\tilde{X}_F = Y$ and $\tilde{X}_{F'} = Y'$, thus

$\mathscr{S} \circ \Pi_1^\tau \simeq id_{\mathbf{SCov}(X)}$ (notice that the components of the natural transformation are identity maps:

$$\begin{array}{ccc} p_F & \xrightarrow{id} & p \\ \downarrow & & \downarrow \\ p_{F'} & \xrightarrow{id} & p' \end{array}$$

and in this case we do not only have an isomorphism of functors $\mathscr{S} \circ \Pi_1^\tau X \simeq id_{\mathbf{SCov}(X)}$ but infact we get a full equality $\mathscr{S} \circ \Pi_1^\tau X = id_{\mathbf{SCov}(X)}$).

Second, we show that $id_{\mathbf{OCovMor}(\Pi_1^\tau X)} \simeq \Pi_1^\tau \circ \mathscr{S}$.

So suppose $F\colon \mathcal{H} \to \Pi_1^\tau X$ is an open covering morphism. By Theorem 4.2.11, $\mathscr{S}(F) = p_F \colon \tilde{X}_F \to X$ is a semicovering which induces an open covering morphism $\Pi_1^\tau p_F \colon \Pi_1^\tau \tilde{X}_F \to \Pi_1^\tau X$. We define a functor $N_F \colon \mathcal{H} \to \Pi_1^\tau \tilde{X}_F$ such that $(N_F)_{ob} = id_{ob(\mathcal{H})}$. Moreover, if $h \in \mathcal{H}(y_1, y_2)$ and $F(h) = [\alpha] \in \Pi_1^\tau X(x_1, x_2)$, let $N_F(h) = [\tilde{\alpha}_{y_1}] \in \Pi_1^\tau \tilde{X}_F(y_1, y_2)$ be the class of the unique lift of α with respect to p_F. So N_F is well defined both on objects and on morphisms of $\mathcal{H}$. Thanks to the description of lifts of paths in Lemma 4.2.6, we immediately get $\tilde{\alpha}_{y_1}(1) = y_1 \cdot F(h) = y_2$.
Since $\Pi_1^\tau p_F \circ N_F = F$ (it is a straight verification), N_F is an open covering morphism by Lemma 3.2.2; but moreover $(N_F)_{ob}$ is the identity on objects and so $N_F \colon \mathcal{H} \to \Pi_1^\tau \tilde{X}_F$ is an isomorphism of **Top**-groupoids.

Thus to end the proof we need to show that N_F is actually a natural transformation of functors such that $id_{\mathbf{OCovMor}(\Pi_1^\tau X)} \simeq \Pi_1^\tau \circ \mathscr{S}$, i.e. that the following diagram of **Top**-functors quasi-commutes for any morphism $K\colon \mathcal{H} \to \mathcal{H}'$ of open covering morphisms F and $F'\colon \mathcal{H}' \to \Pi_1^\tau X$:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{N_F} & \Pi_1^\tau \tilde{X}_F \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle \Pi_1^\tau K_{ob}} \\ \mathcal{H}' & \xrightarrow{N_{F'}} & \Pi_1^\tau \tilde{X}_F. \end{array}$$

Certainly the diagram commutes on objects by definition of N_F and $N_{F'}$. Concerning morphisms, suppose $g \in \mathcal{H}(y_1, y_2)$ and $F(g) = [\alpha]$ so that $N_F(g) = [\tilde{\alpha}_{y_1}]$. Then $\Pi_1^\tau K_{ob}([\tilde{\alpha}_{y_1}])$ is the homotopy class $\left[\tilde{\alpha}_{K(y_1)}\right]$ of the unique lift α with respect to $p_{F'} \colon \tilde{X}_{F'} \to X$ starting at $K(y_1)$. The definition of $N_{F'}$ gives $N_{F'}(K(g)) = \left[\tilde{\alpha}_{K(y_1)}\right]$ because $F'(K(g)) = F(g) = [\alpha]$ and $K(g) \in \mathcal{H}'(K(y_1), K(y_2))$. Hence the diagram commutes and the proof ends. □

As in the $\mu-$equivalence 1.3.36, by defining $(\mu^\tau)^{-1} := (\mathscr{R}^\tau)^{-1} \circ \mathscr{S}$ we get the $\mu^\tau-$equivalence:

Theorem 4.2.13 (ENRICHED $\mu-$EQUIVALENCE)**.** *Let X be locally wep-connected. Then the enriched monodromy functor μ^τ induces an equivalence of categories*

$$\mu^\tau \colon \mathbf{SCov}(X) \simeq \mathbf{TopFct}\left(\Pi_1^\tau X, \mathbf{Set}^e\right).$$

This last theorem together with Theorems 4.2.12 and 3.2.9 closes the triangle and makes it a quasi-commutative diagram of categories where each arrow is an equivalence, proving Theorem 4.2.1.

APPENDIX: an inductive approximation of $\Pi_1^{\tau}(X)$

It may be of some interest to understand how it is possible to build the topological fundamental group $\tau(G)$ starting from a quasitopological group G as defined in Section 2.1 and then how to generalize this construction to quasitopological groupoids. Although annoying in all details, the results here exposed exploit a common transfinite argument. For notations, see Chapter 2.

Thanks to Proposition 2.1.13 it is possible approximate $\tau(G)$ by transfinite induction whenever G is a quasitopological group: let $G = G_0$ be a quasitopological group with topology $\mathcal{T}_{G_0}$. Iterate the action of c by letting $G_\alpha = c(G_{\alpha-1})$ with topology $\mathcal{T}_{G_\alpha}$ for each ordinal α with predecessor. When α is a limit ordinal, let G_α have topology $\mathcal{T}_{G_\alpha} = \bigcap_{\beta<\alpha} \mathcal{T}_{G_\beta}$. One easily shows G_α is a quasitopological group for each ordinal α. Finally we obtain $\tau(G)$ since "the iteration ends" (for the proof, see [**6**, 3.8]):

Theorem. *There exists an ordinal α such that $G_\alpha = \tau(G)$.*

In the next theorem we explicitly generalize this result to groupoids.

Let $\mathcal{G} = \mathcal{G}_0$ be a **qTop**-groupoid (actually, what is laid out here in all generality may be thought in relation with the quasitopological fundamental groupoid: $\mathcal{G}_0 = \Pi_1^{qtop} X$). Construct **qTop**-groupoids $\mathcal{G}_\zeta$ inductively so that if ζ is a successor ordinal, the topology of $\mathcal{G}_\zeta(x,y)$ (if not empty) is the quotient topology with respect to all possible multiplication maps

$$\mu\colon \bigsqcup_{a\in ob(\mathcal{G})} \mathcal{G}_{\zeta-1}(x,a) \times \mathcal{G}_{\zeta-1}(a,y) \to \mathcal{G}_\zeta(x,y).$$

Notice that for each new built groupoid, what changes is the topology on the hom-sets, but objects are the same. If ζ is a limit ordinal, the topology of $\mathcal{G}_\zeta(x,y)$ is the intersection of the topologies of $\mathcal{G}_\eta(x,y)$ for $\eta < \zeta$.

Theorem. *Let $\mathcal{G}$ be a* **qTop**-*groupoid. Then:*

1. *$\mathcal{G}_\zeta$ is a* **qTop**-*groupoid for each ζ.*
2. *The identities $\mathcal{G} \to \mathcal{G}_\zeta \to \mathcal{G}_{\zeta+1} \to \tau(\mathcal{G})$ are morphisms of* **qTop**-*groupoids for each ζ.*
3. *$\tau(\mathcal{G}_\zeta) = \tau(\mathcal{G})$ for each ζ.*
4. *$\mathcal{G}_\zeta$ is a* **Top**-*groupoid if and only if $\mathcal{G}_\zeta = \tau(\mathcal{G})$ if and only if $\mathcal{G}_\zeta(x,y) = \mathcal{G}_{\zeta+1}(x,y)$ for all $x, y \in ob(\mathcal{G})$.*
5. *There is an ordinal number ζ_0 such that $\mathcal{G}_\zeta = \tau(\mathcal{G})$ for e ach $\zeta \geq \zeta_0$.*

Proof. 1. Let us show that, for each ordinal, translations and inversion are continuous. Take for example left translations: $\lambda_g \colon \mathcal{G}_\zeta(w,x) \to \mathcal{G}_\zeta(w,y)$ where $g \colon x \to y$.

- If ζ is a successor ordinal, we have the homeomorphism $\lambda_g^{'} \colon \mathcal{G}_{\zeta-1}(w,x) \to \mathcal{G}_{\zeta-1}(w,y)$ and hence we can build the commuting diagram

$$
\begin{array}{ccc}
\mathcal{G}_{\zeta-1}(w,x) & \xrightarrow{\lambda_g^{'}} & \mathcal{G}_{\zeta-1}(w,y) \\
\downarrow & \searrow & \downarrow \\
\mathcal{G}_\zeta(w,x) & \xrightarrow{\lambda_g} & \mathcal{G}_\zeta(w,y)
\end{array}
$$

  where the left vertical arrow is the continuous multiplication $\mathcal{G}_{\zeta-1}(w,x) \times \{id_x\} \to \mathcal{G}_\zeta(w,x)$ and the left vertical arrow is built similarly. Since the top arrow is continuous, the bottom one is continuous too.

- If ζ is a limit ordinal, λ_g is still continuous by construction of $\mathcal{G}_\zeta$.

Similarly one concludes for inversion and right translations.

2. Again, we procede by transfinite induction.

- If ζ is a successor ordinal, each map $\mathcal{G}_{\zeta-1}(x,y) \to \mathcal{G}_\zeta(x,y)$ is continuous as we noticed above. Thus if $\mathcal{G} \to \mathcal{G}_{\zeta-1}$ is a morphism of **qTop**-groupoids, so is $\mathcal{G} \to \mathcal{G}_\zeta$. Moreover, vertical and top arrows of the following commutative diagram are continuous (notice that the top id is continuous by Remark 2.1.7):

$$
\begin{array}{ccc}
\coprod_{a \in ob(\mathcal{G})} \mathcal{G}_{\zeta-1}(x,a) \times \mathcal{G}_{\zeta-1}(a,y) & \xrightarrow{id} & \coprod_{a \in ob(\mathcal{G})} \tau(\mathcal{G})(x,a) \times \tau(\mathcal{G})(a,y) \\
\downarrow \mu & & \downarrow \mu \\
\mathcal{G}_\zeta(x,y) & \xrightarrow{id} & \tau(\mathcal{G})(x,y)
\end{array}
$$

  hence the bottom identity is continuous too.

- If ζ is a limit ordinal, the result follows by construction of $\mathcal{G}_\zeta$.

3. Since $id \colon \mathcal{G}_\zeta \to \tau(\mathcal{G})$ is a morphism of **qTop**-groupoids, so is $id \colon \tau(\mathcal{G}_\zeta) \to \tau(\tau(\mathcal{G})) = \tau(\mathcal{G})$. By (1), $id \colon \mathcal{G} \to \tau(\mathcal{G}_\zeta)$ is a morphism of **qTop**-groupoids whose adjoint is the inverse $id \colon \tau(\mathcal{G}) \to \tau(\mathcal{G}_\zeta)$. So $\tau(\mathcal{G}_\zeta) = \tau(\mathcal{G})$.

4. The first double implication is essentially (3). For the second one, observe that $\mathcal{G}_\zeta$ is a **Top**-groupoid if and only if $\mu \colon \coprod_{a \in ob(\mathcal{G})} \mathcal{G}_\zeta(x,a) \times \mathcal{G}_\zeta(a,y) \to \mathcal{G}_\zeta(x,y)$ is continuous for all $x, y \in ob(\mathcal{G})$.

5. For each ordinal ζ, let $A_\zeta := \coprod_{x,y \in ob(\mathcal{G})} \mathcal{G}_\zeta(x,y)$ be the disjoint union of the hom-sets (each one with its own topology) and let $\mathcal{T}_\zeta$ be the topology of A_ζ. Point (1) gives that $\mathcal{T}_{\zeta+1} \subseteq \mathcal{T}_\zeta \subseteq \mathcal{T}_0$ for each ζ and point (4) gives that $\mathcal{T}_{\zeta+1} = \mathcal{T}_\zeta$ if and only if $\mathcal{G}_\zeta = \tau(\mathcal{G})$. Suppose $\mathcal{G}_\zeta \neq \tau(\mathcal{G})$ for all ordinals ζ. Then $\mathcal{T}_\zeta \smallsetminus \mathcal{T}_{\zeta+1} \neq \emptyset$ for each ordinal ζ, contradicting the known fact that there is no injection of ordinal numbers into the power set of $\mathcal{T}_0$. Hence there exists an ordinal ζ_0 such that $\mathcal{G}_{\zeta_0} = \tau(\mathcal{G})$. Since $\mathcal{G}_{\zeta_0} \to \mathcal{G}_\zeta \to \tau(\mathcal{G})$ are morphisms of **Top**-groupoids whenever $\zeta \geq \zeta_0$, it follows that $\mathcal{G}_\zeta = \tau(\mathcal{G})$ for all $\zeta \geq \zeta_0$. $\square$

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